Allocating essential inputs*
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Abstract

Regulators must often allocate essential inputs, such as spectrum rights, transmission capacity or slots for trains or plane departures, which can transform the structure of the downstream market. These decisions involve a trade-off, as provisions aimed at fostering competition and lowering prices for consumers also tend to limit the proceeds from the sale of the inputs. We characterize the optimal allocation, from the standpoints of consumer and total welfare. We show that standard auctions yield substantially different outcomes. Finally, we show that the optimal auction design will, at times, be characterized by *bunching*, that is, the allocation being independent of offers of the firms.

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1 Introduction

Over the past decades, regulators have increasingly turned to competition as an alternative to direct price control. Regulators have done this by requiring, via auction,

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divestiture or other means, the allocation of essential inputs to multiple parties. Arguably, the application of game theory to the sale of spectrum rights has been one of the most successful applications of economic analysis to solve a practical problem. And new auction designs are being adapted to the allocation of other formerly regulated upstream inputs, notably departure slots for trains and planes, and in the energy sector for various inputs, such as transmission and capacity. This paper discusses how auctions can adversely affect downstream market concentration. We show that the auction design that optimally addresses this trade-off, will, at times, be characterized by *bunching*, that is, the allocation being independent of the firms' offers.

This is the case, in part, because auctions for reallocation or new tranches of essential inputs often affect the competitors' costs and quality of service and therefore downstream market concentration. Bids will then reflect the impact on market structure, which we call a *foreclosure value* in addition to the *technical value* of the input in providing the good or service. Consumer welfare can then be adversely affected when an auction for an upstream input increases concentration downstream. Thus, an open auction for an essential input cannot always be relied on to allocate the input in a way that maximizes total welfare. For this reason, caps, set-asides and other measures are imposed to limit risk of an auction resulting in consolidation. These restrictions can be on spectrum bandwidth, transmission capacity or airport landing slots, which affect firms' costs or the quality of their offerings and can transform market structure.

In imposing such restrictions in an auction, regulators face a trade-off. On the one hand, they may seek to maximize consumer or social welfare. On the other hand, they may face political pressures to maximize revenue from the sale or lease of such resources. This tension can be particularly acute when the competing firms start with different levels of inputs and market shares, as the more established firms are then likely to be willing to outbid the weaker rivals in order to strengthen their market position. Our results indicate that, to address these concerns, regulators should, at times, include provisions, such as caps, and set-asides, that effectively eliminate competition in the auction.

In most auctions for an upstream input, the ex ante market structure includes one or a few asymmetric incumbents, and a limited set of potential challengers;[1]. To capture

[1] In four-firm markets for wireless services, there is usually a large gap (10% or more) between

the essential features of such auctions we consider a simplified and stylized model in which a single incumbent initially enjoys a cost advantage over a potential entrant. A regulator can allocate a divisible amount of newly released essential input, which can either widen or narrow the cost differential. This two-firm model reflects the regulators' trade-off mentioned above: awarding the newly released input to the incumbent tends to generate higher revenue, whereas awarding it to the challenger promotes competition in the market.

When the regulator's only objective is maximizing consumer surplus, or puts only a moderate weight on auction revenues or profits, the optimal policy is to allocate the resource so as to equalize the costs of the two firms, as consumer prices are lowest in that case. However, equalizing costs tends to minimize profits, and the willingness and ability of the firms to pay for the resource are thus also the lowest. Hence, when the regulator puts a large weight on profits or auction revenues, the optimal allocation limits the cost advantage of the incumbent, but no longer tries to equalize the firms' costs; it instead leaves an advantage to one firm – either one, if the incumbent's advantage can be overcome, and the incumbent otherwise. The winner then pays an amount equal to its operating profit.

Our baseline model is highly stylized; in particular, we assume that firms offer perfect substitutes and only differ in their costs. We show that these insights carry over when the firms offer differentiated goods or services. Specifically, we consider a Hotelling style model of differentiated products competition with elastic individual demands. We show that the optimal allocation is close to that in the baseline model of Bertrand competition when products are not too differentiated.[2]. We also characterize that the optimal outcome when the products are not close substitutes. In some, but not all, cases the optimal outcome will tend to equalize costs.

We then characterize the optimal allocation for the case in which the each firm's cost is private information. We find that the objective of the regulator and of the firms are so conflicting that it is typically impossible to induce the two firms to reveal costs in a

the market shares of the second and third largest operators; in three-firm markets, the dominant firm's market share can exceed 60%, as is the case in Switzerland. Where auctions are used for train departure slots, there is typically one incumbent facing one or a few potential challengers

[2]In a Cournot model with linear demand, the optimal allocation minimizes the sum of the two firms' costs; hence, it leads again to equalizing them when costs are a decreasing convex function of bandwidth.

useful way. Instead, the optimal allocation is similar to the previous one, except that the regulator must base its allocation decision on the expected value of its objective and not on reports, or bids, of the firms.

Next, we contrast this optimal allocation with the outcome that would arise in the types of auctions commonly used to allocate radio spectrum licenses. In particular, we compare the outcomes of sequential auctions, sealed-bid Vickrey-Clarke-Groves ("VCG") auctions, and simultaneous multi-round ascending ("SMRA") or clock auctions. In our framework, all these auctions exacerbate the incumbency advantage, as the incumbent always ends up winning all the newly released input. Moreover, revenues are the same in the VCG and clock auctions, and lower in a sequential auction.

Finally, we consider various measures that the regulator can adopt to promote competition, such as caps and set-asides, and discuss how they can be used to implement the optimal allocation.

The remainder of this section discusses the related literature. Section 2 presents the model; Section 3 characterizes the optimal allocation in the case of perfect information, and Section 4 extends the analysis to the case where the challenger's cost is private information. Section 5 examines the outcome of standard auction formats used in these settings. Section 6 discusses the use of regulatory instruments to implement the optimal allocation. Section 7 concludes.

Related Literature

Our insights are reminiscent of the literature on second-sourcing. Although the focus there was mostly on competition "for the market" rather than "in the market,"it was recognized that the awarding of a contract or a procurement decision could affect the purchaser's ability to switch to alternative suppliers later on, or the suppliers' ability to compete effectively for subsequent contracts.[3] One set of papers have considered the impact of the chosen market structure (e.g., monopoly or duopoly) on prices and welfare.[4] We build on this literature and study how the allocation of a (divisible) essential input can further affect the market structure and the outcome of competition.

[3]For instance, Anton and Yao (1987) show that second-sourcing can be used to reduce suppliers' informational rents. Rob (1986), Laffont and Tirole (1988) and Riordan and Sappington (1989) consider the trade-off between such *ex post* savings and suppliers' *ex ante* R&D incentives.

[4]See, e.g., Dana and Spier (1994), McGuire and Riordan (1995) and Auriol and Laffont (1992).

Our paper relates to the large literature on optimal auction design, starting with Myerson (1981), the classic paper for single-object auctions.[5] Post-auction interaction generates externalities not only between the firms and their customers in the downstream market,[6] but also among the bidders in the auction: each bidder's payoff depends not only on what it wins, but also on what its rivals win. A series of papers, most notably by Jehiel and Moldovanu, have explored single-object auctions with such externalities.[7]

Much of the most closely related literature is concerned with allocation of spectrum licenses - but there are a number of papers which consider other types of upstream inputs such as for arrival and departure slots for planes and trains.[8] More specifically, a European mandate for open access for railways has spurred efforts, and related research to allocate train slots via auction.[9] Jehiel and Moldovanu (2003) consider several examples of auctions of fixed-size spectrum licenses, and discuss the likely market outcomes.[10] Mayo and Sappington (2016) explore a Hotelling model in which a single block of spectrum is available. They show that an auction is unlikely to result in an optimal allocation and consider various corrective handicapping policies. Klemperer (2004) warns regulators against the temptation of taking measures to increase auction revenues at the cost of discouraging entry, and suggests instead the Anglo-Dutch hybrid auction as a way to balance the trade-off between revenues and post-auction concentration. Cramton et al. (2011) note that provisions favoring entrants need not always sacrifice auction revenues; they provide one example, with one incumbent and several symmetric potential entrants, in which setting a license aside for the entrants does not affect auction revenues. Cramton et al. (2011) also argue that, absent provisions to handicap large bidders, entrants and small participants are unlikely to win new spectrum; hence, regulators should concentrate their efforts on achieving an efficient allocation rather than revenue maximization. Finally, Janssen and Karamychev (2009, 2010) study the

[5] See also Maskin et al. (1989) and Armstrong (2000). Also, Milgrom (2004) provides sufficient conditions for the simultaneous ascending auction to result in a Pareto-optimal equilibrium.

[6] Borenstein (1988) shows that the resulting discrepancy between private and social benefits can lead to inefficient outcomes.

[7] See Jehiel and Moldovanu (2001), Jehiel and Moldovanu (2000), and Jehiel et al. (1996). Also, Varma (2003) and Goeree (2003) consider auctions in which bids convey signals that affect rivals' behavior after the auction. See Salant (2014) for a more extensive discussion

[8] See Nilsson (1999) and Cox et al (2002) and on airline landing slots,Rassenti, et al (1982).

[9] See Cherbonnier, Salant and Van der Straten (2021).

[10] See also Hoppe et al. (2006), who show that limiting the number of licenses to be auctioned may foster entry, by exacerbating free-riding among incumbents' preemption strategies.

impact of auctions on *ex post* prices.[11] Their analysis, however, focuses on firms' risk preferences rather than on the impact on market structure.[12]

A couple of papers consider multi-object auctions. Levin and Skrzypacz (2016) examine bidder incentives in a Combinatorial Clock Auction ("CCA"). They show that bidders may bid more aggressively on packages that they anticipate to lose, in order to increase the price paid by rivals. More closely related to our paper is Kasberger (2017), who examines auction designs that can achieve an optimal allocation, in a setting where Cournot competitors bid on the entire allocation across firms.

There is also some empirical work that sheds light on the benefits of competition. Landier and Thesmar (2012) evaluate the macroeconomic impact of the entry of the fourth telecom operator in France, Free. They find that entry benefited the population in several ways. First, it had an immediate effect on consumer prices, which increased the purchasing power of the population. Second, the price shock induced by the enhancement of competition created between 16,000 and 30,000 jobs in France. The authors argue that far from distressing the financial position of incumbents, the increased competition encouraged investments in the sector.[13] Hazlett and Muñoz (2009) conducted a large-scale cross-country analysis of spectrum awards and found a significant positive relation between market concentration and consumer prices. This suggests that the social benefits from encouraging entry can more than offset the loss of auction revenue from spectrum withholding or concentration. Salant and Ershov (2022) examine the interaction of pre-auction concentration and auctions on the auction outcome and post-auction concentration using 10 years of European data. They find that the stronger incumbents tend to win a larger share of low band (below 1 GHz) spectrum and that the impact of ex ante market asymmetries is more pronounced the more asymmetric is a market to start with.

[11] See also Janssen and Karamychev (2007), who show that auctions do not always select the most efficient firm.

[12] Other papers include Moldovanu and Sela (2001) in which a seller is conducting an all pay auction so as to maximize the sum of the bidder payments (or efforts). Eső et al. (2010) examines efficient capacity allocations when there is Cournot competition in the downstream market and Brocas (2013) examines optimal auction design of a single, indivisible object when there are externalities.

[13] Woroch (2018) using US regional data finds higher spectrum concentration is associated with higher penetration. However, he admittedly cannot control for endogeneity of spectrum concentration.

2 Model

For the sake of exposition, we consider the case of mobile communication services, where spectrum constitutes an essential input, and study the optimal allocation of new spectrum. It should be clear that the analysis can be readily transposed to key inputs in other industries.

Two firms, an incumbent I and a new entrant E, compete à la Bertrand for a consumer demand $D(p)$. The operators have constant returns to scale, but their costs depend on how much bandwidth they have: the more spectrum a mobile operator has, the more data it can carry at a given cell-site; it can thus maintain a given network capacity with fewer cells, and thus at lower costs.

Each firm starts with a given amount of bandwidth, $B_i, i = I, E$. Its capacity to offer service will be denote $B_i - \theta_i$. The parameter θ_i is a measure of each firm's handicap, which is NOT observable to the Regulator or the other firm. Absent any new bandwidth allocation, a firm's (constant) marginal cost of serving a customer is

$$\bar{c}_i = c(B_i - \theta_i),$$

where $\bar{c}_i$ denotes its initial unit cost, and $c(\cdot)$ is a strictly decreasing function (i.e., $c'(\cdot) < 0$) that is common to both firms.

We will assume that prior to the allocation of new spectrum, $B_I - \theta_I > B_E - \theta_E$ so that $\bar{c}_I = c(B_I - \theta_I) < \bar{c}_E = c(B_E - \theta_E)$ and the entrant obtains no profit, whereas the incumbent obtains a profit which, assuming that the entrant exerts effective competitive pressure (see condition (4) below), is equal to:

$$\Pi(B_I - \theta_I, B_E - \theta_E) \equiv [c(B_E - \theta_E) - c(B_I - \theta_I)] D(c(B_E - \theta_E)), \qquad (1)$$

which increases with the bandwidth advantage of the incumbent:[14]

[14] In what follows, $\partial_i f(\cdot)$ denotes the partial derivative of $f(\cdot)$ with respect to its i^{th} argument.

$$\partial_1\Pi(B_I - \theta_I), (B_I - \theta_I)$$
$$= - D(c((B_E - \theta_E))c'(B_I - \theta_I) > 0, \quad (2)$$
$$\partial_2\Pi(B_I - \theta_I, B_E - \theta_E)$$
$$=\partial_2\Pi((B_I - \theta_I, (B_E - \theta_E)$$
$$=\Big[D(c(B_E - \theta_E)) + \big[c(B_E - \theta_E) - c(B_I - \theta_I)\big] D'(c(B_E - \theta_E))\Big]$$
$$\times c'(B_E - \theta_E) < 0 \quad (3)$$

Let Δ denote the amount of new spectrum available. Each firm i can thus obtain an additional bandwidth $b_i \geq 0$, subject to $b_I + b_E \leq \Delta$. With this additional bandwidth, the cost of firm i can lie anywhere in the range $[\underline{c}_i, \bar{c}_i]$, where

$$\underline{c}_i = c\left(B_i - \theta_i + \Delta\right)$$

denotes the lowest cost that firm i can achieve with all the additional spectrum. We assume that cost differences are never so drastic that competition is ineffective; that is, the incumbent cannot charge its monopoly price, even if it obtains all the additional spectrum:

$$\bar{c}_E < p^m\left(\underline{c}_I\right), \quad (4)$$

where $p^m(c) \equiv \min_p \{p \mid p \in \arg\max_{\tilde{p}} (\tilde{p} - c) D(\tilde{p})\}$. This assumption ensures that the competitive price is always equal to the higher of the two costs:[15]

- As long as $B_I + b_I - \theta_I > B_E + b_E - \theta_E$, the incumbent maintains a cost advantage (that is, $c_I = c(B_I + b_I - \theta_I) < c_E = c(B_E + b_E - \theta_E)$) and thus wins the downstream market. The profits are then $\pi_E = 0$ and $\pi_I = \Pi(B_I + b_I - \theta_I, B_E + b_E - \theta_E)$, and consumer surplus is equal to $S(c(B_E + b_E - \theta_E))$, where

$$S(p) \equiv \int_p^{+\infty} D(x)\, dx.$$

- If instead $B_I + b_I - \theta_! < B_E + b_E - \theta_E$, the entrant obtains a lower cost; the profits

[15]It also ensures that demand is positive in the relevant range.

of the two firms are then $\pi_I = 0$ and $\pi_E = \Pi(B_E + b_E - \theta_E, B_I + b_I - \theta_I)$, and consumer surplus is equal to $S(c(B_I + b_I - \theta_I))$.

This parameter θ should not be interpreted literally as the difference in spectrum holdings (which is likely to be public information); rather, we use it as a proxy for the initial cost asymmetry between the two firms. In practice, an incumbent benefits from scale economies arising from its existing spectrum holdings and from its denser network of cell sites; it may also benefit from a better bargaining position when dealing with equipment suppliers, and possibly from superior know-how and expertise (due, e.g., from learning-by-doing). Firm i' cost can thus be expressed as $C(A_i)$, where A_i denotes firm i's total asset and is of the form $A_i = K_i + B_i$, where K_i reflects firm i's accumulated capital other than spectrum, and $K_I > K_E$. In this setting, the relevant cost handicap of the entrant is given by $\theta = K_I - K_E + B_I - B_E$, and is likely to be private information even if the spectrum holdings B_I and B_E are public knowledge. For the sake of exposition, and in line with our previous analysis, we simply denote by B_I and B_E the two firms' initial "total assets".

3 Complete Information

As a benchmark, this section characterizes the optimal spectrum allocation when costs are public information, assuming the regulator cannot regulate price directly.[16] So, we assume $B_I > B_E$ and $\theta_E = \theta_I = 0$. We first consider the case where the regulator aims at maximizing consumer surplus, before considering the case where it aims at maximizing social welfare, accounting for a social cost of public funds.

3.1 Consumer Surplus

We first note that a regulator maximizing consumer surplus should seek to minimize the cost asymmetry among the two firms:

[16]Should the regulator be able to regulate price, it can give all the spectrum to the incumbent, and set price equal to cost.

