When Growth leads to Zero-sum Conflict*

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Abstract

In many settings, individuals face a tradeoff between cooperating to expand a collective good (a "collective pie") or competing to expand the share they own of that good. We study this tradeoff as a dynamic public goods problem where the size of the collective pie (and its shares) can only be gradually changed and contributions to the pie's growth are irreversible. Our main result is that the pie's growth eventually halts and zero-sum conflict between the agents ensues forever. In the essentially unique subgame perfect equilibrium, growth ultimately leads to a stage-game that is reminiscent of the prisoner's dilemma. However, unlike a repeated prisoner's dilemma, cooperation is unsustainable in our model for any discount factor. We also explore the relationship between growth and inequality in the agents' shares. We highlight the empirical relevance of our theory to factionalism in organizations, special interest politics, and market competition.

1 Introduction

In many settings, individuals face a tradeoff between pursuing cooperative actions that expand a collective good or competitive actions that expand the share they own of that good. This classical tradeoff is ever present. Members of an organization must choose between investing their time in expanding their organization's resources or trying to redirect existing resources toward their preferred use (Milgrom and Roberts, 1988; Milgrom, 1988). In an industry, firms may expand their industry's market as a whole (e.g., via industry-wide advertising) or compete to increase their market share. Organized interest groups may exert effort toward expanding production—"society's

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pie"—or try to shift a greater share of the existing pie toward their members (see, e.g., Olson, 1982).

Understanding when and why individuals cooperate to grow a collective good or when, instead, they pursue zero-sum conflict over its distribution is crucial for our understanding of the dynamics of organizations, markets, and political competition (Ali, Mihm and Siga, 2024; Bueno de Mesquita and Dziuda, 2023). Yet, the dynamics of growth and conflict are potentially complicated. In these settings, individuals' interactions are rarely one-shot; instead, individuals must repeatedly resolve this tradeoff as they interact in the same market place, organization, or economy. This gives rise to dynamic consequences: Choosing to expand one's share today means starting with a higher share tomorrow; expanding the collective good today leads to a larger good tomorrow. Therefore, present decisions may have long-term consequences for both the evolution of the collective good's size and its distribution. Furthermore, since individuals have long-term interests, there is scope for relational contracts to emerge whereby cooperation is sustained over time by the threat of (self-enforcing) punishment.

In this paper, we study this tradeoff as a dynamic public goods problem. We focus on settings that have two features common to the literature on public goods. First, individuals face constraints imposed by institutions or technology that allow only for *gradual* shifts in the shares and size of the collective good (Acharya and Ortner, 2022; Bowen, Chen and Eraslan, 2014; Bowen, Chen, Eraslan and Zápal, 2017). Second, contributions to expand the collective good are *irreversible* (Battaglini, Nunnari and Palfrey, 2014; Chen, Deng, Fujio and Khan, 2023). Irreversibility appears naturally in many settings. For example, useful ideas contributed by an organization's member cannot be unlearned. A firm's advertisement that makes a new segment of the population aware of the industry's existence can not be easily reversed.

In our benchmark model, two forward-looking agents own complementary shares of a pie, and each period they simultaneously choose whether to expand the pie or expand their share of it. Agents' actions have constant and additive effects on the size of the pie and the shares. In particular, the pie only grows if some agent contributes to it and each agent's contribution adds a fixed amount to the pie; thus, if both agents contribute to it, the pie grows by twice as much as if only one contributes. Similarly, if only one agent attempts to increase her share, it increases by a fixed amount. But if both (or neither) attempt to increase their share, there is a stalemate and the agents' shares remain unchanged. This means that an agent, if she so chooses, can always defend her current share.

Two key forces underscore our model and are derived from the agents' myopic (stage game) incentives. First, an agent's incentive to expand the pie is increasing in her share of it (à la Olson, 1971). Second, her incentive to expand her share is increasing in the size of the pie. Hence, an agent is more tempted to expand her share when the pie is large or her share is small. With myopic agents, a sufficiently large pie leads to perpetual conflict: Both agents fight for their share, producing a stalemate that stops growth. Yet, one may conjecture that cooperation can be sustained when agents are forward-looking and interacting repeatedly.

Our main result shows that this conjecture is wrong: Growth eventually halts and zero-sum conflict follows thereafter. Furthermore, the size of the pie for which conflict ensues is independent of the agents' discount factor (and, hence, coincides with the agents' myopic behavior). To see why, suppose we try to sustain a path of growth via grim-trigger strategies: If any agent deviates to increase her share, the other will punish her by attempting to increase her share, and growth will cease thereafter. By the ensuing stalemate over the shares of the pie, a deviating agent will still guarantee herself the payoff that she appropriated in her deviation. The breakdown of cooperation then follows from the fact that, since the pie grows with constant returns, the total future growth of the pie is independent of its current size. However, each agent's gain from deviating to increase her share naturally increases in the pie's size—in turn, there exists a size of the pie for which agents deviate.[1] The fact that cooperation eventually halts has an unraveling effect. As a result, forward-looking agents are unable to grow the pie beyond what myopic agents can achieve. More generally, all dynamics in our model are unaffected by the agents' discount factors.

Our results on the dynamics of growth and conflict before growth halts connect inequality both to the speed and limit of growth. If agents have equal initial shares, both contribute to expand the pie and eventually stop at the same time. If they have different initial shares, the agent with the smallest share stops cooperating first. This causes the speed of growth to slow and the initial inequality in the agents' shares to decline: The agent who owns a larger share keeps the pie growing while the agent with a smaller share free rides and increases her own share at the expense of the large-share agent. However, because the share of the large-share agent is gradually decreased, this also decreases the large-share agent's incentive to contribute to expanding the pie and, eventually, leads the large-share agent to also compete to increase (or defend) her share of the pie. Perhaps interestingly, this initial inequality leads the pie to grow to a larger long-run size than it would otherwise be attained if the agents had equal shares. Hence, more persistent growth requires initial inequality but growth also gradually erodes inequality. Our dynamics of growth and conflict resemble key observations in the seminal works of Olson (1971) and Hirshleifer (1991). In collective action problems, there is a "tendency for 'exploitation' of the great by the small" (Olson, 1971) and, in

[1] In Section 6, we discuss the robustness of this result to relaxations of key assumptions, including: the possibility that the size of pie depreciates or conflict over the shares destroys part of the pie, and endowing the agents with asymmetric discount factors.

power struggles, "poorer or smaller combatants often end up improving their position relative to richer or larger ones" (Hirshleifer, 1991).

In our framework, initial conditions matter. More extreme initial levels of inequality are more likely to have a persistent effect, which consists of greater long-run inequality but also a greater long-run size of the pie. However, this does not mean that extreme initial inequality is necessarily the utilitarian optimum since, as a result of the "tendency for 'exploitation' of the great by the small," more unequal initial conditions translate into equilibrium paths where only one agent contributes to the collective good, implying slower growth rates.

Institutions and the broader technological environment play an important role in enabling, or hindering, cooperation. Consider the example of market competition and firms' advertising decisions mentioned earlier. The effectiveness of the firms to expand the market or their share might depend on their ability to micro-target new customers outside of the market or customers of their competitor. New technologies, access to richer data, or changes in privacy regulation may affect one or both of these. Similarly, in organizations and politics, interest groups' effectiveness in redirecting resources toward their own interests might depend on institutional structures, such as the stringency of checks and balances. Holding all else equal, if institutions or technology are such that it becomes easier for agents to expand their share or harder to expand the pie, then cooperation becomes harder to sustain, conflict ensues earlier, and the long-run size of the pie is smaller.[2] This result provides insight into when and why demand for more stringent regulations or institutions (e.g., checks and balances) might arise. Our framework also highlights how sophisticated institutions that limit agents' abilities to expand their share as the pie increases may be necessary for sustaining cooperation and growth.

Our setting connects to two classical games: The prisoner's dilemma and the centipede game (Rosenthal, 1981). When the pie has grown enough, the stage game that agents face is a prisoner's dilemma: Each agent's dominant strategy in the stage game is to expand her share but both agents would be better off if they expanded the pie. Therefore, the dynamics of growth in our model imply that regardless of initial conditions, agents end up eventually playing a repeated prisoner's dilemma at perpetuity. However, unlike the folk theorems for repeated games (Friedman, 1971), cooperation cannot be sustained because cooperation leads to a larger pie, which increases the attractiveness of defection and this, in turn, has an unraveling effect that makes cooperation impossible. In this sense, the logic of conflict is reminiscent of a centipede game. Yet, in our model, the unraveling argument of the centipede game

[2] Naturally, changes that make it easier for agents to expand their share might also have spillover effects that make it easier to expand the pie. So long as the spillovers are not too large, the same conclusion holds.

applies only partially: When the pie is small, agents are still able to cooperate and grow the pie despite the anticipation of future conflict.

Finally, we emphasize the empirical relevance of our results by focusing on three applications. First, we consider the emergence of factionalism in social movement organizations. We show that the literatures in organizational behavior and sociology provide support for our underlying mechanism and that a larger pie—more organizational resources—may induce conflict in organizations. Second, we apply our theory to market competition and firms' decisions to invest in industry-wide ("generic") or firm-specific ("branded") advertising, where the former aims to expand market demand for the entire industry and the latter aims to steal, or defend, market share from competitors. In this context, we show that our key predictions for growth and conflict are supported by a range of anecdotal and empirical evidence. We also explain how our mechanism highlights new policy implications of federal laws that mandate industry-wide advertising—in particular, how such laws may inadvertently increase market concentration. Last, we highlight how our theory is consistent with observations made by Olson (1982), which connect the existence of special interest groups (so called "distributional coalitions") to a country's poorer economic performance.

The remainder of the paper is structured as follows. Subsection 1.1 reviews the related literature. Section 2 presents the benchmark model. Section 3 establishes our core result that growth eventually halts. Section 4 characterizes the dynamics of growth and conflict. Section 5 discusses the empirical relevance of our theory with applications to organizations, politics, and markets. Section 6 discusses some relaxations of key modeling choices. Section 7 concludes the paper. All proofs are deferred to the appendix.

1.1 Related literature

We study a classical tradeoff between expanding the pie or expanding one's share of it. This tradeoff appears in Olson (1982) but he does not develop a formal model nor analyze its dynamic consequences. Since Olson, economists have formalized this tradeoff in a variety contexts but have largely approached the problem by considering static settings, the steady states of dynamic systems, dynamic settings with myopic agents, or dynamic settings that prohibit history-dependent strategies (see, e.g., Eggert et al., 2011; Gonzalez, 2007; Gustafsson et al., 2020; Maxwell and Reuveny, 2005; Skaperdas, 1992).[3] We complement this literature by analyzing this classical tradeoff in a dynamic setting with forward-looking agents and allowing for history-dependent strategies.

[3]Our tradeoff differs from the (related) "guns and butter" tradeoff (Grossman and Kim, 1995; Hirshleifer, 1995; Skaperdas and Syropoulos, 2001) because our setting features a collective good: Agents are able to exert effort toward growing a pie that simultaneously benefits both agents.

We connect to the literature on dynamic public goods problem. As is common in this literature, our public good is subject to irreversibility (Battaglini et al., 2014; Chen et al., 2023; Lockwood and Thomas, 2002; Marx and Matthews, 2000; Matthews, 2013; Parihar, 2024).[4] and gradualism. Previous work has focused on the emergence of gradualism in public goods problems (and bargaining) as a means to achieving more efficient outcomes (see, e.g., Admati and Perry, 1991; Compte and Jehiel, 2004). Others have focused on how the irreversibility of a public good contribution can in of itself generate gradualism (Lockwood and Thomas, 2002). We contrast this literature by imposing gradualism as a constraint (similar to Acharya and Ortner (2022)), which captures frictions generated by institutions, technology, or consumer behavior. In addition, in our paper, the public good (the size of the pie) is irreversible but, at the same time, the benefit an agent derives from the public good can be gradually changed—in this sense, our public good has a fungible private component. This latter feature contrasts with the voluntary contributions literature. Our results also differ in important ways from those Lockwood and Thomas (2002) and Compte and Jehiel (2004). These papers illustrate how gradualism is needed to achieve more efficient outcomes and, as the agents become more patient, the inefficiency induced by gradualism decreases. In contrast, gradualism does not allow for more efficient outcomes in our framework; instead, gradualism is barrier: if the agents were able to grow the pie by a larger amount, then they would obtain a more efficient outcome. Furthermore, due to the unraveling effect that emerges in equilibrium, the inefficiency that occurs in equilibrium is independent of the agents' discount factors and, hence, cannot be resolved by more patient agents.

In the context of (dynamic) veto bargaining, a common explanation for inefficient outcomes is the *status quo effect:* bargaining outcomes that are achieved today can increase agents' reservation values in the future (Acharya and Ortner, 2022; Compte and Jehiel, 2004; Duggan and Kalandrakis, 2012; Dziuda and Loeper, 2016; Kalandrakis, 2004).[5] Our model also features a *status quo effect* in the form of "gradualism" whereby only small changes in the size of the pie and its shares can be achieved each period. However, unlike the veto bargaining literature, our agents are not endowed with veto power and, furthermore, there are no synergies in their actions. This implies that the net effect of an agent's action is independent of the other's action. Perhaps closer to our paper is Acharya and Ortner (2022), which studies a setting where two agents engage in veto bargaining with policies arriving stochastically and there is an exogenously-

[4]Irreversibility is also a common assumption in environmental contexts, where the pie corresponds to the amount of resources or the level of environmental exploitation (Harstad, 2023). For such contexts, it may be natural to modify our model to impose an exogenous bound on the maximum level of the pie (e.g., equal to the maximum amount of resources that could plausibly be exploited). Our results on the dynamics of growth and conflict are unchanged when such a bound is imposed (see, Remark 1).

[5]Offering a different perspective, Bowen et al. (2014) show that the status quo effect can improve efficiency (see also, Bowen et al., 2017). For a recent survey, see Eraslan et al. (2022).

assumed (and commonly known) Pareto frontier. Our framework poses an interesting contrast because—despite the potential for unbounded growth—in equilibrium, what would appear as a Pareto frontier emerges endogenously.

Finally, our theory speaks to the emergence of zero-sum competition in markets, politics, and organizations. Recent work has focused on the role of information asymmetries in generating zero-sum competition. Bueno de Mesquita and Dziuda (2023) analyze a dynamic model of politics where the underlying prevalence of common-value issues is known to politicians but unknown to voters. Bueno de Mesquita and Dziuda show the inevitability of "partisan traps," whereby political competition centers around zero-sum (partisan) issues. Politicians exploit their informational advantage to make voters more pessimistic about the prevalence of common-value issues, inducing them to behave as partisans—ultimately, leading to a zero-sum trap. Similarly, Ali, Mihm and Siga (2024) illustrates how an ex-ante Pareto-efficient policy may be rejected by a majority of voters because of the uncertainty about how its zero-sum benefits are distributed. Ash, Morelli and Van Weelden (2017) illustrate how a politician's desire to signal her type to voters can lead her to (inefficiently) allocate resources toward divisive, zero-sum policy issues instead of common-value ones. Our theory, in contrast, features no informational asymmetry.[6] We offer a complementary perspective in which the emergence and inevitability of zero-sum competition is a product of the growth of the collective good itself.

2 Benchmark model

Time is discrete and infinite, $t \in \{1, 2, \ldots\}$. There are two agents, A and B, who own shares s_t and $1 - s_t$, respectively, of a pie π_t. The (exogenous) initial shares and pie are s_0 and π_0. The pie and each agent's share constitute a two-dimensional state of the world $(s_t, \pi_t) \in [0, 1] \times [\pi_0, \infty)$.

At the start of each period t, the agents simultaneously choose a binary action: $a_A^t \in \{0, 1\}$ and $a_B^t \in \{0, 1\}$. These actions $a_t := (a_A^t, a_B^t)$ together with the previous period's state (s_{t-1}, π_{t-1}) determine period-t state, (s_t, π_t). By choosing $a_j^t = 1$ an agent $j \in \{A, B\}$ increases the pie by $\omega > 0$, while choosing $a_j^t = 0$ is an attempt to increase her share by $\kappa > 0$, where $\kappa = 2^{-m} > 0$ for some $m \in \mathbb{N}$. Formally, the states of the world evolve according to the law of motion:

$$s_t = \max\left\{0, \min\left\{1, s_{t-1} + \kappa(a_B^t - a_A^t)\right\}\right\} \qquad \text{and} \qquad \pi_t = \pi_{t-1} + \omega(a_A^t + a_B^t).$$

[6] In a different tradition, Pei (2023) has shown the impossibility of cooperation in a repeated prisoner's dilemma if agents are sufficiently patient and have control over their publicly observable record of behavior.

The initial state of the world is (s_0, π_0), where $s_0 = \ell\kappa$ for some $\ell \in \{1, \ldots, 2^m - 1\}$ and $\pi_0 > 0$. To avoid issues of indifference, we assume π_0 is not a multiple of ω.

Payoffs. Agent A and B's per-period payoffs are $u_A(s_t, \pi_t) := s_t\pi_t$ and $u_B(s_t, \pi_t) := (1 - s_t)\pi_t$, respectively. The agents discount future payoffs at a rate of $\beta \in (0, 1)$; hence, agent j's payoff is $\sum_{t=1}^{\infty} \beta^{t-1} u_j(s_t, \pi_t)$ for $j \in \{A, B\}$.

Equilibrium concept. Our equilibrium concept is pure-strategy Subgame Perfect equilibrium (equilibrium hereafter). A strategy for $j \in \{A, B\}$ is a mapping $a_j(s_{t-1}, \pi_{t-1}, h_{t-1})$ from every feasible state (s_{t-1}, π_{t-1}) and history of past actions h_{t-1} into a binary decision $a_j^t \in \{0, 1\}$. We denote the agents' strategy profile by $\sigma = (a_A, a_B)$ and an equilibrium by $\sigma^* = (a_A^*, a_B^*)$. Given a strategy profile σ, we define agent j's continuation payoff from a state $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with history h_{t-1} as

$$V_j(s, \pi \mid \sigma, h_{t-1}) := \mathbb{E}\Big[\sum_{t'=t}^{\infty} \beta^{t'-t} u_j(s_{t'}, \pi_{t'})\Big| \sigma, s, \pi, h_{t-1}\Big],$$

where $(s_{t'}, \pi_{t'})$ and $h_{t'}$ are determined by σ and the law of motion for every period $t' \geq t$.

3 Growth eventually halts

In this section, we show that there exists a limit to the growth of the pie. We begin with two auxiliary lemmas that highlight the key forces that ultimately limit growth. First, consider the payoff an agent obtains if *perpetual conflict* occurs: Both agents perpetually attempt to increase their shares—guaranteeing that the pie never grows and each agent's share remains fixed. Lemma 1 says that, in every equilibrium, an agent's continuation payoff must be at least as high as her payoff from this perpetual conflict.

Lemma 1 (Perpetual conflict as a lower bound on payoffs.) *Let the state be* $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ *with some history* h_{t-1}*. In every equilibrium,* σ^**, each agent's continuation payoff is bounded below by the continuation payoff of perpetual conflict:*

$$V_A(s, \pi \mid \sigma^*, h_{t-1}) \geq \frac{s\pi}{1-\beta} \quad \textit{and} \quad V_B(s, \pi \mid \sigma^*, h_{t-1}) \geq \frac{(1-s)\pi}{1-\beta}. \tag{1}$$

Since each agent's impact on the share is symmetric, an agent's choice to increase her share has a defensive effect: She ensures that her share does not decrease regardless of the other agent's action. At the same time, growth is irreversible. Therefore, an agent's continuation payoff must not be less than maintaining her current share and the current size of the pie for every future period, which precisely corresponds to the perpetual conflict payoff.

Second, consider the payoff an agent obtains from the opposite extreme: *perpetual cooperation*, whereby in every period the agents choose to expand the pie and thus, the pie grows at its maximum speed: 2ω per period. This maximizes the joint continuation payoffs of the agents—the "surplus" generated by the agents—which is simply the discounted sum of the size of the future pies (since the agents own complementary shares). Intuitively, in every equilibrium, this surplus is bounded above by the surplus generated by perpetual cooperation. Yet, because there are constant returns to cooperation, the surplus generated by perpetual cooperation is decreasing as a fraction of the pie's current size. Lemma 2 formalizes these results.

Lemma 2 (Perpetual cooperation as an upper bound on payoffs.) *Let the state be* $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ *with some history* h_{t-1}*. In every equilibrium,* σ^**, the sum of agents' continuation payoffs is bounded above by the continuation payoff of perpetual cooperation:*

$$V_A(s, \pi \mid \sigma^*, h_{t-1}) + V_B(s, \pi \mid \sigma^*, h_{t-1}) = \sum_{t'=t}^{\infty} \beta^{t'-t} \pi^*_{t'} \leq \frac{\pi}{1-\beta} + \frac{2\omega}{(1-\beta)^2}. \tag{2}$$

Moreover, the surplus from perpetual cooperation is finite and decreasing as a fraction the pie's current size.

Lemmas 1 and 2 present two forces that act against cooperation: that agents are able to defend their existing shares of the pie and the surplus generated by perpetual cooperation is bounded and decreasing as fraction of the pie's size. Consider now, for example, a strategy profile whereby the agents perpetually cooperate, so the surplus generated by the agents is the right hand side of (2). If an agent, say agent A, instead decides to deviate and increase her share, she free rides on the other agent's choice to expand the pie and benefits immediately from a larger share of the now larger pie: $(s + \kappa)(\pi + \omega)$. By Lemma 1, she can (at minimum) retain this same benefit in every future period. Applying the same logic to agent B, it follows that sustaining perpetual cooperation requires a sufficiently large surplus to be available:

$$\begin{aligned} V_A(s, \pi \mid \sigma^*, h_{t-1}) + V_B(s, \pi \mid \sigma^*, h_{t-1}) &\geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta} + \frac{(1-s+\kappa)(\pi+\omega)}{1-\beta} \\ &= \frac{(1+2\kappa)(\pi+\omega)}{1-\beta}. \end{aligned}$$

However, as the pie grows, even the surplus generated by perpetual cooperation—the maximum possible surplus—is insufficient: Cooperation cannot be sustained. Intuitively, when the pie is large enough, since there are constant returns to cooperation, perpetual cooperation generates only an arbitrarily small surplus relative to what perpetual conflict generates. At the same time, the incentive to deviate increases as the pie grows, and eventually some agent will prefer not to cooperate.

Lemma 3 formalizes the above intuition and implications. It says that if each agent has a non-unit share (i.e., $s \neq 0, 1$), then growth eventually halts: For a sufficiently large pie, in every equilibrium, the agents engage in perpetual (zero-sum) conflict and the size of the pie and the agents' shares remain fixed.[7]

Lemma 3 (Limited growth.) *There exists $\bar{\pi}(\kappa, \omega, \beta)$ such that if the state is $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with $\pi > \bar{\pi}(\kappa, \omega, \beta)$, then, in every equilibrium, conflict ensues and growth halts for interior shares, i.e., if $s \notin \{0, 1\}$, then $(s^*_{t'}, \pi^*_{t'}) = (s, \pi)$ for every $t' \geq t$;*

We have seen that conflict ensues when the pie is sufficiently large. However, Lemma 3 and the above discussion suggest that the agents' discount factor plays an important role in determining the limit of growth; indeed, the surplus generated by perpetual cooperation is larger when the agents are more patient. Therefore, one may expect that even if growth must eventually halt, a more patient agent can be incentivized to cooperate for larger pies through the promise of the other's future cooperation. Proposition 1 establishes that this is not the case: The limit of growth is independent of the agents' discount factors.

Proposition 1 (The unraveling effect of limited growth.) *Consider the state be $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that*

$$\pi > \frac{\omega}{\kappa}s - \omega \text{ and } \pi > \frac{\omega}{\kappa}(1-s) - \omega \tag{3}$$

and any history h_{t-1}. In every equilibrium, σ^*, conflict ensues and growth halts for interior shares, that is, if $s \notin \{0, 1\}$, then $(s^*_{t'}, \pi^*_{t'}) = (s, \pi)$ for every $t' \geq t$.

Therefore, the limit of growth depends only on the agents' shares, s, and the extent to which they can expand their share, κ, or the pie, ω, per period. The reason why is that the limits of growth posed in Lemma 3 have an unraveling effect in equilibrium. If

[7]The case where an agent owns a non-interior share (i.e., $s \in \{0, 1\}$) is an edge case that does not arise on the equilibrium path (we show this formally in Section 4). For completeness, Corollary A.1 in Appendix A, states the possible equilibrium outcomes for these subgames. In some equilibria of these subgames, the pie may continue to grow; however, in every equilibrium, the agents' shares remain fixed. It follows because, when one agent has a unit share, the other agent (who has a zero share) is indifferent between having a zero share of the current pie and having a zero share of a larger pie. This allows for the existence of an equilibrium whereby the unit-share agent expands her share (to no effect) and the zero-share agent grows the pie unless the unit-share agent grows the pie, in which case the zero-share agent reverts to expanding her share forever. Yet, this scenario never arises on the equilibrium path since an agent with non-zero share always has a strict incentive to avoid subgames where her share is zero. The possibility of growth in this edge case (and our results more generally), do not rely on the possibility that agents can have zero, or unit, shares. Our essentially unique equilibrium (to be presented in Proposition 2) remains unchanged if agents' shares are bounded between $\underline{s}$ and $1 - \underline{s}$ for some $\underline{s} > 0$ sufficiently small.

agents know that cooperation is impossible tomorrow, they will only cooperate today if it is in their myopic interest. Iterating on this logic, by backward induction shows that the limit of growth unravels to the largest pie for which there exist myopic incentives for growth—which is a bound smaller than the one in Lemma 3. Since this frontier is exclusively determined by myopic incentives, it is independent of agents' discount factor.

This logic is reminiscent of the centipede game, in which two agents fail to grow a common pie because each is afraid of the other appropriating it. In the centipede game, the expectation of the other agent's selfish behavior also triggers an unraveling effect making cooperation altogether impossible. However, unlike the centipede game, in our model agents accrue payoffs in each period—hence, temporary cooperation can offer both agents benefits—and an agent may be better off cooperating today even if she expects that in the future the other agent will selfishly deviate. These crucial differences result from the gradualism with which shares change and give cooperation a greater chance of being sustained in our setting.