Proposition 1 *To maximize consumer surplus, it is optimal to allocate all the additional spectrum among the two firms so as minimize their cost difference. The associated consumer price is*

$$p^S \equiv \max\{\underline{c}_E, \hat{c}\}$$

where $\underline{c}_E = c(B_E + \Delta)$ *and*

$$\hat{c} \equiv c\left(\frac{B_I + B_E + \Delta}{2}\right).$$

Proof. See Appendix A. ■

The intuition is straightforward. Maximizing consumer surplus amounts to minimizing the competitive price, which is equal to the lower of the two costs, $c_I = c(B_I + b_I)$ and $c_E = c(B_E + b_E)$. Hence, it is always optimal to distribute all the additional spectrum, and if there is enough spectrum to offset the initial cost difference, it is optimal to allocate this spectrum so as to equate the two costs (leading to $p = \hat{c}$). If instead it is impossible to do so, then it is optimal to allocate all the additional spectrum to the entrant (leading to $p = \underline{c}_E$), so as to minimize the cost asymmetry. Interestingly, a former FCC Chief Technology Office indeed argued that equalizing spectrum holdings is essential for effective competition among carriers.[17]

3.2 Social Welfare

In practice, industry regulators may need to pay attention to firms' profitability and/or to the revenues generated by the scarce resources that they manage. First, firms would not operate at a loss absent socially costly subsidies. This concern does not affect the findings of Proposition 1, however: the described allocation remains optimal even when taking into account firms' budget constraints, as the entrant always obtains zero profit and the incumbent obtains a non-negative profit. Second, firms' financial contributions (e.g., in the form of – lump-sum – spectrum licensing fees) reduce the public budget deficit and/or lower distortionary taxes. To account for this concern, we now suppose that the regulator aims at maximizing social welfare, defined as the sum of consumer surplus and, with a weight $\lambda \in [0, 1)$, the revenues generated from the allocation of the

[17] See Peha (2017).

resource Δ, subject to firms' viability constraints; that is:

- Any transfer t obtained from the firms generates a social gain λt, representing the social benefit from reducing budget deficit or distortionary taxes; social welfare is thus given by

$$S + \lambda (t_I + t_E),$$

where t_i denotes the transfer obtained from firm $i = I, E$ and, as before, $S = S(p)$ denotes consumer surplus.

- The regulator must accommodate the firms' profitability constraints: for $i = I, E$,

$$\pi_i - t_i \geq 0.$$

It follows that it is optimal to choose $t_i = \pi_i$ for $i = I, E$; social welfare can thus be expressed as

$$S + \lambda \Pi,$$

where $\Pi = \pi_I + \pi_E$ denotes total industry profit.

Obviously, it is again optimal to allocate all the additional bandwidth Δ:

Lemma 1 *It is socially optimal to allocate all the additional spectrum.*

Proof. When $\lambda > 0$, giving any residual bandwidth to the firm with the lower cost (or to either firm, if both have the same cost) would further reduce its cost and increase industry profit, and thus the obtained revenues, without any adverse effect on consumers. When $\lambda = 0$ (i.e., the regulatory objective focuses on consumer surplus), giving any residual bandwidth to the firm with the higher cost (or sharing it equally between the two firms, if both have the same cost) would reduce the price and enhance consumer surplus. ■

Therefore, without loss of generality, we can restrict attention to spectrum allocations of the form $b_I = \Delta - b_E$, for some $b_E \in [0, \Delta]$. Furthermore:

- If $b_E < (B_I - B_E + \Delta)/2$, this spectrum allocation yields a competitive equilibrium of the form

$$p = c_E > \hat{c} > c_I = \gamma(p), \tag{5}$$

where

$$\gamma(p) \equiv c\left(B_I + B_E + \Delta - c^{-1}(p)\right)$$

denotes the lower cost among the two firms, when the bandwidth is allocated so as to set the higher cost to p. The resulting social welfare that can be expressed as:

$$W(p;\lambda) \equiv S(p) + \lambda[p - \gamma(p)] D(p). \quad (6)$$

- If instead $b_E > (B_I - B_E + \Delta)/2$ (in which case the feasibility condition $\Delta \geq b_E$ requires $\Delta > B_I - B_E$), then

$$p = c_I > \hat{c} > c_E = \gamma(p), \quad (7)$$

  which, keeping p constant, generates the same social welfare as the equilibrium described by (5) (the roles of the two firms are simply swapped).

Hence, looking for the optimal spectrum allocation amounts to maximizing $W(p;\lambda)$ in the range

$$p \in \left[p^S, \bar{c}_E\right],$$

where

$$p^S \equiv \max\{\underline{c}_E, \hat{c}\},$$

with the caveat that, if $p^S = \hat{c} > \underline{c}_E$, any price $p \in [\hat{c}, \bar{c}_I]$ can be achieved in two equivalent ways, by conferring the same cost advantage to either firm.

We have:

$$\begin{aligned}\frac{\partial W}{\partial p}(p;\lambda) &= -D(p) + \lambda[1 - \gamma'(p)] D(p) + \lambda[p - \gamma(p)] D'(p) \\ &= \lambda D(p)\left[\rho(p) - \gamma'(p) - \frac{1}{\lambda}\right] \quad (8)\end{aligned}$$

where

$$\rho(p) \equiv 1 - \frac{p - \gamma(p)}{\mu(p)},$$

and

$$\mu(p) \equiv -\frac{D(p)}{D'(p)}$$

denotes the market power function – see Weyl and Fabinger (2013); $\rho(p)$ can be interpreted as a competition index: it is equal to 1 when $p = \gamma(p) (= \hat{c})$, that is, when both firms face the same cost and thus exert perfect competition on each other, and would be instead equal to 0 if $p = p^m(\gamma(c))$, that is, if the firm with the higher cost were no longer exerting any competitive pressure on the other firm – our working assumption rules out this case, implying $\rho(p) < 1$).

To ensure that maximizing $W(p;\lambda)$ yields a unique solution, we will maintain the following regularity conditions:

Assumption A:

1. The unit cost function $c(\cdot)$ is (strictly decreasing and) weakly convex: $c''(B) \geq 0 > c'(B)$ for any $B \geq 0$.

2. The market power function is weakly decreasing in the relevant range: $\mu'(p) \leq 0$ for any $p \in \left[p^S, \bar{c}_E\right]$.

Assumption A.1 asserts that, while using more spectrum enables the firms to reduce their costs (i.e., $c'(\cdot) < 0$), this is less and less so as more bandwidth becomes available. This is indeed often the case in practice. For instance, in his study of mobile network costs, Peha (2017) notes: "(We calculate) capacity in a way that is traditional for a capacity-limited cellular network, where an MNO's capacity increases linearly with the number of cell towers that the MNO uses, and linearly with the amount of spectrum it holds." As capacity requirements are proportional to demand, and capacity per tower is proportional to the bandwidth available, it follows that unit costs per subscriber are inversely proportional to bandwidth.[18] Also, for other applications of this analysis, such as the allocation of airplane landing or train departure slots, average waiting time, and waiting costs, will be a convex decreasing function of the number of slots. As the next Lemma shows, Assumption A.1 ensures that, while $\gamma(p)$ strictly decreases as p increases, it does so at a decreasing rate.

Assumption A.2 amounts to assuming that the demand function is log-concave; it also implies that the monopoly pass-through rate is lower than one; together with Assumption A.1, it ensures that the competition index $\rho(p)$ strictly decreases with p. We

[18] Algebraically, let T denote the number of cell towers needed. Then $T = \alpha\frac{D}{B}$ And then the cost per subscriber is $\frac{\alpha\tau}{B}$ where τ is the cost per tower and α denotes the capacity per tower when $B = 1$.

have:

Lemma 2 *For any* $p \in \left[p^S, \bar{c}_E\right]$*:*

(i) $\gamma(p) \leq \hat{c} \leq p$ *(with strict inequalities for* $p > \hat{c}$*) and, under Assumption A.1:*

$$-1 \leq \gamma'(p) < 0 \leq \gamma''(p).$$

(ii) $0 < \rho(p) < 1$ *and, under Assumption A:*

$$\rho'(p) < 0.$$

Proof. See Appendix B. ■

The following Proposition characterizes the socially optimal allocation:

Proposition 2 *Let*

$$\underline{\lambda} \equiv \frac{1}{\rho(p^S) - \gamma'(p^S)} \text{ and } \bar{\lambda} \equiv \frac{1}{\rho(\bar{c}_E) - \gamma'(\bar{c}_E)}.$$

Under Assumption A, $\underline{\lambda} = 1/2$ *if* $p^S = \hat{c} \geq \underline{c}_E$ *and* $\underline{\lambda} > 1/2$ *otherwise,* $\bar{\lambda} > \underline{\lambda}$*, and the spectrum allocation that maximizes social welfare yields a unique equilibrium price,* p^W*, which is as follows:*

- *if* $\lambda \leq \underline{\lambda}$*, then it is optimal to minimize the cost difference:* $p^W = p^S$*;*
- *if instead* $\lambda \geq \bar{\lambda}$*, then it is optimal to allocate all the additional bandwidth to the incumbent:* $p^W = \bar{c}_E$*;*
- *finally, if* $\underline{\lambda} < \lambda < \bar{\lambda}$*, then* p^W *lies (strictly) between* p^S *and* $\bar{c}_E$*, and is the unique solution to*

$$\rho(p) - \gamma'(p) = \frac{1}{\lambda}. \tag{9}$$

Furthermore:

- *when* $p^S = \hat{c} > \underline{c}_E$*, there exists* $\hat{\lambda} \in \left(1/2, \bar{\lambda}\right)$ *such that, if* $\lambda \leq \hat{\lambda}$*, there are two optimal spectrum allocations, giving the same cost advantage to either firm;*

– *otherwise (i.e., either* $p^S = \underline{c}_E \geq \hat{c}$*, or* $p^S = \hat{c} > \underline{c}_E$ *and* $\lambda > \hat{\lambda}$*), the optimal spectrum allocation is unique and maintains a cost advantage to the incumbent.*

Proof. See Appendix C. ■

As long as the weight on profits remains small, maximizing social welfare still amounts to minimizing cost differences, as is the case when focusing on consumer surplus. This is no longer the case, however, when the weight placed on profit becomes significant. In particular, even if cost equalization is feasible (i.e., the additional bandwidth is large enough to offset the initial asymmetry: $\Delta \geq B_I - B_E$, and so $p^S = \hat{c}$), it is no longer optimal whenever $\lambda > 1/2$, that is, whenever the weight on profits is more than half of that on consumer surplus.[19] Hence, in the particular case where the regulatory objective is "total welfare," measured by the sum of consumer surplus and profits (i.e., $\lambda = 1$), it is optimal to maintain a cost advantage (for the incumbent when $p^W > \bar{c}_I$, for either firm otherwise). This insight is quite robust: in any setting in which firms compete in prices, cost equalization is a local minimum of "total welfare," measured by the sum of consumer surplus and profits. To see this, note that total welfare can be expressed as

$$\Pi + S = S(p) + (p - c) D(p),$$

where $c = \gamma(p)$. Starting from cost equalization, where $p = c = \hat{c}$, and using $S'(p) = -D(p)$, introducing a slight asymmetry ($dp > 0 > dc = \gamma'(\hat{c}) dp$) increases total welfare:

$$d(\Pi + S) = (p - c) D'(c) dp - D(c) dc|_{p=c=\hat{c}} = -D(\hat{c}) dc > 0.$$

As the weight on profit further increases, the socially optimal price increases, up to the point that it may become optimal to give all the additional spectrum to the incumbent, so as to maximize industry profit. This, however, can occur only when the entrant still maintains a significant competitive pressure on the incumbent and/or the weight on profits exceeds that on consumer surplus.[20] By contrast, an increase in

[19] Indeed, when $p^S = \hat{c}\,(> \underline{c}_E)$, $\rho(p^S) = -\gamma'(p^S) = 1$ and, thus, $\underline{\lambda} = 1/2$.

[20] To see this, note that, in the limit case where $\bar{c}_E = p^m(\underline{c}_I)$, we have $\rho(\bar{c}_E) = 0$ and, thus, $\bar{\lambda} = -1/\gamma'(\bar{c}_E) > 1$.

the bandwidth initially available to either firm, or in the additional bandwidth made available, leads to a reduction in the socially optimal price. More precisely:

• If $\lambda \leq \underline{\lambda}$, then $p^W = c\,(B_E + \Delta)$; p^W thus does not depend on B_I, but strictly decreases as B_E or Δ increases.

• If instead $\lambda \geq \bar{\lambda}$, then $p^W = c\,(B_E)$; p^W thus only depends on B_E, and strictly decreases as B_E increases.

• Finally, it is optimal to divide the additional bandwidth, and we have:

Corollary 1 *As long as $\underline{\lambda} < \lambda < \bar{\lambda}$, the socially optimal price strictly increases with λ, but strictly decreases as the total bandwidth, $B_I + B_E + \Delta$, increases.*

Furthermore, as long as $p^S = \underline{c}_E \geq \hat{c}$, or $p^S = \hat{c} > \underline{c}_E$ and $\lambda > \hat{\lambda}$, the unique optimal spectrum allocation maintains a cost advantage to the incumbent and is such that:

- *any increase in λ leads to a re-allocation of the additional bandwidth Δ in favor of the incumbent;*
- *any increase in the additional bandwidth Δ is shared between the two firms;*
- *any increase in the bandwidth initially available to one firm, B_E or B_I, leads to a re-allocation of the additional bandwidth Δ in favor of the other firm, which is however limited so as to ensure that both firms end-up with a larger total bandwidth.*

Proof. See Appendix D. ■

3.3 Product Differentiation

Focusing on perfect substitutes simplifies the analysis and highlights key determinants of the optimal input allocation. One drawback is a winner-takes-all.feature, in which firms fight intensively to corner the market. In practice, firms often differentiate their offerings, which softens competition and enables them to share the market. To check the robustness of our analysis, in this section we extend our analysis to standard models of product differentiation. Specifically, we maintain the assumption that firms face constant unit costs, which are decreasing in bandwidth (i.e., $c_i = c(B_i) > 0$, where $c'(B) < 0$), but, on the demand side we now consider a Hotelling setting of horizontal differentiation, in which (i) a mass M of consumers are uniformly distributed over the segment $[0,1]$ and face transportation costs that are linear in distance, and (ii) the two firms I and E are respectively located at $x_I = 0$ and $x_E = 1$.

To provide a first robustness check we start with the case where, as in our baseline model, consumers have an elastic demand; we show that the optimal allocation converges to that of the baseline model when the firms offer close substitutes. To test further the robustness of our analysis, we then turn to a standard Hotelling setting with unit-demand consumers, and show how our insights carry over for arbitrary degrees of product differentiation. We provide a full analysis in Appendix A, and only sketch the main steps here.

• *A first robustness check.* To allow for an elastic demand, as in the baseline model, we assume here that consumers have an elastic demand $d(p) > 0$, where p is the price charged per unit, and $d'(p) < 0$. For the sake of exposition, transportation costs are assumed to be independent of the quantity bought. Hence, a consumer located at x derives a net surplus $s(p_i) - t(x - x_i)$ from patronizing firm i, where

$$s(p) \equiv \int_p^{+\infty} d(v)\, dv$$

denotes individual consumer surplus, and t denotes the transportation cost per unit distance. Firm i's profit is therefore $(p_i - c_i)\, \hat{x}_i(p_I, p_E)\, D(p_i)$, where $\hat{x}_i(p_I, p_E)$denotes denotes firm i's market share and $D(p) \equiv M d(p)$ denotes total demand at price p. Intuitively, competition becomes tougher as t tends to vanish, and costs must therefore be almost equal for the market to be shared. If instead a firm faces a substantially higher cost c, then the other firm (with cost $\gamma(c) < c$) corners the market and charges a price $\hat{p}(c) \in (\gamma(c), c)$ such that:

$$s(\hat{p}(c)) = s(c) + t.$$

The more efficient firm thus obtains a profit equal to $[\hat{p}(c) - \gamma(c)]\, D(\hat{p}(c))$, whereas total welfare can be expressed as

$$\hat{W}(c; \lambda) \equiv S(\hat{p}(c)) + \lambda\, [\hat{p}(c) - \gamma(c)]\, D(\hat{p}(c)),$$

where $S(p) \equiv M s(p)$ denotes total consumer surplus. As t tends to vanish, $p = \hat{p}(c) \simeq c$

and thus $\hat{W}(c;\lambda) \simeq W(p;\lambda)$, the welfare function studied in the baseline model of Bertrand competition (see (6)). If the regulator wants instead to maintain a shared-market equilibrium outcome, then costs should be almost equalized ($c_I \simeq c_E \simeq \hat{c}$, and thus $p_I \simeq p_E \simeq \hat{c}$), and total welfare thus converges to $S(\hat{c}) = W(\hat{c};\lambda)$. Hence, in both types of equilibrium (shared-market or cornered-market), total welfare converges to $W(p;\lambda)$; it follows that the optimal allocation converges towards that of our baseline model of perfect substitutes:

Proposition 3 *In the Hotelling model in which consumer demand is elastic and transportation costs, t, are linear in distance, for t sufficiently small, the welfare maximizing spectrum allocation is arbitrarily close to that which maximizes welfare in the baseline model with perfect substitutes, and the resulting market equilibrium price is arbitrarily close to p^W.*

Proof. See Appendix A.1. ■

• *Arbitrary degree of product differentiation.* We now consider an arbitrary given transportation parameter, t, and show how our qualitative insights carry over. For tractability, we focus on the standard Hotelling model, in which consumers have unit-demand (and "high enough" reservation prices, so that the entire consumer segment is always served in equilibrium). On the supply side, we maintain Assumption A.1 from our baseline model, according to which unit costs will be a convex decreasing function of bandwidth. Without loss of generality, we focus on the case in which $c_I \leq c_E$.[21]

As is well-known, in equilibrium firms share the market as long as their costs do not differ too much—namely, $c_E - c_I \leq 3t$—in which case the industry profit is then

$$\Pi(c_I, c_E) = t + \frac{(c_E - c_I)^2}{9t},$$

whereas consumer surplus is

$$S(c_I, c_E) = r + \frac{5t}{4} - \frac{c_I + c_E}{2} + \frac{(c_E - c_I)^2}{36t}.$$

[21]As seen in the previous Section, any feasible cost allocation (c_i, c_j) can be achieved with $c_I = \min\{c_i, c_j\}$ and $c_E = \max\{c_i, c_j\}$.