It is interesting to note that at the point at which growth ceases, the agents' stage game payoffs correspond to a *prisoners' dilemma*: In the stage game, both agents' dominant strategy is to increase their share but they would be strictly better off if both expanded the pie. More generally, at any point where cooperation to expand the pie fails, the stage game is a prisoners' dilemma. Yet, unlike a repeated prisoners' dilemma, this is a dynamic game: Cooperation generates a larger pie, which in turn increases agents' incentives to defect. This difference makes cooperation impossible, regardless of the agents' discount factors. This conclusion is perhaps surprising given that our equilibrium concept allows for history-dependant strategies, so agents could potentially sustain cooperation through relational incentives (see, e.g. Friedman, 1971).[8]

Proposition 1 leaves open a number of questions: Can growth be achieved when the pie is small? How does the size of the pie and the agents' shares evolve in the face of limited growth? What is the relationship between the inequality of agents' shares and the long-run size of the pie? We turn to these questions in the next section.

4 Dynamics of growth and conflict and its consequences

We now characterize the equilibrium dynamics of growth and conflict. These dynamics are underscored by a simple logic: An agent's marginal benefit from increasing her share is (roughly) in proportion to the size of the pie, and her marginal benefit from increasing the pie is (roughly) in proportion to the size of her share. This suggests that a small pie is more likely to incentivize agents to expand the pie; while a large pie is

[8]Related folk theorems for dynamic games, such as those in Dutta (1995), do not apply to our setting because of the irreversibility constraint.

more likely to incentive agents to expand their shares—inducing perpetual zero-sum conflict. Conversely, it also suggests that, for any given size of the pie, the agent with the smallest share will have the greatest incentive to expand her share.

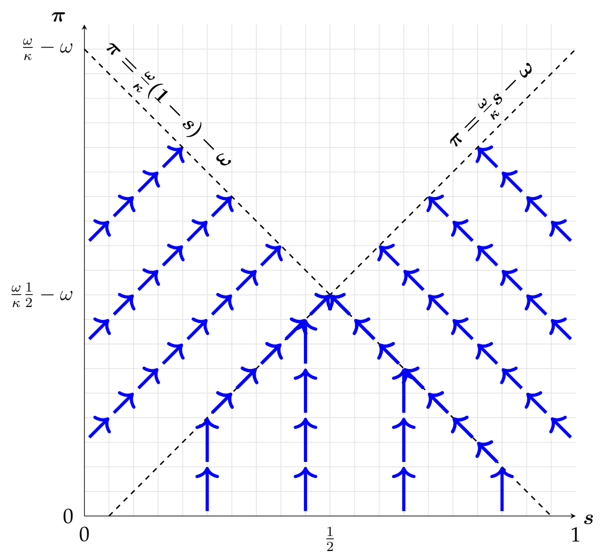

Figure 1: The dynamics induced by equilibrium behavior with respect to (s, π).

Proposition 2 says that there exists an essentially unique equilibrium that is Markovian (history-independent); it is depicted in Figure 1. The dynamics induced by the equilibrium have two key features that follow from the logic described above:

(i) When the agents' initial shares are not too unequal (and the initial pie is not too large), both agents initially cooperate to expand the pie. However, as the pie grows, cooperation ceases: The agent with the smaller share chooses to expand her share, while the agent with the larger share continues to expand the pie. This implies that the state evolves along a diagonal in the (s, π) space: The pie expands (albeit at a slower rate) and agents' shares gradually equalize to $s_t = \frac{1}{2}$. Once any initial inequality has been removed, perpetual zero-sum conflict ensues and growth halts with the long-run size of the pie attaining a minimal value.

(ii) When the agents' initial shares are sufficiently unequal (and the pie is not too large), the agents never cooperate. The agent with the smaller share immediately chooses to expand her share, and the agent with the larger initial share expands the pie. Again, this process temporarily leads to an expanding pie and a gradually more equal distribution of shares between the agents. But once the pie has grown sufficiently, perpetual conflict ensues. However, at this point—and unlike in case (i)—the agents' shares are not equalized, the initial inequality persists (although reduced) and the long-run size of the pie is larger than the minimal value attained in (i).

Proposition 2 (Dynamics of growth and conflict.) *There exists a Subgame Perfect equilibrium σ^* whereby actions are Markovian on and off the equilibrium path as follows: For any state $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ and any history h_{t-1}*

$$a_A^*(s, \pi, h_{t-1}) = \begin{cases} 0 & \text{if } \pi > \frac{\omega}{\kappa}s - \omega \\ 1 & \text{if } \pi < \frac{\omega}{\kappa}s - \omega. \end{cases} \text{ and } a_B^*(s, \pi, h_{t-1}) = \begin{cases} 0 & \text{if } \pi > \frac{\omega}{\kappa}(1-s) - \omega \\ 1 & \text{if } \pi < \frac{\omega}{\kappa}(1-s) - \omega. \end{cases} \tag{4}$$

Furthermore, this Subgame Perfect equilibrium is essentially unique: For any initial state (s_0, π_0) with $s_0 \in (0, 1)$, the state evolves according to a unique equilibrium path, $\{(s_t^*, \pi_t^*)\}_{t=0}^{\infty}$, as described by (4).

The dynamics exhibited by Proposition 2 could be mistaken as diminishing marginal returns: In equilibrium, as the pie grows, the amount by which it grows is (weakly) decreasing, and similarly, as an agent's share increases, she is less likely to further expand her share. Yet, in our setting, there are constant returns to both actions.[9] Our dynamics are, instead, endogenously driven by the payoff complementarity between an agent's share and the size of the pie.

As in Proposition 1, the equilibrium dynamics in Proposition 2 are independent of the agents' discount factors and follow precisely the (Markovian) dynamics that would arise if the agents were myopic. In equilibrium, therefore, the agents' dynamic concerns play no role. Intuitively, this feature arises via an unraveling argument. The inevitability of perpetual conflict (recall Proposition 1) makes it impossible for the agents to develop a relational contract that improves over their myopically preferred strategies.

Our model allows for the possibility of an unbounded growth of the pie. However, in some settings, it may be natural to consider some upper limit, say $\bar{\pi}$, to the size of the pie. For example, the upper bound may represent the optimal level of growth after which further growth is detrimental or the (finite) amount of resources available; alternatively, π_t may map into the probability for which agents will capture some fixed prize and this probability is bounded by $\bar{\pi} = 1$. The equilibrium behavior and dynamics presented in Proposition 1 are unaffected if the maximum size of the pie is bounded. Intuitively, a bounded pie necessarily means that growth must halt—in particular, when $\pi_t = \bar{\pi}$, perpetual conflict must ensue. But then an unraveling argument (per the results presented in Section 3) applies: In any subgame that could lead to $\pi_{t+1} = \bar{\pi}$, the agents anticipate that perpetual conflict will ensue if the pie is expanded and, in turn, resort to their myopic incentives. The remainder of the argument then follows identically to that of the benchmark model.

[9]If there are diminishing marginal returns to agents' actions, our results are strengthened—more limited growth and earlier conflict. We return to these assumptions in Section 6.

Remark 1 (Extension with a bounded pie.) *Suppose the pie is bounded above by some (exogenous) value $\bar{\pi} \geq 1$; i.e., the law of motion of π_t is $\pi_t = \min\{\bar{\pi}, \pi_{t-1} + \omega(a_A^t + a_B^t)\}$. Then Proposition 1 holds verbatim.*

An exogenous upper bound on the pie, $\bar{\pi}$, may also naturally arise in settings where the collective good, π, is, in fact, a collective bad, such as a pile of trash or pollution. The agents' actions would then correspond to a decision to reduce the pile of trash (reducing the collective bad) or attempting to shift some share of the trash toward the other agent. Such a model is isomorphic to our model[10] and, hence, Remark 1 can be applied.

4.1 Inequality and the long-run size of the pie

We now turn to the consequences of the equilibrium dynamics for long-run outcomes and the optimal level of (in)equality. Corollary 1 summarizes the implications of the dynamics of growth and conflict (Proposition 2) for the relationship between initial levels of inequality and long-run outcomes. The corollary focuses on cases where the initial size of the pie is intermediate. When the initial inequality of shares is low, initial conditions do not matter: In the long-run, there is perfect equality and the long-run pie attains its minimum long-run value of (approximately) $\frac{\omega}{\kappa}\frac{1}{2} - \omega$. However, when initial inequality is high, initial conditions matter: In the long-run, inequality persists and the long-run pie is greater than $\frac{\omega}{\kappa}\frac{1}{2} - \omega$. Further, both the long-run inequality and the long-run pie are increasing in the initial inequality.[11]

Corollary 1 (Initial conditions, long-run inequality, and the long-run pie.) *Suppose the initial size of the pie is intermediate: $\frac{\omega}{\kappa}(1-3\kappa) - \omega < \pi_0 < \frac{\omega}{\kappa}\frac{1}{2} - \omega$. Only when initial inequality is large enough, the initial conditions, (s_0, π_0), matter for long-run outcomes, $(s_\infty^*, \pi_\infty^*)$. In particular:*

(i) *If $\left|s_0 - \frac{1}{2}\right| < \frac{1}{2} - (\pi_0 + \omega)\frac{\kappa}{\omega}$, then initial conditions do not matter. There is long-run equality $s_\infty = \frac{1}{2}$, and the long-run pie is minimized, $\pi_\infty^* \in (\frac{\omega}{\kappa}\frac{1}{2} - \omega, \frac{\omega}{\kappa}\frac{1}{2} + \omega)$.*

(ii) *Otherwise, initial conditions matter. Inequality is persistent, long-run inequality is increasing in the initial inequality, and the long-run pie is increasing in the initial inequality: $s_\infty^* \gtrless \frac{1}{2} \iff s_0 \gtrless \frac{1}{2}$; and $\left|s_\infty^* - \frac{1}{2}\right|$ and $\pi_\infty^* > \frac{\omega}{\kappa}\frac{1}{2} + \omega$ are increasing in $\left|s_0 - \frac{1}{2}\right|$.*

[10] To be more precise, the agents' per period payoffs should be rewritten as $u_A(s_t, \pi_t) = \bar{\pi} - \pi_t s_t$ and $u_B(s_t, \pi_t) = \bar{\pi} - \pi_t(1 - s_t)$ and the law of motion for the agents' shares rewritten as $s_t = \max\{0, \min\{1, s_{t-1} - \kappa(a_B^t - a_A^t)\}\}$.

[11] When the initial size of the pie is large, there is no growth for intermediate ranges of initial inequality—the initial conditions are the same as the long-run outcomes. When the initial size of the pie is sufficiently small, all levels of initial inequality with $s_0 \in \{\kappa, \ldots, 1-\kappa\}$ lead to long-run equality $s_\infty = \frac{1}{2}$ and a minimally-sized long-run pie $\pi_\infty^* \in (\frac{\omega}{\kappa}\frac{1}{2} - \omega, \frac{\omega}{\kappa}\frac{1}{2} + \omega)$.

Although greater initial inequality is needed for a larger long-run pie, it does not follow that inequality is optimal in the utilitarian sense. This is because when the initial size of the pie is intermediate there is a tradeoff between growing faster or growing into a larger long-run pie. When there is little inequality, the pie develops quickly toward its minimal long-run size; when there is extreme inequality, the pie develops slowly toward its maximal long-run size.[12] Therefore, the utilitarian optimal level of inequality depends on the agents' discount factor: when the agents are more patient, maximal inequality is optimal; when the agents are impatient, minimal inequality is optimal. Proposition 3 formalizes this result. The proof appears in Supplemental Appendix S.4.

Proposition 3 (Utilitarian-optimal inequality.) *Suppose the initial size of the pie is intermediate: $\frac{\omega}{\kappa}(1-3\kappa)-\omega<\pi_0<\frac{\omega}{\kappa}\frac{1}{2}-\kappa$.*

(i) *If the agents are sufficiently impatient (β sufficiently close to 0), the utilitarian-optimal initial condition features minimal inequality: $s_0=\frac{1}{2}$ or $s_0=\frac{1}{2}\pm\kappa$.*

(ii) *If the agents are sufficiently patient (β sufficiently close to 1), the utilitarian-optimal initial condition features maximal inequality: $s_0=\kappa$ or $s_0=1-\kappa$.*

Part (ii) of Corollary 1 and Proposition 3 suggest that when agents are sufficiently patient, they may have an incentive to agree to redistribute their initial shares increasing inequality as a way so to induce a larger long-run size of the pie and higher utilitarian payoff. For example, for a fixed initial size of the pie, the small-share agent may agree to redistribute part of her share to the large-share agent. However, in the absence of transferable utility, the small-share agent would not agree to such a redistribution plan. Intuitively, the plan hurts the small-share agent in the short-run because she must sacrifice part of their share. Therefore, the agreement could only possibly be reached if the long-run state induced by greater inequality is beneficial to the small-share agent, but this is not the case. Greater inequality induces a long-run state that lies higher along the diagonal illustrated in Figure 1 (and per Proposition 2). However, moving the state to a higher point along this diagonal hurts the small-share agent: they strictly prefer lower states along this diagonal (and the high-share agent strictly prefers the opposite). Thus, there is no hope for a redistributive agreement.

4.2 Institutions and technology

So far, we have fixed agents' effectiveness at expanding the pie or their shares. In this section, we explore the implications of changes in the agents' effectiveness: ω and κ.

[12]When the initial size of the pie is large or small, this tradeoff does not arise. For a large initial pie, greater inequality is the utilitarian optimum (since there is no growth for low levels of inequality). For a small initial pie, minimal inequality is optimal because greater inequality does not generate a larger long-run pie and leads to slower growth—this is partly due to the coarseness of our state space.

These parameters may naturally differ between contexts and depend on institutions and the broader technological environment. Consider the example of market competition and firms' advertising decisions mentioned in the introduction. The effectiveness of the firms to expand the market or their share might depend on their ability to micro-target new customers outside of the market or customers of their competitor. New technologies, access to richer data, or changes in privacy regulation may affect one or both of these.[13]

Agent's incentives depend on their effectiveness at each action. Intuitively, as the agents become more effective at expanding their shares, they are also less willing to expand the pie, and the perpetual-conflict region becomes larger (i.e., as κ increases, the upper triangle in Figure 1 expands). In turn, the long-run size of the pie is smaller and long-run inequality is larger. The reverse holds when the agents become more effective at growing the pie. Corollary 2 states this result and highlights an important role of institutions and technology in fostering greater growth and reducing conflict.

Corollary 2 (More effective agents.) *As κ increases or ω decreases,*

(i) *the set of parameters that induce perpetual conflict (weakly) increases in the set-inclusion order;*

(ii) *for any initial condition, (s_0, π_0), the long-run inequality (weakly) increases, and the long-run size of the pie (weakly) decreases.*

We can apply this result to contexts in which reducing agents' effectiveness in expanding their share has spillover effects such that agents are also less effective at growing the pie. For example, in the context of legislative bargaining, stronger checks and balances and constraints on legislative majorities may limit politicians' ability to achieve partisan as well as bipartisan reforms (Acharya and Ortner, 2022; Dziuda and Loeper, 2016; Gratton and Morelli, 2022; Lee, 2022). Formally, these spillovers can be analyzed by taking the agent's effectiveness at expanding the pie to be an increasing function of their ability to expand their share: $\omega := H(\kappa)$, where $H(\kappa) > 0$ and $H'(\kappa) \geq 0$ for all κ. Intuitively, as long as the spillover effect, $H'(\kappa)$, is not too large, Corollary 2 will continue to hold. This suggests that fostering greater growth requires institutions that limit agents' effectiveness in expanding their shares but are also carefully crafted as to avoid stifling agents' abilities to grow the pie.

Finally, Corollary 2 provides some insights into when and why agents may demand—and even agree—on reforms that modify their ability to expand their shares. For example, consider an instance where the state is such that, in equilibrium, agents

[13] Similarly, in organizations and politics, interest groups' effectiveness in redirecting resources toward their own interests might depend on institutional structures, such as the stringency of checks and balances.

engage in perpetual conflict. In such cases, a reform that reduces their effectiveness expanding their shares may facilitate a (temporary) end of conflict and a larger pie. Such a reform constitutes a Pareto improvement so, if available, is likely to be agreed to by the agents. Applied to the context of democratic politics, this suggests that countries with higher levels of development and state capacity may be more likely to strengthen checks and balances.[14] Now, instead, consider states such that, in equilibrium, the agents are not in perpetual conflict (i.e., the size of the pie is not too large). In such cases, it is possible that reducing the agents' abilities to expand their shares is not Pareto optimal and, hence, is less likely to be demanded by both agents. Returning again to the context of democratic politics, checks and balances are less likely to be strengthened in countries with lower levels of development and state capacity.

5 Empirical relevance

Our theory points toward a type of organizational "resource curse," whereby the prevalence (or sudden influx) of resources produces an inefficient halt in cooperation and the emergence of zero-sum conflict. Our resource curse stems neither from agents being short-sighted (or subjected to turnover) nor from decreasing returns in the technology used to expand the pie. Instead, our resource curse is grounded in the simple fact that avoiding conflict becomes harder as growth increases the victor's spoils.[15] Indeed, our theory also highlights that the temptation for an agent to stop cooperating is strongest for the agent with a smaller share, since she has relatively more to gain from expanding her own share of the existing pie compared to expanding the pie as a whole.

Our organizational resource curse resonates with several applications. In the following subsections, we explore three applications: factionalism within social movement organizations, competition in markets, and distributive conflict within economies.

5.1 Factionalism in social movement organizations

Within a social movement organization (and sometimes between social movement organizations), there is often a collective goal that unites its members, e.g., progress toward social justice or environmental goals. Nonetheless, organization scholars and sociologists widely agree that, with progress and growth, *factionalism* arises, leading to internal conflict over "movement resources, direction, and media attention" (Kretschmer, 2024). Indeed, in his seminal book Gamson (1975) refers to factionalism as the "nature of

[14]We share this feature with Karakas (2017).

[15]A related dynamic appears in Acemoglu et al. (2008) where, as the economy grows, a politician's temptation to steal production also grows. In this electoral accountability setting, voters avoid this "resource curse" by providing relational incentives to the politician.

the beast" and Kretschmer (2024) concludes that factionalism is "probably inevitable." As in our theory, factionalism also imposes an opportunity cost as "resources are channeled into internal disputes rather than toward pressing the [organization's] demands" (Balser, 1997).

Our theory highlights that, as a social movement organization becomes more prominent and accumulates more resources and power, factionalism will arise. This is consistent with Gamson's and Kretschmer's observations that factionalism is inevitable. Further, in our theory it is the expansion of resources itself that generates factionalism and conflict. This mechanism, which runs contrary to some other arguments, is nevertheless supported by existing work.[16] First, in summarizing the literature, Kretschmer (2024) writes "evidence across a variety of movements shows us that in boom times, when movements are popular, public participation is high, and resources are flowing in, factionalism [...] within and between groups is likely to be high." Kretschmer (2013) hints at our mechanism by asking "is factionalism more likely in prosperous economic times because there are more resources to fight over?" For evidence, Kretschmer points to Balser's (1997) case study of the SNCC (Student Nonviolent Coordinating Committee) in the United States. Balser documents how the influx of new resources from the Kennedy administration in 1961 "instigated a serious conflict" between members with competing views over the organization's direction. More systematic empirical studies also come to a similar conclusion. Soule and King (2008) study competition between social movement organizations within the same "social movement industry" (i.e., sharing a common goal). They analyze the organizational tactics adopted by protest organizations supporting peace, women's, and environmental movements in New York State between 1960 and 1986. Contrary to their own hypothesis, they document that tactics associated with competition over resources are more likely when "resources are prevalent." Although the causes of factionalism are surely multi-faceted, our mechanism appears empirically relevant across a variety of settings within and between social movement organizations.

5.2 Competition in growing markets

In market competition, firms face a tradeoff between taking actions that expand the market as a whole—benefiting all firms in the industry—and actions that expand their market share—at the expense of their competitors. For example, and as mentioned in the introduction, firms can choose between engaging in industry-wide ("generic") advertising or in firm-specific ("branded") advertising. Industry-wide advertising aims

[16]Other arguments include that organization growth often entails a more diverse body of members or that, once an organization comes close to achieving its goals, it can struggle to reorient and establish consensus for a new collective goal (see, e.g., case studies and cited literature within Balser, 1997; Miller, 1999).

to expand the market for the entire industry, so it is cooperative in nature, generating a collective good that benefits all firms in the industry.[17] In contrast, firm-specific advertising is competitive and primarily aims to steal, or defend, market share from competing firms in the same industry without affecting the size of the market as a whole (Bass et al., 2005; Krishnamurthy, 2000; Ward et al., 1985).

In this setting, our theory makes three key predictions that are supported by anecdotal and empirical evidence. First, we predict that firms will collectively underinvest in industry-wide advertising. Indeed, this is a key argument used by proponents of the many federal laws that require firms to fund industry-wide advertising campaigns (Messer et al., 2008; Varian, 2006). As put by Ward (2006),

> "Removing potential free-riders and creating a pool of funds earmarked for generic advertising messages is precisely the intent of the national legislation for supporting commodity checkoff programs and an important objective of many federal and state marketing orders."

The claim that, absent mandatory requirements, firms underinvest in industry-wide advertising is also supported by a range of empirical and experimental studies (Depken et al., 2002; Liaukonyte et al., 2015; Messer et al., 2008; Tchumtchoua and Cotterill, 2010).

Second, our theory predicts that firms are more likely to engage in industry-wide advertising when the industry's market is relatively small but will switch to firm-specific advertising as the market grows. Indeed, when federal law does not require industry-wide advertising, marketing scholars often describe industry-wide advertising as a feature of emerging markets and firm-specific advertising as a feature of mature markets (Beard, 2010, 2011; Yoo and Mandhachitara, 2003). Quoting a contemporary source, Beard's (2011) historical analysis describes the steamship industry's strategy to engage in industry-wide advertising in the early 1900s:

> "It dawned upon both lines that it was work meet [i.e., suitable or fitting] for even two advertisers to educate the local public to travel more by water. The patronage was not fixed; it could be increased as the public came to understand the pleasures and the benefits of a trip by steamer (Pickett 1910, 29)."

Beard explains that similar attitudes appeared in industries where there was "near-unanimous confidence in market expansion." Yet, these cooperative behaviors eventually end. For example, in the early 1980s, the fast food industry matured and firms

[17] In some industries, the amount of expenditure on industry-wide advertising is large and significant. As described by Chakravarti and Janiszewski (2004), "In 2002 alone, the annual domestic expenditure on generic advertising for cheese, beef, and Florida orange juice was $47 million, $45 million, and $24 million, respectively. By comparison, advertising budgets for Kraft cheese, Hormel Foods fresh meat products, and Tropicana orange juice were approximately $26.6 million, $2.6 million, and $32.4 million, respectively."

switched to combative and firm-specific advertising. Again quoting a contemporary source, Beard explains the switch:

> "fast-food marketers are dealing with slow-growth industry, where market share gains are the driving force behind any expansion (Kreisman and Mashall, 1982, 1)."

Another example comes from the cigarette industry. Tremblay and Tremblay (2012) notes that between 1914 and 1940, the market was "taking off" and cigarette advertising was mostly constructive (increasing market demand for all firms). However, in the 1960s, advertising turned combative (attempting to "steal customers from another firm").

Third, our theory predicts that, as a market matures, firms with smaller market shares are more likely to engage in firm-specific advertising—free riding off the (temporarily) more cooperative advertising made by large market share firms. Suggestive evidence for this prediction comes from the observation that, when federal laws mandate producers to contribute to industry-wide advertising campaigns, large firms appear content with the mandatory programs whereas small firms tend to complain (Hamilton et al., 2013; Ward, 2006; Zheng et al., 2010). Indeed, in some cases, small firms are formally exempt from the mandatory program—in effect, allowing them to free ride on the contributions of larger firms (Zheng et al., 2010, p. 752). In other cases, when voluntary industry-wide advertising programs have broken down or never existed at all, it has been observed that larger firms may nonetheless continue to fund industry-wide advertising regardless of other firms' decisions (see, e.g., Section 6 of Krishnamurthy, 2000, for examples from the life insurance and butter industry).

Beyond being consistent with the above stylized facts, we provide an additional insight that—to the best of our knowledge—is novel to the literature. Our theory suggests that federal laws that mandate firms to contribute to industry-wide advertising may be a double-edged sword. On the one hand, and as is well-established, such mandatory programs allow for a larger expansion of markets than would otherwise be achievable. But on the other hand, these programs are likely to insulate larger firms from having their market shares eroded over time by smaller firms' competitive and firm-specific advertising.[18] Thus, mandatory programs pose a dynamic tradeoff between efficiency (a larger market) and market concentration.

5.3 Distributive conflict and economic growth

At a macroeconomic level, our theory connects to a long-running debate about the sources of economic growth. Olson (1982) describes a framework that shares many

[18] Although it has been noted in literature that these mandatory programs have distributional (and possibly unequal) consequences on small and large firms (see, e.g., Liaukonyte et al., 2015; Zheng et al., 2010), the focus has not been on the evolution of market concentration.

similarities with ours: Society is comprised of special interest groups ("distributional coalitions") that can benefit their members by either "making the pie the society produces larger, so that its members would get larger slices even with the same shares as before, or alternatively by obtaining larger shares or slices of the social pie." Olson argues that distributional coalitions are costly for economic growth (society's pie) as they direct resources toward distributive conflict and away from efficient uses. Indeed, Olson documents that: mature societies have slower economic growth and more special interest groups than less mature societies, and these special interest groups are "overwhelmingly orientated to struggles over the distribution of income and wealth rather than to the production of additional output." Olson aimed to explain the "economic miracles" of Germany and Japan after World War II. Differently from us, his explanation focused on periods of economic and political stability, which—he argues—allows distributional coalitions to emerge, and not on growth per se.[19]

Our theory also speaks to the dynamic competition between special interest groups. It predicts that smaller interest groups will be initially more active (and, hence, effective) in distributive conflict. Furthermore, this asymmetry will generate greater distributional conflict in the long run as larger interest groups will gradually lose incentives to expand society's pie and, hence, will also pursue distributional conflict.

6 Discussion: robustness and the way out

Before concluding the paper, we return to our core result, namely, that growth eventually halts. We discuss now the role played by key assumptions in our benchmark model and explore extensions of the benchmark to illustrate the robustness of this result when certain assumptions are relaxed. In doing so, we also provide insights into institutional solutions will allow for continual growth—a way out of limited growth.