A particular feature of the Hotelling model with inelastic participation is that a uniform increase in costs is fully passed-through to consumers; it follows that increasing firms' average cost has no impact on industry profit but harms consumers. By contrast, keeping the average cost constant, exacerbating the asymmetry between the two firms benefits both firms and consumers. Indeed, doing so weakens the competitive pressure exerted by the high-cost firm and further reduces the cost of the low-cost firm; as the low-cost firm has a larger market share, the overall effect on industry profit is positive. Furthermore, as usual consumer surplus is a convex function of prices: keeping average cost (and, thus, price) constant, asymmetric prices benefit consumers, as a larger proportion of them can then benefit from the lower price.

Finally, total welfare is given by:

$$W\left(c_I, c_E\right) = r - \frac{rt}{4} + \lambda t - \frac{c_I + c_E}{2} + \beta \left(\frac{c_I - c_E}{2}\right)^2,$$

where

$$\beta \equiv \frac{1 + 4\lambda}{9t}.$$

When instead $c_E - c_I > 3t$, then the incumbent corners the market; we then have: $\Pi\left(c_I, c_E\right) = \left(\pi_I =\right) c_E - c_I - t, S(c_I, c_E) = r - c_E - t/2$, and:

$$W\left(c_I, c_E\right) = r - \frac{t}{2} - c_E + \lambda(c_E - c_I - t).$$

Hence, in the range where I corners the market, welfare decreases as $\left(1 - \lambda\right) c_E + \lambda c_I$ increases; the convexity of the cost function $c\left(B\right)$ then ensures that, as long as $\lambda \leq 1/2$ (i.e., the weight on profit is not too large), it is not optimal for further increase the cost asymmetry between the two firms.

To go further, we consider two cases in more detail, in which unit costs are either inversely proportional to bandwidth, or quadratic in bandwidth; that is:

$$c_i = c(B_i) = \frac{\alpha}{B_i} \tag{10}$$

and

$$c_i = c(B_i) = \gamma_0 - \gamma_1 B_i + \gamma_2 B_i^2 \tag{11}$$

We have:

Proposition 4 *Consider the Hotelling model unit-demand consumers and linear transportation costs. If $\lambda \leq 1/2$:*

(i) It is never optimal to maintain a cost asymmetry exceeding what enables a firm to corner the market.

(ii) There exists $\hat{\beta}(\alpha)$ and $\tilde{\beta}(\gamma_1, \gamma_2)$, where $\hat{\beta}(\alpha)$ is decreasing in α and $\tilde{\beta}(\gamma_1, \gamma_2)$ is decreasing in γ_1 and increasing in γ_2, such that the welfare maximizing spectrum allocation minimizes the cost difference between the two firms whenever $\beta \leq \hat{\beta}(\alpha)$ if $c(B)$ is given by (10) and whenever $\beta \leq \tilde{\beta}(\gamma_1, \gamma_2)$ if $c(B)$ is given by (11). If instead $\beta > \hat{\beta}(\alpha)$ or $\beta > \tilde{\beta}(\gamma_1, \gamma_2)$, then the welfare maximizing spectrum allocation gives the incumbent a cost advantage equal to the transport cost, t, which enables the incumbent to just corner the market.

Proof. See Appendix A.2. ■

Proposition 4 first extends the previous robustness check: as long as the weight of profit in the welfare objective function is not too large (namely, $\lambda \leq 1/2$), it is never optimal to exacerbate the cost asymmetry beyond what enables a firm to corner the market; it follows that the optimal cost asymmetry tends to vanish as the products become close substitutes (as an arbitrarily small cost asymmetry suffices to generate a cornered market equilibrium when the transportation cost t tends to 0).

In addition, as in our baseline model of perfect substitutes, it remains optimal to minimize the cost asymmetry when the parameter β is small enough, that is, when the weight of profits, λ, is small, and/or products are largely differentiated (i.e., the transport cost, t, is high, implying that competition between firms is limited). Otherwise, it is optimal to maintain a bandwidth gap that (i) is large enough to allow the low-cost firm to corner the market, but (ii) maximizes in this range the competitive pressure from the other firm: this is achieved by making both firms equally capable of serving the consumers who most favor the low-cost firm, that is, by maintaining a cost difference equal to the transport cost, t.[22]

[22]It can further be checked that, even when $\lambda > 1/2$, it remains optimal to minimize the cost difference as long as β is small enough; however, for larger values of β it can become optimal to increase the cost difference beyond what enables a firm to corner the market.

4 Incomplete information

This section studies how the above allocation must be adjusted when (both) firms' costs are private information.

We focus on a simple setting in which each firms can have a high or low ex ante handicap, which we denote $\theta_i^h, i \in \{I, E\}$ and $h \in \{H, L\}$. We show that the allocation and transfers from the firms that maximize welfare can be characterized by "bunching"; the allocation and transfers are independent of information reported, or bids, by the firms.

4.1 Assumptions and Notation

As above we let $\mathbf{b} \equiv (b_I, b_E)$ denote an allocation in the set $\mathcal{B} \equiv \left\{\mathbf{b} \in \mathbb{R}_+^2 \mid b_I + b_E \leq \Delta\right\}$ of feasible bandwidth allocations, and $\mathbf{t} \equiv (t_I, t_E) \in \mathbb{R}^2$ denote the transfer payments made by the two firms. Thus, firm i's gross profit is given by

$$\pi_i\left(b_i, b_j, \theta_i, \theta_j\right) \equiv \max\left\{\underline{\pi}_i\left(b_i, b_j, \theta_i, \theta_j\right), 0\right\},$$

where $j \neq i \in \mathcal{I}$ denotes i's rival and

$$\underline{\pi}_i\left(b_i, b_j, \theta_i, \theta_j\right) \equiv C\left(b_j - \theta_j\right) - C\left(b_i - \theta_i\right).$$

For $i \in \mathcal{I}$ and $(h, k) \in \mathcal{T}^2$, let b_i^{hk} and t_i^{hk} denote the allocated bandwidth and transfer designed for firm i when $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$.

For $i \neq j \in \mathcal{I}$ and $(h, k) \in \mathcal{T}^2$, let

$$\sigma_i^{hk} \equiv b_i^{hk} + b_j^{kh} \text{ and } \beta_i^{hk} \equiv b_i^{hk} - b_j^{kh}$$

respectively denote the total allocated bandwidth and the difference between the firms i and j's allocations designed for $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$, and

$$\gamma_i^{hk} \equiv \theta_i^h - \theta_j^k$$

denote the gap between the two firms' handicaps.[23]

For $i \neq j \in \mathcal{I}$ and $(h,k) \in \mathcal{T}^2$, let

$$\underline{\pi}_i^{hk} \equiv \underline{\pi}_i \left(b_i^{hk}, b_j^{kh}, \theta_i^h, \theta_j^k\right), \; \pi_i^{hk} \equiv \pi_i \left(b_i^{hk}, b_j^{kh}, \theta_i^h, \theta_j^k\right) \text{ and } \Pi_i^{hk} \equiv \pi_i^{hk} - t_i^{hk}$$

respectively denote firm i's cost advantage (or disadvantage) over firm j, i's gross profits and its net payoff under truthtelling when $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$, and

$$\tilde{\underline{\pi}}_i^{hk} \equiv \underline{\pi}_i(b_i^{\tilde{h}k}, b_j^{k\tilde{h}}, \theta_i^h, \theta_j^k), \; \tilde{\pi}_i^{hk} \equiv \pi_i(b_i^{\tilde{h}k}, b_j^{k\tilde{h}}, \theta_i^h, \theta_j^k) \text{ and } \tilde{\Pi}_i^{hk} \equiv \tilde{\pi}_i^{hk} - t_i^{\tilde{h}k}$$

respectively denote firm i's cost advantage (or disadvantage) over firm j, i's gross profits and its net payoff from misreporting its type (i.e., reporting $\theta_i^{\tilde{h}}$ instead of θ_i^h) when $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$.

Here we describe the optimal auction design problem facing a regulator seeking to maximize a weighted sum of consumer surplus and revenue transfers from the firms. We first describe the set of feasible allocations

4.2 Feasible mechanisms

A *direct mechanism* (DM) is a mapping that assigns an allocation $(\mathbf{b}, \mathbf{t})$ to any reported $\theta \in \Theta$. From the revelation principle, we can restrict attention to direct *incentive compatible* mechanisms (DICMs) that are *feasible* and *individually rational*, where the feasibility, individual rationality and incentive compatibility constraints are as follows[24]:

- Feasible bandwidth allocations: for any $(h,k) \in \mathcal{T}^2$,

$$\mathbf{b}^{hk} \equiv \left(b_I^{hk}, b_E^{kh}\right) \in \mathcal{B} \equiv \left\{\mathbf{b} \in \mathbb{R}_+^2 \mid b_I + b_E \leq \Delta\right\}.$$

[23]Thus, by construction, $\beta_j^{kh} = -\beta_i^{hk}$ and $\gamma_j^{kh} = -\gamma_i^{hk}$; furthermore, under truthtelling, θ_i^h strictly wins the competition against θ_j^k if and only if $\beta_i^{hk} > \gamma_i^{hk} (\Longleftrightarrow \beta_j^{kh} < \gamma_j^{kh})$.

[24]We focus on ex-post incentive compatibiity (see Yamashita and Zhu (2022)).

- Feasible transfers: for any $(h,k) \in \mathcal{T}^2$,

$$\mathbf{t}^{hk} \equiv \left(t_I^{hk}, t_E^{kh}\right) \in \mathbb{R}^2.$$

- Individual rationality: for $i \neq j \in \mathcal{I}$ and $(h,k) \in \mathcal{T}^2$, the individual rationality constraint is given by:

$$\Pi_i^{hk} \geq 0. \tag{$IR_i^h k$}$$

- Incentive compatibility: for $i \neq j \in \mathcal{I}$, $h \neq \tilde{h} \in \mathcal{T}$ and $k \in \mathcal{T}$, the incentive compatibility constraint is given by:

$$\Pi_i^{hk} \geq \tilde{\Pi}_i^{hk}. \tag{$IC_i^h k$}$$

Recall that the regulator will want to maximize a weighted sum of consumer surplus (which is a negative constant $\times$ price) and the total transfers from the firms. Thus, the regulator's problem can be expressed as:

$$\max_{(\mathbf{b}^{hk},\mathbf{t}^{hk})\in(\mathcal{B}\times\mathbb{R}^2)^4} \left\{\mu^{LL}W^{LL} + \mu^{LH}W^{LH} + \mu^{HL}W^{HL} + \mu^{HH}W^{HH}\right\}$$

subject to $\left\{\left(IR_i^{hk}\right), \left(IC_i^{hk}\right)\right\}_{i\in\mathcal{I},(h,k)\in\mathcal{T}^2}$where μ^{hk} denotes the probability of state $(h,k) \in \mathcal{T}^2$ and

$$W^{hk} = -\max_{i\neq j}\left\{\left(C(b_i^{hk} - \theta_i^h\right), C\left(b_j^{kh} - \theta_j^k\right)\right\} + \lambda\left(t_I^{hk} + t_E^{kh}\right)$$

We first provide a few useful lemmas.[25]

As a first step in the proof of bunching, we show that an individual rationality constraints must bind for firms with the highest handicap.

Lemma 3 (IC and IR Constraints) *The optimal DICM is such that, for any $i \in \mathcal{I}$ and any $k \in \mathcal{T}$:*

(i) $\left(IR_i^{Hk}\right)$ *and* $\left(IC_i^{Lk}\right)$ *are both binding;*

(ii) $\pi_i^{Lk} - \tilde{\pi}_i^{Lk} \geq \tilde{\pi}_i^{Hk} - \pi_i^{Hk}$.

[25] In what follows we will denote W^{hk} by W_i^{hk} if $b_i^{hk} - \theta_i^h \geq b_j^{kh} - \theta_j^h$ and by W_j^{hk} if $b_i^{hk} - \theta_i^h < b_j^{kk} - \theta_j^h$.

Conversely, and DM satisfying (i) *and* (ii) *is individually rational and incentive compatible.*

Proof. Fix $i \in \mathcal{I}$ and $k \in \mathcal{T}$. We first show that any optimal DICM satisfies (i) and (ii). Note that t_i^{Lk} and t_i^{Hk} appear in (only) four constraints, namely:

$$\pi_i^{Hk} - t_i^{Hk} \geq 0, \tag{$IR_i^H k$}$$

$$\pi_i^{Lk} - t_i^{Lk} \geq 0, \tag{$IR_i^L k$}$$

and

$$\pi_i^{Hk} - t_i^{Hk} \geq \tilde{\pi}_i^{Hk} - t_i^{Lk}, \tag{$IC_i^H k$}$$

$$\pi_i^{Lk} - t_i^{Lk} \geq \tilde{\pi}_i^{Lk} - t_i^{Hk}, \tag{$IC_i^L k$}$$

Furthermore, as $\theta_i^H > \theta_i^L$, we have (with j denoting i's rival):

$$\tilde{\pi}_i^{Lk} = \pi_i \left(b_i^{Hk}, b_j^{kH}, \theta_i^L, \theta_j^k\right) \geq \pi_i \left(b_i^{Hk}, b_j^{kH}, \theta_i^H, \theta_j^k\right) = \pi_i^{Hk}. \tag{12}$$

Combining this condition with $\left(IC_i^{Lk}\right)$ and $\left(IR_i^{Hk}\right)$ yields $\left(IR_i^{Lk}\right)$, which can therefore be ignored. Furthermore, among the remaining relevant constraints:

- the only one that can limit a unilateral increase in t_i^{Lk} is $\left(IC_i^{Lk}\right)$;
- and the only that can limit a uniform increase in both t_i^{Hk} and t_i^{Lk} is $\left(IR_i^{Hk}\right)$.

Increasing transfers being socially desirable, $\left(IC_i^{Lk}\right)$ and $\left(IR_i^{Hk}\right)$ must be binding, which establishes part (i). Finally, combining $\left(IC_i^{Hk}\right)$ and $\left(IC_i^{Lk}\right)$ yields:

$$\pi_i^{Lk} - \tilde{\pi}_i^{Lk} \geq t_i^{Lk} - t_i^{Hk} \geq \tilde{\pi}_i^{Hk} - \pi_i^{Hk},$$

which establishes part (ii).

Conversely, suppose that a DRM satisfies (i) and (ii). From (i), it (weakly) satisfies $\left(IR_i^{Hk}\right)$ and $\left(IC_i^{Lk}\right)$, which together with (12), imply $\left(IR_i^{Lk}\right)$. Furthermore, together with (ii), that $\left(IC_i^{Lk}\right)$ is binding yields:

$$t_i^{Hk} - t_i^{Lk} = \tilde{\pi}_i^{Lk} - \pi_i^{Lk} \leq \pi_i^{Hk} - \tilde{\pi}_i^{Hk},$$

which amounts to $\left(IC_i^{Hk}\right)$. ■

Corollary 2 *The optimal DICM is such that, for any $i \in \mathcal{I}$ and any $k \in \mathcal{T}$:*

$$t_i^{Hk} = \pi_i^{Hk} \geq 0, \tag{13}$$

and

$$t_i^{Lk} = \pi_i^{Lk} - \left(\tilde{\pi}_i^{Lk} - \pi_i^{Hk}\right) \in \left[0, \pi_i^{Lk}\right]. \tag{14}$$

In particular:

(i) $t_i^{hk} = 0$ *whenever* $\beta_i^{hk} \leq \gamma_i^{hk}$;

(ii) $t_i^{Hk} > 0$ *whenever* $\beta_i^{Hk} > \gamma_i^{Hk}$;

(iii) $t_i^{Lk} > 0$ *whenever* $\beta_i^{Lk} > \gamma_i^{Hk}$.

Proof. See Appendix B. ■

4.3 Properties of the Optimal Direct Revelation Mechanism

This section provides the main properties of the optimal DM. In Appendix B we show that if a firm's bandwidth allocation is less than its rival (i.e., it *loses* in the market), when it has a low handicap, then it must also lose when it has a high handicap. This implies that all profits are taxed away, and that high handicap firms earn zero gross (or net) profits.

From Lemma 6 that three configurations may potentially be optimal:

(a) Firm i always loses.

(b) Firm i loses only when it has a low handicap and its rival has a high handicap.

(c) Low handicap firm i loses and high handicap firm i wins.

The following lemma shows that for λ, the weight on transfers low enough.

1. that in all cases when the firms have unequal costs, a small increase in the bandwidth allocated to the higher cost firm and an offsetting decrease in the bandwidth to the lower cost firm will increase welfare, and
2. the optimal mechanisms always allocates all the bandwidth.

Lemma 4 (Cost Equalization) *Consider any DICM such that* $b_i^{hk} - \theta_i^h > b_j^{kh} - \theta_j^k$. *Then there exists* $\hat{\lambda} \in (0, \frac{1}{2})$ *such that DICM defined by*

$$\tilde{b}_i^{hk} = b_i^{hk} - \eta$$

and

$$\tilde{b}_j^{kh} = b_j^{kh} + \eta,$$

with $\tilde{t}_i^{hk}$ *and* $\tilde{t}_j^{kh}$ *defined by* (14) *and* (13)*, will increase welfare whenever* $\lambda < \hat{\lambda}$ *where* μ_I^{hk} *is the probability that* $(\theta_i, \theta_j) = (\theta_i^{hk}, \theta_j^{kh})$.