Asymmetric discounting. Our benchmark model assumes that agents discount time in the same way. But if agents have asymmetric discount factors, our key result that growth eventually halts continues to hold. We analyze this extension formally in Supplemental Appendix S.5. Furthermore, we show that—as in the benchmark model—the point at which growth halts is independent of either agent's discount factor. It is straightforward to see then that all the dynamics of our benchmark model are identical in this extension.[20] Thus, whether one agent is more patient than the other has no bearing on the dynamics of growth and conflict—contrasting Rubinstein (1982), where

[19] There is debate surrounding the explanatory power of Olson's argument (see, e.g., Coates and Heckelman, 2003; Knack, 2003; Mokyr and Nye, 2007; Weede, 1987), although recent work continues to support Olson's core predictions (see, e.g. Heckelman, 2007).

[20] In particular, the same proof arguments used to characterize the dynamics of growth and conflict in the benchmark model can be applied.

more patient agents extract a larger share of the surplus.

Exponential growth of pie. A key assumption of our benchmark model is that of constant returns: An agent's decision to expand the pie contributes a fixed amount ω, regardless of the size of the pie. Constant returns implies that, as the pie grows, the relative (discounted) payoff from continually growing the pie compared to pursuing conflict is decreasing. Hence, for a large enough pie, the agents prefer conflict. Naturally, and unsurprisingly, this result also holds if there are *decreasing returns* to the growth of the pie (for details, see Supplemental Appendix S.6).[21] With exponential returns, however, continual growth may be possible. We can formalize this intuition by assuming that the period-t pie is given by $\pi_t = \pi_{t-1}(1 + r(a_A^t + a_B^t))$ for $r > 0$. For a sufficiently small exponential growth rate r, growth will halt. However, for larger growth rates, it is possible for the agents to sustain an equilibrium with continual growth.[22]

The assumption of constant (or decreasing) returns is consistent with existing work on the dynamic provision of public goods. For example, Battaglini et al. (2014) effectively assumes decreasing returns since an agent's private consumption can be transformed into a public good contribution at a constant rate but the agents' benefit from the public good exhibits diminishing returns. Similarly, Marx and Matthews (2000) assume agents' benefit from the public good's level of provision is piece-wise linear with initially positive and constant marginal returns and then, above a certain threshold, zero marginal returns. More generally, a growing body of empirical research (see, e.g., Bloom et al., 2020) calls into question the exponential returns assumption that is traditionally present in growth models.

Ability to expand one's share as a function of the pie. Our benchmark model assumes that an agent's ability to expand her share is independent of the size of the pie. Naturally, our result—that growth eventually halts—continues to hold if an agent's ability to expand her share *increases* with the size of the pie.[23] If, instead, an agent's ability to expand her share *decreases* with the size of the pie, our result is robust if the rate of decrease is not too fast. Formally, we can modify the law of motion of the agents' shares so that κ is replaced with $\tilde{\kappa}(\pi)$, where $\tilde{\kappa}(\cdot)$ is a positive and decreasing function of π. If the agent's ability to expand their share does not decline too fast, then growth eventually halts for all (interior) distributions of shares. Formally, the condition we

[21]Formally, this would mean that an agent's decision to expand the pie contributes an amount $\tilde{\omega}(\pi)$, where $\tilde{\omega}(\cdot)$ is a positive and decreasing function of π.

[22]This result requires that the growth rate is also not too large since a sufficiently large growth rate leads to the agents' discounted payoffs being possibly unbounded. In such cases, the one-shot deviation principle does not apply due to a violation of the "continuity at infinity" condition and, hence, proving equilibrium existence becomes intractable.

[23]Formally, this would mean that, in the law of motion of s_t, κ is replaced with $\tilde{\kappa}(\pi)$, where $\tilde{\kappa}(\cdot)$ is a positive and increasing function of π.

require is that $\tilde{\kappa}(\pi)\pi$ is increasing and unbounded as π grows, e.g., $\tilde{\kappa}(\pi) = \frac{1}{\sqrt{\pi}}$ (for details, see Supplemental Appendix S.7). Conversely, this suggests that, in the absence of exponential growth, continual growth requires "sophisticated" institutions that increasingly—and quickly—constrain agents' abilities to expand their share as the pie grows.

Depreciation of the pie or destructive (negative-sum) conflict. In some settings, it may be natural to assume that, in the absence of growth, the size of pie decreases. This may arise because of depreciation or because of conflict being destructive: *negative-sum* conflict. Our benchmark model can be extended to incorporate this feature by assuming that, whenever conflict ensues, the period-t pie is $\pi_t = (1-\delta)\pi_{t-1}$, where $\delta \in (0,1)$ is the rate of depreciation (or destruction). If δ is not too large relative to the discount factor, growth eventually halts for all (interior) distributions of shares (for details, see Supplemental Appendix S.8). Consistent with the folk theorem results of Dutta (1995) for stochastic games, this result implies that there still exists an upper bound on the growth of the pie as long the agents are not too patient; however, in this extension, periods of growth and decline may still arise periodically. Conversely, when the level of depreciation (or destruction caused by conflict) is large, the folk theorems of Dutta (1995) will apply and growth need not halt. Thus, institutions that allow for more severe—and necessarily negative-sum—punishments can provide a way out of the growth trap presented in the benchmark model.

Continuous actions. Our benchmark model illustrates our tradeoff of interest in the simplest possible: agents face a binary choice between growing the pie or expanding their share. In some settings, it is more realistic that agents can flexibly choose how to allocate a budget of resources (or time) between the two actions, i.e., agents face a continuous action set. This would allow an agent to allocate some fraction of her budget toward growing the pie and the remaining fraction toward defending (or expanding) her share. By allocating some resources toward defending her share, she will limit the gain that the other agent would obtain from deviating and attempting to expand her share, thus making defection less attractive. A priori, this might suggest that the agents could escape our benchmark model's growth trap via continual cooperation but at decreasing levels. Via a simple example in Supplemental Appendix S.9 (and with a proof argument that resembles the one in the benchmark model), we show that cooperation of this form cannot be sustained in equilibrium.

7 Conclusion

We proposed a dynamic theory of growth and conflict with collective goods. Our focus centers on each agent's tradeoff between expanding a collective pie and attempting to expand her share of it. We showed that, as the pie grows, an individual's incentives to expand her share of the pie eventually dominates her incentive to expand the pie. Ultimately, this leads the pie's growth to halt and produces an (endogenous) upper bound on the pie's size. This bound on the growth of the pie is unaffected by the individuals' patience levels and persists despite the possibility of unbounded growth of the pie, the presence of constant returns of individuals' actions to expand the pie, and the possibility of relational contracts.

When the pie is not too large, growth is possible. We characterized the dynamics of growth and conflict prior to growth halting, illustrating how individuals' incentives to expand the pie depends on their shares and the pie's size. When the pie is small and individuals have relatively equal shares, both contribute to expanding the pie. However, when individuals have sufficiently unequal shares, as per Olson (1971), there is a "tendency for 'exploitation' of the great by the small." The individual with the largest share contributes to the pie's expansion, while the small-share individual not only free rides but also expands her own share. These dynamics connect inequality to the speed and limits of growth: Greater inequality in the shares slows down growth but leads to a larger long-run size of the pie—yet, growth also gradually reduces inequality. In evaluating the benefits and costs of inequality, our result highlights a novel tradeoff between the speed at which a collective good grows and the long-run size of such good.

A novel feature of our theory is that it provides a rationale for the ubiquity of the prisoners' dilemma. In our theory, given any initial condition, the pie and its share evolve along a path that leads to a prisoners' dilemma stage-game: Both agents would be better off expanding the pie but their (stage-game) dominant strategy is to attempt to expand their share. Furthermore, we shed light on why cooperation may be unsustainable even if agents are patient and repeatedly play a prisoners' dilemma. Since cooperation generates a larger pie, it leads to greater incentives for defection in the future. Anticipating the inevitability of future defection has an unraveling effect that makes even temporary cooperation impossible. Instead, zero-sum conflict emerges over the shares of the pie.

Although our theory finds support in empirical studies, its pessimistic conclusion—that growth eventually halts—gives an important role for institutions and technology. A solution to the halting of growth is that agents achieve exponential growth. However, as previously noted, a growing body of empirical research calls into question the realism of exponential growth (Bloom et al., 2020). Alternatively, institutions that make it more difficult for agents to expand their shares may facilitate greater (albeit still temporary) cooperation. Indeed, Olson (1982) provides suggestive evidence of such institutions in his cross-country assessment of economic outcomes and interest groups' prevalence. Similarly, in the context of market competition, the presence federal laws that mandate industry-wide advertising and their continued support on the side of producers suggest an important role for regulation. Nonetheless, as discussed in Section 6, *sustained* growth requires that each agent's effectiveness in expanding their share is not only limited but also declines rapidly as the pie grows. Therefore, growth requires "sophisticated" institutions (and regulation) that dynamically adjust to a changing environment.

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A Proofs from Section 3

Proof of Lemma 1. For sake of a contradiction, suppose there exists an equilibrium σ^* such that (1) does not hold. Without loss of generality, suppose $V_A(s,\pi \mid \sigma^*, h_{t-1}) < \frac{s\pi}{1-\beta}$ and consider A's strategy $a'_A(s_{t'}, \pi_{t'}, h_{t'}) = 0$ for all possible $s_{t'}, \pi_{t'}, h_{t'}$ and $t' \geq t$. Under a'_A, we have $s_{t'} \geq s$ and $\pi_{t'} \geq \pi$ for all $t' \geq t$. Therefore, A's payoff from a'_A is at least $\frac{s\pi}{1-\beta}$, which is strictly higher than the equilibrium payoff, $V_A(s,\pi \mid \sigma^*, h_{t-1})$. Thus, we have a contradiction: σ^* is not a Nash equilibrium of the subgame and, hence, is not subgame perfect. ■

Proof of Lemma 2. The agents' payoffs sum to π_t in every period t and, by the law of motion, $\pi_{t'} \leq \pi_{t'-1} + 2\omega$ for all t'. It is then immediate that the sum of the agents' equilibrium payoffs is:

$$V_A(s,\pi \mid \sigma^*) + V_B(s,\pi \mid \sigma^*) = \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \leq \sum_{j=0}^{\infty} \beta^j (\pi + 2(j+1)\omega) = \frac{\pi(1-\beta) + 2\omega}{(1-\beta)^2}.$$

■

Proof of Lemma 3. We prove the following (slightly stronger) corollary.

Corollary A.1 ***(Lemma 3 extended)*** *There exists $\bar{\pi}(\kappa,\omega,\beta)$ as defined in Lemma A.2 such that if $(s_{t-1}, \pi_{t-1}) = (s,\pi)$ with $\pi > \bar{\pi}(\kappa,\omega,\beta)$, then in every equilibrium: (i) if $s \notin \{0,1\}$, then $s_{t'} = s$ and $\pi_{t'} = \pi$ for all $t' \geq t$; (ii) if $s \in \{0,1\}$, then $s_{t'} = s$ and $\pi_{t'} \in \{\pi_{t'-1}, \pi_{t'-1} + \omega\}$ for all $t' \geq t$.*

The above corollary follows immediately from two auxiliary lemmas: Lemmas A.1 and A.2, which we prove below.

Lemma A.1 *Let σ^* be an equilibrium. There exists a threshold $\tilde{\pi}(\kappa,\omega,\beta)$ such that if $\pi_{t-1} > \tilde{\pi}(\kappa,\omega,\beta)$, then it cannot be that $a^*_A(s_{t-1}, \pi_{t-1}, h_{t-1}) = a^*_B(s_{t-1}, \pi_{t-1}, h_{t-1}) = 1$ for any history h_{t-1}.*

Proof. For sake of a contradiction, suppose there exists an equilibrium σ^* such that, for some $(s_{t-1}, \pi_{t-1}) = (s,\pi)$ with some history h_{t-1} and π arbitrarily large, $a^*_A(s,\pi,h_{t-1}) = a^*_B(s,\pi,h_{t-1}) = 1$. We consider 2 cases.

Case 1: Suppose $s \in \{\kappa, \ldots, 1-\kappa\}$. Agent A's payoff from deviating is

$$(s+\kappa)(\pi+\omega) + \beta V_A(s+\kappa, \pi+\omega \mid \sigma^*) \geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta},$$

where the inequality follows from Lemma 1. Hence, a necessary condition for σ^* to be an equilibrium is

$$V_A(s,\pi \mid \sigma^*) \geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta}. \tag{A.1}$$

A similar argument for Agent B gives the necessary condition:

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1 - s + \kappa)(\pi + \omega)}{1 - \beta}. \tag{A.2}$$

Therefore, to sustain equilibrium σ^*, it must be that the sum of the RHS of (A.1) and (A.2), which yields $\frac{(\pi+\omega)(1+2\kappa)}{1-\beta}$, is weakly less than the maximum discounted sum of payoffs given by the RHS of (2) in Lemma 2—otherwise, at least one of the agents would have an incentive to deviate. That is, we require

$$\frac{(\pi + \omega)(1 + 2\kappa)}{1 - \beta} \leq \frac{\pi(1 - \beta) + 2\omega}{(1 - \beta)^2} \iff 2\kappa\pi + \omega(1 + 2\kappa) - \frac{2\omega}{1 - \beta} \leq 0,$$

which does not hold if π is sufficiently large. Let $\tilde{\pi}_1(\kappa, \omega, \beta)$ denote the first value for which the above inequality fails.

Case 2: Suppose $s \in \{0, 1\}$. We focus on $s = 0$ (the argument for $s = 1$ is similar). By the same argument as in Case 1, we have that Inequality (A.1) must hold with $s = 0$ and, by Lemma 1, Agent B's equilibrium payoff at $(0, \pi)$ must satisfy

$$V_B(0, \pi \mid \sigma^*) \geq \frac{(\pi + 2\omega)}{1 - \beta}. \tag{A.3}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (A.1) at $s = 0$ and (A.3), which yields $\frac{\kappa(\pi+\omega)+(\pi+2\omega)}{1-\beta}$, is weakly less than the maximum discounted sum of payoffs given by the RHS of (2) in Lemma 2—otherwise, at least one of the agents would have an incentive to deviate. That is, we require

$$\frac{\kappa(\pi + \omega) + (\pi + 2\omega)}{1 - \beta} \leq \frac{\pi(1 - \beta) + 2\omega}{(1 - \beta)^2} \iff \kappa\pi + \omega(\kappa + 2) - \frac{2\omega}{(1 - \beta)} \leq 0$$

which does not hold if π is sufficiently large. Let $\tilde{\pi}_2(\kappa, \omega, \beta)$ denote the first value for which the above inequality fails.

By combining the bounds from Case 1 and Case 2, i.e., $\tilde{\pi}(\kappa, \omega, \beta) := max\{\tilde{\pi}_1(\kappa, \omega, \beta), \tilde{\pi}_2(\kappa, \omega, \beta)\}$, and taking $\pi > \tilde{\pi}(\kappa, \omega, \beta)$, we obtain our desired contradiction. ■

Lemma A.2 *There exists a threshold $\bar{\pi}(\kappa, \omega, \beta)$ such that if $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with $\pi > \bar{\pi}(\kappa, \omega, \beta)$, then there is no equilibrium σ^* such that any of the following is true: (i) for some $s \in \{0, \ldots, 1 - \kappa\}$ and history h_{t-1}, $a^*_B(s, \pi, h_{t-1}) = 1$; (ii) for some $s \in \{\kappa, \ldots, 1\}$ and history h_{t-1}, $a^*_A(s, \pi, h_{t-1}) = 1$.*

Proof. We will prove the statement in Part (i)—the proof of Part (ii) is similar. For sake of a contradiction, let the state be $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with $s \in \{0, \ldots, 1 - \kappa\}$, and suppose that an equilibrium σ^* exists such that, for π arbitrarily large and some history h_{t-1}, $a_B^*(s, \pi, h_{t-1}) = 1$. By Lemma A.1, for any state (s_{t-1}, π_{t-1}) with $\pi_{t-1} > \tilde{\pi}(\kappa, \omega, \beta)$, it cannot be that $a_A^*(s_{t-1}, \pi_{t-1}, h_{t-1}) = a_B^*(s_{t-1}, \pi_{t-1}, h_{t-1}) = 1$. Therefore, taking $\pi > \tilde{\pi}(\kappa, \omega, \beta)$, σ^* must be such that at most one agent chooses action $a_j^{t'} = 1$ and, hence, $\pi_{t'} \leq \pi_{t'-1} + \omega$ for all $t' \geq t$. Using a similar argument as in the proof of Lemma 2, it follows that

$$V_A(s, \pi \mid \sigma^*) + V_B(s, \pi \mid \sigma^*) \leq \sum_{j=0}^{\infty} \beta^j (\pi + (j+1)\omega) = \frac{\pi(1-\beta) + \omega}{(1-\beta)^2}. \tag{A.4}$$

By construction of σ^*, $a_A^*(s, \pi, h) = 0$ and $a_B^*(s, \pi, h) = 1$. Thus, using Lemma 1, Agent A's equilibrium payoff can be bounded from below as follows:

$$V_A(s, \pi \mid \sigma^*) = (s+\kappa)(\pi+\omega) + \beta V_A(s+\kappa, \pi+\omega \mid \sigma^*) \geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta}. \tag{A.5}$$

Now consider Agent B's payoff from deviating: $(1-s)\pi + \beta V_B^*(s, \pi \mid \sigma^*) \geq \frac{(1-s)\pi}{1-\beta}$, where the inequality follows from Lemma 1. Hence, a necessary condition for σ^* to be an equilibrium is

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1-s)\pi}{1-\beta}. \tag{A.6}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (A.5) and (A.6), which yields $\frac{(s+\kappa)(\pi+\omega)+(1-s)\pi}{1-\beta}$, is weakly less than (A.4). That is, we require

$$\frac{(s+\kappa)(\pi+\omega)+(1-s)\pi}{1-\beta} \leq \frac{\pi(1-\beta)+\omega}{(1-\beta)^2} \iff \kappa\pi + \omega(s+\kappa) \leq \frac{\omega}{(1-\beta)},$$

which does not hold if π is sufficiently large. Let $\tilde{\pi}'(\kappa, \omega, \beta)$ denote the first value such that, for all $s \in \{0, 1, \ldots, 1-\kappa\}$, the above inequality does not hold. Taking π to be greater than both $\tilde{\pi}'(\kappa, \omega, \beta)$ and $\tilde{\pi}(\kappa, \omega, \beta)$ (as given in Lemma A.1) delivers a contradiction and completes the proof. ■ ■

Proof of Proposition 1. Proposition 1 is implied by Proposition 2, which is proven without reference to Proposition 1. For completeness, we note that a direct proof of Proposition 1 can be obtained by combining Lemmas S.1.1 and B.1. ■

B Proof of (essential) uniqueness in Proposition 2

The proof of essential uniqueness follows from Lemmas B.1, B.2, and B.3, which we present and prove below. Supplemental Appendix S.2 includes auxiliary lemmas used in the proof arguments of this appendix.

Lemma B.1 *There exists an equilibrium σ^* whereby for state $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that*

$$\pi > \frac{\omega}{\kappa}s - \omega \quad \textit{and} \quad \pi > \frac{\omega}{\kappa}(1-s) - \omega, \tag{B.1}$$

conflict occurs: $a^*_A(s, \pi, h_{t-1}) = a^*_B(s, \pi, h_{t-1}) = 0$. *In any equilibrium:*

(i) *for any* $s \in \{\kappa, \dots, 1-\kappa\}$ *such that* (B.1) *holds, the equilibrium behavior and, hence, equilibrium path is unique with* $s_{t'} = s$ *and* $\pi_{t'} = \pi$ *for any* $t' \geq t$.

(ii) *for any* $s \in \{0, 1\}$ *such that* (B.1) *holds, the equilibrium path is such that* $s_{t'} = s$ *for any* $t' \geq t$.

Proof. Existence is proven in Supplemental Appendix S.1. We now prove Parts (i) and (ii) of the lemma via an induction argument. First we introduce some notation. Let $\bar{\pi}$ be the smallest value such that $\bar{\pi} = \pi_0 + j\omega$ for some positive integer j and $\bar{\pi} > \bar{\pi}(\kappa, \omega, \beta)$, where $\bar{\pi}(\kappa, \omega, \beta)$ is as defined in Corollary A.1 (equivalently, Lemma 3).

Our inductive argument parameterizes π as $\pi = \bar{\pi} - m\omega$, where m is an integer. By Corollary A.1, Parts (i) and (ii) of the lemma statement hold for all $m \leq 0$. Now, our inductive assumption is that there exists an integer m such that the lemma holds for any (s, π) satisfying (B.1) and $\pi = \bar{\pi} - m'\omega$ for integer $m' \leq m - 1$.

Consider the case of $\pi = \bar{\pi} - m\omega$ for arbitrary $m > 0$. Suppose $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that $s = 0$ (the case of $s = 1$ is similar and, hence, omitted). For sake of a contradiction, suppose there exists an equilibrium such that $s_{t'} \neq 0$ for some $t' \geq t$. If $s_{t'} \neq 0$, it must be that at some $\tilde{t} \in \{t, \dots, t'\}$, $a^{\tilde{t}}_A = 0$ and $a^{\tilde{t}}_B = 1$ — let $\tilde{t}$ be the smallest such value. Given the inductive argument, in this equilibrium, B's payoff from period $\tilde{t}$ onward is $\frac{(1-\kappa)(\pi+\omega)}{1-\beta}$. But if B deviated to $a^{\tilde{t}}_B = 0$, then, by Lemma 1, she would obtain a payoff of at least $\frac{\pi}{1-\beta}$, which is strictly higher than her equilibrium payoff because (B.1) holds. Thus, we have a contradiction.

Now suppose $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that $s \in \{\kappa, \dots, 1-\kappa\}$. Hence, for all possible actions of either agent at period t, we have $s_t \in \{s-\kappa, s, s+\kappa\} \subseteq [0, 1]$. We explore now A's best response to each of B's possible strategies at (s, π). Suppose $a^*_B(s, \bar{\pi} - m\omega, h_{t-1}) = 0$. By the inductive argument, A's payoff from $a^t_A = 1$ is $\frac{(s-\kappa)(\pi+\omega)}{1-\beta}$ and A's payoff from $a^t_A = 0$ is $s\pi + \beta V_A(s, \pi \mid \sigma^*, h_t) \geq \frac{s\pi}{1-\beta}$, where the inequality follows from Lemma 1. Therefore, $a^*_A(s, \bar{\pi} - m\omega, h_{t-1}) = 0$ is the unique best response to $a^*_B(s, \pi, h_{t-1}) = 0$ if $\frac{s\pi}{1-\beta} > \frac{(s-\kappa)(\pi+\omega)}{1-\beta} \iff \pi > \frac{\omega}{\kappa}s - \omega$, which holds for all (s, π) satisfying (B.1). Now suppose, instead, that $a^*_B(s, \bar{\pi} - m\omega, h_{t-1}) = 1$. By the inductive argument, A's payoff from $a^t_A = 1$ is $\frac{s(\pi+2\omega)}{1-\beta}$ and, by Lemma 1, A's payoff from $a^t_A = 0$ is at least as large as $\frac{(s+\kappa)(\pi+\omega)}{1-\beta}$. Therefore, $a^*_A(s, \bar{\pi} - m\omega, h_{t-1}) = 0$ is the unique best response to $a^*_B(s, \pi, h_{t-1}) = 1$ if $\frac{(s+\kappa)(\pi+\omega)}{1-\beta} > \frac{s(\pi+2\omega)}{1-\beta} \iff \pi > \frac{\omega}{\kappa}s - \omega$, which holds for all (s, π) satisfying (B.1). By a similar argument, we conclude that $a^*_B(s, \bar{\pi} - m\omega, h_{t-1}) = 0$. Thus, in any equilibrium, Parts (i) and (ii) of the lemma hold. ■

Lemma B.2 *There exists an equilibrium σ^* whereby for any state $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that*

$$\pi > \frac{\omega}{\kappa} s - \omega \quad \textit{and} \quad \pi < \frac{\omega}{\kappa}(1-s) - \omega, \tag{B.2}$$

$a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 1$. And symmetrically, if $\pi < \frac{\omega}{\kappa} s - \omega$ and $\pi > \frac{\omega}{\kappa}(1-s) - \omega$, then $a_A^*(s, \pi, h_{t-1}) = 1$ and $a_B^*(s, \pi, h_{t-1}) = 0$. Furthermore, the equilibrium behavior and, hence, equilibrium path is unique at any such state with $s \in \{\kappa, \dots, 1-\kappa\}$.

Proof. Existence has been proven in Supplemental Appendix S.1. In what follows, σ^* is taken to be a strategy profile satisfying Lemma B.1, and the state $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ satisfies (B.2). The proof of the symmetric case is similar and, hence, omitted. For a given (s, π), we define $\bar{m}$ to be the smallest integer-value $m = \bar{m}$ such that

$$\pi + m\omega > \frac{\omega}{\kappa}(1-s) - \omega. \tag{B.3}$$

We will often utilize the observation that (B.3) holds at $m = \bar{m}$ if and only if

$$\pi + (\bar{m} - m')\omega > \frac{\omega}{\kappa}(1 - (s + m'\kappa)) - \omega \qquad \forall m' \in \{0, 1, \dots, \bar{m}\}. \tag{B.4}$$

We will prove the lemma via a 2-dimensional induction argument where a generic state (s, π) is parameterized by $(\ell, \bar{m}) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $s = \ell\kappa$ and π is such that $\bar{m} \geq 1$, where $\bar{m}$ is defined in (B.3). Abusing notation slightly, given another generic state (s', π') we will denote the corresponding parameterization by the pair $(\ell', \bar{m}')$. We proceed with the following steps for (s, π) satisfying (B.2):

Step 1. For $\ell = 1$ and $\bar{m} = 1$, the unique equilibrium action is $a_A^t = 0$ and $a_B^t = 1$.

Step 2. For $\ell > 1$ and $\bar{m} = 1$, the unique equilibrium action is $a_A^t = 0$ and $a_B^t = 1$.

Step 3. For $\ell = 1$ and $\bar{m} = 2$, the unique equilibrium action is $a_A^t = 0$ and $a_B^t = 1$.

Step 4. For $\ell > 1$ and $\bar{m} = 2$, the unique equilibrium action is $a_A^t = 0$ and $a_B^t = 1$.

Step 5. For $\ell = 1$ and $\bar{m} > 2$, the unique equilibrium action is $a_A^t = 0$ and $a_B^t = 1$.

Step 6. For $\ell > 1$ and $\bar{m} > 2$, the unique equilibrium action is $a_A^t = 0$ and $a_B^t = 1$.