Proof. See online Appendix B. ■

A key intermediate result is that all bandwidth must be allocation. An implication of (Myerson (1981)) is that an optimal auction need not allocate all the bandwidth.

Lemma 5 (Full Allocation) *The optimal DICM is such that* $b_I^{hk} + b_E^{kh} = \Delta$ *for any* $(h, k) \in \mathcal{T}^2$.

Proof. See online Appendix B. ■

4.4 Bunching

For the both firms to be relevant, we assume here that the two types of each firm are sufficiently different that a low-type firm necessarily wins against a high-type rival; that is, δ is sufficiently large, namely:

Assumption H:

$$\delta > \gamma + \Delta.$$

This assumption indeed implies that $\gamma_i^{LH} < \beta < \gamma_i^{HL}$ for any $i \in \mathcal{I}$ and any feasible $\beta \in [-\Delta, \Delta]$. Let:

$$\mathbf{b}^* = (b_I^*, b_E^*) \equiv \left(\frac{\Delta - \gamma}{2}, \frac{\Delta + \gamma}{2} \right) \tag{15}$$

denote the bandwidth allocation that allocates all the available spectrum so as to equalize costs when both firms have the same type of handicap (i.e., in states (H, H) and (L, L)), but not in states (H, L) and (L, H). We have:

Proposition 5 (optimal allocation) *The bandwidth allocation given by* (15) *regardless of the handicaps (full bunching) is always optimal.*

Proof. See Appendix B ■

One reason for the bunching result is that the optimal DM should not assign less bandwidth to a higher handicap firm than it does to a lower handicap firm. But, giving high handicap firms strictly more bandwidth and requiring no transfers from them will violate incentive compatibility constraints for low handicap firms. Additionally, a firm that a higher handicap firm that is losing the market will have a lower willingness to pay for a block of spectrum than would a firm with a slightly lower handicap. But, the converse is true if the firm is winning the market to start with. Further, all these constraints must hold for both firms at the same time. In an on-line appendix we show our results generalize with one-sided imperfect information and multiple types.

The lessons from this analysis are two-fold. On the one hand, the qualitative insights of the previous section for the case of complete information appear robust: as long as the weight on revenues is not too large, it is socially desirable to allocate some of the bandwidth to the entrant (even if the incumbent ends up serving the market), and to do so in a way that limits the cost asymmetry between the two firms. On the other hand, accounting for firms? private information leads to the adoption of a somewhat coarser mechanism, because of the conflict of interests that arises between the regulator and the firms. In particular, under incomplete information the optimal mechanism exhibits full "bunching": the allocation is the same, regardless of the magnitudes of the handicaps. Hence, the previous insights carry over, but rely on the overall distribution of the cost handicap, rather than on its actual realization.

5 Standard Auctions

Regulators use different auction formats for allocating multiple blocks of spectrum (which may be heterogenous). Most common are clock and simultaneous multi-round ascending auctions (SMRAs); regulators have at times used the combinatorial clock auction (CCAs), which is a Vickrey auction variant. And, often spectrum blocks are not auctioned all at one time, but in a sequence of auctions.[26] To compare these auction formats, we adopt a discretized version of our framework and assume that the additional spectrum Δ is divided into k equal blocks of size $\delta \equiv \Delta/k$; for the sake of exposition, we only consider the case of complete information among the two bidders.

As regulators often allocate different tranches of spectrum at different times, we first consider sequential auctions (Section 5.1). To approximate CCAs, we then turn to simultaneous Vickrey-Clarke-Groves (VCG) auctions (Section 5.2).[27] Finally, to approximate SMRAs, we discuss the case of simultaneous clock auctions (Section 5.3).[28]

We show below that, in our setting, the outcomes of these auction formats depart drastically from the optimal allocations characterized above: while all the available spectrum is allocated, it is likely to go to the incumbent, thus reinforcing its initial advantage.

5.1 Sequential Auctions

We start with the case of sequential auctions. Specifically, we assume that k successive auctions are organized, one for each of block, and that the outcome of each auction is publicly announced before the next auction takes place. As bidders have perfect information about each other, all classic auction formats (first-price or second-price sealed-bid auctions, as well as ascending or descending auctions) yield the same outcome. For the sake of exposition, we will refer to these auctions as "classic auctions". It is well-known that these auctions can generate multiple equilibria, as the losing bidder may

[26] See Myers (2023) and Salant (2014).

[27] A CCA starts with multiple clock rounds, each involving single package bids, and ends with a supplementary round with multipe package bids. VCG allocation rules then apply, except when the VCG outcome would not be in the core.

[28] In SMRAs and clock auctions, prices increase across multiple rounds of bidding. The main difference is that prices are set by bidders in SMRAs, and by the auctioneer in clock auctions. With homogenous blocks and symmetric information, the two auctions result in essentially the same outcome.

bid more than its value without incurring a loss; to address this issue, we focus on coalition-proof Nash equilibria – see Bernheim et al. (1987); in our two-bidder setting, this amounts to focusing on Pareto-undominated Nash equilibria.

The following proposition shows that the incumbent wins all the auctions, and may even do so at zero price if the initial handicap of the entrant exceeds the size of the individual blocks:

Proposition 6 *Suppose that k blocks are sold sequentially using any classic auction format. At any coalition-proof Nash equilibrium, the incumbent wins all k auctions; furthermore, if $B_I - B_E > \Delta/k$, then the incumbent acquires each block for free.*

Proof. See Appendix F. ■

The intuition is simple, and reminiscent of the insight of Vickers (1986) for patent races.[29] When a monopolistic incumbent bids against a potential entrant for a better technology, the incumbent's gain from preserving its monopoly position (and enjoying the better technology) exceeds the profit that the entrant would obtain in a duopoly situation, even when the new technology would enable the entrant to win the competition for the market. Likewise, here the incumbent, who by assumption is the initial leader, gains more from maintaining its incumbency advantage than an entrant gains from overtaking the incumbent.

Proposition 6 shows that, with sequential auctions, the incumbent can obtain all the additional spectrum for free when this spectrum is divided into sufficiently many small blocks. However, when the size of the blocks exceeds the initial handicap of the entrant, the incumbent pays for the first block a price equal to:

$$p^S(B_I, B_E) = \begin{cases} \sum_{h=0}^{m-1} \phi_h^k(B_E, B_I) - \sum_{h=1}^{m} \phi_h^k(B_I, B_E) & \text{if } k = 2m, \\ \sum_{h=0}^{m} \phi_h^k(B_E, B_I) - \sum_{h=1}^{m} \phi_h^k(B_I, B_E) & \text{if } k = 2m+1. \end{cases}$$

where

$$\phi_h^k(B_1, B_2) \equiv \Pi(B_1 + (k-h)\delta, B_2 + h\delta).$$

In the particular case where $B_I = B_E = B$, both firms bid the full benefit generated by

[29] Also see Gilbert and Newbery (1982) and Riordan and Salant (1994).

the additional spectrum and the equilibrium price is thus equal to

$$p^S(B, B) = \Pi(B + \Delta, B).$$

5.2 VCG Auctions

This section considers a single, simultaneous VCG auction for all k blocks, in which each bidder submits a sealed bid demand schedule specifying how much it would offer for every number of blocks it may wish to purchase. That is:

- Each firm $i = I, E$ submits a bid of the form[30]

$$\beta_i = \{\beta_i(n_I, n_E)\}_{(n_I, n_E) \in \mathcal{A}},$$

  where $n_i \in \mathcal{K} \equiv \{0, 1, 2, ..., k\}$ denotes the number of blocks assigned to firm $i \in \{I, E\}$, and

$$\mathcal{A} \equiv \{(n_I, n_E) \in \mathcal{K} \times \mathcal{K} \mid n_I + n_E \leq k\}.$$

- The resulting spectrum allocation, $n^V(\beta_I, \beta_E) = \left(n_I^V(\beta_I, \beta_E), n_E^V(\beta_I, \beta_E)\right)$ maximizes the sum of the offers over feasible allocations, i.e.,

$$n^V(\beta_I, \beta_E) = \arg\max_{n \in \mathcal{A}} \{\beta_I(n) + \beta_E(n)\}.$$

- Finally, the price paid by each bidder i is the value that the other bidder would offer for bidder i's blocks, and is thus equal to (where the subscript "$-i$" refers to firm i's rival)

$$p_i^V(\beta_I, \beta_E) = \max_{n \in \mathcal{A}} \{\beta_{-i}(n)\} - \beta_{-i}\left(n^V(\beta_I, \beta_E)\right).$$

It is well-known that it is a dominant strategy for each firm to bid its full value for each package. The following proposition shows that, in this equilibrium, the incum-

[30]In theory, bids should be made for each entire allocation (n_I, n_E). In practice, firm i is often asked to submit bids for the various combinations of slots assigned to it (that is, $\beta_i = \{\beta_i(n_i)\}_{n_i \in \mathcal{K}}$). However, in our simple two-bidder setting, in which all k blocks are always allocated, the distinction is moot.

bent again wins all the blocks. However, it always pays a positive price whenever the additional spectrum is large enough to offset the handicap of the entrant:

Proposition 7 *In a simultaneous VCG auction for the k blocks, the incumbent wins all the blocks and pays a price equal to the entrant's profits from winning all the blocks:*

$$p^V(B_I, B_E) = \begin{cases} \Pi(B_E + \Delta, B_I) & \text{if } \Delta > B_I - B_E, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See Appendix G. ■

The underlying logic is the same as for sequential auctions, and results again in allocating all the additional spectrum to the incumbent. Furthermore, if the additional bandwidth is not large enough to fully offset the initial handicap of the entrant, then revenues are zero in both auctions. Otherwise, Proposition 7 shows that the incumbent must now pay a positive price, independent of the block size. The following proposition shows that this price is typically higher than in sequential auctions:

Proposition 8 *Revenues are always at least as high in a VCG auction than in a sequential auction, and strictly higher in the case where $\Delta > B_I - B_E$; furthermore, VCG revenues are independent of the block size, whereas a sequential auction brings no revenue if the size of the blocks is sufficiently small.*

Proof. See Appendix H. ■

5.3 Clock Auctions

Now consider a two-bidder clock auction. The auctioneer posts a price per block, and bidders announce how many blocks they want at that price. The posted price is initially set to zero and increases by increments as long as there is excess demand; as price increases, bidders can maintain or reduce the number of blocks they demand, but not augment it. When the market clears, each bidder obtains its desired number of blocks at the clearing price.[31]

[31] When demand abruptly drops below the clearing level, various tie-breaking rules apply, which often involve a random element.

In this auction, it is a dominant strategy for the entrant to bid for all k blocks as long as the clock price p satisfies

$$p < p^E \equiv \frac{\Pi(B_E + \Delta, B_I)}{k},$$

and to drop out once the posted price tops p^E. By contrast, the dominant strategy of the incumbent is to bid for all k blocks as long as the clock price p satisfies

$$p < p^I \equiv \frac{\Pi(B_I + \Delta, B_E)}{k}.$$

As $p^I > p^E$ whenever $B_I > B_E$, the incumbent wins the clock auction at price p^E, which is the same outcome as with a VCG auction. Thus, we have:

Proposition 9 *Auction outcomes are the same with a simultaneous VCG auction and a clock auction.*

6 Policy Implications

The above analysis characterizes the optimal allocation of essential inputs and compares it with the outcomes of standard auctions, with and without measures (such as caps and set-asides) designed to maintain ex post competition. We show that the outcome of standard auctions typically differs drastically from the optimal allocation. With pure Bertrand competition, the leading firm tends to win all the bandwidth, whereas the optimal allocation limit the cost differences of the firms whenever the weight on revenues is not too large. With product differentiation, the optimal outcome is again to limit the bandwidth asymmetries when products are not too differentiated.[32] More generally, any auction which maximizes revenue is unlikely to maximize social welfare unless maximizing industry profit happens to coincide with maximizing consumer welfare.

While we focus for simplicity on a setting in which a single incumbent faces a single challenger, the insights can shed some light on regulators' actual policy choices. In a

[32] In a Cournot model with linear inverse demand $P(Q) = a - bQ$, total consumer and producer surplus is $W = \frac{(2a-c_1-c_2)^2}{18b}$; it is therefore maximized when firms have equal bandwidth whenever unit costs are weakly convex in bandwidth. There, as well, standard auctions will not achieve this result – see Kasberger (2017).

Correlation of MS and total spectrum won	0.674
Correlation of MS and HF spectrum won	0.537
Correlation of MS and LF spectrum won	0.696

Table 1: Correlation of market shares (MS) and total, high and low band frequency (HF,LF) winnings

number of recent spectrum auctions, regulators have faced quite explicit choices between running a competitive auction and avoiding consolidation in the market. This was the case in a 2019 auction in Switzerland, in which three competitors are vying for the 30 MHz available in the 700 MHz band (a relevant band for 5G networks).[33] The smallest challenger, SALT, stated that "with less than 10 MHz [in that band] no competitive and nationwide 5G network can be operated", thus suggesting that, in order to maintain competition, the regulator should allocate one third of the spectrum to each incumbent.[34]

In practice, as shown in Table (1), the larger incumbents tend to win more spectrum, despite the fact that most European auctions have had caps or set-asides to favor challengers.[35] This suggests that increased concentration in spectrum holdings, and in downstream market power is a risk that tends to increase over time. This section discusses regulatory instruments and policy to address this issue.

6.1 Regulatory Instruments

Regulators can adopt different measures to promote competition – the most common being spectrum caps, set-asides and bidding preferences or discounts. A spectrum cap is a limit on the total amount of spectrum a firm can have. A set-aside reserves some spectrum for target groups such as "challengers" or entrants. A bidding preference provides a discount off the final auction price to such challengers. Each of these provisions can

[33] A similar situation arose in the US Incentive Auction, where the regulator (the FCC) had to determine how much spectrum to reserve for Sprint and T-Mobile, the two firms with limited holdings; the FCC reserved a spectrum adequate for one, but not both, of the two firms: the reserve was limited to three blocks, and each laggard needed to win two blocks in order to become economically efficient. In a 2013 auction, the Austrian regulator had to decide on caps in the allocation of 28 blocks among three incumbents. The adopted cap of 14 blocks resulted in very high revenues and nearly forced market consolidation.

[34] See https://www.handelszeitung.ch/unternehmen/telekommunikation-vertragswidrig-salt-pruft-klage-gegen-upc.

[35] See the Appendix for a list of the auctions included in this measure and further details about these auctions.

take different forms. For example, a cap can be on the amount available in the auction or on overall spectrum holdings, including what bidders have prior to the auction. A set-aside can also take the form of floors, i.e., minimum spectrum packages.[36]

Caps and set-asides impose similar restrictions on the set of feasible allocations. To see this, let $B_E \geq 0$ and $B_I > B_E$ denote, as before, the amount of spectrum initially owned by the entrant and the incumbent, and by Δ the additional amount made available. Introducing an overall cap K limits the additional bandwidth b_i that firm i can obtain, that is:

$$B_i + b_i \leq K.$$

Obviously, a cap K has no effect if it exceeds $B_I + \Delta\,(> B_E + \Delta)$. Introducing instead a "binding" cap $K < B_I + \Delta$ *de facto* reserves a bandwidth

$$S(K) \equiv \Delta - (K - B_I)$$

for the entrant. However, compared with the imposition of such a cap K, introducing a set-aside $S(K)$ further restricts the set of feasible allocations, as it also prevents the entrant from having less than $S(K)$ of additional bandwidth; that is, both instruments can be used to put the same *upper* bound on I's share of the additional spectrum, b_I, but in addition a set-aside puts a *lower* bound on E's share of this spectrum, b_E. As a result, as discussed below, caps and set-asides have different impacts on the outcome of an auction.

Some auctions, such as the early US auctions, included spectrum caps on overall spectrum holdings, of the form $B_i + b_i \leq K$, as discussed above.[37] Other auctions have used caps on the amount of spectrum each firm can acquire in the auction, of the form $b_i \leq k$. However, these two variants can have very different effects on product market competition. Indeed, whereas an overall cap can be effective in limiting the acquisition of additional spectrum by the incumbent, an auction cap puts more stringent limitations on the entrant, and may actually keep it from overtaking the incumbent.

Set-asides are often accompanied by reserve prices, which tend to discourage entry –

[36]For instance, floors have been used in the UK – see https://www.ofcom.org.uk/about-ofcom/latest/media/media-releases/2011/ofcom-prepares-for-4g-mobile-auction; for a discussion of this case, see Myers (2013).

[37]See https://www.fcc.gov/node/189694 for a discussion of US spectrum caps.

in our Bertrand setting, any positive reserve price would deter entry. Entry was indeed discouraged in the 2013 4G auctions in Austria and limited in the UK, which relied on various forms of set-asides accompanied by reserve prices. In the same vein, while our analysis suggests that all spectrum should be allocated, this does not always occur in practice, as reserve prices sometimes result in unsold blocks. For instance, this has been the case in 4G auctions in Spain and Portugal, in which some of the most valuable (900 MHz) spectrum remained unsold.[38]

When the initial handicap of the entrant is very large, it is optimal to allocate all the additional spectrum to the entrant. Overall caps, set-asides and bidding credits can all be used to accomplish this.[39] In what follows, we concentrate on the more interesting case where the regulator finds it optimal to share the additional spectrum between the two firms.

Consider first the case where the regulator focuses on consumer surplus. From Proposition 1, it is optimal to equalize the costs of the two firms; that is, the optimal allocation $\left(b_I^S, b_I^S\right)$ is such that:

$$c_I\left(B_I + b_I^S\right) = c_E\left(B_E + b_E^S\right) = \hat{c} = c\left(\frac{B_I + B_E + \Delta}{2}\right).$$

This could be achieved with an overall cap set to $K^S = \left(B_I + B_E + \Delta\right)/2$. By contrast, a set-aside $S^S = S\left(K^S\right)$ would not work, as it would put the entrant ahead of the incumbent (as $B_E + S^S \left(= K^S\right) > B_I$), and thus result in the entrant winning all the additional spectrum. A bidding credit would not be effective either, as it would result in either firm winning all the additional spectrum (I if the bidding credit is too low, and E otherwise).