Step 1: $\bar{\ell} = 1$ and $\bar{m} = 1$. Let (s, π) be such that (B.2) holds, $\ell = 1$, and $\bar{m} = 1$ (recall that $\bar{m}$ is defined as per (B.3)). We consider three alternative strategies at period $t-1$ and derive a contradiction in each case, which allows us to conclude that the only possible equilibrium behavior is $a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 1$.

(a) Suppose there exists an equilibrium σ^* such that $a_A^*(s, \pi, h_{t-1}) = 1$ and $a_B^*(s, \pi, h_{t-1}) = 0$. Then, the state evolves to $(s_t, \pi_t) = (0, \pi + \omega)$. By Lemma S.2.1, the state must (eventually) evolve for some $t' \geq t+1$ to $(s_{t'}, \pi_{t'}) = (0, \pi + 2\omega)$, $(0, \pi + 3\omega)$, or $(\kappa, \pi + 2\omega)$. In the first two cases (and by Lemma B.1, Part (ii)), in equilibrium,

$s_{\tilde{t}} = 0$ at perpetuity and, hence, A obtains a future discounted payoff stream equal to zero. This can't be an equilibrium since, by deviating to $a_A^t = 0$, A obtains a strictly positive payoff. Therefore, the third case must hold: in equilibrium the state must (eventually) evolve to $(s_{t'}, \pi_{t'}) = (\kappa, \pi + 2\omega)$ for some $t' \geq t + 1$. But by Lemma S.2.3, it must be that $t' = t + 1$ and then, by Lemma S.2.2, we have a contradiction.

(b) Suppose $a_A^*(s, \pi, h_{t-1}) = 1$ and $a_B^*(s, \pi, h_{t-1}) = 1$. By Lemma S.2.4, this can't occur in equilibrium.

(c) Suppose $a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 0$. By Lemma S.2.1, there exists some future time period $t' \geq t + 1$ such that, in equilibrium, $a_j^{t'} = 1$ for some $j \in \{A, B\}$. Let t' be the smallest such value. Given Parts (a) and (b) above, it follows that $(a_A^{t'}, a_B^{t'}) = (0, 1)$ and hence, $(s_{t'}, \pi_{t'}) = (s + \kappa, \pi + \omega)$. Applying Lemma S.2.3 then gives the desired contradiction.

Step 2: $\ell > 1$ and $\bar{m} = 1$. Let (s, π) be such that (B.2) holds, $\ell > 1$, and $\bar{m} = 1$. Our inductive assumption is that the lemma statement holds for any (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $1 \leq \ell' < \ell$ and $\bar{m}' = 1$. We consider three alternative strategies at period $t - 1$ and derive a contradiction in each case, which allows us to conclude that $a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 1$.

(a) Suppose $a_A^*(s, \pi, h_{t-1}) = 1$ and $a_B^*(s, \pi, h_{t-1}) = 0$. By Lemma S.2.2, this can't occur in equilibrium.

(b) and (c) Repeating the arguments of Step 1(b) and Step 1(c), neither $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$ or $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 0$ can be an equilibrium.

Step 3: $\ell = 1$ and $\bar{m} = 2$. Let (s, π) be such that (B.2) holds, $\ell = 1$, and $\bar{m} = 2$. Our inductive assumption is that the lemma statement holds for any (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $\ell' \geq 1$ and $\bar{m}' = 1$. We consider three alternative strategies at period $t - 1$ and derive a contradiction in each case, which allows us to conclude that $a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 1$.

(a) Suppose $a_A^*(s, \pi, h_{t-1}) = 1$ and $a_B^*(s, \pi, h_{t-1}) = 0$. Then, the state evolves to $(s_t, \pi_t) = (0, \pi + \omega)$. And by Lemma S.2.1 the state must (eventually) evolve to $(0, \pi + 3\omega)$, $(\kappa, \pi + 2\omega)$, or $(s_{t'}, \pi_{t'}) = (0, \pi + 2\omega)$ for some $t' \geq t + 1$. Let t' be the smallest such value.

We consider each of these cases separately. Suppose $(s_{t'}, \pi_{t'}) = (0, \pi + 3\omega)$. Because $\bar{m} = 2$ and applying Lemma B.1, in equilibrium A obtains payoff equal to zero in every period. However, by deviating to $a_A^t = 0$, A obtains a strictly positive payoff. Therefore, this first case cannot occur. Now suppose $(s_{t'}, \pi_{t'}) = (\kappa, \pi + 2\omega)$. By Lemma S.2.3, it must be that $t' = t + 1$, but then Lemma S.2.2 leads to a contradiction.

Therefore, the third case $(s_{t'}, \pi_{t'}) = (0, \pi + 2\omega)$ must hold. For $a^*_A(s, \pi, h_{t-1}) = 1$ and $a^*_B(s, \pi, h_{t-1}) = 0$ to be an equilibrium, it must be that A obtains strictly positive payoff in equilibrium; therefore, from $(s_{t'}, \pi_{t'})$ it must be that the state (eventually) evolves to $(\kappa, \pi + 3\omega)$ in some period $\tilde{t} \geq t' + 1$. However, by applying Lemma S.2.3, we can conclude that it must be that the state evolves *immediately* from $(s_{t'}, \pi_{t'})$ to $(s_{t'+1}, \pi_{t'+1}) = (\kappa, \pi + 3\omega)$, i.e., $\tilde{t} = t' + 1$. Now consider period t' at which the state evolves from $(0, \pi + \omega)$ to $(s_{t'}, \pi_{t'}) = (0, \pi + 2\omega)$. At this point it must be that $(a^{t'}_A, a^{t'}_B) = (1, 0)$. Using Lemma B.1 and because $\bar{m} = 2$, in equilibrium, B's payoff is

$$\pi + 2\omega + \beta \frac{(1-\kappa)(\pi + 3\omega)}{1 - \beta}.$$

However, if B deviates to $a^{t'}_B = 1$, then the state evolves to $(s_{t'}, \pi_{t'}) = (0, \pi + 3\omega)$ and, by Lemma 1, they obtain payoff at least $\frac{\pi+3\omega}{1-\beta}$, which is strictly higher than their equilibrium payoff—a contradiction.

(b) and (c) Repeating the arguments of Step 1(b) and Step 1(c), neither $a^*_A(s, \pi, h_{t-1}) = a^*_B(s, \pi, h_{t-1}) = 1$ or $a^*_A(s, \pi, h_{t-1}) = a^*_B(s, \pi, h_{t-1}) = 0$ can be an equilibrium.

Step 4: $\ell > 1$ and $\bar{m} = 2$. Let (s, π) be such that (B.2) holds, $\ell > 1$, and $\bar{m} = 2$. Our inductive assumptions are two-fold:

(i) the lemma statement holds for any (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $\ell' \geq 1$ and $\bar{m}' = 1$;

(ii) the lemma statement holds for any (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $1 \leq \ell' \leq \ell - 1$ and $\bar{m}' = 2$.

The arguments used for Step 2 apply verbatim here; thus, the only possible equilibrium behavior is $a^*_A(s, \pi, h_{t-1}) = 0$ and $a^*_B(s, \pi, h_{t-1}) = 1$.

Step 5: $\ell = 1$ and $\bar{m} > 2$. Let (s, π) be such that (B.2) holds, $\ell = 1$, and $\bar{m} > 2$. Our inductive assumption is:

(i) the lemma statement holds for all (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $\ell' \geq 1$ and $\bar{m}' < \bar{m}$.

We proceed with the proof argument as usual by considering three alternative strategies at period $t - 1$ and derive a contradiction in each case, which allows us to conclude that $a^*_A(s, \pi, h_{t-1}) = 0$ and $a^*_B(s, \pi, h_{t-1}) = 1$.

(a) Suppose $a^*_A(s, \pi, h_{t-1}) = 1$ and $a^*_B(s, \pi, h_{t-1}) = 0$. Then, the state moves from $(s_{t-1}, \pi_{t-1}) = (\kappa, \pi)$ to $(s_t, \pi_t) = (0, \pi + \omega)$. There are then 2 cases to consider:

Case 1. the state remains with $s_{t'} = 0$ for all $t' \geq t + 1$;

Case 2. the state moves to $s_{t'} = \kappa$ for some $t' \geq t + 1$.

Case 1 cannot occur since in such an equilibrium A obtains payoff zero in every period, but from deviating A obtains a strictly positive payoff. Therefore, Case 2 must hold: for some $t' \geq t+1$, we have $(s_{t'}, \pi_{t'}) = (\kappa, \pi_{t'-1} + \omega)$. Let t' be the smallest such value, which implies that $(s_{t'-1}, \pi_{t'-1}) = (0, \pi_{t'-1})$. Note that $\pi_{t'-1}$ need not equal $\pi + \omega$.

Now consider period $t' - 1$, where the state is $(s_{t'-1}, \pi_{t'-1}) = (0, \pi_{t'-1})$ and then in the next period evolves to $(s_{t'}, \pi_{t'}) = (\kappa, \pi_{t'-1} + \omega)$. By Lemma S.2.3, it cannot be that $(a_A^{t'-1}, a_B^{t'-1}) = (0, 0)$ and, hence, $(s_{t'-2}, \pi_{t'-2}) \neq (0, \pi_{t'-1})$. Therefore, by the law of motion and because t' is the smallest value such that Case 2 occurs, it must be that either $(s_{t'-2}, \pi_{t'-2}) = (0, \pi_{t'-1} - \omega)$ or $(s_{t'-2}, \pi_{t'-2}) = (0, \pi_{t'-1} - 2\omega)$.

Suppose $(s_{t'-2}, \pi_{t'-2}) = (0, \pi_{t'-1} - 2\omega)$. Then, at period $t' - 1$, the agents must have choosen actions $(a_A^{t'-1}, a_B^{t'-1}) = (1, 1)$. Thus, A's equilibrium payoff at period t' is determined by their discounted payoff from the path of states:

$$(j\kappa, \pi_{t'-1} + j\omega) \qquad \text{for } j \geq 0,$$

until (B.1) is satisfied, at which point the state remain fixed for all future periods. Using the inductive assumption, if instead A deviates to $a_A^{t'-1} = 0$, A's payoff is determined by their discounted payoff from the path of states:

$$((1+j)\kappa, \pi_{t'-1} + (j+1-2)\omega) \qquad \text{for } j \geq 0,$$

until (B.1) is satisfied, at which point the state remain fixed for all future periods. A's payoff from this deviation is strictly profitable because the number of steps before (B.1) is satisfied is the same regardless of whether A deviates or follows equilibrium play, and furthermore at every point along the deviation path A obtains strictly higher stage payoff than under the equilibrium path:

$$((1+j)\kappa)(\pi_{t'-1} + (j+1-2)\omega) > j\kappa(\pi_{t'-1} + j\omega)) \iff \pi_{t'-1} > \omega,$$

which is (trivially) true. Therefore, it cannot be that $(s_{t'-2}, \pi_{t'-2}) = (0, \pi_{t'-1} - 2\omega)$.

Suppose instead that $(s_{t'-2}, \pi_{t'-2}) = (0, \pi_{t'-1} - \omega)$. Then, by the law of motion, at period $t' - 1$, it must be that the agents chose $(a_A^{t'-1}, a_B^{t'-1}) = (1, 0)$ and $(s_{t'-1}, \pi_{t'-1}) = (0, \pi_{t'-1})$. B's equilibrium payoff is thus determined by their discounted payoff from the path of states

$$(j\kappa, \pi_{t'-1} + j\omega) \qquad \text{for } j \geq 0,$$

until (B.1) is satisfied, at which point the state remains fixed for all future periods. If instead B deviates to $a_B^{t'-1} = 1$, the state evolves to $(s_{t'-1}, \pi_{t'-1}) = (0, \pi_{t'-1} + \omega)$. By

Lemma S.2.5, B's payoff is weakly larger than the discounted payoff from the path

$$(j\kappa, \pi_{t'-1} + (j+1)\omega) \qquad \text{for } j \geq 0,$$

until (B.1) is satisfied, at which point the state remains fixed for all future periods. B's payoff from this deviation is strictly profitable. To see why, note that the number of steps before (B.1) is satisfied under the equilibrium is at most one more than that under the deviation and—in any case—at each step along the path, we have that B's stage payoff from the deviation is strictly higher than the equilibrium payoff, i.e.,

$$(1 - j\kappa)(\pi_{t'-1} + (1+j)\omega) > (1 - j\kappa)(\pi_{t'-1} + j\omega) \qquad \text{and}$$
$$(1 - j\kappa)(\pi_{t'-1} + (1+j)\omega) > (1 - (j+1)\kappa)(\pi_{t'-1} + (j+1)\omega).$$

Thus, we have a contradiction. It cannot be that $a_A^*(s, \pi, h_{t-1}) = 1$ and $a_B^*(s, \pi, h_{t-1}) = 0$.

(b) and (c) Repeating the arguments of Step 1(b) and Step 1(c), neither $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$ or $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 0$ can be an equilibrium.

Step 6: $\ell > 1$ and $\bar{m} > 2$. Let (s, π) be such that (B.2) holds, $\ell > 1$, and $\bar{m} > 2$. Our inductive assumptions are two-fold:

(i) the lemma statement holds for all (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $\ell' \geq 1$ and $\bar{m}' < \bar{m}$;

(ii) the lemma statement holds for (s', π') satisfying (B.2) with $(\ell', \bar{m}')$ such that $1 \leq \ell' \leq \ell - 1$ and $\bar{m}' = \bar{m}$.

Applying the same proof argument as in Step 2(a)-(c), it follows that the only possible equilibrium behavior is for $a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 1$. ∎

Lemma B.3 *There exists an equilibrium σ^*, whereby for any $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that*

$$\pi < \frac{\omega}{\kappa}s - \omega \quad \textit{and} \quad \pi < \frac{\omega}{\kappa}(1-s) - \omega, \tag{B.5}$$

$a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$. *Furthermore, the equilibrium behavior and, hence, equilibrium path is unique.*

Proof. Existence has been proven in Supplemental Appendix S.1. We prove the uniqueness result via a 2-dimensional inductive argument where (s, π) satisfying (B.5) is parameterized via $(\ell, m) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{>0}$ such that $s = \frac{1}{2} \pm \ell\kappa$ and m is the smallest positive integer such that $\pi + m\omega > \frac{\omega}{\kappa}(\frac{1}{2} - \kappa\ell) - \omega$. We proceed via the following steps:

Step 1. For $\ell = 0$ and $m \in \{1, 2\}$, the unique equilibrium is $a_A^t = a_B^t = 1$.

Step 2. For $\ell \geq 1$ and $m \in \{1,2\}$, the unique equilibrium is $a_A^t = a_B^t = 1$.

Step 3. For $\ell = 0$ and $m > 2$, the unique equilibrium is $a_A^t = a_B^t = 1$.

Step 4. For $\ell \geq 1$ and $m > 2$, the unique equilibrium is $a_A^t = a_B^t = 1$.

Steps 1 and 2 are straightforward and, hence, omitted. For completeness, we present the details in Supplemental Appendix S.3. We now proceed with Steps 3 and 4.

Step 3. Suppose (ℓ, m) such that $\ell = 0$ and $m > 2$. Our inductive assumption is that the lemma holds for all (s', π') satisfying (B.5) with (ℓ', m') where $\ell' \in \mathbb{N}_{\geq 0}$ and $m' < m$. We consider three alternative strategies at period $t-1$ and derive a contradiction in each case, which allows us to conclude that $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$.

(a) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0,1)$ or $(a_A^t, a_B^t) = (1,0)$. Since $s = \frac{1}{2}$ and applying a symmetry argument, we can focus on the first case without loss of generality. By Lemmas B.1 and B.2 and the inductive assumption, A's equilibrium payoff is

$$\sum_{j=0}^{\bar{j}} \beta^j (s+\kappa)(\pi + (1+2j)\omega) + \frac{\beta^{\bar{j}+1}}{1-\beta} s(\pi + (1+2\bar{j})\omega + \omega),$$

where $\bar{j} = \lceil \frac{m}{2} \rceil - 1$.[24] If A deviates to $a_A^{t\prime} = 1$, then because of the inductive assumption and $\ell = 0$, A's payoff is

$$\sum_{j=0}^{\bar{j}} \beta^j s(\pi + 2(j+1)\omega) + \frac{\beta^{\bar{j}+1}}{1-\beta} s(\pi + 2(\bar{j}+1)\omega),$$

where $\bar{j}$ is as previously defined, $\bar{j} = \lceil \frac{m}{2} \rceil - 1$.[25] Therefore, A's payoff from deviating is strictly higher than their equilibrium payoff if and only if

$$\sum_{j=0}^{\bar{j}} \beta^j s(\pi + 2(j+1)\omega) > \sum_{j=0}^{\bar{j}} \beta^j (s+\kappa)(\pi + (1+2j)\omega),$$

which is true because $s(\pi + 2(j+1)\omega) > (s+\kappa)(\pi + (1+2j)\omega) \iff \pi + 2j\omega < \frac{\omega}{\kappa}s - \omega$ and $j \leq \bar{j} = \lceil \frac{m}{2} \rceil - 1$. Thus, we have a contradiction.

(b) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0,0)$. By Lemma S.2.1, we know that at some point $\bar{t} \geq t+1$, the state must evolve. By Point (a) above, it must be that

[24] Note that $j = \bar{j}$ is the smallest integer j such that $\pi + (1+2j)\omega > \frac{\omega}{\kappa}(1-(s+\kappa)) - \omega$.

[25] Note that $j = \bar{j}$ is the smallest integer j such that $\pi + 2(j+1)\omega > \frac{\omega}{\kappa}(1-s) - \omega = \frac{\omega}{\kappa}s - \omega$, where the equality follows because $s = \frac{1}{2}$.

$(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1,1)$. Consider period $\bar{t}-1$, A's equilibrium payoff is

$$\sum_{j=0}^{\bar{j}-1} \beta^j s(\pi + 2j\omega) + \frac{\beta^{\bar{j}}}{1-\beta} s(\pi + 2\bar{j}\omega),$$

where $\bar{j} = \lceil \frac{m}{2} \rceil$.[26] If A deviates to $a_A^{\bar{t}-1\prime} = 1$, then their payoff is

$$\begin{aligned}&\sum_{j=0}^{\bar{j}-1} \beta^j (s-\kappa)(\pi + (1+2j)\omega) + \frac{\beta^{\bar{j}}}{1-\beta} s(\pi + (1+2(\bar{j}-1))\omega + \omega) \\ &= \sum_{j=0}^{\bar{j}-1} \beta^j (s-\kappa)(\pi + (1+2j)\omega) + \frac{\beta^{\bar{j}}}{1-\beta} s(\pi + 2\bar{j}\omega),\end{aligned}$$

where $\bar{j}$ is as previously defined, $\bar{j} = \lceil \frac{m}{2} \rceil$.[27] Therefore, A's payoff from deviating is strictly higher than their equilibrium payoff if and only if

$$\sum_{j=0}^{\bar{j}-1} \beta^j (s-\kappa)(\pi + (1+2j)\omega) > \sum_{j=0}^{\bar{j}-1} \beta^j s(\pi + 2j\omega),$$

which is true because $(s-\kappa)(\pi + (1+2j)\omega) > s(\pi + 2j\omega) \iff \pi + 2j\omega < \frac{\omega}{\kappa} s - \omega$ and $j \le \bar{j} - 1 = \lceil \frac{m}{2} \rceil - 1$. Thus, we have a contradiction.

Step 4. Suppose (ℓ, m) such that $\ell \ge 1$ and $m > 2$. Our inductive assumption is two-fold: first, that the proposition holds for all (s', π') satisfying (B.5) with (ℓ', m') where $\ell' \le \ell - 1$ and $m' = m$; second, that the proposition holds for all (s', π') satisfying (B.5) with (ℓ', m') where $\ell' \in \mathbb{N}_{\ge 0}$ and $m' < m$. By a symmetry argument, we focus on the case of $s = \frac{1}{2} - \ell\kappa$ (the case of $s = \frac{1}{2} + \ell\kappa$ is similar and hence omitted). We consider three alternative strategies at period $t-1$ and derive a contradiction in each case, which allows us to conclude that $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$.

(a) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0, 1)$. By Lemmas B.1 and B.2, $\ell \ge 1$ and the inductive assumption, A's equilibrium payoff is

$$\sum_{j=0}^{\bar{j}} \beta^j (s+\kappa)(\pi + (1+2j)\omega) + \beta^{\bar{j}+1} V_A(s+\kappa, \pi + (1+2\bar{j})\omega),$$

where $\bar{j} = \lceil \frac{m}{2} \rceil$.[28] If A deviates to $a_A^{t\prime} = 1$, then because of the inductive assumption

[26] Note that $j = \bar{j}$ is the smallest integer j such that $\pi + 2j\omega > \frac{\omega}{\kappa}(1-s) - \omega = \frac{\omega}{\kappa} s - \omega$, where the equality holds because $s = \frac{1}{2}$.
[27] Note that $j = \bar{j} - 1$ is the smallest integer j such that $\pi + (1+2j)\omega > \frac{\omega}{\kappa}(s-\kappa) - \omega$.
[28] Note that $j = \bar{j}$ is the smallest integer j such that $\pi + (1+2j)\omega > \frac{\omega}{\kappa}(s+\kappa) - \omega$.

and $\ell \geq 1$, her payoff is

$$\sum_{j=0}^{\bar{j}-1} \beta^j s(\pi + 2(j+1)\omega) + \beta^{\bar{j}}(s+\kappa)(\pi + 2\bar{j}\omega + \omega) + \beta^{\bar{j}+1} V_A(s+\kappa, \pi + (1+2\bar{j})\omega),$$

where $\bar{j}$ is as previously defined, $\bar{j} = \lceil \frac{m}{2} \rceil$.[29] Therefore, A's payoff from deviating is strictly higher than their equilibrium payoff if and only if

$$\sum_{j=0}^{\bar{j}-1} \beta^j s(\pi + 2(j+1)\omega) + \beta^{\bar{j}}(s+\kappa)(\pi + 2\bar{j}\omega + \omega) > \sum_{j=0}^{\bar{j}} \beta^j (s+\kappa)(\pi + (1+2j)\omega)$$
$$\iff \sum_{j=0}^{\bar{j}-1} \beta^j s(\pi + 2(j+1)\omega) > \sum_{j=0}^{\bar{j}-1} \beta^j (s+\kappa)(\pi + (1+2j)\omega),$$

which is true because $s(\pi + 2(j+1)\omega) > (s+\kappa)(\pi + (1+2j)\omega) \iff \pi + (1+2j)\omega < \frac{\omega}{\kappa}(s+\kappa) - \omega$ and $j \leq \bar{j} - 1 = \lceil \frac{m}{2} \rceil - 1$. Thus, we have a contradiction.

(b) Suppose that in equilibrium, $(a_A^t, a_B^t) = (1, 0)$. By Lemmas B.1 and B.2 and $\ell \geq 1$ and the inductive assumption, B's equilibrium payoff is

$$\sum_{j=0}^{\bar{j}} \beta^j (1-s+\kappa)(\pi + (1+2j)\omega) + \beta^{\bar{j}+1} V_B(s-\kappa, \pi + (1+2\bar{j})\omega),$$

where $\bar{j} = \lceil \frac{m}{2} \rceil - 1$.[30] If B deviates to $a_B^{t\prime} = 1$, then because of the inductive assumption, B's payoff is

$$\sum_{j=0}^{\bar{j}} \beta^j (1-s)(\pi + 2(j+1)\omega) + \beta^{\bar{j}+1} V_B(s, \pi + 2(\bar{j}+1)\omega),$$

where $\bar{j}$ is as previously defined, $\bar{j} = \lceil \frac{m}{2} \rceil - 1$.[31] Using Lemma S.2.6, we obtain the following lower bound on B's payoff from deviating

$$\sum_{j=0}^{\bar{j}} \beta^j (1-s)(\pi + 2(j+1)\omega) + \beta^{\bar{j}+1} V_B(s-\kappa, \pi + (1+2\bar{j})\omega).$$

It follows that B's deviation is strictly greater than their equilibrium payoff if

$$\sum_{j=0}^{\bar{j}} \beta^j (1-s)(\pi + 2(j+1)\omega) > \sum_{j=0}^{\bar{j}} \beta^j (1-s+\kappa)(\pi + (1+2j)\omega).$$

[29]Note that $j = \bar{j} - 1$ is the smallest integer j such that $\pi + 2(j+1)\omega > \frac{\omega}{\kappa} s - \omega$.
[30]Note that $j = \bar{j}$ is the smallest integer j such that $\pi + (1+2j)\omega > \frac{\omega}{\kappa}(s-\kappa) - \omega$.
[31]Note that $j = \bar{j}$ is the smallest integer j such that $\pi + 2(j+1)\omega > \frac{\omega}{\kappa} s - \omega$.

This inequality holds if

$$(1-s)(\pi+2(j+1)\omega) > (1-s+\kappa)(\pi+(1+2j)\omega) \iff \pi+2j\omega < \frac{\omega}{\kappa}(1-s)-\omega.$$

But notice that $j \leq \bar{j} = \lceil \frac{m}{2} \rceil - 1$ and, by construction of $\bar{j}$, for any $j \leq \bar{j}$ we have $\pi + 2j\omega < \frac{\omega}{\kappa}s - \omega \implies \pi + 2j\omega < \frac{\omega}{\kappa}(1-s) - \omega$ because $s < \frac{1}{2}$ (i.e., $\ell \geq 1$). Therefore, B has a strict incentive to deviate — a contradiction.

(c) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0,0)$. By Lemma S.2.1, we know that at some point $\bar{t} \geq t+1$, the state must evolve. By Parts (a) and (b) above, it must be that $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1,1)$. Then, A's equilibrium payoff in period $\bar{t}-1$ is $\sum_{j=0}^{\bar{j}} \beta^j s(\pi + 2j\omega) + \beta^{\bar{j}+1} V_A(s, \pi + 2\bar{j}\omega)$, where $\bar{j} = \lceil \frac{m}{2} \rceil$. Note that $j = \bar{j}$ is the smallest integer j such that $\pi + 2j\omega > \frac{\omega}{\kappa}s - \omega$. If A deviates to $a_A^{\bar{t}-1'} = 1$ her payoff is

$$\sum_{j=0}^{\bar{j}-1} \beta^j (s-\kappa)(\pi+(2j+1)\omega) + \beta^{\bar{j}} s(\pi+2\bar{j}\omega) + \beta^{\bar{j}+1} V_A(s, \pi+2\bar{j}\omega),$$

where $\bar{j}$ is as previously defined, $\bar{j} = \lceil \frac{m}{2} \rceil$.[32] Therefore, A's payoff from deviating is strictly higher than her equilibrium payoff if and only if

$$\begin{aligned}
\sum_{j=0}^{\bar{j}-1} \beta^j (s-\kappa)(\pi+(2j+1)\omega) + \beta^{\bar{j}} s(\pi+2\bar{j}\omega) &> \sum_{j=0}^{\bar{j}} \beta^j s(\pi+2j\omega) \\
\iff \sum_{j=0}^{\bar{j}-1} \beta^j (s-\kappa)(\pi+(2j+1)\omega) &> \sum_{j=0}^{\bar{j}-1} \beta^j s(\pi+2j\omega),
\end{aligned}$$

which is true because $(s-\kappa)(\pi+(2j+1)\omega) > s(\pi+2j\omega) \iff \pi+2j\omega < \frac{\omega}{\kappa}s - \omega$ and $j \leq \bar{j} - 1 = \lceil \frac{m}{2} \rceil - 1$. Thus, we have a contradiction.

■

[32] Note that $j = \bar{j} - 1$ is the smallest integer j such that $\pi + (2j+1)\omega > \frac{\omega}{\kappa}(s-\kappa) - \omega$.

Supplemental appendix for "When Growth leads to Zero-sum Conflict" by Álvaro Delgado-Vega and Barton E. Lee

S.1 Proof of existence in Proposition 2

The proof of existence follows from Lemmas S.1.1, S.1.2 and S.1.3, which we present and prove below.

Lemma S.1.1 *The strategy profile σ^* described in Proposition 2 is an equilibrium for any subgame with state (s, π) such that*

$$\pi > \frac{\omega}{\kappa}(1-s) - \omega \quad \textit{and} \quad \pi > \frac{\omega}{\kappa}s - \omega. \tag{S.1.1}$$

Proof. Let (s, π) be a state satisfying (S.1.1) and suppose σ^* is the strategy profile described in the lemma. Under σ^*, $(a_A^t, a_B^t) = (0, 0)$ and Agent A obtains payoff $\frac{s\pi}{1-\beta}$. If A makes a one-step deviation at period t, then $(a_A^t, a_B^t) = (1, 0)$ and A's payoff is $\frac{(s-\kappa)(\pi+\omega)}{1-\beta}$. This deviation is not strictly profitable because (S.1.1) holds: $\frac{s\pi}{1-\beta} > \frac{(s-\kappa)(\pi+\omega)}{1-\beta} \iff \pi > \frac{\omega}{\kappa}s - \omega$. A similar argument shows that B also does not have a strictly profitable deviation since $\pi > \frac{\omega}{\kappa}(1-s) - \omega$. We conclude that σ^* is an equilibrium for any subgame satisfying (S.1.1). ■

Lemma S.1.2 *The strategy profile σ^* described in Proposition 2 is an equilibrium for any subgame with state (s, π) such that*

$$\pi < \frac{\omega}{\kappa}(1-s) - \omega \quad \textit{and} \quad \pi > \frac{\omega}{\kappa}s - \omega, \textit{ or} \tag{S.1.2}$$

$$\pi > \frac{\omega}{\kappa}(1-s) - \omega \quad \textit{and} \quad \pi < \frac{\omega}{\kappa}s - \omega. \tag{S.1.3}$$

Proof. We prove the lemma for case where (S.1.2) holds (the case of (S.1.3) is similar and, hence, omitted). Let (s, π) be a state satisfying (S.1.2) and suppose σ^* is the strategy profile described in Proposition 2. Then, σ^* prescribes $(a_A^t, a_B^t) = (0, 1)$ and Agent B's payoff is given by the discounted payoff from the path

$$(s_{t-1+j}, \pi_{t-1+j}) = \begin{cases} (s + j\kappa, \pi + j\omega) & \text{for } j \in \{1, \dots, \bar{j}\}, \\ (s + \bar{j}\kappa, \pi + \bar{j}\omega) & \text{for } j > \bar{j}, \end{cases} \tag{S.1.4}$$

where $\bar{j}$ is defined as the smallest integer such that $\pi + \bar{j}\omega > \frac{\omega}{\kappa}(1 - (s + \bar{j}\kappa)) - \omega$. Graphically, Figure 1 illustrates this path as a temporary evolution of the state in the diagonal direction. If B makes a one-step deviation at period t, then $(a_A^t, a_B^t) = (0, 0)$

and B's payoff is given by the discounted payoff from the path

$$(s_{t-1+j}, \pi_{t-1+j}) = \begin{cases} (s+(j-1)\kappa, \pi+(j-1)\omega) & \text{for } j \in \{1, \ldots, \bar{j}+1\}, \\ (s+\bar{j}\kappa, \pi+\bar{j}\omega) & \text{for } j > \bar{j}+1, \end{cases} \tag{S.1.5}$$

where $\bar{j}$ is as previously defined. This deviation is not strictly profitable for B because:

(i) B's payoff for any $j \in \{1, \ldots, \bar{j}\}$ along the equilibrium path (S.1.4) provides strictly higher per-period payoff than the path (S.1.5):

$$(1-(s+j\kappa))(\pi+j\omega) > (1-(s+(j-1)\kappa))(\pi+(j-1)\omega) \iff \pi+(j-1)\omega < \frac{\omega}{\kappa}(1-(s+(j-1)\kappa))-\omega$$

which is true by construction of $\bar{j}$, and

(ii) B's payoff for any $j > \bar{j}$ is the same along both paths (S.1.4) and (S.1.5) since the state remains fixed at $(s+\bar{j}\kappa, \pi+\bar{j}\omega)$.

Now consider A's incentive to deviate. Under σ^*, $(a_A^t, a_B^t) = (0,1)$ and Agent A's payoff is given by the discounted payoff from the path (S.1.4). If A makes a one-step deviation at period t, then $(a_A^t, a_B^t) = (1,1)$ and A's payoff is given by the discounted payoff from the path

$$(s_{t-1+j}, \pi_{t-1+j}) = \begin{cases} (s+(j-1)\kappa, \pi+(j+1)\omega) & \text{for } j \in \{1, \ldots, \bar{j}\}, \\ (s+(\bar{j}-1)\kappa, \pi+(\bar{j}+1)\omega) & \text{for } j > \bar{j}, \end{cases} \tag{S.1.6}$$

where $\bar{j}$ is as previously defined.[33] This deviation is not strictly profitable for A because: A's payoff for any $j \geq 1$ along the equilibrium path (S.1.4) provides strictly higher per-period payoff than the path (S.1.6):

$$(s+j\kappa)(\pi+j\omega) > (s+(j-1)\kappa)(\pi+(j+1)\omega) \iff \pi > \frac{\omega}{\kappa}s - \omega,$$

which is true by assumption that (s,π) satisfies (S.1.2). We conclude that σ^* is an equilibrium for any subgame satisfying (S.1.2). ■

Lemma S.1.3 *The strategy profile σ^* described in Proposition 2 is an equilibrium for any subgame with state (s,π) such that*

$$\pi < \frac{\omega}{\kappa}(1-s) - \omega \quad \textit{and} \quad \pi < \frac{\omega}{\kappa}s - \omega. \tag{S.1.7}$$

[33] To see why, notice that $\bar{j}$ was previously defined as the smallest integer such that $\pi + \bar{j}\omega > \frac{\omega}{\kappa}(1 - (s+\bar{j}\kappa)) - \omega$. The analogous condition for the path that follows from A's deviation is that $\bar{j}'$ is smallest integer such that $\pi + (\bar{j}'+1)\omega > \frac{\omega}{\kappa}(1-(s+(\bar{j}'-1)\kappa)) - \omega$, but this condition simplifies to $\pi + \bar{j}'\omega > \frac{\omega}{\kappa}(1-(s+\bar{j}'\kappa)) - \omega$, i.e., the same condition that defined $\bar{j}$.

Proof. Suppose σ^* is the strategy profile described in the lemma. Let (s, π) be a state satisfying (S.1.2). Without loss of generality, we assume that $s \leq \frac{1}{2}$. In equilibrium, $(a_A^t, a_B^t) = (1, 1)$ and, following the graphical illustration in Figure 1, Agent A's payoff is given by the discounted payoff from the path

$$(s_{t-1+j}, \pi_{t-1+j}) = \begin{cases} (s, \pi + 2j\omega) & \text{for } j \in \{1, \ldots, \bar{j}\}, \\ (s + (j - \bar{j})\kappa, \pi + (2\bar{j} + (j - \bar{j}))\omega) & \text{for } j \in \{\bar{j} + 1, \ldots, \bar{\ell}\}, \\ (s + (\bar{\ell} - \bar{j})\kappa, \pi + (2\bar{j} + (\bar{\ell} - \bar{j}))\omega) & \text{for } j > \bar{\ell}, \end{cases} \tag{S.1.8}$$

where $\bar{j}$ is defined as the smallest integer such that $\pi + 2\bar{j}\omega > \frac{\omega}{\kappa}s - \omega$ and $\bar{\ell}$ is defined as the smallest integer such that $\pi + (2\bar{j} + (\bar{\ell} - \bar{j}))\omega > \frac{\omega}{\kappa}(1 - (s + (\bar{\ell} - \bar{j})\kappa)) - \omega$. Note that if $\bar{\ell} = \bar{j}$, we use the convention that $\{\bar{j} + 1, \ldots, \bar{\ell}\} = \emptyset$, i.e., we exclude the middle case in (S.1.8). The case of $\bar{\ell} = \bar{j}$ occurs if and only if $s = \frac{1}{2}$. We proceed by considering two cases depending on whether $s < 1/2$ or $s = 1/2$.

Case of $s < 1/2$. If A makes a one-step deviation at period t, then $(a_A^t, a_B^t) = (0, 1)$ and A's payoff is given by the discounted payoff from the path

$$\begin{aligned} &(s_{t-1+j}, \pi_{t-1+j}) \\ &= \begin{cases} (s + \kappa, \pi + (2j - 1)\omega) & \text{for } j \in \{1, \ldots, \bar{j} + 1\}, \\ (s + (1 + j - (\bar{j} + 1))\kappa, \pi + (2(\bar{j} + 1) - 1 + (j - (\bar{j} + 1)))\omega) & \text{for } j \in \{\bar{j} + 2, \ldots, \bar{\ell} - 1\}, \\ (s + (\bar{\ell} - \bar{j})\kappa, \pi + (2\bar{j} + (\bar{\ell} - \bar{j}))\omega) & \text{for } j > \bar{\ell}, \end{cases} \\ &= \begin{cases} (s + \kappa, \pi + (2j - 1)\omega) & \text{for } j \in \{1, \ldots, \bar{j} + 1\}, \\ (s + (j - \bar{j})\kappa, \pi + (2\bar{j} + (j - \bar{j}))\omega) & \text{for } j \in \{\bar{j} + 2, \ldots, \bar{\ell} - 1\}, \\ (s + (\bar{\ell} - \bar{j})\kappa, \pi + (2\bar{j} + (\bar{\ell} - \bar{j}))\omega) & \text{for } j > \bar{\ell}, \end{cases} \end{aligned} \tag{S.1.9}$$

where $\bar{j}$ and $\bar{\ell}$ are as previously defined.[34] This deviation is not strictly profitable for A because:

(i) A's payoff for any $j \in \{1, \ldots, \bar{j}\}$ along the equilibrium path (S.1.8) provides strictly higher per-period payoff than the path (S.1.9):

$$s(\pi + 2j\omega) > (s + \kappa)(\pi + (2j - 1)\omega) \iff \pi + 2(j - 1)\omega < \frac{\omega}{\kappa}s - \omega,$$

which is true by construction of $\bar{j}$, and

[34] To see why, recall that $\bar{j}$ was defined as the smallest integer such that $\pi + 2\bar{j}\omega > \frac{\omega}{\kappa}s - \omega$. The analogous condition for the path that follows from A's deviation is that $\bar{j}'$ is the smallest integer such that $\pi + (2\bar{j}' - 1)\omega > \frac{\omega}{\kappa}(s + \kappa) - \omega$, but this condition simplifies to $\pi + 2(\bar{j}' - 1)\omega > \frac{\omega}{\kappa}s - \omega$ and, hence, $\bar{j}' = \bar{j} + 1$. A similar argument shows that $\bar{\ell}' = \bar{\ell} - 1$.

(ii) A's per-period payoff for any $j > \bar{j}$ is equal for the paths (S.1.8) and (S.1.9) since the states evolve along the same path for $j > \bar{j}$ (including for $j > \bar{\ell}$).

Now consider B's incentive to deviate. If B makes a one-step deviation at period t, then $(a_A^t, a_B^t) = (1, 0)$ and B's payoff is the discounted payoff from the path

$$(s_{t-1+j}, \pi_{t-1+j}) = \begin{cases} (s-\kappa, \pi + (2j-1)\omega) & \text{for } j \in \{1, \ldots, \bar{j}\}, \\ (s + (j-1-\bar{j})\kappa, \pi + (2\bar{j} - 1 + (j-\bar{j}))\omega) & \text{for } j \in \{\bar{j}+1, \ldots, \bar{\ell}+1\}, \\ (s + (\bar{\ell}-\bar{j})\kappa, \pi + (2\bar{j} + (\bar{\ell}-\bar{j}))\omega) & \text{for } j > \bar{\ell}, \end{cases} \tag{S.1.10}$$

where $\bar{j}$ and $\bar{\ell}$ are as previously defined.[35] This deviation is not strictly for B because:

(i) B's per-period payoff for any $j \in \{1, \ldots, \bar{j}\}$ along the equilibrium path (S.1.8) is strictly higher than along (S.1.10):

$$(1-s)(\pi + 2j\omega) > (1-(s-\kappa))(\pi + (2j-1)\omega) \iff \pi + 2(j-1)\omega < \frac{\omega}{\kappa}(1-s) - \omega$$

which is true because $s \leq 1/2$ and by construction of $\bar{j}$, and

(ii) B's per-period payoff for any $j \in \{\bar{j}+1, \ldots, \bar{\ell}\}$ is strictly higher along the equilibrium path than along (S.1.10)

$$(1-(s+(j-\bar{j})\kappa))(\pi + (2\bar{j} + (j-\bar{j}))\omega) > (1-(s+(j-1-\bar{j})\kappa))(\pi + (2\bar{j} - 1 + (j-\bar{j}))\omega)$$
$$\iff \pi + 2(\bar{j}-1)\omega < \frac{\omega}{\kappa}(1-s) - \omega,$$

which is true because $s \leq 1/2$ and by construction of $\bar{j}$, and

(iii) B's per-period payoff for any $j > \bar{\ell}$ is equal along the equilibrium path and the deviation since the states evolve along the same path for $j > \bar{\ell}$

We conclude that σ^* is an equilibrium for any subgame satisfying (S.1.7).

Case of $s = 1/2$. Note that in (S.1.8), the middle case is excluded because $\bar{\ell} = \bar{j}$ and, hence, $\{\bar{j}+1, \ldots, \bar{\ell}\} = \emptyset$. By a symmetry argument, it suffices to show that just one of the agents, say Agent B, has no incentive to deviate. The proof argument is then similar to the above case of Agent B deviating when $s < 1/2$ (in fact, the proof argument above only utilizes the weak inequality $s \leq 1/2$). ■

[35] To see why, recall that $\bar{j}$ was defined as the smallest integer such that $\pi + 2\bar{j}\omega > \frac{\omega}{\kappa}s - \omega$. The analogous condition for the path that follows from B's deviation is that $\bar{j}'$ is the smallest integer such that $\pi + (2\bar{j}' - 1)\omega > \frac{\omega}{\kappa}(s-\kappa) - \omega$, but this condition simplifies to $\pi + 2\bar{j}'\omega > \frac{\omega}{\kappa}s - \omega$ and, hence, $\bar{j}' = \bar{j}$. A similar argument shows that $\bar{\ell}' = \bar{\ell} + 1$.

S.2 Auxiliary lemmas for Appendix B

Lemma S.2.1 *If (B.1) is not satisfied for* $(s_{t-1}, \pi_{t-1}) = (s, \pi)$*, then there is no equilibrium whereby, for any history* h_{t-1}*,* $(a_A^{t'}, a_B^{t'}) = (0, 0)$ *for all* $t' \geq t$*.*

Proof. Suppose for a contradiction that (B.1) is not satisfied and for some equilibrium σ^* and history h_{t-1}, $(a_A^{t'}, a_B^{t'}) = (0, 0)$ for all $t' \geq t$. Then the equilibrium payoffs for A and B are $\frac{s\pi}{1-\beta}$ and $\frac{(1-s)\pi}{1-\beta}$, respectively. We begin with the case of $s \notin \{0, 1\}$. If A deviates to $a_A^t = 1$, her payoff is

$$(s - \kappa)(\pi + \omega) + \beta V_A(s - \kappa, \pi + \omega \mid \sigma^*, h_t) \geq \frac{(s - \kappa)(\pi + \omega)}{1 - \beta},$$

where the inequality follows from Lemma 1. Therefore, to sustain the equilibrium, it must be that $\frac{(s-\kappa)(\pi+\omega)}{1-\beta} \leq \frac{s\pi}{1-\beta} \iff \pi \geq \frac{\omega}{\kappa}s - \omega$. Similarly, by considering B's deviation to $a_B^t = 1$, we require that $\frac{(1-(s+\kappa))(\pi+\omega)}{1-\beta} \leq \frac{(1-s)\pi}{1-\beta} \iff \pi \geq \frac{\omega}{\kappa}(1 - s) - \omega$. Since (B.1) is not satisfied and π is not a multiple of ω, we have a contradiction.

Now suppose $s = 1$ (the case of $s = 0$ is similar and, hence, omitted). Because (B.1) is not satisfied and π_0 is positive and not a multiple of ω, it must be that $\pi < \frac{\omega}{\kappa} - \omega$. Using the same argument as the case of $s \notin \{0, 1\}$, if A deviates to $a_A^t = 1$, her payoff is at least $\frac{(1-\kappa)(\pi+\omega)}{1-\beta}$. This is strictly higher than her equilibrium payoff of $\frac{\pi}{1-\beta}$ because $\pi < \frac{\omega}{\kappa} - \omega$. Thus, we have a contradiction. ■

Lemma S.2.2 *Let* σ^* *be an equilibrium such that for some* $(s_t, \pi_t) = (s, \pi)$*, where* $\pi < \frac{\omega}{\kappa}(1 - s) - \omega$ *and, for some history* h_t*, the state evolves to* $(s_{t+1}, \pi_{t+1}) = (s + \kappa, \pi + \omega)$*. Furthermore, suppose equilibrium play under* σ^* *follows Proposition 2 for any state* $(s_{t'}, \pi_{t'}) = (s + \kappa, \pi + \omega)$ *and any state reachable thereafter. Then it cannot be that at* $(s_{t-1}, \pi_{t-1}) = (s + \kappa, \pi - \omega)$*, the agents choose* $a_A^*(s + \kappa, \pi - \omega, h_{t-1}) = 1$ *and* $a_B^*(s + \kappa, \pi - \omega, h_{t-1}) = 0$ *for any history* h_{t-1}*.*

Proof. Suppose for the sake of a contradiction that such behavior is prescribed by an equilibrium σ^*. Then, at $(s_{t-1}, \pi_{t-1}) = (s + \kappa, \pi - \omega)$ B obtains equilibrium payoff

$$(1 - s)\pi + \beta\Big((1 - s - \kappa)(\pi + \omega) + \beta V_B(s + \kappa, \pi + \omega \mid \sigma^*)\Big). \tag{S.2.1}$$

If B deviates to $a_B^t = 1$, B obtains payoff

$$(1 - s - \kappa)(\pi + \omega) + \beta V_B(s + \kappa, \pi + \omega \mid \sigma^*). \tag{S.2.2}$$

Note that, by the lemma's supposition, the continuation payoffs in (S.2.1) and (S.2.2) are history independent. After rearranging (S.2.1) and (S.2.2), it follows that this deviation is strictly profitable if and only if

$$\frac{(1 - s)\pi}{1 - \beta} < (1 - s - \kappa)(\pi + \omega) + \beta V_B(s + \kappa, \pi + \omega \mid \sigma^*).$$

But, by Lemma 1, the RHS has lower bound $\frac{(1-s-\kappa)(\pi+\omega)}{1-\beta}$. Therefore, a strictly profitable deviation exists if $\frac{(1-s)\pi}{1-\beta} < \frac{(1-s-\kappa)(\pi+\omega)}{1-\beta}$, which is true because $\pi < \frac{\omega}{\kappa}(1-s) - \omega$. Thus, we have a contradiction. ■

Lemma S.2.3 *Let σ^* be an equilibrium such that for some $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ where $\pi < \frac{\omega}{\kappa}(1-s) - \omega$ and some history h_{t-1}, the state evolves to $(s_{t'}, \pi_{t'}) = (s+\kappa, \pi+\omega)$ for some $t' \geq t$. Furthermore, suppose equilibrium play under σ^* follows Proposition 2 for any state $(s_{t'}, \pi_{t'}) = (s+\kappa, \pi+\omega)$ and for any state reachable thereafter. Then it cannot be that at (s, π), $a_A^*(s, \pi, h_{t-1}) = 0$ and $a_B^*(s, \pi, h_{t-1}) = 0$ for any history h_{t-1}.*

Proof. Suppose for a contradiction that such behavior is prescribed by an equilibrium σ^*. Without loss of generality, we consider the case where $t' = t+1$ (i.e., we focus on the period immediately before the state evolves in the event that the state does not evolve for multiple periods). Since σ^* is an equilibrium, it must be that B best responds to $a_A^*(s, \pi, h_{t-1}) = 0$ at period t:

$$\begin{aligned}&(1-s)\pi + \beta\Big((1-(s+\kappa))(\pi+\omega) + \beta V_B(s+\kappa, \pi+\omega \mid \sigma^*)\Big) \\ &\geq (1-(s+\kappa))(\pi+\omega) + \beta V_B(s+\kappa, \pi+\omega \mid \sigma^*),\end{aligned} \tag{S.2.3}$$

where the continuation payoffs are history independent by the lemma's supposition. Rearranging (S.2.3) gives the following condition:

$$\frac{(1-s)\pi}{1-\beta} \geq (1-(s+\kappa))(\pi+\omega) + \beta V_B(s+\kappa, \pi+\omega \mid \sigma^*) \geq \frac{(1-(s+\kappa))(\pi+\omega)}{1-\beta},$$

where the last inequality follows from Lemma 1. However, this condition holds if and only if $\pi \geq \frac{\omega}{\kappa}(1-s) - \omega$, which is not true—hence, we have a contradiction. ■

Lemma S.2.4 *Let $\tilde{\sigma}$ be an equilibrium and let $(s_{t-1}, \pi_{t-1}) = (s, \pi)$, where (B.2) holds and the minimum value of $m \in \mathbb{N}$ such that (B.3) holds is some $m = \bar{m} \geq 1$. Furthermore, suppose equilibrium play under $\tilde{\sigma}$ follows Proposition 2 for any state $(s_{t'}, \pi_{t'}) \in \Big\{(s, \pi+2\omega), (s+\kappa, \pi+\omega)\Big\}$ and for any state reachable thereafter. Then it cannot be that, at $(s_{t-1}, \pi_{t-1}) = (s, \pi)$, $\tilde{\sigma}$ prescribes $a_A^t = 1$ and $a_B^t = 1$ for any history h_{t-1}.*

Proof. For sake a contradiction, suppose that such an equilibrium $\tilde{\sigma}$ exists. We begin with two observations. First, note that by the lemma's supposition:

(i) if the state reaches $(s_{t'}, \pi_{t'}) = (s, \pi+2\omega)$ for some $t' \geq t$, then it evolves deterministically along the path $(s_{t'+j}, \pi_{t'+j}) = (s+j\kappa, \pi+(2+j)\omega)$ for $j \in \mathbb{N}_{\geq 0}$ until $j = \bar{j}$, where $\bar{j}$ is the smallest integer such that

$$\pi_{t'+\bar{j}} > \frac{\omega}{\kappa}(1-s_{t'+\bar{j}}) - \omega.$$

at which point conflict ensues and the state does not evolve further;

(ii) if the state reaches $(s_{t'}, \pi_{t'}) = (s+\kappa, \pi+\omega)$ for some $t' \geq t$, then it evolves deterministically along the path $(s_{t'+j}, \pi_{t'+j}) = (s+(1+j)\kappa, \pi+(1+j)\omega)$ for $j \in \mathbb{N}_{\geq 0}$ until $j = \bar{j}$, where $\bar{j}$ is the smallest integer such that

$$\pi_{t'+\bar{j}} > \frac{\omega}{\kappa}(1 - s_{t'+\bar{j}}) - \omega,$$

at which point conflict ensues and the state does not evolve further.

Second, note that the value $\bar{j}$ is the same in both Parts (i) and (ii) above—to see this, simply observe that

$$(\pi + (2+j)\omega) > \frac{\omega}{\kappa}(1 - (s+j\kappa)) - \omega \iff \pi + (1+j)\omega > \frac{\omega}{\kappa}(1 - (s+(1+j)\kappa)) - \omega.$$

We now proceed with the proof argument. In equilibrium, by Part (i) above, Agent A's payoff at (s, π):

$$\sum_{j=0}^{\bar{j}-1} \beta^j (s+j\kappa)(\pi + (2+j\omega)) + \beta^{\bar{j}} \frac{(s+\bar{j}\kappa)(\pi+(2+\bar{j}\omega))}{1-\beta},$$

where the first summation is taken to equal zero if $\bar{j} = 0$. Now suppose A deviates to $a_A^t = 0$. Then, by Part (ii) above, Agent A's payoff is

$$\sum_{j=0}^{\bar{j}-1} \beta^j (s+(1+j)\kappa)(\pi + (1+j\omega)) + \beta^{\bar{j}} \frac{(s+(1+\bar{j})\kappa)(\pi+(1+\bar{j}\omega))}{1-\beta},$$

which is strictly higher than their equilibrium payoff—a contradiction. To see this, notice that, for any $j \geq 0$ (including $j > \bar{j}$), we have

$$(s+(1+j)\kappa)(\pi+(1+j\omega)) > (s+j\kappa)(\pi+(2+j\omega)) \iff \pi > \frac{\omega}{\kappa}s - \omega,$$

which is true by (B.2). ■

Lemma S.2.5 *Let $\tilde{\sigma}$ be an equilibrium such that for any $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ where $s \in \{\kappa, \dots, 1-\kappa\}$ and for some $m \in \mathbb{N}_{>0}$:*

$$\pi < \frac{\omega}{\kappa}(1-s) - \omega, \qquad \pi > \frac{\omega}{\kappa}s - \omega \quad \textit{and} \quad \pi + m\omega > \frac{\omega}{\kappa}(1-s) - \omega, \tag{S.2.4}$$

the state's equilibrium evolution at (s, π) and any state reachable thereafter is as per σ^* in Proposition 2. Then at any state $(s_{t-1}, \pi_{t-1}) = (0, \pi)$ such that

$$\pi + m\omega > \frac{\omega}{\kappa} - \omega \quad \textit{and} \quad \pi + (m-1)\omega < \frac{\omega}{\kappa} - \omega, \tag{S.2.5}$$

we have

$$V_B(0,\pi \mid \tilde{\sigma}, h_{t-1}) \geq \sum_{j=0}^{\bar{j}} \beta^j (1-(1+j)\kappa)(\pi+(1+j)\omega) + \beta^{\bar{j}+1}\frac{(1-(1+\bar{j})\kappa)(\pi+(1+\bar{j})\omega)}{1-\beta}, \tag{S.2.6}$$

where $\bar{j} = \lfloor \frac{m-1}{2} \rfloor$.