When instead the regulator also cares about revenues, Proposition 2 applies, and two cases arise.

[38] See http://www.minetad.gob.es/telecomunicaciones/es-ES/ResultadosSubasta/Informe_Web_29072011_fin_de_subasta.pdf and http://www.anacom.pt/streaming/Final_Report_Auction.pdf?contentId=1115304&field=ATTACHED_FILE for results of the 2011 Spanish and Portuguese multi-band auctions. Also, in France, the regulator took a number of years to reduce the reserve price before awarding a fourth 3G license. See https://www.arcep.fr/?id=8562.

[39] Specifically, an overall cap $K = B_I$, a set-aside $S = \Delta$, or a bidding credit reflecting I's profit with the additional spectrum, would all result in E winning all the additional spectrum. When the regulator cares about revenues as well, however, this would need to be complemented with a tax (e.g., an unconditional tax on I's equilibrium profit).

• If the handicap of the entrant and/or the weight placed on revenues is not too large, then there are two optimal allocations, which: (i) confer a competitive advantage to one firm and lead to the same consumer price, p^W – more specifically, one firm (the "loser") ends-up with an overall holding equal to $B_l^W \equiv c^{-1}\left(p^W\right)$, whereas the other firm (the "winner") accumulates a total amount of spectrum equal to $B_w^W \equiv c^{-1}\left(\gamma\left(p^W\right)\right) > B_l^W$; and: (ii) appropriate the winner's equilibrium profit, $\pi^W \equiv \left(p^W - c\left(K^W\right)\right) D\left(p^W\right)$. This optimal allocation could be achieved by setting aside an amount $S^W = B_I - B_E$ for the entrant, and introducing an overall cap set to $K^W = B_w^W$. The set-aside is designed to offset the initial handicap of the entrant; as a result, both firms are willing to bid up to π^W to reach the overall cap.[40] Interestingly, neither instrument alone suffices to achieve a desired outcome. Relying only on a set-aside would again result in one firm (either one) winning all the additional spectrum (the same would be obtained with a bidding credit). Relying only on a cap could achieve the desired spectrum allocation (by setting the cap to K^W), but it would leave a positive rent to the incumbent;[41] a second instrument would be required to deal with this issue.[42]

• If the handicap of the entrant and/or the weight placed on revenues is large enough, the unique optimal allocation is such that $c_E^W = c\left(B_l^W\right) > \max\{\bar{c}_I, \underline{c}_E\}$, and thus satisfies: $B_E^W = B_l^W < B_I$ – that is, it is no longer optimal to offset the initial handicap of the entrant. In this case, setting aside an amount $S^W = B_l^W - B_E$ for the entrant, or alternatively introducing a cap set to $K^W = B_w^W$, would both achieve the desired spectrum allocation. However, either instrument would again need to be complemented with another instrument designed to limit the rent left to the incumbent.[43]

[40]Consider, for instance, a simple clock auction with the set-aside S^W and the overall cap K^W (with the rule that any unbid spectrum is allocated to the losing bidder). As E secures an amount $B_I - B_E$ for free, both firms are then willing to bid up to $\pi^W / \left(K^W - B_I\right)$ to obtain the additional amount $K^W - B_I$. As a result, either firm wins and accumulates an overall holding of $K^W = B_w^W$, and the other firm obtains an overall holding of B_l^W, leading to the market price p^W.

[41]For instance, in a simple clock auction I would be willing to bid up to $\pi^W / \left(K^W - B_I\right)$, whereas E is only willing to bid up to $\pi^W / \left(K^W - B_E\right)$. The conclusion follows from $B_E < B_I$.

[42]This, for instance, could be an unconditional tax on I's profit, as described in footnote 39. Alternatively, it could take the form of a reserve price – a non-linear or discriminatory reserve price may however be needed.

[43]Combining a set-aside with a cap no longer suffices here to achieve the desired allocation *and* to appropriate the profit π^W; that is, the additional instrument should again be a tax on the incumbent's equilibrium profit, and/or some form of a reserve price.

6.2 Regulatory Experience and Policy

Most countries have adopted auctions and other spectrum assignment procedures, including caps and set-asides, designed to promote *ex post* competition in the market for mobile communications services. However, these provisions have tended to have limited long-run impact, once the first set of spectrum allocations was completed.

The three initial waves of spectrum allocations resulted in 4 - 5 mobile operators in most European countries and at least 4 operators in almost all of the US and Canada,[44] and often as many as 5 or 6.[45] Since the 3G auctions, however, consolidation has been the rule in much of Europe, including in Austria, Germany, Italy, the Netherlands, Switzerland and the UK, despite various measures employed to promote competition.[46] The Netherlands set aside two prime, low frequency 4G blocks for entrants,[47] which did attract two new bidders; however, only one entrant won any blocks, and after the auction it signed a network-sharing agreement with the one incumbent (T-Mobile) that failed to win any low frequency blocks, and eventually merged with it. Austria failed to attract any bidders for the two blocks provisionally set aside for entrants. The UK's provisions for a floor mentioned in footnote 36 also failed to induce meaningful changes in the competitive structure.

The US, too, has seen continuing consolidation.[48] Since the FCC abandoned overall spectrum caps in 2003, the two largest MNOs have acquired most of the spectrum that has been auctioned. The HHI has increased from 2151 in 2003 to 3027 at the end of

[44]For the first generation of analog cellular in the Americas and in Europe, regulators tended to award one license to the incumbent local exchange carrier and one to an entrant. Additional operators entered during the second and third waves of spectrum allocations, starting in the late 1980's and running through the early 2000's (first for 2G, and then for 3G spectrum).

[45]The US and Canada, unlike European countries, awarded regional and not national licenses.

[46]The UK had 5 mobile network operators ("MNOs"), but even after set-asides in the recent 4G auction, only 4 remain, and there is talk of further consolidation. Germany had 6 winners after their 3G auction, but two winners abandoned their licenses, and subsequent to a recent merger, there are now only 3 MNOs left. The Netherlands had 5 MNOs, but mergers resulted in 3 players. The Austrian 3G auction had 6 winners. By the time the Austrians auctioned off 4G spectrum, a little over 10 years after the 3G auctions, there were only three MNOs remaining in the market. Finally, the 2013 re-auction of legacy spectrum in Austria nearly left Austria with two viable MNOs. See Salant (2014)) for a discussion.

[47]By contrast, Germany turned down requests of entrants for set-asides.

[48]Among the large regional and national carriers that at one time existed: (i) Cingular, BellSouth, Ameritech, and Leapwireless have all been absorbed by AT&T; (ii) BellAtlantic, NYNEX, USWest, Airtouch , GTE, Cincinnati Bell, and Alltel have been absorbed by Verizon; (iii) Western Wireless, Voicestream, Omnipoint and Powertel formed T-Mobile, which subsequently acquired MetroPCS; (iv) Sprint merged with Nextel; and (v) US Cellular is still independent, but has sold off most of its larger markets.

2013. In a very recent auction for AWS-3 spectrum, AT&T and Verizon spent 6 and 10 times as much as the third largest MNO in the auction (T-Mobile), and no other MNO spent even 1% of what AT&T spent. In the most recent 600 MHz auction, the reserve price of $1.25 per MHzPOP[49] has deterred the weakest incumbent, Sprint, from even participating.[50]

Finally, Canada at one time had 4 national operators, which was reduced to 3 via merger. Despite having conducted a series of auctions with provisions including caps and regulations on wholesale prices, in the hope of attracting more competitors, no fourth national operator has emerged.

Regulatory agencies face a great deal of uncertainty. Experience in the past several years suggests that set-asides are not very likely to attract new entrants when incumbents start with a large amount of spectrum, and significant sunk investments that an entrant would need to duplicate. This suggests that caps that limit further consolidation may be preferable to set-asides or other measures to favor entrants. That is, the goal is limiting risk of excessive consolidation, as attracting entrants has proven difficult. A few auctions, the UK 4G auction, the US Incentive Auction, and the Canadian 600 MHz auction, have included what are effectively set-asides for smaller players.[51]

7 Conclusion

This paper characterizes the optimal allocation of a scarce resource (e.g., spectrum rights) between an incumbent and a challenger, for a regulator seeking to maximize the social surplus. The main insight is that the regulator wants to limit the dominance of the incumbent, and ensure that the challenger exerts an effective competitive pressure. More specifically, when the regulator focuses on consumer surplus, and does not care about auction revenues, it tries to equalize firms' competitiveness. When instead the

[49]The term "per MHzPOP" is used to compare prices of different sized blocks in different countries or regions. Literally, one MHzPOP is a license of one MHz covering an area with a population of one.

[50]In addition, the spectrum reserved for challengers sold eventually for essentially the same price than the non-reserved spectrum - less than 1% overall difference, and, in nearly 20% of the PEAs the reserved spectrum was more expensive than the non-reserved spectrum.

[51]The UK had spectrum "floors" for both entrants and smaller players. The US Incentive Auction included "reserved" and "non-reserved" spectrum. Interestingly, in that auction, Verizon never bid, AT&T stopped bidding in the after the first of four stages, and there was on average a 1% difference in the average price of set-aside and non-set-aside licenses. The 2019 Canadian auction includes set-asides for "those bidders that are registered ... as facilities-based providers ."

regulator seeks to maximize social welfare, taking into account auction revenues, it finds it optimal to maintain some asymmetry among the competitors.

Further, we find that there is a tension between the regulator's objective and the challenger's incentives to report its handicap. More specifically, a regulator may want to provide a weaker challenger more spectrum than a stronger challenger. However, doing so gives the stronger challenger an incentive to try to act as if it is weak. As a result of this tension, the optimal auction is likely to exhibit "bunching", in that the challenger ends up with the same allocation, regardless of its initial handicap.

The finding that the regulator wishes to limit dominance contrasts sharply with the outcome of standard types of auctions, such as sequential, clock and VCG auctions, which all result in increasing dominance: in the Bertrand competition setting that we consider, the incumbent always obtains all the additional spectrum.[52] Furthermore, while the spectrum allocations are the same, revenues are lower in a sequential auction.

Finally, we examine some policy implications. When the regulator's objective only includes consumer surplus, a cap on firms' overall spectrum holdings can suffice to achieve the desired allocation. By contrast, neither a cap on the amount of spectrum that any firm can win in the auction, nor a set-aside reserved for the challenger are helpful – auction-specific caps could actually be counter-productive, as they may limit the challenger's ability to reduce its handicap. When the regulator also cares about auction revenues, an overall cap needs to be complemented with a set-aside or with another instrument designed to limit the incumbent's rent.

While this paper has focused on spectrum auctions, similar issues arise in many other sectors. We discuss a few below.

Sports broadcasting rights.

Sports broadcasting rights, and, in particular, soccer rights are often regulated, especially in Europe. The reason for competition authorities to intervene is the concern that a concentration of broadcasting rights could create or reinforce the dominance of the rights holder.[53] Indeed, the fraction of rights owned by a provider affects the perceived

[52]This may not always hold when firms compete à la Cournot competition, in which case the packaging of blocks can also affect the outcome.

[53]See, e.g., http://ec.europa.eu/competition/elojade/isef/case_details.cfm?proc_code=

value of its offering relative to that of its rivals. Suppose, for example, that the value a consumer derives from provider i's offering is of the form:

$$u + v(r_i),$$

where r_i denotes the fraction of rights own by firm i. Formally, the increase in firm i's perceived quality stemming from an increase in r_i then has the same effect as the cost reduction resulting from an increase in the bandwidth b_i in our model.

Train and plane slots

Another application is the allocation of plane take-off and landing and train slots. The frequency of service offered on a given route will affect the average wait time, and thus the cost imposed on customers. Many passengers, e.g., because they purchased their tickets in advance or because they benefit from affinity programs, will therefore favor the carrier offering the most frequent service. Suppose, for example, that the net value offered by firm i is of the form:

$$u - c(s_i),$$

where s_i denotes the number of departure slots allocated to firm i. The reduction in customer waiting time, reduce overall travel costs stemming from an increase in s_i then has the same effect as the reduction in firm i's own operating cost in our model.[54]

Electric transmission.

Electricity transmission rights raises similar issues in regions in which energy supply is limited and relatively costly.[55] The particular details of electricity markets differ quite a bit from spectrum,[56] but the basic message that the allocation of transmission rights can affect post-auction competition applies.

Similar issues arise with the allocation of many other scarce resources, such as landing

1_38173.

[54] For a discussion of plane slot auctions see Rassenti, Smith and Bulfin (1982). Affuso and Newbery (2000) provide a summary of some of the issues encountered with auctions of train slots.

[55] See Joskow and Tirole (2000) and Loxley and Salant (2004).

[56] See for example `http://pjm.com/markets-and-operations/ftr.aspx`, and also Salant (2005).

slots (and other airport facilities, such as gates, kerosene tanks, and so forth), gas pipeline capacity, concessions to operate in given areas (e.g., highway service stations), or when zoning regulations put constraints on commercial activities or on the number (and/or the size) of supermarkets.[57]

[57]In France, for instance, zoning regulations have prevented the entry of new retail chains.

Appendix

A Proof of Proposition 1

It is obviously optimal to allocate all the additional spectrum: allocating any residual bandwidth equally among the two firms reduces for sure the resulting competitive price and thus benefits consumers.

Without loss of generality, we can thus restrict attention to spectrum allocations of the form $b_E \in [0, \Delta]$, $b_I = \Delta - b_I$, yielding a competitive price equal to

$$\begin{aligned} p &= \max \{c(B_E + b_E), c(B_I + b_I)\} \\ &= \max \{c(B_E + b_E), c(B_I + \Delta - b_E)\}. \end{aligned}$$

Maximizing consumer surplus amounts to minimizing this competitive price; therefore:

- If $\Delta \geq B_I - B_E$, there is enough spectrum to offset the initial cost asymmetry; the optimal spectrum allocation thus equates the costs of the two firms:

  $$b_E = \hat{b} \equiv \frac{B_I - B_E + \Delta}{2} \text{ and } b_I = \Delta - \hat{b},$$

  leading to an equilibrium price equal to (where the superscript S refers to consumer surplus):

  $$p^S = c_I = c_E = \hat{c}.$$

- If instead $\Delta < B_I - B_E$, there is not enough spectrum to offset the initial cost disadvantage of the entrant; to minimize this disadvantage, it it then optimal to give all the additional spectrum to the entrant:

  $$b_E = \Delta \text{ and } b_I = 0,$$

  leading to

  $$p^S = \underline{c}_E = c(B_E + \Delta) > \bar{c}_I = c(B_I).$$

B Proof of Lemma 2

Part (i). Any $p \in \left[p^S, \bar{c}_E\right]$ can be supported by an equilibrium by choosing $b_E \in \left[0, \hat{b}\right]$ and $b_I = \Delta - b_E$ such that $c_E = c\left(B_E + b_E\right) = p$ and $c_I = c\left(B_I + b_I\right) = \gamma\left(p\right)$. We have:

$$\gamma'\left(p\right) = \gamma'\left(c_E\right) = -\frac{c'\left(B_I + B_E + \Delta - c^{-1}\left(c_E\right)\right)}{c'\left(c^{-1}\left(c_E\right)\right)} = -\frac{c'\left(B_I + b_I\right)}{c'\left(B_E + b_E\right)},$$

and

$$\gamma''\left(p\right) = \gamma''\left(c_E\right) = \frac{c''\left(B_I + b_I\right) + \frac{c'\left(B_I + b_I\right)}{c'\left(B_E + b_E\right)} c''\left(B_E + b_E\right)}{\left[c'\left(B_E + b_E\right)\right]^2}.$$

Assumption A.1 then readily yields $\gamma'\left(p\right) < 0 < \gamma''\left(p\right)$; together with $b_E \leq \hat{b}$ (which implies $B_I + b_I \geq B_E + b_E$), it yields $\gamma'\left(p\right) \geq -1$, with a strict inequality if $b_E < \hat{b}$ and $c''\left(\cdot\right) > 0$.

Part (ii). For any $p \in \left[p^S, \bar{c}_E\right]$, $p \geq \gamma\left(p\right)$ and thus $\rho\left(p\right) \leq 1$ (with strict inequalities for $p > p^S$). Furthermore, for any $c \geq 0$, the monopoly profit function

$$\pi^m\left(p; c\right) \equiv \left(p - c\right) D\left(p\right)$$

satisfies:

$$\frac{\partial \pi^m}{\partial c}\left(p; c\right) = -D'\left(p\right)\left[c + \mu\left(p\right) - p\right],$$

where, from Assumption A.2, the expression in brackets strictly decreases as p increases. It follows that $\pi^m\left(p; c\right)$ is strictly quasi-concave and is maximal for $p = p^m\left(c\right)$, characterized by the first-order condition:

$$p^m = c + \mu\left(p^m\right).$$

As is well-known, the monopoly price $p^m\left(c\right)$ is moreover (weakly) increasing in c. As $\gamma\left(p\right) \geq \underline{c}_I$, the assumption $\bar{c}_E < p^m\left(\underline{c}_I\right)$ thus ensures that, for any $p \in \left[p^S, \bar{c}_E\right]$, we have:

$$p \leq \bar{c}_E < p^m\left(\underline{c}_I\right) \leq p^m\left(\gamma\left(p\right)\right).$$

The strict quasi-concavity of the profit function $\pi^m(p;c)$ then yields:

$$\gamma(p) + \mu(p) - p > 0,$$

that is, $\rho(p) > 0$. Finally, we have:

$$\rho'(p) = \frac{\mu'(p)[p - \gamma(p)] - \mu(p)[1 - \gamma'(p)]}{\mu^2(p)} < 0,$$

where the inequality stems from the part (i) of the Lemma and Assumption A.2.

C Proof of Proposition 2

The derivative of the welfare function $W(p;\lambda)$ can be expressed as $\partial W(p;\lambda)/\partial p = \lambda D(p)\phi(p;\lambda)$, where

$$\phi(p;\lambda) \equiv \rho(p) - \gamma'(p) - \frac{1}{\lambda}.$$

From Lemma 2, $\phi(p;\lambda)$ is strictly decreasing in p. Let

$$\underline{\lambda} \equiv \frac{1}{\rho(p^S) - \gamma'(p^S)} \text{ and } \bar{\lambda} \equiv \frac{1}{\rho(\bar{p}) - \gamma'(\bar{p})}.$$

When the additional bandwidth is large enough to overcome the handicap, $p^S = \hat{c}\,(\geq \underline{c}_E)$ and thus $\rho(p^S) = -\gamma'(p^S) = 1$; hence, $\underline{\lambda} = 1/2$. Otherwise (i.e., when $\Delta < B_I - B_E$), $p^S = \underline{c}_E > \hat{c} > \gamma(\underline{c}_E) = \bar{c}_I$ and thus:

$$\rho(p^S) = 1 - \frac{\underline{c}_E - \bar{c}_I}{\mu(\underline{c}_E)} < 1 \text{ and } -\gamma'(p^S) = \frac{c'(B_I)}{c'(B_E + \Delta)} < 1;$$

hence, $\underline{\lambda} > 1/2$.

Three cases can be distinguished:

• Case a: $\lambda \leq \underline{\lambda}$. We then have $\phi(p;\lambda) \leq 0$ in the relevant range $p \in [p^S, \bar{c}_E]$, implying that the optimal price is $p^W = p^S$; that is, it is still optimal to allocate all the additional bandwidth to the entrant, as when maximizing consumer surplus.

• Case b: $\lambda \geq \bar{\lambda}$. We then have $\phi(p;\lambda) \geq 0$ in the relevant range $p \in [p^S, \bar{c}_E]$, implying that the optimal price is $p^W = \bar{c}_E$; that is, it is instead optimal to allocate all the additional bandwidth to the incumbent, as if the objective were to maximize industry profit.

• Case a: $\underline{\lambda} < \lambda < \bar{\lambda}$. The optimal price, p^W, is then the unique solution in p to (9); this optimal price lies strictly between p^S and $\bar{c}_E$ and it is therefore optimal to share the additional bandwidth between the two firms. Furthermore:

- When $p^S = \underline{c}_E \geq \hat{c}$, there is not enough (or just enough, in case of equality) additional bandwidth to offset the initial cost asymmetry; E's cost thus remains (weakly) higher than I's cost, and so there is a unique optimal spectrum allocation, which consists in giving $b_E^W = c^{-1}\left(p^W\right) - B_E$ to the entrant and $b_I^W = \Delta - b_E^W$ to the incumbent.

- When instead $p^S = \hat{c} > \underline{c}_E$, there is again a unique optimal spectrum allocation whenever $p^W > \bar{c}_I$; when instead $p^W \leq \bar{c}_I$, there are two optimal bandwidth allocations, which consist in sharing the additional bandwidth so as to give a cost equal to p^W to one operator and a cost equal to $\gamma\left(p^W\right)$ to the other operator. As $\phi(p;\lambda)$ increases with λ and decreases in p, the solution to (9), p^W, increases with λ, from $p^S = \hat{c} < \bar{c}_I$ for $\lambda = \underline{\lambda}$ to $\bar{c}_E > \bar{c}_I$ for $\lambda = \bar{\lambda}$. Hence, there exists $\hat{\lambda} \in \left(\underline{\lambda}, \bar{\lambda}\right)$ such that $p^W > \bar{c}_I$ for $\lambda > \hat{\lambda}$.

D Proof of Corollary 1

Building on the previous analysis, the optimal price, $p^W = p^W(\lambda)$, is the unique solution to $\phi(p;\lambda) = 0$, where $\phi(p;\lambda)$ strictly increases with λ and, from Lemma 2, strictly decreases in p. It follows that $p^W(\lambda)$ strictly increases with λ. When the optimal spectrum allocation maintains a cost advantage to the incumbent, this implies a re-allocation of Δ which further favors the incumbent.

Turning to the impact of bandwidth, and using

$$\gamma(p;B) \equiv c\left(B - c^{-1}(p)\right),$$

$$\rho(p;B) \equiv 1 - \frac{p - \gamma(p;B)}{\mu(p)},$$

and

$$\phi(p;B) \equiv \rho(p;B) - \frac{\partial \gamma}{\partial p}(p;B) - \frac{1}{\lambda},$$

the optimal price can be expressed as $p^W = p^W(B)$, where $p_\phi(B)$ is the unique solution to

$$\phi(p; B) = 0. \tag{16}$$

The optimal price thus only depends on total available bandwidth, $B = B_I + B_E + \Delta$. In addition,

$$\frac{\partial p^W}{\partial B} = -\frac{\frac{\partial \phi}{\partial B}}{\frac{\partial \phi}{\partial p}}\left(p^W, B\right),$$

where

$$\begin{aligned}
\frac{\partial \phi}{\partial B}\left(p^W, B\right) &= \frac{1}{\mu\left(p^W\right)}\frac{\partial \gamma}{\partial B}\left(p^W; B\right) - \frac{\partial^2 \gamma}{\partial B \partial p}\left(p^W; B\right) \\
&= \frac{c'\left(B - c^{-1}\left(p^W\right)\right)}{\mu\left(p^W\right)} + \frac{c''\left(B - c^{-1}\left(p^W\right)\right)}{c'\left(c^{-1}\left(p^W\right)\right)}\mu\left(p^W\right) \\
&= \frac{c'\left(S_I\right)}{\mu\left(c\left(S_E\right)\right)} + \frac{c''\left(S_I\right)}{c'\left(S_E\right)},
\end{aligned}$$

where $S_I = B_I + b_I$ and $S_E = B_E + b_E$, respectively, denote the overall amount of spectrum eventually assigned to the incumbent and to the entrant, assuming that the incumbent is favored when the optimal price p^W can be achieved in two symmetric ways, and

$$\begin{aligned}
\frac{\partial \phi}{\partial p}\left(p^W, B\right) =& \frac{\partial \rho}{\partial p}(p; B) - \frac{\partial^2 \gamma}{\partial p^2}(p; B) \\
=& \frac{\mu'(p)\left[p - c\left(B - c^{-1}(p)\right)\right]}{\mu^2(p)} - \frac{1 + \frac{c'\left(B - c^{-1}(p)\right)}{c'\left(c^{-1}(p)\right)}}{\mu(p)} \\
& - \frac{c''\left(B - c^{-1}(p)\right)}{\left[c'\left(c^{-1}(p)\right)\right]^2} - \frac{c'\left(B - c^{-1}(p)\right)c''\left(c^{-1}(p)\right)}{\left[c'\left(c^{-1}(p)\right)\right]^3} \\
=& \frac{\mu'\left(c\left(S_E\right)\right)\left[c\left(S_E\right) - c\left(S_I\right)\right]}{\mu^2\left(c\left(S_E\right)\right)} - \frac{1 + \frac{c'\left(S_I\right)}{c'\left(S_E\right)}}{\mu\left(c\left(S_E\right)\right)} - \frac{c''\left(S_I\right)}{\left[c'\left(S_E\right)\right]^2} - \frac{c'\left(S_I\right)c''\left(S_E\right)}{\left[c'\left(S_E\right)\right]^3},
\end{aligned}$$

where the last equality uses the fact that, by construction, $p^W = c_E = c\left(S_E\right)$ and $\gamma\left(c_E\right) = c_I = c\left(S_I\right)$. It follows that the derivative of p^W with respect to total bandwidth can be expressed as:

$$
\begin{aligned}
\frac{\partial p^W}{\partial B} &= -\frac{\frac{\partial \phi}{\partial B}}{\frac{\partial \phi}{\partial p}}\left(p^W, B\right) \\
&= \frac{\frac{c'(S_I)}{\mu(c(S_E))} + \frac{c''(S_I)}{c'(S_E)}}{\frac{c'(S_I)}{\mu(c(S_E))c'(S_E)} + \frac{c''(S_I)}{[c'(S_E)]^2} + \frac{1}{\mu(c(S_E))} + \frac{c'(S_I)c''(S_E)}{[c'(S_E)]^3} - \frac{\mu'(c(S_E))[c(S_E)-c(S_I)]}{\mu^2(c(S_E))}} \\
&= c'\left(S_E\right)\frac{A}{A+B},
\end{aligned}
$$

with

$$
\begin{aligned}
A &= \frac{c'\left(S_I\right)}{\mu\left(c\left(S_E\right)\right)c'\left(S_E\right)} + \frac{c''\left(S_I\right)}{\left[c'\left(S_E\right)\right]^2} > 0, \\
B &= \frac{1}{\mu\left(c\left(S_E\right)\right)} + \frac{c'\left(S_I\right)c''\left(S_E\right)}{\left[c'\left(S_E\right)\right]^3} - \frac{\mu'\left(c\left(S_E\right)\right)\left[c\left(S_E\right) - c\left(S_I\right)\right]}{\mu^2\left(c\left(S_E\right)\right)} > 0,
\end{aligned}
$$

where the inequalities follows from $c'(\cdot) > 0 > c''(\cdot)$, $\mu(\cdot) > 0 > \mu'(\cdot)$ and $c(S_E) > c(S_I)$. Using

$$p^W = c\left(S_E\right),$$

we thus have:

$$0 < \frac{\partial S_E}{\partial B} = \frac{A}{A+B} < 1.$$

Therefore:

- An increase in Δ leads to an increase in both b_E (as $\partial S_E/\partial B > 0$) and b_I (as $\partial S_E/\partial B < 1$).
- An increase in B_I leads to an increase in b_E (as $\partial S_E/\partial B > 0$) and a reduction in b_I (as $\partial S_E/\partial B < 1$).
- An increase in B_E leads to an increase in b_I (as $\partial S_E/\partial B < 1$) and thus to a reduction in b_E (as $b_E + b_I = \Delta$).

E Proof of Proposition 4

E.1 Shared-market equilibria

We first study shared market equilibria, in which both firms attract some consumers. The location $\hat{x}$ of the customer who is indifferent between patronizing the two firms is determined by:

$$s(p_1) - t\hat{x} = s(p_2) - t(1 - \hat{x}),$$

where p_1 and p_2 denote firms' prices,

$$s(p) \equiv \int_p^{+\infty} d(v)\, dv$$

denotes individual consumer surplus, and t denotes transportation costs per unit distance. Consumers located at $x < \hat{x}$ then purchase from firm 1 and those consumers located at $x > \hat{x}$ purchase from firm 2. Firm i's market share is therefore

$$\hat{x}_i(p_i, p_j) \equiv \frac{1}{2} + \frac{s(p_i) - s(p_j)}{2t},$$

where $i \neq j \in \{1, 2\}$, and its profit is

$$\Pi_i(p_i, p_j) \equiv M\hat{x}_i(p_i, p_j)\, \pi_i(p_i),$$

where

$$\pi_i(p_i) \equiv (p_i - c_i)\, d(p_i)$$

denotes firm i's per consumer profit. The equilibrium prices, p_1^* and p_2^*, and the associated equilibrium variables satisfy the first-order conditions, which, using $s(p) = -d(p)$, can be written as:

$$d(p_i^*)\, \pi_i(p_i^*) = 2t\hat{x}_i\left(p_i^*, p_j^*\right) \pi_i'(p_i^*) \leq 2t\hat{x}_i\left(p_i^*, p_j^*\right) d(p_i^*),$$

where the inequality follows from $\pi_i'(p_i^*) = d(p_i^*) + (p_i^* - c_i)\, d'(p_i^*) \leq d(p_i^*)$, as $d'(\cdot) < 0$ and active firms never sell below costs (i.e., $p_i^* \geq c_i$). Dividing by $d(p_i^*)$ and adding the resulting inequalities for the two firms yields $\pi_1(p_1^*) + \pi_2(p_2^*) < 2t$. As active firms never make a loss (i.e., $\pi^* (p_i^*) \geq 0$), this condition in turn implies that each firm i obtains a total profit lower than $2t$:

$$\Pi_i^* \equiv \Pi_i \left(p_i^*, p_j^*\right) = M\hat{x}_i(p_i^*, p_j^*)\pi_i (p_i^*) \leq 2tM.$$

It follows that, when firms face different costs, the market cannot remain shared as t tends to vanish. To see this, suppose that $c_i < c_j$. From the above analysis, in any shared-market equilibrium, $\Pi_i^* \leq 2tM$ and $p_j^* \geq c_j$. Hence, firm i's profit tends to 0 as t tends to vanish. But then, firm i could corner the market by charging $p_i (t)$ such that $s (p_i) = s (c_j) + t$; as t goes to 0, $p_i (t)$ tends to c_j and firm i could thus secure in this way (close to) $M (c_j - c_i) \pi_i (c_j)$, which is bounded away from 0, a contradiction.

E.2 Cornered-market equilibria

We now characterize cornered market equilibria, in which one firm, say firm i, attracts all consumers; the other firm, j, thus makes zero profit. We first note that this requires asymmetric costs. To see this, suppose instead that both firms face the same cost c. Firm i cannot be pricing below c, otherwise it would make a loss and profitably deviate by raising its price. But then, firm j could profitably deviate by pricing slightly above cost, which, thanks to product differentiation, would enable it to gain a positive market share and earn a small but positive margin.

The two firms must therefore be facing different costs. A standard Bertrand argument ensures that the more efficient firm wins the market, and that the other firm does not charge more than its cost; that is, firm j faces some cost c and firm i faces a lower cost, of the form $\gamma (c) < c$, and $p_j^* \leq c$. As usual, we will focus on trembling-hand perfect equilibria, and thus discard those equilibria in which the losing firm would price below its own cost. It follows that the candidate equilibrium is:

$$p_j^* = c \text{ and } p_i^* = p^* (c), \tag{17}$$

where $p^* (c)$ is such that $\hat{x}_i (p^* (c), c) = 1$, that is:

$$s (p^* (c)) = s (c) + t.$$

It thus satisfies $p^*(c) < c$ (as $s(p^*(c)) > s(c)$) and

$$0 < p^{*\prime}(c) = \frac{d(c)}{d(p^*(c))} < 1. \tag{18}$$

In this candidate equilibrium, firm i obtains a profit equal to:

$$\Pi^*(c) \equiv [p^*(c) - \gamma(c)] D(p^*(c)),$$

where $D(p) \equiv Md(p)$ denotes total demand at price p.

For this to be an equilibrium, firm i should not benefit from increasing its price (in which case it would share the market with firm j); we have:

$$\begin{aligned} \frac{1}{M}\frac{\partial \Pi_i}{\partial p_i}(p_i, p_j)\bigg|_{p_j^*=c, p_i^*=p^*(c)} &= d(p^*(c)) + [p^*(c) - \gamma(c)] d'(p^*(c)) - \frac{d(p^*(c))}{2t}\pi^*(c) \\ &\leq \frac{d(p^*(c))}{2t}[2t - \pi^*(c)]. \end{aligned}$$

Hence, whenever:

$$\pi^*(c) \geq 2t,$$

there exists indeed a cornered market equilibrium, in which the firm with cost c prices at cost, and the other firm, with cost $\gamma(c)$, charges $p^*(c)$ and obtains a profit equal to $\pi^*(c)$.

E.3 Welfare analysis

In a cornered market equilibrium, social welfare is equal to:

$$W^*(c;\lambda) \equiv S(p^*(c)) + \lambda \Pi^*(c),$$

where $S(p) \equiv Ms(p)$ denotes total consumer surplus.

When the regulator focuses on consumer surplus, we thus have:

$$\frac{\partial W^*}{\partial c}(c;\lambda) = -D(p^*(c)) p^{*\prime}(c) < 0,$$

and so it is never optimal to have a cost handicap larger than what is needed for one firm to "barely" corner the market. It follows that, as t vanishes, the optimal allocation converges to cost equalization.

When $\lambda > 0$, we have:

$$W^*(c;\lambda) \equiv S(p^*(c)) + \lambda\left[p^*(c) - \gamma(c)\right] D(p^*(c)).$$

As t tends to vanish, $p^*(c) \simeq c$ converges to c, and thus $W^*(c;\lambda) \simeq W(c;\lambda)$, where $W(p;\lambda)$ is the welfare function studied in the baseline model of Bertrand competition, given by (6). Furthermore, maintaining a shared-market equilibrium outcome as t tends to vanish requires cost equalization (that is, $c_I = c_E = \hat{c}$); as the equilibrium price converges to cost (i.e., $p_I^* \simeq p_E^* \simeq \hat{c}$), it follows that total welfare converges to $S(\hat{c}) = W(\hat{c};\lambda)$. Hence, in both types of equilibrium (shared-market or cornered-market), the equilibrium prices tend to cost and total welfare converges to $W(c;\lambda)$; it follows that the optimal allocation converges towards that of the Bertrand baseline model (perfect substitutes).

F Proof of Proposition 6

As is well-known, in each (classic) auction, the higher-valuation bidder wins and pays a price equal to the lower-valuation bidder, where all valuations take into account the expected equilibrium outcome of subsequent auctions.