Proof. We begin by proving the result for $m=1$ and $m=2$. We then prove the result for arbitrary $m>2$ via an induction argument.

Case of $m \in \{1,2\}$**.** Let $\tilde{\sigma}$ be an equilibrium and suppose the conditions of the lemma statement are satisfied for $m \in \{1,2\}$. For state $(0,\pi)$ satisfying (S.2.5), we wish to prove (S.2.6), which simplifies to

$$V_B(0,\pi \mid \tilde{\sigma}, h_{t-1}) \geq \frac{(1-\kappa)(\pi+\omega)}{1-\beta}.$$

Suppose the state is $(0,\pi)$ with some history h_{t-1} and satisfies (S.2.5) for $m \in \{1,2\}$. We know that B's equilibrium payoff must be weakly larger than the minimum payoff obtained by choosing $a_B^t = 1$; that is,

$$\begin{aligned} V_B(0,\pi \mid \tilde{\sigma}, h_{t-1}) \geq \min \Big\{ &(\pi+2\omega) + \beta V_B(0,\pi+2\omega \mid \tilde{\sigma}, h_{t-1}, (a_A^t, a_B^t) = (1,1)), \\ &(1-\kappa)(\pi+\omega) + \beta V_B(\kappa, \pi+\omega \mid \tilde{\sigma}, h_{t-1}, (a_A^t, a_B^t) = (0,1)) \Big\}. \end{aligned}$$

Because $m \in \{1,2\}$ and by Lemma 1, it follows that the above inequality simplifies to

$$V_B(0,\pi \mid \tilde{\sigma}, h_{t-1}) \geq \min \left\{ \frac{(\pi+2\omega)}{1-\beta}, \frac{(1-\kappa)(\pi+\omega)}{1-\beta} \right\} \geq \frac{(1-\kappa)(\pi+\omega)}{1-\beta},$$

as required.

Case of $m>2$**.** We now extend the result to general m. Let $\tilde{\sigma}$ be an equilibrium and suppose $m>2$ is such that the conditions of the lemma statement are satisfied. Suppose the state is $(0,\pi)$ with some history h_{t-1} and satisfies (S.2.5) for m.

We will prove the result via an induction argument. Note that the proof argument above for $m \in \{1,2\}$ immediately implies that (S.2.6) holds for any state $(0,\pi')$ satisfying (S.2.5) for $\bar{j} = \lfloor \frac{m'-1}{2} \rfloor$ and $m' \in \{1,2\}$. Thus, we make the inductive assumption that the lemma statement holds for $m' \in \{m-2, m-1\}$. We know that B's equilibrium payoff must be weakly larger than the minimum payoff obtained by choosing $a_B^t = 1$; that is,

$$V_B(0, \pi \mid \tilde{\sigma}, h_{t-1}) \geq \min \Big\{ (\pi + 2\omega) + \beta V_B(0, \pi + 2\omega \mid \tilde{\sigma}, h_{t-1}, (a_A^t, a_B^t) = (1, 1)), \\ (1 - \kappa)(\pi + \omega) + \beta V_B(\kappa, \pi + \omega \mid \tilde{\sigma}, h_{t-1}, (a_A^t, a_B^t) = (0, 1)) \Big\}. \tag{S.2.7}$$

Because the conditions in the lemma statement are satisfied for m and $s > 0$, the second term in the minimization is equal to the RHS of (S.2.6), i.e., the discounted payoff from the path

$$(j\kappa, \pi + j\omega) \tag{S.2.8}$$

for $j \geq 1$ until $j = \lfloor \frac{m-1}{2} \rfloor$, at which point the state does not evolve any further. Thus, it suffices to show that the first term in the minimization is weakly larger than (S.2.6). Using the inductive argument for $m' = m - 2$, we have that the first term in the minimization is weakly larger than

$$(\pi + 2\omega) + \beta \left(\sum_{j=0}^{\bar{j}'} \beta^j (1 - (1 + j)\kappa)(\pi + (3 + j)\omega) + \beta^{\bar{j}'+1} \frac{(1 - (1 + \bar{j}')\kappa)(\pi + (3 + \bar{j}')\omega)}{1 - \beta} \right),$$

where $\bar{j}' = \lfloor \frac{m-3}{2} \rfloor$. This payoff equals the discounted payoff from the path

$$((j - 1)\kappa, \pi + (1 + j)\omega) \tag{S.2.9}$$

for $j \geq 1$ until $j = 2 + \lfloor \frac{m-3}{2} \rfloor = \lfloor \frac{m-1}{2} \rfloor$, at which point the state does not evolve any further. But notice that, for any $j \in \{1, \ldots, \lfloor \frac{m-1}{2} \rfloor\}$, B's payoff from the state (S.2.9) is strictly greater than the payoff from the state (S.2.8). Furthermore, the length of the paths in (S.2.8) and (S.2.9) are equal to $\lfloor \frac{m-1}{2} \rfloor$. Thus, B's discounted payoff from (S.2.9) is strictly greater than their payoff from (S.2.8). It follows that (S.2.7) implies

$$\begin{aligned} V_B(0, \pi \mid \tilde{\sigma}, h_{t-1}) &\geq (\pi + 2\kappa) + \beta V_B(0, \pi + 2\omega \mid \tilde{\sigma}, h_{t-1}, (a_A^t, a_B^t) = (1, 1)) \\ &= \sum_{j=0}^{\bar{j}} \beta^j (1 - (1 + j)\kappa)(\pi + (1 + j)\omega) + \beta^{\bar{j}+1} \frac{(1 - (1 + \bar{j})\kappa)(\pi + (1 + \bar{j})\omega)}{1 - \beta}, \end{aligned}$$

where $\bar{j} = \lfloor \frac{m-1}{2} \rfloor$, as required. ■

Lemma S.2.6 *Let σ^* be an equilibrium. For any $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ such that $s > \kappa$ and*

$$\pi > \frac{\omega}{\kappa} s - \omega \qquad \textit{and} \qquad \pi < \frac{\omega}{\kappa}(1 - s) - \omega,$$

we have

$$V_B(s,\pi \mid \sigma^*) > V_B(s-\kappa,\pi-\omega \mid \sigma^*). \tag{S.2.10}$$

Proof. By Lemma B.2, (S.2.10) simplifies to

$$\begin{aligned}
&\sum_{j=0}^{\bar{j}} \beta^j (1-s-(j+1)\kappa)(\pi+(j+1)\omega) + \frac{\beta^{\bar{j}+1}}{1-\beta}(1-s-(\bar{j}+1)\kappa)(\pi+(\bar{j}+1)\omega) \\
&\qquad > \sum_{j=0}^{\bar{j}+1} \beta^j (1-s-j\kappa)(\pi+j\omega) + \frac{\beta^{\bar{j}+2}}{1-\beta}(1-s-(\bar{j}+1)\kappa)(\pi+(\bar{j}+1)\omega),
\end{aligned}$$

where $\bar{j}$ is defined as the smallest integer $j=\bar{j}$ such that $\pi+(j+1)\omega > \frac{\omega}{\kappa}(1-(s+(j+1)\kappa))-\omega$. Simplifying further gives

$$\begin{aligned}
&\sum_{j=0}^{\bar{j}} \beta^j (1-s-(j+1)\kappa)(\pi+(j+1)\omega) + \beta^{\bar{j}+1}(1-s-(\bar{j}+1)\kappa)(\pi+(\bar{j}+1)\omega) \\
&\qquad > \sum_{j=0}^{\bar{j}} \beta^j (1-s-j\kappa)(\pi+j\omega) + \beta^{\bar{j}+1}(1-s-(\bar{j}+1)\kappa)(\pi+(\bar{j}+1)\omega) \\
\iff &\sum_{j=0}^{\bar{j}} \beta^j (1-s-(j+1)\kappa)(\pi+(j+1)\omega) > \sum_{j=0}^{\bar{j}} \beta^j (1-s-j\kappa)(\pi+j\omega).
\end{aligned}$$

This inequality is true because

$$\begin{aligned}
&(1-s-(j+1)\kappa)(\pi+(j+1)\omega) > (1-s-j\kappa)(\pi+j\omega) \\
\iff &\pi+j\omega < \frac{\omega}{\kappa}(1-(s+j\kappa))-\omega
\end{aligned}$$

and $j \leq \bar{j}$. ■

S.3 Omitted steps from Proof of Lemma B.3

In this appendix, we provide the omitted steps (Step 1 and Step 2) from Proof of Lemma B.3.

Step 1. Suppose (ℓ, m) such that $\ell = 0$ and $m \in \{1,2\}$. We consider three alternative strategies at period $t-1$ and derive a contradiction in each case, which allows us to conclude that $a_A^*(s,\pi,h_{t-1}) = a_B^*(s,\pi,h_{t-1}) = 1$.

(a) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0,1)$ or $(a_A^t, a_B^t) = (1,0)$. Since $s = \frac{1}{2}$ and via a symmetry argument, we can focus on the first case without loss of generality. Then, by Lemmas B.1 and B.2 and because $\ell = 0$ and $m \in \{1, 2\}$, A has a strict incentive to deviate to $a_A^{t\prime} = 1$ if and only if

$$(s+\kappa)(\pi+\omega) + \frac{\beta}{1-\beta}s(\pi+2\omega) < \frac{1}{1-\beta}s(\pi+2\omega),$$

which holds because $s(\pi + 2\omega) > (s + \kappa)(\pi + \omega) \iff \pi < \frac{\omega}{\kappa}s - \omega$; which is a contradiction.

(b) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0, 0)$. We know by Lemma S.2.1 that at some point $\bar{t} \geq t + 1$, the state must evolve. By Point 1 above, it must be that $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1, 1)$. Consider period $\bar{t} - 1$, A has a strict incentive to deviate to $a_A^{\bar{t}-1\prime} = 1$ if and only if $s\pi + \frac{\beta}{1-\beta}s(\pi+2\omega) < (s-\kappa)(\pi+\omega) + \frac{\beta}{1-\beta}s(\pi+2\omega)$, which holds because $\pi < \frac{\omega}{\kappa}s - \omega$; which is a contradiction.

Step 2. Suppose (ℓ, m) such that $\ell \geq 1$ and $m \in \{1, 2\}$. For $m = 1$, our inductive assumption is that the lemma holds for all (s', π') satisfying (B.5) with $(\ell', m') = (\ell', m)$ for all $\ell' \leq \ell - 1$.[36] For $m = 2$, our inductive assumption additionally includes that the lemma holds for all (s', π') satisfying (B.5) with $(\ell', m') = (\ell', 1)$ where $\ell' \in \mathbb{N}_{\geq 0}$. By a symmetry argument, we focus on the case of $s = \frac{1}{2} - \ell\kappa$ (the case of $s = \frac{1}{2} + \ell\kappa$ is similar and hence omitted). We consider three alternative strategies at period $t - 1$ and derive a contradiction in each case, which allows us to conclude that $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$.

(a) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0, 1)$. By Lemmas B.1 and B.2, $m \leq 2$ and $\ell \geq 1$ and the inductive argument, A's equilibrium payoff is given by

$$(s+\kappa)(\pi+\omega)+\beta(s+\kappa)(\pi+3\omega)+\beta^2 V_A(s+\kappa, \pi+3\omega \mid \sigma^*) = (s+\kappa)(\pi+\omega)+\beta V_A(s, \pi+2\omega \mid \sigma^*),$$

where the equality follows by Lemma B.2. If A deviates to $a_A^{t\prime} = 1$, their payoff is $s(\pi + 2\omega) + \beta V(s, \pi + 2\omega \mid \sigma^*)$, which is strictly higher than their equilibrium payoff because $\pi < \frac{\omega}{\kappa}s - \omega$. This is a contradiction.

(b) Suppose that in equilibrium, $(a_A^t, a_B^t) = (1, 0)$. By Lemmas B.1 and B.2, B's equilibrium payoff is $(1 - s + \kappa)(\pi + \omega) + \beta V_B(s - \kappa, \pi + \omega)$. If B deviates to $a_B^{t\prime} = 1$, their payoff is

$$(1-s)(\pi+2\omega) + \beta V_B(s, \pi+2\omega \mid \sigma^*) > (1-s)(\pi+2\omega) + \beta V_B(s-\kappa, \pi+\omega \mid \sigma^*),$$

where the lower bound follows by Lemma S.2.6. This lower bound is strictly higher than the equilibrium payoff because $\pi < \frac{\omega}{\kappa}(1 - s) - \omega$; this is a contradiction.

[36] When combined with the fact that $m = 1$ and (s', π') satisfying (B.5), this inductive assumption simply reduces to the lemma holding for $m' = 1$ and $\ell' = \ell - 1$.

(c) Suppose that in equilibrium, $(a_A^t, a_B^t) = (0, 0)$. By Lemma S.2.1, we know that at some point $\bar{t} \geq t+1$, the state must evolve. By Parts (a) and (b) above, it must be that $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1, 1)$. Consider period $\bar{t}-1$, A's equilibrium payoff is

$$s\pi + \beta s(\pi + 2\omega) + \beta^2 V_A(s, \pi + 2\omega) = s\pi + \beta V_A(s-\kappa, \pi+\omega),$$

where the equality follows by Lemma B.2. If A deviates to $a_A^{\bar{t}-1\prime} = 1$, their payoff is

$$(s-\kappa)(\pi+\omega) + \beta V_A(s-\kappa, \pi+\omega),$$

which is strictly greater than their equilibrium payoff because $\pi < \frac{\omega}{\kappa}s - \omega$. This is a contradiction.

S.4 Proof of Proposition 3

Proof of Proposition 3. Given $\pi_0 < \frac{\omega}{\kappa}\frac{1}{2} - \omega$, define $\underline{s}$ (resp., $\overline{s}$) as the smallest (resp., largest) value of s_0 such that, in equilibrium, $a_A^t = a_B^t = 1$ for $t = 1$ (per Proposition 2). Formally, $\underline{s}$ and $\overline{s}$ are the values that satisfy

$$\pi_0 = \frac{\omega}{\kappa}\underline{s} - \omega \qquad \text{and} \qquad \pi_0 = \frac{\omega}{\kappa}(1-\overline{s}) - \omega.$$

Note that $0 < \underline{s} < \overline{s} < 1$ because $0 < \pi_0 < \frac{\omega}{\kappa}\frac{1}{2} - \omega$.

We begin with two auxiliary claims.

Claim 1: For any $\beta \in [0, 1)$, the maximum utilitarian payoff from an initial condition $s_0 \in \{\frac{1}{2}-\kappa, \frac{1}{2}, \frac{1}{2}+\kappa\}$ is strictly higher than the utilitarian payoff from any other feasible $s_0' \in (\underline{s}, \overline{s}) \backslash \{\frac{1}{2}-\kappa, \frac{1}{2}, \frac{1}{2}+\kappa\}$.

Proof of Claim 1. Without loss of generality, assume $s_0' > \frac{1}{2}+\kappa$ and $s_0' \in (\underline{s}, \overline{s})$. By Proposition 2, both agents contribute to expanding the pie, $a_A^t = a_B^t = 1$, until $\pi_t > \frac{\omega}{\kappa}(1-s_0') - \omega$. At this point, the large-share agent continue to expand the pie and the small-share agent expands their share until $\pi_t > \frac{\omega}{\kappa}s_t - \omega$; at this point, π_t remains fixed forever. Therefore, the utilitarian payoff from any such s_0' can be directly calculated as

$$\sum_{j=0}^{\bar{j}} \beta^j\left(\pi_0 + 2\omega(1+j)\right) + \sum_{j=\bar{j}+1}^{\bar{j}'} \beta^j \Big(\pi_0 + \big(2(1+\bar{j}) + j - \bar{j}\big)\omega\Big) \\ + \frac{\beta^{\bar{j}'+1}}{1-\beta}\Big(\pi_0 + \big(2(1+\bar{j}) + \bar{j}' - \bar{j}\big)\omega\Big), \tag{S.4.1}$$

where $\bar{j}$ is the smallest positive integer j such that

$$\pi_0 + 2\omega(1+j) > \frac{\omega}{\kappa}(1 - s_0') - \omega$$

and $\bar{j}'$ is the smallest positive integer $j \geq \bar{j}$ such that

$$\pi_0 + \big(2(1+\bar{j}) + j - \bar{j}\big)\omega > \frac{\omega}{\kappa}\big(s_0' - \kappa(j - \bar{j})\big) - \omega.$$

Note that $\bar{j}' \geq \bar{j} + 2$ because $s_0' \geq \frac{1}{2} + 2\kappa$.

Now consider the utilitarian payoff from $s_0 = s_0' - 2\kappa$. For this initial condition, the agents will choose $a_A^t = a_B^t = 1$ for one additional period before the small-share agent decides to expand their share ($a_j^t = 0$); furthermore, the long-run size of the pie will be the same with initial condition s_0'. Thus, the initial condition s_0 provides faster growth without any cost to the long-run size of the pie and, hence, provides higher utilitarian payoff. Formally, and similar to (S.4.1), the utilitarian payoff from s_0 is

$$\sum_{j=0}^{\tilde{j}} \beta^j \big(\pi_0 + 2\omega(1+j)\big) + \sum_{j=\tilde{j}+1}^{\tilde{j}'} \beta^j \Big(\pi_0 + \big(2(1+\tilde{j}) + j - \tilde{j}\big)\omega\Big) \\ + \frac{\beta^{\tilde{j}'+1}}{1-\beta}\Big(\pi_0 + \big(2(1+\tilde{j}) + \tilde{j}' - \tilde{j}\big)\omega\Big), \tag{S.4.2}$$

where $\tilde{j}$ is the smallest positive integer j such that

$$\pi_0 + 2\omega(1+j) > \frac{\omega}{\kappa}(1 - (s_0' - 2\kappa)) - \omega \iff \pi_0 + 2\omega(1+j-1) > \frac{\omega}{\kappa}(1 - s_0') - \omega,$$

and $\tilde{j}'$ is the smallest positive integer $j \geq \tilde{j}$ (equality will hold if and only if $s_0 = \frac{1}{2}$) such that

$$\pi_0 + \big(2(1+\tilde{j}) + j - \tilde{j}\big)\omega > \frac{\omega}{\kappa}\big((s_0' - 2\kappa) - \kappa(j - \tilde{j})\big) - \omega.$$

Note that in (S.4.2) we use the convention that if $\tilde{j}' < \tilde{j} + 1$, then the summation is equal to zero. Comparing the definitions of $\tilde{j}$ and $\bar{j}$, it is immediate that $\tilde{j} = \bar{j} + 1$. Substituting this into the definition for $\tilde{j}'$ gives

$$\begin{aligned} &\pi_0 + \big(2(1+\bar{j}+1) + j - (\bar{j}+1)\big)\omega > \frac{\omega}{\kappa}\big((s_0' - 2\kappa) - \kappa(j - (\bar{j}+1))\big) - \omega \\ &\qquad \iff \pi_0 + \big(2(1+\bar{j}) + (j+2) - \bar{j}\big)\omega > \frac{\omega}{\kappa}\big(s_0' - \kappa(j - \bar{j})\big) - \omega \\ &\iff \pi_0 + \big(2(1+\bar{j}) + (j+1) - \bar{j}\big)\omega > \frac{\omega}{\kappa}\big(s_0' - \kappa(j + 1 - \bar{j})\big) - \omega; \end{aligned}$$

thus, $\tilde{\bar{j}}' = \bar{j}' - 1$. Returning to (S.4.2), we can rewrite the utilitarian payoff as

$$\sum_{j=0}^{\bar{j}+1} \beta^j \big(\pi_0 + 2\omega(1+j)\big) + \sum_{j=\bar{j}+1+1}^{\bar{j}'-1} \beta^j \Big(\pi_0 + \big(2(1+\bar{j}+1) + j - (\bar{j}+1)\big)\omega\Big)$$
$$+ \frac{\beta^{\bar{j}'-1+1}}{1-\beta} \Big(\pi_0 + \big(2(1+\bar{j}+1) + \bar{j}' - 1 - (\bar{j}+1)\big)\omega\Big)$$
$$= \sum_{j=0}^{\bar{j}} \beta^j \big(\pi_0 + 2\omega(1+j)\big) + \beta^{\bar{j}+1}\Big(\pi_0 + \big(2(1+\bar{j}) + 2\big)\omega\Big)$$
$$+ \sum_{j=\bar{j}+2}^{\bar{j}'-1} \beta^j \Big(\pi_0 + \big(2(1+\bar{j}) + (j+1) - \bar{j}\big)\omega\Big) + \beta^{\bar{j}'}\Big(\pi_0 + \big(2(1+\bar{j}) + \bar{j}' - \bar{j}\big)\omega\Big)$$
$$+ \frac{\beta^{\bar{j}'+1}}{1-\beta}\Big(\pi_0 + \big(2(1+\bar{j}) + \bar{j}' - \bar{j}\big)\omega\Big). \tag{S.4.3}$$

Note that the first summation and the final term in (S.4.3) and (S.4.1) are equal, whereas the second summation in (S.4.1) is strictly smaller than the middle terms in (S.4.3). Thus, it follows that (S.4.3) is strictly greater than (S.4.1), and the utilitarian payoff is higher with initial condition s_0. ■

Claim 2: For any $\beta \in [0,1)$ and any $s_0 \notin (\underline{s}, \bar{s})$, the utilitarian payoff is strictly increasing as $|\frac{1}{2} - s_0|$ increases.

Proof of Claim 2. Without loss of generality, assume $s_0 > \frac{1}{2}$. Notice that when $s_0 \notin (\underline{s}, \bar{s})$ and $s_0 > \frac{1}{2}$, one agent contributes to expanding the pie ($a_j^t = 1$) and the other agent expands their share ($a_j^t = 0$) until $\pi_t > \frac{\omega}{\kappa} s_t - \omega$; at which point, π_t remains fixed forever (as per Proposition 2). Therefore, the utilitarian payoff from any such s_0 can be directly calculated as

$$\sum_{j=0}^{\bar{j}} \beta^j (\pi_0 + \omega(1+j)) + \frac{\beta^{\bar{j}+1}}{1-\beta}(\pi_0 + \omega(1+\bar{j})), \tag{S.4.4}$$

where $\bar{j}$ is the smallest (positive) integer j such that $\pi_0 + \omega(1+j) > \frac{\omega}{\kappa}(s_0 - \kappa(1+j)) - \omega$. It is immediate that (S.4.4) is strictly increasing in $\bar{j}$. Furthermore, because $\bar{j}$ is strictly increasing in s_0, the claim follows. ■

Together Claims 1 and 2 imply that, for the purpose of maximizing utilitarian payoffs for $s_0 \in \{\kappa, \ldots, 1-\kappa\}$, it is without loss of generality to consider $s_0 \in \{\kappa, \frac{1}{2} - \kappa, \frac{1}{2}, \frac{1}{2} + \kappa, 1-\kappa\}$. We now proceed with the proof argument. Again, for simplicity and without loss of generality, we assume $s_0 \geq \frac{1}{2}$ and, thus, consider initial conditions: $s_0 \in \{\frac{1}{2}, \frac{1}{2}+\kappa, 1-\kappa\}$. We now characterize the utilitarian payoff for each initial condition.

Suppose $s_0 = 1 - \kappa$. By Proposition 2, the utilitarian payoff is

$$\sum_{t=0}^{\tilde{t}} \beta^t(\pi_0 + (2+t)\omega) + \frac{\beta^{\tilde{t}+1}}{1-\beta}\tilde{\pi}_\infty, \tag{S.4.5}$$

where $\tilde{\pi}_\infty = \pi_0 + (2+\tilde{t})\omega$ for some positive integer $\tilde{t}$. Because $\pi_0 > \frac{\omega}{\kappa}(1-3\kappa) - \omega$, we have $\tilde{\pi}_\infty > \frac{\omega}{\kappa}\frac{1}{2} + \omega$. To see this, notice that if m is the largest positive integer such that $\pi_0 > \frac{\omega}{\kappa}(1-m\kappa) - \omega$, then, for initial condition $s_0 = 1 - m\kappa$, the value of π_t for which conflict ensues is contained in $(\frac{\omega}{\kappa}\frac{1}{2} - \omega, \frac{\omega}{\kappa}\frac{1}{2})$. Furthermore, initial condition $s_0 = 1-(m-1)\kappa$, the value of π_t for which conflict ensures is contained in $(\frac{\omega}{\kappa}\frac{1}{2}, \frac{\omega}{\kappa}\frac{1}{2} + \omega)$, i.e., the interval of values is shifted up by ω. Repeating this argument and noting that it can be applied at least 3 times (since $\pi_0 > \frac{\omega}{\kappa}(1-3\kappa) - \omega$), shows that $\tilde{\pi}_\infty > \frac{\omega}{\kappa}\frac{1}{2} + \omega$.

Suppose $s_0 = \frac{1}{2}$ or $s_0 = \frac{1}{2} + \kappa$. By Proposition 2, the utilitarian payoff is

$$\sum_{t=0}^{\hat{t}} \beta^t(\pi_0 + 2(1+t)\omega) + \frac{\beta^{\hat{t}+1}}{1-\beta}\hat{\pi}_\infty, \qquad \text{or} \tag{S.4.6}$$

$$\sum_{t=0}^{\hat{t}'} \beta^t(\pi_0 + 2(1+t)\omega) + \beta^{\hat{t}'+1}(\pi_0 + 2(1+\hat{t}'+1)\omega + \omega) + \frac{\beta^{\hat{t}'+1}}{1-\beta}\hat{\pi}'_\infty, \tag{S.4.7}$$

respectively, where $\hat{\pi}_\infty = (\pi_0 + 2(1+\hat{t})\omega)$ and $\hat{\pi}'_\infty = (\pi_0 + 2(1+\hat{t}')\omega + \omega)$, and $\hat{t}$ and $\hat{t}'$ are positive integers. Given Proposition 2, it is immediate that $\hat{\pi}_\infty, \hat{\pi}'_\infty \in (\frac{\omega}{\kappa}\frac{1}{2} - \omega, \frac{\omega}{\kappa}\frac{1}{2} + \omega)$.