The proof proceeds by induction. We will label "auction h", for $h = 1, ..., k$, the auction taking place when h blocks remain to be allocated (hence, auction "k" is the first auction, and auction "1" is the auction for the last block). Let $p_0(B_L, B_l) \equiv 0$ and $\Pi_0(B_L, B_l) \equiv \Pi(B_L, B_l)$, where $\Pi(\cdot,\cdot)$ is given by (1), and for every $h = 1, ..., k$, let L_h and l_h respectively denote the leader and the laggard (i.e., the firm with the larger and with the smaller bandwidth) at the beginning of auction h – if both firms have the same bandwidth at the beginning of auction h, then select either firm as leader with probability 1/2.

We will use the following induction hypothesis H_h:

1. If $B_{L_h} > B_{l_h}$, then L_h wins auction h and obtains an expected net profit equal to

$\Pi_h\left(B_{L_h}, B_{l_h}\right) = \Pi\left(B_{L_h} + h\delta, B_{l_h}\right) - p_h\left(B_{L_h}, B_{l_h}\right)$, where

$$p_h\left(B_{L_h}, B_{l_h}\right) = \begin{cases} \Pi_{h-1}\left(B_{l_h} + \delta, B_{L_h}\right) & \text{if } B_{L_h} - B_{l_h} < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

whereas l_h obtains zero expected net profit.

2. If $B_{L_h} = B_{l_h}$, then either firm wins auction h and pays a price

$$p_h\left(B_{L_h}, B_{L_h}\right) = \Pi\left(B_{L_h} + h\delta, B_{L_h}\right).$$

Both firms obtain zero expected net profit.

We first check that H_1 holds:

- If $B_{L_1} \geq B_{l_1} + \delta$, then the laggard cannot obtain any profit in the product market, regardless of whether it wins the auction; hence, the leader obtains the last block for free.
- If instead $B_{L_1} < B_{l_1} + \delta$, then winning the auction gives the laggard a profit (gross of the price paid in the last auction) equal to $\Pi_0\left(B_{l_1} + \delta, B_{L_1}\right) = \Pi\left(B_{l_1} + \delta, B_{L_1}\right)$, and gives the leader a (gross) profit equal to $\Pi_0\left(B_{L_1} + \delta, B_{l_1}\right) = \Pi\left(B_{L_1} + \delta, B_{l_1}\right)$. Therefore:
  - If $B_{L_1} > B_{l_1}$, then the leader has a greater willingness to pay, as

  $$\Pi\left(B_{L_1} + \delta, B_{l_1}\right) > \Pi\left(B_{l_1} + \delta, B_{l_1}\right) > \Pi\left(B_{l_1} + \delta, B_{L_1}\right),$$

  where the first and second inequalities respectively stem from (2) and (3). Hence, the leader obtains the last block for a price equal to $p_1\left(B_{L_1}, B_{l_1}\right) = \Pi\left(B_{l_1} + \delta, B_{L_1}\right)$.
  - If instead $B_{L_1} = B_{l_1}$, then both firms obtains the same (gross) profit from winning the auction, and thus bid the same amount, equal to this profit. Hence, $p_1\left(B_{L_1}, B_{L_1}\right) = \Pi\left(B_{L_1} + \delta, B_{L_1}\right)$, either firm wins at that price, and both firms obtain zero net profit.

Suppose now that H_t holds for $t = 1, ..., h$, and consider auction $h+1$. If the leading firm L_{h+1} wins, then it will be again the leader in the next round, and will enjoy a bandwidth advantage of at least δ; therefore, according to the induction hypothesis, its profit from winning (gross of the price paid in auction $h+1$) is given by (taking into account that $p_h\left(B_{L_{h+1}} + \delta, B_{l_{h+1}}\right) = 0$, as $\left(B_{L_{h+1}} + \delta\right) - B_{l_{h+1}} \geq \delta$):

$$\hat{\Pi}_L = \Pi\left(B_{L_{h+1}} + (h+1)\,\delta, B_{l_{h+1}}\right).$$

If instead the laggard firm l_{h+1} wins auction $h+1$, it then becomes the leader in the next round if $B_{L_{h+1}} - B_{\ell_{h+1}} < \delta$, and otherwise remains the laggard (or becomes equally efficient as its rival, in which case it also obtains zero profit in the product market); therefore, according to the induction hypothesis, it obtains a profit (gross of the price paid in auction $h+1$) equal to:

$$\hat{\Pi}_l = \begin{cases} \Pi\left(B_{l_{h+1}} + (h+1)\,\delta, B_{L_{h+1}}\right) - p_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right) & \text{if } B_{L_{h+1}} - B_{l_{h+1}} < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore:

- If $B_{L_{h+1}} = B_{l_{h+1}}$,

  $$\Pi\left(B_{l_{h+1}} + (h+1)\,\delta, B_{L_{h+1}}\right) = \Pi\left(B_{L_{h+1}} + (h+1)\,\delta, B_{l_{h+1}}\right) = \Pi\left(B_{L_{h+1}} + (h+1)\,\delta, B_{L_{h+1}}\right)$$

  and

  $$p_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right) = p_h\left(B_{L_{h+1}} + \delta, B_{L_{h+1}}\right) = 0,$$

  and thus $\hat{\Pi}_L = \hat{\Pi}_l$. Hence, both firms bid

  $$p_{h+1} = \Pi\left(B_{L_{h+1}} + (h+1)\,\delta, B_{L_{h+1}}\right),$$

  either firm wins, and both firms obtain zero net profit.

- If instead $B_{L_{h+1}} > B_{l_{h+1}}$, then $\hat{\Pi}_L > \hat{\Pi}_l$, as $p_h(\cdot) \geq 0$ and

  $$\Pi\left(B_{L_{h+1}} + (h+1)\,\delta, B_{l_{h+1}}\right) > \Pi\left(B_{l_{h+1}} + (h+1)\,\delta, B_{l_{h+1}}\right) > \Pi\left(B_{l_{h+1}} + (h+1)\,\delta, B_{L_{h+1}}\right),$$

where the first and second inequalities respectively stem again from (2) and (3), and so the leading firm L_{h+1} wins auction $h+1$. Furthermore:

– When $B_{L_{h+1}} - B_{l_{h+1}} > \delta$, the lagging firm l_{h+1} would remain behind and thus obtain zero profit even if it were to win; hence, it bids zero, that is, $p_{h+1} = 0$.

– When instead $B_{L_{h+1}} - B_{l_{h+1}} < \delta$, the lagging firm l_{h+1} is willing to bid up to

$$\Pi\left(B_{l_{h+1}} + (h+1)\,\delta, B_{L_{h+1}}\right) - p_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right) = \Pi_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right).$$

The equilibrium price is thus equal to $p_{h+1} = \Pi_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right)$, as in the induction hypothesis. It follows that the equilibrium payoffs are also as in the induction hypothesis.

Therefore, H_{h+1} holds when H_t holds for $t = 1, ..., h$. It follows that the incumbent firm I wins all successive rounds. Furthermore, if $B_I - B_E \geq \delta$, then it obtains all the bandwidth at zero price. If instead $B_I - B_E < \delta$, then using the induction hypothesis we have:

$$p_k\left(B_I, B_E\right) = \begin{cases} \sum_{h=0}^{m-1} \phi_h^k\left(B_E, B_I\right) - \sum_{h=1}^{m} \phi_h^k\left(B_I, B_E\right) & \text{if } k = 2m, \\ \sum_{h=0}^{m} \phi_h^k\left(B_E, B_I\right) - \sum_{h=1}^{m} \phi_h^k\left(B_I, B_E\right) & \text{if } k = 2m+1. \end{cases}$$

where

$$\phi_h^k\left(B_1, B_2\right) \equiv \Pi\left(B_1 + (k-h)\,\delta, B_2 + h\delta\right).$$

Note that when $B_I = B_E = B$,

$$\phi_m^{2m}\left(B, B\right) = \Pi\left(B + m\delta, B + m\delta\right) = 0$$

and thus the equilibrium price is equal to

$$p_k\left(B, B\right) = \phi_0^k\left(B, B\right) = \Pi\left(B + \Delta, B\right).$$

G Proof of Proposition 7

We now show that, for each firm $i = I, E$, it is a dominant strategy to bid $\beta_i^*(n) = \pi_i(n)$, where (using the subscript "$-i$" to refer to firm i's rival):

$$\pi_i(n) \equiv \begin{cases} \Pi(B_i + n_i\delta, B_{-i} + n_{-i}\delta) & \text{if } B_i + n_i\delta > B_{-i} + n_{-i}\delta, \\ 0 & \text{otherwise.} \end{cases}$$

with $\Pi(\cdot, \cdot)$ given by (1).

To see this, consider an alternative strategy $\hat{\beta}_i$, and suppose that, for some bidding strategy of the other firm, β_{-i}, the bidding strategies β_i^* and $\hat{\beta}_i$ lead to different outcomes. As the payments only depend on the bids through the spectrum allocation, this implies that β_i^* and $\hat{\beta}_i$ lead to different spectrum allocations, which we will respectively denote by n^* and $\hat{n}$. Likewise, let Π_i^* and $\hat{\Pi}_i$ denote the net payoffs of firm i associated with the bidding strategies β_i^* and $\hat{\beta}_i$. We have:

$$\begin{aligned} \Pi_i^* - \hat{\Pi}_i &= \left\{\pi_i(n^*) - p_i^V(\beta_i^*, \beta_{-i})\right\} - \left\{\pi_i(\hat{n}) - p_i^V\left(\hat{\beta}_i, \beta_{-i}\right)\right\} \\ &= \left\{\pi_i(n^*) - \left[\max_{n \in \mathcal{A}}\{\beta_{-i}(n)\} - \beta_{-i}(n^*)\right]\right\} - \left\{\pi_i(\hat{n}) - \left[\max_{n \in \mathcal{A}}\{\beta_{-i}(n)\} - \beta_{-i}(\hat{n})\right]\right\} \\ &= \pi_i(n^*) - \pi_i(\hat{n}) - [\beta_{-i}(\hat{n}) - \beta_{-i}(n^*)] \\ &= \beta_i^*(n^*) - \beta_i^*(\hat{n}) - [\beta_{-i}(\hat{n}) - \beta_{-i}(n^*)]. \end{aligned}$$

But, by construction, as the bidding strategy β_i^* leads to n^*, it must be the case that

$$\beta_i^*(n^*) + \beta_{-i}(n^*) \geq \beta_i^*(\hat{n}) + \beta_{-i}(\hat{n}).$$

It follows that $\Pi_i^* \geq \hat{\Pi}_i$, establishing that bidding $\beta_i^*(n) = \pi_i(n)$ is a dominant strategy for firm i.

Given these bidding strategies, the outcome maximizes

$$\max_{n \in \mathcal{A}} \Pi(B_I + n_I\delta, B_E + n_E\delta),$$

which is achieved for $n_I = k$ and $n_E = 0$. That is, the incumbent firm I obtains all k

blocks, and pays a price equal to:

$$\max_{n \in \mathcal{A}} \{\beta_E(n)\} - \beta_E(k,0) = \begin{cases} \Pi(B_E + \Delta, B_I) & \text{if } \Delta > B_I - B_E, \\ 0 & \text{otherwise.} \end{cases}$$

H Proof of Proposition 8

When the handicap of the entrant is too large to be offset by the additional spectrum (i.e., when $B_I - B_E \geq \Delta$), the equilibrium prices are zero in both types of auctions. We now focus on the more interesting case where $B_I - B_E < \Delta$. In the case of a multi-unit VCG auction, the price is then always positive and equal to

$$p^V = \Pi(B_E + \Delta, B_I) > 0.$$

By contrast, in the case of a sequential auction, the price remains zero when the lagging firm cannot catch-up with a single block of size $\delta = \Delta/k$. Hence, for any given $\Delta > 0$, the price remains zero when the spectrum is divided in sufficiently many blocks, namely, when

$$k \geq \bar{k} = \frac{\Delta}{B_I - B_E}.$$

Finally, when instead the lagging firm could catch up with a single block (i.e., $B_I - B_E < \delta = \Delta/k$), the price is of the form (using the induction hypothesis):

$$p_k(B_I, B_E) = \Pi_{k-1}(B_E + \delta, B_I) = \Pi(B_E + \Delta, B_I) - p_{k-1}(B_E + \delta, B_I),$$

where $p_{k-1}(B_E + \delta, B_I) > 0$. Hence, the revenue is again lower with sequential auctions.

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Online Appendix - 1
(Not for Publication)

A Product differentiation

A.1 Proof of Proposition 3

We consider here the Hotelling model with elastic individual demands.

A.1.1 Shared-market equilibria

We first study shared market equilibria, in which both firms attract some consumers. The customer who is indifferent between patronizing the two firms is located at a distance $\hat{x}_i$ from firm i that is determined by:

$$s(p_i) - t\hat{x}_i = s(p_j) - t(1 - \hat{x}_i),$$

where p_i and p_j denote firms' prices,

$$s(p) \equiv \int_p^{+\infty} d(v)\, dv$$

denotes individual consumer surplus, and t denotes transportation costs per unit distance. Firm i's market share is therefore

$$\hat{x}_i(p_i, p_j) \equiv \frac{1}{2} + \frac{s(p_i) - s(p_j)}{2t},$$

where $i \neq j \in \{I, E\}$, and its profit is

$$\Pi_i(p_i, p_j) \equiv M\hat{x}_i(p_i, p_j)\, \pi_i(p_i),$$

where

$$\pi_i(p_i) \equiv (p_i - c_i)\, d(p_i)$$

denotes firm i's per consumer profit. The equilibrium prices, p_I^* and p_E^*, and the associated equilibrium variables satisfy the first-order conditions, which, using $s(p) = -d(p)$, can be written as:

$$d\left(p_i^*\right)\pi_i\left(p_i^*\right) = 2t\hat{x}_i\left(p_i^*, p_j^*\right)\pi_i'\left(p_i^*\right) \leq 2t\hat{x}_i\left(p_i^*, p_j^*\right)d\left(p_i^*\right),$$

where the inequality follows from $\pi_i'\left(p_i^*\right) = d\left(p_i^*\right) + \left(p_i^* - c_i\right)d'\left(p_i^*\right) \leq d\left(p_i^*\right)$, as $d'\left(\cdot\right) < 0$ and active firms never sell below costs (i.e., $p_i^* \geq c_i$). Dividing by $d\left(p_i^*\right)$ and adding the resulting inequalities for the two firms yields $\pi_I\left(p_I^*\right) + \pi_E\left(p_E^*\right) < 2t$. As active firms never make a loss (i.e., $\pi^*\left(p_i^*\right) \geq 0$), this condition in turn implies that each firm i obtains a total profit lower than $2t$:

$$\Pi_i^* \equiv \Pi_i\left(p_i^*, p_j^*\right) = M\hat{x}_i(p_i^*, p_j^*)\pi_i\left(p_i^*\right) \leq 2tM.$$

It follows that, when firms face different costs, the market cannot remain shared as t tends to vanish. To see this, suppose that $c_i < c_j$. From the above analysis, in any shared-market equilibrium, $\Pi_i^* \leq 2tM$ and $p_j^* \geq c_j$. Hence, firm i's profit tends to 0 as t tends to vanish. But then, firm i could corner the market by charging $p_i\left(t\right)$ such that $s\left(p_i\right) = s\left(c_j\right) + t$; as t goes to 0, $p_i\left(t\right)$ tends to c_j and firm i could thus secure in this way (close to) $M\left(c_j - c_i\right)\pi_i\left(c_j\right)$, which is bounded away from 0, a contradiction.

A.1.2 Cornered-market equilibria

We now characterize cornered market equilibria, in which one firm, say firm i, attracts all consumers; the other firm, j, thus makes zero profit. We first note that this requires asymmetric costs. To see this, suppose instead that both firms face the same cost c. Firm i cannot be pricing below c, otherwise it would make a loss and profitably deviate by raising its price. But then, firm j could profitably deviate by pricing slightly above cost, which, thanks to product differentiation, would enable it to gain a positive market share and earn a small but positive margin.

The two firms must therefore be facing different costs. A standard Bertrand argument ensures that the more efficient firm wins the market, and that the other firm does not charge more than its cost; that is, firm j faces some cost c and firm i faces a lower cost, of the form $\gamma\left(c\right) < c$, and $p_j^* \leq c$. As usual, we will focus on trembling-hand perfect equilibria, and thus discard those equilibria in which the losing firm would price below its own cost. It follows that the candidate equilibrium is:

$$p_j^* = c \text{ and } p_i^* = \hat{p}(c),$$

where $\hat{p}(c)$ is such that $\hat{x}_i(\hat{p}(c), c) = 1$, that is:

$$s(\hat{p}(c)) = s(c) + t.$$

It thus satisfies $\hat{p}(c) < c$ (as $s(\hat{p}(c)) > s(c)$) and

$$0 < \hat{p}'(c) = \frac{d(c)}{d(\hat{p}(c))} < 1.$$

In this candidate equilibrium, firm i obtains a profit equal to:

$$\Pi^*(c) \equiv [\hat{p}(c) - \gamma(c)] D(\hat{p}(c)),$$

where $D(p) \equiv Md(p)$ denotes total demand at price p.

For this to be an equilibrium, firm i should not benefit from increasing its price (in which case it would share the market with firm j); we have:

$$\begin{aligned} \frac{1}{M}\frac{\partial \Pi_i}{\partial p_i}(p_i, p_j)\bigg|_{p_j^*=c, p_i^*=\hat{p}(c)} &= d(\hat{p}(c)) + [\hat{p}(c) - \gamma(c)] d'(\hat{p}(c)) - \frac{d(\hat{p}(c))}{2t}\pi^*(c) \\ &\leq \frac{d(\hat{p}(c))}{2t}[2t - \pi^*(c)]. \end{aligned}$$

Hence, whenever:

$$\pi^*(c) \geq 2t,$$

there exists indeed a cornered market equilibrium, in which the firm with cost c prices at cost, and the other firm, with cost $\gamma(c)$, charges $\hat{p}(c)$ and obtains a profit equal to $\pi^*(c)$.