For different discount factors, β, we now compare the payoff (S.4.5) to the maximum or minimum of (S.4.6) and (S.4.7). Clearly, for β sufficiently small, (S.4.6) and (S.4.7) are strictly greater than (S.4.5). Thus, $s_0 \in \{\frac{1}{2}, \frac{1}{2} + \kappa\}$ maximizes the utilitarian payoff when β is small. Now consider β sufficiently close to 1. Because $\tilde{\pi}_\infty$ in (S.4.5) is strictly greater than $\hat{\pi}_\infty$ and $\hat{\pi}'_\infty$ in (S.4.6) and (S.4.7), it follows that for large enough β, (S.4.5) is strictly greater than (S.4.6) and (S.4.7). Thus, $s_0 \in \{1-\kappa\}$ maximizes the utilitarian payoff when β is close to 1. ■

S.5 Asymmetric discount factors

In this appendix, we consider an extension of the benchmark model where the agents have (possibly) different discount factors: $\beta_A \in [0,1)$ and $\beta_B \in [0,1)$ for agent A and B, respectively. Our key result that growth eventually halts continues to hold. We begin with two auxiliary lemmas (Lemmas S.5.1 and S.5.2). Proposition S.5.1 then presents the main result.

Lemma S.5.1 *There exists $\bar{\pi}(\beta_A, \beta_B)$ such that, for all $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with $\pi > \bar{\pi}(\beta_A, \beta_B)$, there is no equilibrium σ^* such that on the equilibrium path: $\pi_{t'}$ grows unboundedly and $a_A^{t'} = a_B^{t'} \in \{0,1\}$ for all $t' \geq t$.*

Proof. For sake of a contradiction, suppose that there exists an equilibrium σ^* that grows π_{t-1} unboundedly on the equilibrium path and such that, for π_{t-1} arbitrarily large, $a_A^{t'} = a_B^{t'} \in \{0, 1\}$ for all $t' \geq t$. Because $\pi_{t'}$ grows unboundedly, on the equilibrium path there exists $\pi_{t'}$ arbitrarily large such that $a_A^{t'} = a_B^{t'} = 1$ (and, hence, $\pi_{t'+1} = \pi_{t'} + 2\omega$). Abusing notation slightly, we denote this $\pi_{t'}$ by simply π.

We now focus on agent A's incentive to deviate, and suppose $s \in \{0, \ldots, 1-\kappa\}$ (the case of $s = 1$ follows similarly by focusing on agent B's incentive to deviate and, hence, is omitted). On the equilibrium path $a_A^{t'} = a_B^{t'}$ for all $t' \geq t$; thus, agent A's equilibrium payoff is bounded above by

$$s \sum_{j=0}^{\infty} \beta_A^j (\pi + 2(j+1)\omega)) = s \frac{\pi(1-\beta_A) + 2\omega}{(1-\beta_A)^2}.$$

Agent A's payoff from deviating to $a_A^{t'} = 0$ delivers payoff

$$(s+\kappa)(\pi+\omega) + \beta_A V_A(s+\kappa, \pi+\omega \mid \sigma^*, h_{t'}) \geq \frac{(s+\kappa)(\pi+\omega)}{(1-\beta_A)},$$

where the inequality follows from Lemma 1. Note that Lemma 1 (with the same proof argument) continues to hold in this extension. Therefore, to sustain the equilibrium σ^*, we require that

$$\frac{(s+\kappa)(\pi+\omega)}{(1-\beta_A)} \leq s \frac{\pi(1-\beta_A) + 2\omega}{(1-\beta_A)^2},$$

which does not hold for sufficiently large π. Denote the first value for which the inequality does not hold by $\bar{\pi}(s, \beta_A)$. In a similar way, define $\bar{\pi}(1, \beta_B)$ as first value for which the analagous inequality does not hold for agent B when $s = 1$. The lemma is then completed by defining $\bar{\pi}(\beta_A, \beta_B) := \max\{\max_s \bar{\pi}(s, \beta_A), \bar{\pi}(1, \beta_B)\}$. ■

Lemma S.5.2 *Suppose $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with $s \notin \{0, 1\}$. There is no equilibrium σ^* such that, on the equilibrium path, $\pi_{t'}$ grows unboundedly.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. Let $\bar{s}$ (resp., $\underline{s}$) denote the limit superior (resp., limit inferior) values of $s_{t'}$ that occur on the equilibrium path. That is, $\bar{s} := \lim_{t' \to \infty} \sup s_{t'}$ and $\underline{s} := \lim_{t' \to \infty} \inf s_{t'}$. We consider three cases.

Case 1: Suppose $\underline{s} > 0$ or $\bar{s} < 1$. We focus on the latter scenario (the former is similar and, hence, we omit the proof). Consider $t' \geq t$ such that $s_{t'-1} = \bar{s}$ and $\bar{s} \geq s_{t''}$ for all $t'' \geq t'$ and $\pi_{t'-1} > \bar{\pi}(\beta_A, \beta_B)$ (as defined in Lemma S.5.1). Note that such a time period exists by definition of $\bar{s}$ and because σ^* is such that $\pi_{t'}$ grows unboundedly. Applying Lemma S.5.1, we can then conclude that it must be that, on the equilibrium path,

$(s_{t''-1}, \pi_{t''-1}) = (\bar{s}, \pi_{t''-1})$ and $(a_A^{t''}, a_B^{t''}) = (0, 1)$ or $(a_A^{t''}, a_B^{t''}) = (1, 0)$ at some time period $t'' \geq t'$. However, since $\bar{s} \geq s_{t''}$ for all $t'' \geq t'$, only the latter can occur: $(a_A^{t''}, a_B^{t''}) = (1, 0)$.

Consider now agent A's incentive to deviate at time period t''. Agent A's equilibrium payoff is bounded above by

$$(\bar{s} - \kappa)(\pi + \omega) + \beta_A \Big(\bar{s} \sum_{j=0}^{\infty} \beta_A^j (\pi + 2(j+1)\omega) \Big) = (\bar{s} - \kappa)(\pi + \omega) + \beta_A \bar{s} \frac{\pi(1 - \beta_A) + 2\omega}{(1 - \beta_A)^2}.$$

Agent A's payoff from deviating to $a_A^{t'} = 0$ is

$$\bar{s}\pi + \beta_A V_A(\bar{s}, \pi \mid \sigma^*, h_{t'}) \geq \frac{\bar{s}\pi}{(1 - \beta_A)},$$

where the inequality follows from Lemma 1. Therefore, to sustain the equilibrium σ^*, we require that

$$\frac{\bar{s}\pi}{(1 - \beta_A)} \leq (\bar{s} - \kappa)(\pi + \omega) + \beta_A \bar{s} \frac{\pi(1 - \beta_A) + 2\omega}{(1 - \beta_A)^2},$$

which does not hold for sufficiently large π. This contradicts the existence of σ^* because, on the equilibrium path, $\pi_{t'}$ grows unboundedly and, since $\bar{s}$ is the limit superior of $s_{t'}$, the state must repeatedly reach a state $(s_{t''}, \pi_{t''})$ with $\pi_{t''}$ unboundedly large.

Case 2: Suppose $\underline{s} = 0$ and $\bar{s} = 1$. Consider $t' \geq t$ such that $s_{t'-1} = \bar{s}$ and $\bar{s} \geq s_{t''}$ for all $t'' \geq t'$ and $\pi_{t'-1} > \bar{\pi}(\beta_A, \beta_B)$ (as defined in Lemma S.5.1). Note that such a time period exists by definition of $\bar{s}$ and because σ^* is such that $\pi_{t'}$ grows unboundedly. Because $\underline{s} < \bar{s}$, it must be that, on the equilibrium path, $(s_{t''-1}, \pi_{t''-1}) = (\bar{s}, \pi_{t''-1})$ and $(a_A^{t''}, a_B^{t''}) = (1, 0)$ at some time period $t'' \geq t'$. A contradiction then follows via the same argument (and inequalities) presented in Case 1.

Case 3: Suppose $\underline{s} = \bar{s} \in \{0, 1\}$. Without loss of generality, suppose $\underline{s} = \bar{s} = 1$. Consider the first value of $t' \geq t$ such that $s_{t'-1} = \underline{s} = 1$ and $\underline{s} = 1 \leq s_{t''}$ for all $t'' \geq t'$. In other words, $s_{t''} = 1$ for all $t'' \geq t'$. Note that such a time period exists by definition of $\underline{s} = 1$. By the law of motion and because $s_{t-1} = s \notin \{0, 1\}$, at period $t' - 1$, it must have been that $(s_{t'-2}, \pi_{t'-2}) = (1 - \kappa, \pi_{t'-1} - \omega)$ and, on the equilibrium path, the agents chose actions: $(a_A^{t'-1}, a_B^{t'-1}) = (0, 1)$, which lead to state: $(s_{t'-1}, \pi_{t'-1}) = (1, \pi_{t'-1})$.

Consider now agent B's incentive to deviate at time period $t' - 1$. Agent B's equilibrium payoff is zero (by construction of the time period t'). Agent B's payoff from deviating to $a_B^{t'-1} = 0$ is

$$\kappa(\pi_{t'-1} - \omega) + \beta_B V_B(1 - \kappa, \pi_{t'-1} \mid \sigma^*, h_{t'}) \geq \frac{\kappa(\pi_{t'-1} - \omega)}{1 - \beta_B} > 0.$$

Clearly, agent B's deviation is strictly profitable and, hence, we have a contradiction. ■

Proposition S.5.1 *Suppose $(s_0, \pi_0) = (s, \pi)$ with $s \notin \{0, 1\}$, and let σ^* be an equilibrium. If (s_{t-1}, π_{t-1}) is such that*

$$\pi_{t-1} > \max\left\{\frac{\omega}{\kappa}s_{t-1} - \omega, \frac{\omega}{\kappa}(1 - s_{t-1}) - \omega\right\} \tag{S.5.1}$$

and (s_{t-1}, π_{t-1}) is on the equilibrium path, then conflict ensues and $(s_{t'}, \pi_{t'}) = (s, \pi)$ for every $t' \geq t$.

Proof. Let σ^* be an equilibrium. By Lemma S.5.2, the growth of $\pi_{t'}$ is bounded; thus, on the equilibrium path, there exists $\bar{\pi}$ and $\bar{t} \geq t$ such that $\pi_{\bar{t}} = \bar{\pi}$, $(s_{t'}, \pi_{t'}) = (s_{\bar{t}}, \bar{\pi})$ for all $t' \geq \bar{t}$, and $\pi_{\bar{t}-1} < \bar{\pi}$. Furthermore, it must be that $s_{\bar{t}} \notin \{0, 1\}$. To see this, suppose that $s_{\bar{t}} = 1$ (the case of $s_{\bar{t}} = 0$ is similar). Because $s \neq 1$ and using the law of motion, there exists some time period $\tilde{t} \,:\, 0 \leq \tilde{t} < \bar{t}$ such that $s_{\tilde{t}} = 1 - \kappa$, $s_{t'} = 1$ for all $t' > \tilde{t}$, and $(a_A^{\tilde{t}}, a_B^{\tilde{t}}) = (0, 1)$. At this point, agent B has a strictly profitable deviation since their equilibrium continuation payoff is zero but deviating to $a_B^{\tilde{t}+1} = 1$ guarantees strictly positive payoff. Thus, such an equilibrium cannot exist and it must be that $s_{\bar{t}} \notin \{0, 1\}$.

We now show that a necessary condition for σ^* to exist is that

$$\pi_{\bar{t}-1} \leq \max\left\{\frac{\omega}{\kappa}s_{\bar{t}-1} - \omega, \frac{\omega}{\kappa}(1 - s_{\bar{t}-1}) - \omega\right\}; \tag{S.5.2}$$

hence, on the equilibrium path, conflict must ensue whenever the above inequality fails, i.e., (S.5.1) holds. We prove this by considering the 2 possible cases that are consistent with the equilibrium, i.e., growth of $\pi_{t'}$ at period t.

Case 1: $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1, 1)$ and, hence, $(s_{\bar{t}-1}, \pi_{\bar{t}-1}) = (s_{\bar{t}}, \bar{\pi} - 2\omega)$. Consider agent A's incentive to deviate at period $\bar{t}$. Agent A's equilibrium payoff is $\frac{s_{\bar{t}}\bar{\pi}}{1-\beta_A}$. Agent A's payoff from deviating to $a_A^{\bar{t}} = 0$ is

$$(s_{\bar{t}} + \kappa)(\bar{\pi} - \omega) + \beta_A V_A(s_{\bar{t}} + \kappa, \bar{\pi} - \omega \mid \sigma^*, h_{t'}) \geq \frac{(s_{\bar{t}} + \kappa)(\bar{\pi} - \omega)}{(1 - \beta_A)},$$

where the inequality follows from Lemma 1. Thus, for $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1, 1)$ to be an equilibrium, it must be that

$$\frac{(s_{\bar{t}} + \kappa)(\bar{\pi} - \omega)}{(1 - \beta_A)} \leq \frac{s_{\bar{t}}\bar{\pi}}{1 - \beta_A} \iff \bar{\pi} \leq \frac{\omega}{\kappa}s_{\bar{t}} + \omega. \tag{S.5.3}$$

By considering agent B's incentives, it similarly follows that

$$\bar{\pi} \leq \frac{\omega}{\kappa}(1 - s_{\bar{t}}) + \omega. \tag{S.5.4}$$

Combining (S.5.3) and (S.5.4), it must be that $\bar{\pi} \leq \min\left\{\frac{\omega}{\kappa}s_{\bar{t}} + \omega, \frac{\omega}{\kappa}(1-s_{\bar{t}}) + \omega\right\}$ and, hence,

$$\pi_{\bar{t}-1} \leq \min\left\{\frac{\omega}{\kappa}s_{\bar{t}} - \omega, \frac{\omega}{\kappa}(1-s_{\bar{t}}) - \omega\right\} = \min\left\{\frac{\omega}{\kappa}s_{\bar{t}-1} - \omega, \frac{\omega}{\kappa}(1-s_{\bar{t}-1}) - \omega\right\}.$$

Case 2: $(a_A^{\bar{t}}, a_B^{\bar{t}}) \in \{(0,1),(1,0)\}$ **and, hence,** $(s_{\bar{t}-1}, \pi_{\bar{t}-1}) = (s_{\bar{t}} \pm \kappa, \bar{\pi} - \omega)$**.** First, suppose $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1,0)$ and, hence, $(s_{\bar{t}-1}, \pi_{\bar{t}-1}) = (s_{\bar{t}} + \kappa, \bar{\pi} - \omega)$. Consider agent A's incentive to deviate at period $\bar{t}$. Agent A's equilibrium payoff is $\frac{s_{\bar{t}}\bar{\pi}}{1-\beta_A}$. Agent A's payoff from deviating to $a_A^{\bar{t}} = 0$ is

$$(s_{\bar{t}} + \kappa)(\bar{\pi} - \omega) + \beta_A V_A(s_{\bar{t}} + \kappa, \bar{\pi} - \omega \mid \sigma^*, h_{t'}) \geq \frac{(s_{\bar{t}} + \kappa)(\bar{\pi} - \omega)}{(1-\beta_A)},$$

where the inequality follows from Lemma 1. Thus, for $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (1,0)$ to be an equilibrium, it must be that

$$\frac{(s_{\bar{t}} + \kappa)(\bar{\pi} - \omega)}{(1-\beta_A)} \leq \frac{s_{\bar{t}}\bar{\pi}}{1-\beta_A} \iff \bar{\pi} \leq \frac{\omega}{\kappa}s_{\bar{t}} + \omega.$$

Hence,

$$\pi_{\bar{t}-1} \leq \frac{\omega}{\kappa}s_{\bar{t}} = \frac{\omega}{\kappa}s_{\bar{t}-1} - \omega.$$

Second, suppose $(a_A^{\bar{t}}, a_B^{\bar{t}}) = (0,1)$ and, hence, $(s_{\bar{t}-1}, \pi_{\bar{t}-1}) = (s_{\bar{t}} - \kappa, \bar{\pi} - \omega)$. By considering agent B's incentive, a similar argument shows that

$$\frac{(1 - s_{\bar{t}} + \kappa)(\bar{\pi} - \omega)}{(1-\beta_A)} \leq \frac{(1-s_{\bar{t}})\bar{\pi}}{1-\beta_A} \iff \bar{\pi} \leq \frac{\omega}{\kappa}(1-s_{\bar{t}}) + \omega.$$

Hence,

$$\pi_{\bar{t}-1} \leq \frac{\omega}{\kappa}(1-s_{\bar{t}}) = \frac{\omega}{\kappa}(1-s_{\bar{t}-1}) - \omega.$$

Combining the conclusions presented in Cases 1 and 2, shows the necessary condition (S.5.2) and completes the proof. ■

S.6 Diminishing returns to expanding the pie

In this appendix, we consider an extension of the benchmark model whereby an agent's ability to expand the pie features diminishing returns. Formally, the law of motion of the size of the pie has ω replaced by $\tilde{\omega}(\pi)$, where $\tilde{\omega}(\cdot)$ is positive and decreasing function. We show that our key result that growth eventually halts continues to hold. We now proceed with the proof.

Lemma S.6.1 *In every equilibrium σ^*, if the state is $(s_{t-1}, \pi_{t-1}) = (s, \pi)$, the players' continuation payoffs are such that: for any history h_{t-1},*

$$V_A(s, \pi \mid \sigma^*, h_{t-1}) \geq \frac{s\pi}{1-\beta} \quad \text{and} \quad V_B(s, \pi \mid \sigma^*, h_{t-1}) \geq \frac{(1-s)\pi}{1-\beta}. \tag{S.6.1}$$

Proof. The proof argument is similar to that of Lemma 1. ■

Lemma S.6.2 *Let σ^* be an equilibrium and let $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with some history h_{t-1}. There exists a threshold $\tilde{\pi}(\omega, \beta)$ such that: if π sufficiently large, $\pi > \tilde{\pi}(\omega, \beta)$, then it cannot be that $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. Using the fact that the parties' payoffs sum to π_t in every period t (and $\pi_{t'} \leq \pi_{t'-1} + 2\tilde{\omega}(\pi_{t-1})$ for all t'), we know that the sum of the parties' equilibrium payoffs is:

$$V_A(s, \pi \mid \sigma^*) + V_B(s, \pi \mid \sigma^*) = \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \leq \sum_{j=0}^{\infty} \beta^j (\pi + 2(j+1)\tilde{\omega}(\pi)) = \frac{\pi(1-\beta) + 2\tilde{\omega}(\pi)}{(1-\beta)^2}. \tag{S.6.2}$$

We consider 2 cases.

Case 1: Suppose $s \in \{\kappa, \ldots, 1-\kappa\}$. Party A's payoff from deviating is

$$(s+\kappa)(\pi + \tilde{\omega}(\pi)) + \beta V_A(s+\kappa, \pi + \tilde{\omega}(\pi) \mid \sigma^*) \geq \frac{(s+\kappa)(\pi + \tilde{\omega}(\pi))}{1-\beta},$$

where the inequality follows from Lemma S.6.1. Hence, for σ^* to be an equilibrium, a necessary condition is

$$V_A(s, \pi \mid \sigma^*) \geq \frac{(s+\kappa)(\pi + \tilde{\omega}(\pi))}{1-\beta}. \tag{S.6.3}$$

A similar argument for Party B implies that, for σ^* to be an equilibrium, a necessary condition is

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1-s+\kappa)(\pi + \tilde{\omega}(\pi))}{1-\beta}. \tag{S.6.4}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.6.3) and (S.6.4), which yields $\frac{(\pi+\tilde{\omega}(\pi))(1+2\kappa)}{1-\beta}$, is weakly less than (S.6.2) — otherwise, at least one of the parties would have an incentive to deviate. That is, we require

$$\begin{aligned} \frac{(\pi+\tilde{\omega}(\pi))(1+2\kappa)}{1-\beta} &\leq \frac{\pi(1-\beta)+2\tilde{\omega}(\pi)}{(1-\beta)^2} \\ \iff \pi(1+2\kappa)+\tilde{\omega}(\pi)(1+2\kappa) &\leq \pi+\frac{2\tilde{\omega}(\pi)}{1-\beta} \\ 2\kappa\pi+\tilde{\omega}(\pi)(1+2\kappa)-\frac{2\tilde{\omega}(\pi)}{1-\beta} &\leq 0. \end{aligned}$$

For sufficiently large π, the above inequality does not hold since $\lim_{\pi\to\infty}\tilde{\omega}(\pi)$ equals some finite value independent of π. Let $\tilde{\pi}_1(\kappa,\beta)$ be the first such value for which for the inequality does not hold for all $\pi > \tilde{\pi}_1(\kappa,\beta)$.

Case 2: Suppose $s \in \{0,1\}$. We focus on $s=0$ (argument for $s=1$ is similar). By the same argument as in Case 1, we have that Inequality (S.6.3) must hold at $s=0$, i.e.,

$$V_A(s,\pi \mid \sigma^*) \geq \frac{\kappa(\pi+\tilde{\omega}(\pi))}{1-\beta}. \tag{S.6.5}$$

And, by Lemma S.6.1, Party B's equilibrium payoff at $(0,\pi)$ must satisfy

$$V_B(s,\pi \mid \sigma^*) \geq \frac{(\pi+2\tilde{\omega}(\pi))}{1-\beta}. \tag{S.6.6}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.6.5) and the RHS of (S.6.6), which yields $\frac{(1+\kappa)(\pi+\tilde{\omega}(\pi))+\tilde{\omega}(\pi)}{1-\beta}$, is weakly less than (S.6.2) — otherwise, at least one of the parties would have an incentive to deviate. That is, we require

$$\begin{aligned} \frac{(1+\kappa)(\pi+\tilde{\omega}(\pi))+\tilde{\omega}(\pi)}{1-\beta} &\leq \frac{\pi(1-\beta)+2\tilde{\omega}(\pi)}{(1-\beta)^2} \\ \iff \kappa(\pi+\tilde{\omega}(\pi))+2\tilde{\omega}(\pi) &\leq \frac{2\tilde{\omega}(\pi)}{1-\beta}. \end{aligned}$$

For sufficiently large π, the above inequality does not hold since $\lim_{\pi\to\infty}\tilde{\omega}(\pi)$ equals some finite value independent of π. Let $\tilde{\pi}_2(\kappa,\beta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}_2(\kappa,\beta)$ and for any $s \in [0,1]$.

By combining the bounds from Case 1 and Case 2, i.e., $\tilde{\pi}(\kappa,\beta) > \max\{\tilde{\pi}_1(\kappa,\beta), \tilde{\pi}_2(\kappa,\beta)\}$, we obtain the desired result. ∎

Lemma S.6.3 *There exists $\bar{\pi}(\kappa,\beta)$ such that, for any $(s_{t-1},\pi_{t-1}) = (s,\pi)$ with history h_{t-1} such that π is sufficiently large: $\pi > \bar{\pi}(\kappa,\beta)$. There is no equilibrium σ^* such that:*

(i) *for some $s \in \{\kappa,\ldots,1\}$, π and history h_{t-1}, $a_A^*(s,\pi,h_{t-1}) = 1$*

(ii) *for some* $s \in \{0, \ldots, 1-\kappa\}$*,* π *and history* h_{t-1}*,* $a_B^*(s, \pi, h_{t-1}) = 1$*.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. By Lemma S.6.2 and if $\pi > \tilde{\pi}(\kappa, \beta)$, it cannot be that $a_A^*(s, \pi, h) = a_B^*(s, \pi, h) = 1$. Therefore, at most one of the parties chooses action $a_j^t = 1$ and, hence, $\pi_t \leq \pi_{t-1} + \tilde{\omega}(\pi_{t-1})$ for all t. Combining this with the fact that the parties' payoffs sum to π_t in every period, we know that the sum of the parties' equilibrium payoffs is:

$$V_A(s, \pi \mid \sigma^*) + V_B(s, \pi \mid \sigma^*) = \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \leq \sum_{j=0}^{\infty} \beta^j (\pi + (j+1)\tilde{\omega}(\pi)) = \frac{\pi(1-\beta) + \tilde{\omega}(\pi)}{(1-\beta)^2}. \tag{S.6.7}$$

Without loss of generality assume that $a_A^*(s, \pi, h) = 0$ and $a_B^*(s, \pi, h) = 1$. We will prove the statement in Part (ii) and, hence, assume $s \in \{0, \ldots, 1-\kappa\}$ — the proof of Part (i) is similar. Party A's equilibrium payoff can be bounded from below as follows:

$$V_A(s, \pi \mid \sigma^*) = (s+\kappa)(\pi + \tilde{\omega}(\pi)) + \beta V_A(s+\kappa, \pi + \omega \mid \sigma^*) \geq \frac{(s+\kappa)(\pi + \tilde{\omega}(\pi))}{1-\beta}, \tag{S.6.8}$$

where the inequality follows from Lemma S.6.1.

Now consider Party B. Party B's payoff from deviating is

$$(1-s)\pi + \beta V_B^*(s, \pi \mid \sigma^*) \geq \frac{(1-s)\pi}{1-\beta},$$

where the inequality follows from Lemma S.6.1. To be an equilibrium, a necessary condition is

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1-s)\pi}{1-\beta}. \tag{S.6.9}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.6.8) and (S.6.9), which yields $\frac{(s+\kappa)(\pi+\tilde{\omega}(\pi))+(1-s)\pi}{1-\beta}$, is weakly less than (S.6.7) — otherwise, party B would have an incentive to deviate at time t. That is, we require

$$\begin{aligned} \frac{(s+\kappa)(\pi + \tilde{\omega}(\pi)) + (1-s)\pi}{1-\beta} &\leq \frac{\pi(1-\beta) + \tilde{\omega}(\pi)}{(1-\beta)^2} \\ \iff \kappa\pi + \tilde{\omega}(\pi)(s+\kappa) &\leq \frac{\tilde{\omega}(\pi)}{(1-\beta)}. \end{aligned}$$

For sufficiently large π, the above inequality does not hold for any s since $\lim_{\pi \to \infty} \tilde{\omega}(\pi)$ equals some finite value independent of π. Let $\tilde{\pi}'(\kappa, \beta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}'(\kappa, \beta)$ and any $s \in \{0, \ldots, 1\}$. Combining this bound on π with that given in Lemma S.6.2 completes the proof. ■

Corollary S.6.1 *There exists $\bar{\pi}(\kappa,\beta)$ as defined in Lemma S.6.3 such that if $(s_{t-1},\pi_{t-1}) = (s,\pi)$ with $\pi > \bar{\pi}(\kappa,\beta)$, then in every equilibrium*

(i) *If $s \in \{\kappa, 1-\kappa\}$, then $s_{t'} = s$ and $\pi_{t'} = \pi$ for all $t' \geq t$;*

(ii) *If $s \in \{0,1\}$, then $s_{t'} = s$ and $\pi_{t'} \in \{\pi_{t'-1}, \pi_{t'-1} + \tilde{\omega}(\pi_{t'-1})\}$ for all $t' \geq t$.*

S.7 Ability to expand one's share depends on the pie

In this appendix, we consider an extension of the benchmark model whereby an agent's ability to expand their share depends on the size of the pie. Formally, the law of motion of the agents' shares has κ replaced by $\tilde{\kappa}(\pi)$, where $\tilde{\kappa}(\cdot)$ is positive and decreasing function. Our key result that growth eventually halts will be shown to hold under Assumption S.7.1.

Assumption S.7.1 *The ability to expand one's share does not decline too fast as the pie grows: $\tilde{\kappa}(\pi)\pi$ is increasing and unbounded as π grows.*

We now proceed with the proof.

Lemma S.7.1 *In every equilibrium σ^*, if the state is $(s_{t-1},\pi_{t-1}) = (s,\pi)$, the players' continuation payoffs are such that: for any history h_{t-1},*

$$V_A(s,\pi \mid \sigma^*, h_{t-1}) \geq \frac{s\pi}{1-\beta} \quad \text{and} \quad V_B(s,\pi \mid \sigma^*, h_{t-1}) \geq \frac{(1-s)\pi}{1-\beta}. \tag{S.7.1}$$

Proof. The proof argument is similar to that of Lemma 1. ■

Lemma S.7.2 *Suppose Assumption S.7.1 holds. Let σ^* be an equilibrium and let $(s_{t-1},\pi_{t-1}) = (s,\pi)$ with some history h_{t-1}. There exists a threshold $\tilde{\pi}(\omega,\beta)$ such that: if π sufficiently large, $\pi > \tilde{\pi}(\omega,\beta)$, then it cannot be that $a_A^*(s,\pi,h_{t-1}) = a_B^*(s,\pi,h_{t-1}) = 1$.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. Using the fact that the parties' payoffs sum to π_t in every period t (and $\pi_{t'} \leq \pi_{t'-1} + 2\omega$ for all t'), we know that the sum of the parties' equilibrium payoffs is:

$$V_A(s,\pi \mid \sigma^*) + V_B(s,\pi \mid \sigma^*) = \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \leq \sum_{j=0}^{\infty} \beta^j (\pi + 2(j+1)\omega) = \frac{\pi(1-\beta)+2\omega}{(1-\beta)^2}. \tag{S.7.2}$$

We consider 2 cases.

Case 1: Suppose $s \in [\tilde{\kappa}(\pi), 1 - \tilde{\kappa}(\pi)]$. Party A's payoff from deviating is

$$(s + \tilde{\kappa}(\pi))(\pi + \omega) + \beta V_A(s + \tilde{\kappa}(\pi), \pi + \omega \mid \sigma^*) \geq \frac{(s + \tilde{\kappa}(\pi))(\pi + \omega)}{1 - \beta},$$

where the inequality follows from Lemma S.7.1. Hence, for σ^* to be an equilibrium, a necessary condition is

$$V_A(s, \pi \mid \sigma^*) \geq \frac{(s + \tilde{\kappa}(\pi))(\pi + \omega)}{1 - \beta}. \tag{S.7.3}$$

Similarly, Party B's payoff from deviating is

$$(1 - s + \tilde{\kappa}(\pi))(\pi + \omega) + \beta V_B(s - \tilde{\kappa}(\pi), \pi + \omega \mid \sigma^*) \geq \frac{(1 - s + \tilde{\kappa}(\pi))(\pi + \omega)}{1 - \beta},$$

and for σ^* to be an equilibrium, a necessary condition is

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1 - s + \tilde{\kappa}(\pi))(\pi + \omega)}{1 - \beta}. \tag{S.7.4}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.7.3) and (S.7.4), which yields $\frac{(\pi+\omega)(1+2\tilde{\kappa}(\pi))}{1-\beta}$, is weakly less than (S.7.2) — otherwise, at least one of the parties would have an incentive to deviate. That is, we require

$$\begin{aligned} \frac{(\pi + \omega)(1 + 2\tilde{\kappa}(\pi))}{1 - \beta} &\leq \frac{\pi(1 - \beta) + 2\omega}{(1 - \beta)^2} \\ \iff \pi(1 + 2\tilde{\kappa}(\pi)) + \omega(1 + 2\tilde{\kappa}(\pi)) &\leq \pi + \frac{2\omega}{1 - \beta} \\ 2\tilde{\kappa}(\pi)\pi + \omega(1 + 2\tilde{\kappa}(\pi)) - \frac{2\omega}{1 - \beta} &\leq 0. \end{aligned}$$

Under Assumption S.7.1, this inequality does not hold for sufficiently large π. Let $\tilde{\pi}_1(\omega, \beta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}_1(\omega, \beta)$.

Case 2: Suppose $s \in [0, \tilde{\kappa}(\pi)) \cup (1 - \tilde{\kappa}(\pi), 1]$. We focus on $s \in [0, \tilde{\kappa}(\pi))$ (argument for $s \in (1 - \tilde{\kappa}(\pi), 1]$ is similar). By the same argument as in Case 1, we have that Inequality (S.7.3) must hold at s

$$V_A(s, \pi \mid \sigma^*) \geq \frac{(s + \tilde{\kappa}(\pi))(\pi + \omega)}{1 - \beta}. \tag{S.7.5}$$

And, by Lemma S.7.1, Party B's equilibrium payoff at (s, π) must satisfy

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1 - s)(\pi + 2\omega)}{1 - \beta}. \tag{S.7.6}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.8.3) and the RHS of (S.7.6), which yields $\frac{(1+\tilde{\kappa}(\pi))(\pi+\omega)+(1-s)\omega}{1-\beta}$, is weakly less than (S.7.2) — otherwise, at least one of the parties would have an incentive to deviate. That is, we require

$$\frac{(1+\tilde{\kappa}(\pi))(\pi+\omega)+(1-s)\omega}{1-\beta} \leq \frac{\pi(1-\beta)+2\omega}{(1-\beta)^2}$$
$$\iff \tilde{\kappa}(\pi)(\pi+\omega)+(2-s)\omega \leq \frac{2\omega}{1-\beta}.$$

Under Assumption S.7.1, this inequality does not hold for sufficiently large π and any value of $s \in [0,1]$. Let $\tilde{\pi}_2(\omega,\beta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}_2(\omega,\beta)$ and for any $s \in [0,1]$.

By combining the bounds from Case 1 and Case 2, i.e., $\tilde{\pi}(\omega,\beta) > \max\{\tilde{\pi}_1(\omega,\beta), \tilde{\pi}_2(\omega,\beta)\}$, we obtain the desired result. ■

Lemma S.7.3 *Suppose Assumption S.7.1 holds. There exists $\bar{\pi}(\omega,\beta)$ such that, for any $(s_{t-1},\pi_{t-1}) = (s,\pi)$ with history h_{t-1} such that π is sufficiently large: $\pi > \bar{\pi}(\omega,\beta)$. There is no equilibrium σ^* such that:*

(i) *for some $s \in [\tilde{\kappa}(\pi),1]$, π and history h_{t-1}, $a_A^*(s,\pi,h_{t-1}) = 1$*

(ii) *for some $s \in [0,1-\tilde{\kappa}(\pi)]$, π and history h_{t-1}, $a_B^*(s,\pi,h_{t-1}) = 1$.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. By Lemma S.7.2 and if $\pi > \tilde{\pi}(\omega,\beta)$, it cannot be that $a_A^*(s,\pi,h) = a_B^*(s,\pi,h) = 1$. Therefore, at most one of the parties chooses action $a_j^t = 1$ and, hence, $\pi_t \leq \pi_{t-1}+\omega$ for all t. Combining this with the fact that the parties' payoffs sum to π_t in every period, we know that the sum of the parties' equilibrium payoffs is:

$$V_A(s,\pi \mid \sigma^*) + V_B(s,\pi \mid \sigma^*) = \sum_{j=0}^{\infty}\beta^j\pi_{t+j} \leq \sum_{j=0}^{\infty}\beta^j(\pi+(j+1)\omega) = \frac{\pi(1-\beta)+\omega}{(1-\beta)^2}. \tag{S.7.7}$$

Without loss of generality assume that $a_A^*(s,\pi,h) = 0$ and $a_B^*(s,\pi,h) = 1$. We will prove the statement in Part (ii) and, hence, assume $s \in [0,1-\tilde{\kappa}(\pi)]$ — the proof of Part (i) is similar. Party A's equilibrium payoff can be bounded from below as follows:

$$V_A(s,\pi \mid \sigma^*) = (s+\tilde{\kappa}(\pi))(\pi+\omega) + \beta V_A(s+\tilde{\kappa}(\pi),\pi+\omega \mid \sigma^*) \geq \frac{(s+\tilde{\kappa}(\pi))(\pi+\omega)}{1-\beta}, \tag{S.7.8}$$

where the inequality follows from Lemma S.7.1.

Now consider Party B. Party B's payoff from deviating is

$$(1-s)\pi + \beta V_B^*(s,\pi \mid \sigma^*) \geq \frac{(1-s)\pi}{1-\beta},$$

where the inequality follows from Lemma S.7.1. To be an equilibrium, a necessary condition is

$$V_B(s,\pi \mid \sigma^*) \geq \frac{(1-s)\pi}{1-\beta}. \tag{S.7.9}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.7.8) and (S.7.9), which yields $\frac{(s+\tilde{\kappa}(\pi))(\pi+\omega)+(1-s)\pi}{1-\beta}$, is weakly less than (S.7.7) — otherwise, party B would have an incentive to deviate at time t. That is, we require

$$\begin{aligned}
\frac{(s+\tilde{\kappa}(\pi))(\pi+\omega)+(1-s)\pi}{1-\beta} &\leq \frac{\pi(1-\beta)+\omega}{(1-\beta)^2} \\
\iff (1+\tilde{\kappa}(\pi))\pi + \omega(s+\tilde{\kappa}(\pi)) &\leq \pi + \frac{\omega}{(1-\beta)} \\
\iff \tilde{\kappa}(\pi)\pi + \omega(s+\tilde{\kappa}(\pi)) &\leq \frac{\omega}{(1-\beta)}.
\end{aligned}$$

Under Assumption S.7.1 and for any $s \in [0,1]$, this inequality does not hold for sufficiently large π. Let $\tilde{\pi}'(\omega,\beta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}'(\omega,\beta)$ and any $s \in [0,1]$. Combining this bound on π with that given in Lemma S.7.2 completes the proof. ■

Corollary S.7.1 *Suppose Assumption S.7.1 holds. There exists $\bar{\pi}(\omega,\beta)$ as defined in Lemma S.7.3 such that if $(s_{t-1},\pi_{t-1}) = (s,\pi)$ with $\pi > \bar{\pi}(\omega,\beta)$, then in every equilibrium*

(i) *If $s \in [\tilde{\kappa}(\pi), 1-\tilde{\kappa}(\pi)]$, then $s_{t'} = s$ and $\pi_{t'} = \pi$ for all $t' \geq t$;*

(ii) *If $s \notin [\tilde{\kappa}(\pi), 1-\tilde{\kappa}(\pi)]$, then $s_{t'} = s$ and $\pi_{t'} \in \{\pi_{t'-1}, \pi_{t'-1}+\omega\}$ for all $t' \geq t$.*

S.8 Depreciation (or destruction) of the pie

In this appendix, we consider an extension of the benchmark model whereby the size of pie is subject to depreciation (or destruction) if the agents do not expand it. Formally, the law of motion of the pie is such that if both agents choose to expand their share, then $\pi_t = (1-\delta)\pi_{t-1}$ and $s_t = s_{t-1}$, where $\delta \in (0,1)$. Our key result that growth eventually halts will be shown to hold under Assumption S.8.1.

Assumption S.8.1 *The rate of depreciation (or destruction) is small: $\delta \in [0, \frac{\kappa(1-\beta)}{\beta+(1-\beta)\kappa})$.*

We now proceed with the proof.

Lemma S.8.1 *In every equilibrium σ^*, if the state is $(s_{t-1}, \pi_{t-1}) = (s, \pi)$, the players' continuation payoffs are such that: for any history h_{t-1},*

$$V_A(s, \pi \mid \sigma^*, h_{t-1}) \geq \frac{(1-\delta)s\pi}{1-\beta(1-\delta)} \quad \text{and} \quad V_B(s, \pi \mid \sigma^*, h_{t-1}) \geq \frac{(1-\delta)(1-s)\pi}{1-\beta(1-\delta)}. \tag{S.8.1}$$

Proof. The proof argument is similar to that of Lemma 1. ■

Lemma S.8.2 *Suppose Assumption S.8.1 holds. Let σ^* be an equilibrium and let $(s_{t-1}, \pi_{t-1}) = (s, \pi)$ with some history h_{t-1}. There exists a threshold $\tilde{\pi}(\kappa, \omega, \beta, \delta)$ such that: if π sufficiently large, $\pi > \tilde{\pi}(\kappa, \omega, \beta, \delta)$, then it cannot be that $a_A^*(s, \pi, h_{t-1}) = a_B^*(s, \pi, h_{t-1}) = 1$.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. Using the fact that the parties' payoffs sum to π_t in every period t (and $\pi_{t'} \leq \pi_{t'-1} + 2\omega$ for all t'), we know that the sum of the parties' equilibrium payoffs is:

$$V_A(s, \pi \mid \sigma^*) + V_B(s, \pi \mid \sigma^*) = \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \leq \sum_{j=0}^{\infty} \beta^j (\pi + 2(j+1)\omega) = \frac{\pi(1-\beta) + 2\omega}{(1-\beta)^2}. \tag{S.8.2}$$

We consider 2 cases.

Case 1: Suppose $s \in \{\kappa, \ldots, 1-\kappa\}$. Party A's payoff from deviating is

$$(s+\kappa)(\pi+\omega) + \beta V_A(s+\kappa, \pi+\omega \mid \sigma^*) \geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta(1-\delta)},$$

where the inequality follows from Lemma S.8.1. Hence, for σ^* to be an equilibrium, a necessary condition is

$$V_A(s, \pi \mid \sigma^*) \geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta(1-\delta)}. \tag{S.8.3}$$

Similarly, Party B's payoff from deviating is

$$(1-s+\kappa)(\pi+\omega) + \beta V_B(s-\kappa, \pi+\omega \mid \sigma^*) \geq \frac{(1-s+\kappa)(\pi+\omega)}{1-\beta(1-\delta)},$$

and for σ^* to be an equilibrium, a necessary condition is

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1-s+\kappa)(\pi+\omega)}{1-\beta(1-\delta)}. \tag{S.8.4}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.8.3) and (S.8.4), which yields $\frac{(\pi+\omega)(1+2\kappa)}{1-\beta(1-\delta)}$, is weakly less than (S.7.2) — otherwise, at least one of the parties would have an incentive to deviate. That is, we require

$$
\begin{aligned}
& \frac{(\pi+\omega)(1+2\kappa)}{1-\beta(1-\delta)} \leq \frac{\pi(1-\beta)+2\omega}{(1-\beta)^2} \\
\iff \; & \pi\Big(\frac{2\kappa}{1-\beta(1-\delta)}+\frac{1}{1-\beta(1-\delta)}-\frac{1}{1-\beta}\Big)+\frac{\omega(1+2\kappa)}{1-\beta(1-\delta)} \leq \frac{2\omega}{(1-\beta)^2} \\
\iff \; & \pi\Big(\frac{2\kappa}{1-\beta(1-\delta)}-\frac{\beta\delta}{(1-\beta(1-\delta))(1-\beta)}\Big)+\frac{\omega(1+2\kappa)}{1-\beta(1-\delta)} \leq \frac{2\omega}{(1-\beta)^2}.
\end{aligned}
$$

Under Assumption S.8.1, the coefficient of π is strictly positive. Therefore, for sufficiently large π, the inequality does not hold. Let $\tilde{\pi}_1(\kappa,\omega,\beta,\delta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}_1(\kappa,\omega,\beta,\delta)$.

Case 2: Suppose $s \in \{0,1\}$. We focus on $s=0$ (argument for $s=1$ is similar). By the same argument as in Case 1, we have that Inequality (S.8.3) must hold with $s=0$ and, by Lemma S.8.1, Party B's equilibrium payoff at $(0,\pi)$ must satisfy

$$V_B(s,\pi \mid \sigma^*) \geq \frac{(\pi+2\omega)}{1-\beta(1-\delta)}. \tag{S.8.5}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.8.3) at $s=0$ and (S.8.5), which yields $\frac{\kappa(\pi+\omega)+(\pi+2\omega)}{1-\beta(1-\delta)}$, is weakly less than (S.8.2) — otherwise, at least one of the parties would have an incentive to deviate. That is, we require

$$
\begin{aligned}
& \frac{\kappa(\pi+\omega)+(\pi+2\omega)}{1-\beta(1-\delta)} \leq \frac{\pi(1-\beta)+2\omega}{(1-\beta)^2} \\
\iff \; & \pi\Big(\frac{1+\kappa}{1-\beta(1-\delta)}-\frac{1}{1-\beta}\Big)+\frac{2\omega+\omega\kappa}{1-\beta(1-\delta)} \leq \frac{2\omega}{(1-\beta)^2} \\
\iff \; & \pi\Big(\frac{\kappa(1-\beta)-\delta\beta}{(1-\beta(1-\delta))(1-\beta)}\Big)+\frac{2\omega+\omega\kappa}{1-\beta(1-\delta)} \leq \frac{2\omega}{(1-\beta)^2}.
\end{aligned}
$$

Under Assumption S.8.1, the coefficient of π is strictly positive. Therefore, for sufficiently large π, the inequality does not hold. Let $\tilde{\pi}_2(\kappa,\omega,\beta,\delta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}_2(\kappa,\omega,\beta,\delta)$.

By combining the bounds from Case 1 and Case 2, i.e., $\tilde{\pi}(\kappa,\omega,\beta,\delta) > \max\{\tilde{\pi}_1(\kappa,\omega,\beta,\delta),\tilde{\pi}_2(\kappa,\omega,\beta,\delta$ we obtain the desired result. ■

Lemma S.8.3 *Suppose Assumption S.8.1 holds. There exists $\bar{\pi}(\kappa,\omega,\beta,\delta)$ such that, for any $(s_{t-1},\pi_{t-1})=(s,\pi)$ with history h_{t-1} such that π is sufficiently large: $\pi > \bar{\pi}(\kappa,\omega,\beta,\delta)$. There is no equilibrium σ^* such that:*

(i) *for some $s \in \{\kappa,\ldots,1\}$, π and history h_{t-1}, $a_A^*(s,\pi,h_{t-1})=1$*

(ii) *for some $s \in \{0,\ldots,1-\kappa\}$, π and history h_{t-1}, $a_B^*(s,\pi,h_{t-1})=1$.*

Proof. For sake of a contradiction, suppose that such an equilibrium σ^* exists. By Lemma S.8.2 and if $\pi > \tilde{\pi}(\kappa, \omega, \beta, \delta)$, it cannot be that $a_A^*(s, \pi, h) = a_B^*(s, \pi, h) = 1$. Therefore, at most one of the parties chooses action $a_j^t = 1$ and, hence, $\pi_t \leq \pi_{t-1} + \omega$ for all t. Combining this with the fact that the parties' payoffs sum to π_t in every period, we know that the sum of the parties' equilibrium payoffs is:

$$V_A(s, \pi \mid \sigma^*) + V_B(s, \pi \mid \sigma^*) = \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \leq \sum_{j=0}^{\infty} \beta^j (\pi + (j+1)\omega) = \frac{\pi(1-\beta) + \omega}{(1-\beta)^2}. \tag{S.8.6}$$

Without loss of generality assume that $a_A^*(s, \pi, h) = 0$ and $a_B^*(s, \pi, h) = 1$. We will prove the statement in Part (ii) and, hence, assume $s \in \{0, \ldots, 1-\kappa\}$ — the proof of Part (i) is similar. Party A's equilibrium payoff can be bounded from below as follows:

$$V_A(s, \pi \mid \sigma^*) = (s+\kappa)(\pi+\omega) + \beta V_A(s+\kappa, \pi+\omega \mid \sigma^*) \geq \frac{(s+\kappa)(\pi+\omega)}{1-\beta(1-\delta)}, \tag{S.8.7}$$

where the inequality follows from Lemma S.8.1.

Now consider Party B. Party B's payoff from deviating is

$$(1-\delta)(1-s)\pi + \beta V_B^*(s, \pi \mid \sigma^*) \geq \frac{(1-\delta)(1-s)\pi}{1-\beta(1-\delta)},$$

where the inequality follows from Lemma S.8.1. To be an equilibrium, a necessary condition is

$$V_B(s, \pi \mid \sigma^*) \geq \frac{(1-\delta)(1-s)\pi}{1-\beta(1-\delta)}. \tag{S.8.8}$$

Therefore, to sustain the equilibrium σ^*, it must be that the sum of the RHS of (S.8.7) and (S.8.8), which yields $\frac{(s+\kappa)(\pi+\omega)+(1-\delta)(1-s)\pi}{1-\beta(1-\delta)}$, is weakly less than (S.8.6) — otherwise, party B would have an incentive to deviate at time t. That is, we require

$$\begin{aligned} & \frac{(s+\kappa)(\pi+\omega) + (1-\delta)(1-s)\pi}{1-\beta(1-\delta)} \leq \frac{\pi(1-\beta)+\omega}{(1-\beta)^2} \\ \iff \; & \pi\Big(\frac{1+\kappa-\delta(1-s)}{1-\beta(1-\delta)} - \frac{1}{1-\beta}\Big) + \frac{(s+\kappa)\omega}{1-\beta(1-\delta)} \leq \frac{\omega}{(1-\beta)^2}. \end{aligned} \tag{S.8.9}$$

The coefficient on π is positive if and only if

$$(1-\beta)(1+\kappa-\delta(1-s))-(1-\beta(1-\delta))>0$$
$$\iff (1-\beta)(1+\kappa)-(1-\beta)>\delta\Big(\beta+(1-\beta)(1-s)\Big)$$
$$\iff \frac{(1-\beta)\kappa}{\beta+(1-\beta)(1-s)}>\delta.$$

But because $s \leq 1-\kappa$, a sufficient condition for the coefficient to be positive is

$$\delta < \frac{(1-\beta)\kappa}{\beta+(1-\beta)\kappa},$$

which holds by Assumption S.8.1. Therefore, Inequality (S.8.9) does not hold for sufficiently large π. Let $\tilde{\pi}'(\kappa,\omega,\beta,\delta)$ be the first such value for which the inequality does not hold for all $\pi > \tilde{\pi}'(\kappa,\omega,\beta,\delta)$ and any $s \in [0,1]$. Combining this bound on π with that given in Lemma S.8.2 completes the proof. ■

Corollary S.8.1 *There exists $\bar{\pi}(\kappa,\omega,\beta,\delta)$ as defined in Lemma S.8.3 such that if $(s_{t-1},\pi_{t-1}) = (s,\pi)$ with $\pi > \bar{\pi}(\kappa,\omega,\beta,\delta)$, then in every equilibrium*

(i) *If $s \notin \{0,1\}$, then $s_{t'} = s$ and $\pi_{t'} = \pi$ for all $t' \geq t$;*

(ii) *If $s \in \{0,1\}$, then $s_{t'} = s$ and $\pi_{t'} \in \{\pi_{t'-1}, \pi_{t'-1}+\omega\}$ for all $t' \geq t$.*

S.9 Continuous action extension

Example 1 *Let $(s_0,\pi_0) = (s,\pi)$ such that $s \in [\kappa, 1-\kappa]$ and π is arbitrarily large. Suppose, for sake of a contradiction, that there exists an equilibrium σ^* such that the agents continually and symmetrically cooperate at a decreasing level. That is, for $j \in \{A,B\}$, $a_A^t = a_B^t > 0$ and $a_j^t \geq a_j^{t+1}$ for all $t \geq 1$.*

Via a similar argument applied in the benchmark model, we now show that this is not an equilibrium. Without loss of generality, we'll focus on agent A's incentive to deviate. Defining $\bar{a}_\pi = a_A^1 = a_B^1$, agent A's equilibrium payoff is bounded above by

$$\sum_{t=1}^{\infty}\beta^{t-1}(\pi+\omega\sum_{t=1}^{\infty}2a_j^t)s \leq \sum_{t=1}^{\infty}\beta^{t-1}(\pi+2t\bar{a}_\pi\omega)s = \frac{\pi s}{1-\beta}+\frac{2\omega\bar{a}_\pi s}{(1-\beta)^2}, \tag{S.9.1}$$

where the inequality follows because $a_j^t \geq a_j^{t+1}$ for all $t \geq 1$. Now suppose agent A deviates to $a_A^1 = 0$. Applying a similar argument per Lemma 1, a lower bound on A's payoff from this defection is

$$\frac{(\pi+\omega\bar{a}_\pi)(s+\kappa\bar{a}_\pi)}{1-\beta} = \frac{\pi s}{1-\beta}+\bar{a}_\pi\frac{\pi\kappa+\omega(s+\kappa\bar{a}_\pi)}{1-\beta}. \tag{S.9.2}$$

This deviation is strictly profitable if (S.9.2) exceeds the right-hand side of (S.9.1): after rearranging, this gives

$$\bar{a}_\pi \frac{\pi\kappa + \omega(s + \kappa\bar{a}_\pi)}{1-\beta} > \frac{2\omega\bar{a}_\pi s}{(1-\beta)^2} \iff \frac{\pi\kappa + \omega(s + \kappa\bar{a}_\pi)}{1-\beta} > \frac{2\omega s}{(1-\beta)^2},$$

which holds for sufficiently large π *(and any* $\bar{a}_\pi > 0$*). Thus, we have a contradiction and no such equilibrium exists.*