A.1.3 Welfare analysis

In a cornered market equilibrium, social welfare is equal to:

$$W^*(c;\lambda) \equiv S(\hat{p}(c)) + \lambda \Pi^*(c),$$

where $S(p) \equiv Ms(p)$ denotes total consumer surplus.

When the regulator focuses on consumer surplus, we thus have:

$$\frac{\partial W^*}{\partial c}(c;\lambda) = -D(\hat{p}(c))\,\hat{p}'(c) < 0,$$

and so it is never optimal to have a cost handicap larger than what is needed for one firm to "barely" corner the market. It follows that, as t vanishes, the optimal allocation converges to cost equalization.

When $\lambda > 0$, we have:

$$W^*(c;\lambda) \equiv S(\hat{p}(c)) + \lambda[\hat{p}(c) - \gamma(c)]\,D(\hat{p}(c)).$$

As t tends to vanish, $\hat{p}(c) \simeq c$ converges to c, and thus $W^*(c;\lambda) \simeq W(c;\lambda)$, where $W(p;\lambda)$ is the welfare function studied in the baseline model of Bertrand competition, given by (6). Furthermore, maintaining a shared-market equilibrium outcome as t tends to vanish requires cost equalization (that is, $c_I = c_E = \hat{c}$); as the equilibrium price converges to cost (i.e., $p_I^* \simeq p_E^* \simeq \hat{c}$), it follows that total welfare converges to $S(\hat{c}) = W(\hat{c};\lambda)$. Hence, in both types of equilibrium (shared-market or cornered-market), the equilibrium prices tend to cost and total welfare converges to $W(c;\lambda)$, the welfare expression of the baseline model; it follows that the optimal allocation converges towards that of the baseline model (perfect substitutes).

A.2 Proof of Proposition 4

We now turn to the standard Hotelling setting with unit-demand consumers; for the sake of exposition, we focus on the case in which $c_I \leq c_E$.

Firms share the market as long as their prices, p_1 and p_2, do not differ too much; consumers located at a distance $x < \hat{x}_i$ from firm i then purchase from that firm whereas those located at $x > \hat{x}_i$ purchase from firm j, where $\hat{x}_i$ is determined by:

$$\hat{x}_i = \frac{1}{2} + \frac{p_j - p_i}{2t}.$$

As long as the costs are not too different, namely, $|c_I - c_E| \leq 3t$, firms do share the market in equilibrium; the prices and market shares are then given by:

$$p_i^* (c_i, c_j) = t + \frac{2c_i + c_j}{3},$$

and

$$\hat{x}_i^* (c_i, c_j) = \frac{1}{2} \left(1 - \frac{c_i - c_j}{3t} \right),$$

and the resulting profits are:

$$\pi_i^* (c_i, c_j) = (p_i^* (c_i, c_j) - c_i) \, \hat{x}_i^* (c_i, c_j) = \frac{1}{2t} \left(t - \frac{c_i - c_j}{3} \right)^2.$$

A uniform increase in both costs is entirely passed on to consumers; hence, it does not affect total industry profit, which is given by

$$\Pi^* (c_I, c_E) = t + \frac{(c_E - c_I)^2}{9t},$$

which increases with cost asymmetry. By contrast, a uniform increase in both costs reduces consumer surplus, given by:

$$S^* (c_I, c_E) = r + \frac{5t}{4} - \frac{c_E + c_I}{2} + \frac{(c_E - c_I)^2}{36t}.$$

Finally, total welfare is given by:

$$W_S (c_I, c_E) = S^* (c_I, c_E) + \Pi^* (c_I, c_E) = W_0 - \frac{c_E + c_I}{2} + \beta \left(\frac{c_E - c_I}{2} \right)^2, \tag{19}$$

where the subscript S stands for a *Shared-market equilibrium*, and

$$W_0 \equiv r - \frac{rt}{4} + \lambda t \text{ and } \beta \equiv \frac{1 + 4\lambda}{9t}.$$

When instead $c_E - c_I > 3t$, then the incumbent corners the market; we then have:

$p_I = c_E - t, \pi_I = c_E - c_I - t, S(c_I, c_E) = r - c_E - \frac{t}{2}$, and welfare is

$$W_C(c_I, c_E) = r - \frac{t}{2} - c_E + \lambda(c_E - c_I - t),$$

where the subscript C stands for a *Cornered-market equilibrium.*

Let

$$\hat{B} = \frac{B_I + B_E + \Delta}{2}$$

denote the average available bandwidth, and

$$\delta = \hat{B} - (B_E + b_E)$$

denote E's ex post bandwidth handicap. The costs of the two firms can then be expressed as:

$$c_I = c\left(\hat{B} + \delta\right) \text{ and } c_E = c\left(\hat{B} - \delta\right),$$

where the relevant relevant range for $|\delta|$ (reflecting the fact that $B_E + b_E$ can vary from B_E to $B_E + \Delta$) is given by

$$|\delta| \in \left[\underline{\delta} \equiv \max\left\{0, \hat{B} - (B_E + \Delta)\right\}, \bar{\delta} \equiv \min\left\{\hat{\delta}, \hat{B} - B_E\right\}\right],$$

where $\hat{\delta}$ is the bandwidth handicap that would enable the low-cost firm to corner the market.[58] Note that, by construction, $\bar{\delta} < \hat{B}$.

I corners the market when $\delta \geq \tilde{\delta}$, where $\tilde{\delta} > 0$ is implicitly defined by

$$c\left(\hat{B} - \delta\right) - c\left(\hat{B} + \delta\right) = 3t,$$

and in this range the equilibrium welfare, $W_C(c_I, c_E)$, can be expressed as

$$W_C(\delta) \equiv W_0 - (1 - \lambda)\, c\left(\hat{B} - \delta\right) - \lambda c\left(\hat{B} + \delta\right).$$

[58]That is, $\hat{\delta}$ is implicitly characterized by the condition $c\left(\hat{B} - \hat{\delta}\right) - c\left(\hat{B} + \hat{\delta}\right) = 3t$.

For $\lambda \leq 1/2$, we thus have:

$$W_C'(\delta) = (1-\lambda)\, c'\left(\hat{B}-\delta\right) - \lambda c'\left(\hat{B}+\delta\right) < 0,$$

where the inequality follows from $\lambda \leq 1/2$, $\delta \geq \tilde{\delta} > 0$ and $c''(B) > 0 > c'(B)$.[59] Hence, when $\lambda \leq 1/2$, it is never optimal to increase the cost difference beyond what enables a firm to corner the market.

A.2.1 Unit Costs Quadratic in Bandwidth

Suppose now that firm i's unit cost is given by:

$$c_i = c(B_i) = \gamma_0 - \gamma_1 B_i + \gamma_2 B_i^2,$$

where $\gamma_0, \gamma_1 \in \mathbb{R}_+$ and $\gamma_2 \in \mathbb{R}_+^*$. To ensure that the unit cost is decreasing in bandwidth over the relevant range $[B_E, B_I + \Delta]$, we must have:

$$c'(B_I + \Delta) = -\gamma_1 + 2\gamma_2 (B_I + \Delta) < 0,$$

or:

$$\gamma_1 > 2\gamma_2 (B_I + \Delta) \left(> 2\gamma_2 \hat{B}\right).$$

We then have:

$$\frac{c_I - c_E}{2} = \left(\gamma_1 - 2\gamma_2 \hat{B}\right)\delta,$$

which is increasing in δ (as $\gamma_1 > 2\gamma_2 \hat{B}$), and:

$$\frac{c_I + c_E}{2} = \gamma_0 - \gamma_1 \hat{B} + \gamma_2 \left(\hat{B}^2 + \delta^2\right),$$

which is also increasing in δ. It follows that, in the range where the firms share the market, the equilibrium welfare $W_S(c_I, c_E)$ characterized by (19) is of the form

$$\tilde{W}_S\left(\delta^2\right) = \tilde{W}_0 - \tilde{W}_1 \delta^2,$$

[59] We have $W_C'(0) = (1-2\lambda)\, c'\left(\hat{B}\right) \leq 0$ (as $\lambda \leq 1/2$ and $c'(\cdot) < 0$) and $W_C''(\delta) = -(1-\lambda)\, c''\left(\hat{B}-\delta\right) - \lambda c''\left(\hat{B}+\delta\right) < 0$ (as $\lambda \in [0,1]$ and $c''(\cdot) > 0$); hence, $W_C'(\delta) < 0$ for any $\delta > 0$.

where $\tilde{W}_0 \equiv W_0 - \gamma_0 + \gamma_1 \hat{B} - \gamma_2 \hat{B}^2$ and

$$\tilde{W}_1 = \gamma_2 - \beta \left(\gamma_1 - 2\hat{B}\gamma_2 \right)^2 .$$

Hence, minimizing the cost difference (i.e., setting $|\delta| = \underline{\delta}$) is optimal if and only if $\tilde{W}_1 \geq 0$, which amounts to:

$$\beta \leq \frac{\gamma_2}{\left(\gamma_1 - 2\hat{B}\gamma_2 \right)^2}.$$

If instead $\tilde{W}_1 < 0$, then it is optimal to increase the cost difference up to the point where I either obtains the entire additional bandwidth Δ, or corners the market; that is, it is optimal to set $\delta = \bar{\delta}$.

A.2.2 Unit Costs Inversely Proportional to Bandwidth

Suppose now that unit costs are inversely proportional to bandwidth, as follows:

$$c_i = \frac{\alpha}{B_i}.$$

We then have:

$$\frac{c_I - c_E}{2} = \frac{\alpha}{2} \left(\frac{1}{\hat{B} - \delta} - \frac{1}{\hat{B} + \delta} \right) = \frac{\alpha \delta}{\hat{B}^2 - \delta^2},$$

and:

$$\frac{c_I + c_E}{2} = \frac{\alpha}{2} \left(\frac{1}{\hat{B} - \delta} + \frac{1}{\hat{B} + \delta} \right) = \frac{\alpha \hat{B}}{\hat{B}^2 - \delta^2}.$$

Hence, in the range where the firms share the market, the equilibrium welfare $W_S (c_I, c_E)$ characterized by (19) is of the form

$$\hat{W}_S \left(\delta^2 \right) \equiv W_0 - \frac{\alpha \hat{B}}{\hat{B}^2 - \delta^2} + \beta \frac{\alpha^2 \delta^2}{\left(\hat{B}^2 - \delta^2 \right)^2}.$$

The first-order derivative of the function $\hat{W}_S (\cdot)$ is equal to:[60]

$$\hat{W}_S' \left(\delta^2 \right) = \frac{\alpha}{\left(\hat{B}^2 - \delta^2 \right)^2} \left(\alpha\beta - \hat{B} + \frac{2\alpha\beta\delta^2}{\hat{B}^2 - \delta^2} \right).$$

[60] We are referring here ot the derivative of $\hat{W}(x) = \hat{W}_0 - \frac{\alpha \hat{B}}{\hat{B}^2 - x} + \beta \frac{\alpha^2 x}{\left(\hat{B}^2 - x \right)^2}$.

If $\alpha\beta \geq \hat{B}$, then $\hat{W}'_S(\delta^2) > 0$ for any $\delta \in [0, \bar{\delta}]$;[61] it is therefore optimal to increase the cost difference and set $\delta = \bar{\delta}$. In what follows, we now focus on the case $\alpha\beta < \hat{B}$.

The second-order derivative of the function $\hat{W}_S(\cdot)$ is given by

$$\hat{W}''_S(\delta^2) = \frac{2\alpha}{\left(\hat{B}^2 - \delta^2\right)^4}\phi(\delta^2),$$

where

$$\phi(x) \equiv \hat{B}^2\left(2\alpha\beta - \hat{B}\right) + \left(\hat{B} + \alpha\beta\right)x.$$

The sign of $\hat{W}''(\delta)$ is therefore the same as of $\phi(\delta^2)$, which may be negative for δ^2 small (if $\alpha\beta < \hat{B}/2$) but is always positive for δ^2 large enough (as $\phi'(\cdot) = \hat{B} + \alpha\beta > 0$). It follows that, as $|\delta|$ increases, $\hat{W}(\delta)$ (i) is first decreasing and (ii) may be concave for $|\delta|$ small enough but is always convex for larger values of $|\delta|$. It follows that minimizing cost difference (i.e., setting $\delta = \underline{\delta}$) is optimal if and only if

$$\hat{W}_S\left(\underline{\delta}^2\right) \geq \hat{W}_S\left(\bar{\delta}^2\right),$$

and that maximizing the cost difference up to enabling a firm to corner the market (i.e., setting $\delta = \bar{\delta}$) is otherwise optimal. The former condition amounts to

$$|\delta| \in \left[\underline{\delta} \equiv \max\left\{0, \hat{B} - (B_E + \Delta)\right\}, \bar{\delta} \equiv \min\left\{\hat{\delta}, \hat{B} - B_E\right\}\right],$$

Therefore, assuming that it is feasible (i.e., $\Delta \geq B_I - B_E$), cost equalization (i.e., $\delta = 0$) is optimal if and only if it dominates giving the entire bandwidth Δ to $\psi(\Delta) \geq 0$, where:

$$\begin{aligned}\psi(\Delta) &\equiv \hat{W}_0 - \frac{\alpha\hat{B}}{\hat{B}^2 - \delta^2} + \frac{\alpha^2\beta\delta^2}{\left(\hat{B}^2 - \delta^2\right)^2}\Bigg|_{\delta=\underline{\delta}} - \hat{W}_0 - \frac{\alpha\hat{B}}{\hat{B}^2 - \delta^2} + \frac{\alpha^2\beta\delta^2}{\left(\hat{B}^2 - \delta^2\right)^2}\Bigg|_{\delta=\bar{\delta}} \\ &= \frac{\alpha\left(\bar{\delta}^2 - \underline{\delta}^2\right)}{\left(\hat{B}^2 - \underline{\delta}^2\right)^2\left(\hat{B}^2 - \bar{\delta}^2\right)^2}\left[\hat{B}\left(\hat{B}^2 - \bar{\delta}^2\right)\left(\hat{B}^2 - \underline{\delta}^2\right) - \alpha\beta\left(\hat{B}^4 - \underline{\delta}^2\bar{\delta}^2\right)\right].\end{aligned}$$

It follows that setting $\delta = \underline{\delta}$ is optimal if the parameters α and β are not too large,

[61] Recall that $\bar{\delta} < \hat{B}$.

namely, if:

$$\alpha\beta \leq \frac{\left(\hat{B}^2 - \bar{\delta}^2\right)\left(\hat{B}^2 - \underline{\delta}^2\right)}{\hat{B}^4 - \bar{\delta}^2\underline{\delta}^2}\hat{B}.$$

If this condition is not satisfied, setting $\delta = \bar{\delta}$ is instead optimal.

Remark: larger weight on profits. When $\lambda > 1/2$, the analysis of the welfare expression remains valid in the range where the two firms share the market, and thus readily carries over when $\hat{\delta} \geq \hat{B} - B_E$, as it is then impossible to let I corner the market. However, if $\hat{\delta} < \hat{B} - B_E$ (in which case $\bar{\delta} = \hat{\delta}$), then, in the range where I corners the market (i.e., $\delta \in \left[\hat{\delta}, \hat{B} - B_E\right]$), it may become optimal to increase the cost difference. More precisely, in this range the equilibrium welfare, $W_C(\delta)$, satisfies

$$W_C'(0) = (1 - 2\lambda)\, c'\left(\hat{B}\right) > 0,$$

as $\lambda > 1/2$ and $c'(\cdot) < 0$, and

$$W_C''(\delta) = -(1 - \lambda)\, c''\left(\hat{B} - \delta\right) - \lambda c''\left(\hat{B} + \delta\right) < 0,$$

as $\lambda \in [0, 1]$ and $c''(\cdot) > 0$. Hence, $W_C(\cdot)$ is maximal for $\delta^C > 0$, which is the unique solution to

$$0 = W_C'\left(\delta^C\right) = (1 - \lambda)\, c'\left(\hat{B} - \delta\right) - \lambda c'\left(\hat{B} + \delta\right).$$

If $\delta^C \leq \hat{\delta}$, then the previous analysis remains again valid, as $W_C(\delta)$ is decreasing in the relevant range $\delta \in \left[\hat{\delta}, \hat{B} - B_E\right]$. By contrast, if $\delta^C \leq \hat{\delta}$, then in this relevant range $W_C(\delta)$ is maximal for

$$\hat{\delta}_C \equiv \min\left\{\delta^C, \hat{B} - B_E\right\}.$$

In that case, $\bar{\delta} = \hat{\delta}$ must be replaced with $\hat{\delta}_C$ in the previous analysis. For instance, when unit costs are inversely proportional to bandwidth, setting $\delta = \underline{\delta}$ remains optimal if

$$\alpha\beta \leq \frac{\left(\hat{B}^2 - \hat{\delta}_C^2\right)\left(\hat{B}^2 - \underline{\delta}^2\right)}{\hat{B}^4 - \underline{\delta}^2\hat{\delta}_C^2}\hat{B},$$

otherwise it is optimal to set $\delta = \hat{\delta}_C$.

B Online Appendix-2

Proof of Corollary 2.

Proof. Fix $i \in \mathcal{I}$ and $k \in \mathcal{T}$. From Lemma 3, $\left(IR_i^{Hk}\right)$ and $\left(IC_i^{Lk}\right)$ are both binding, which respectively yields $\Pi_i^{Hk} = 0$, or:

$$t_i^{Hk} = \pi_i^{Hk} = \pi_i\left(b_i^{Hk}, b_j^{kH}, \theta_i^H, \theta_j^k\right) \geq 0,$$

and $\Pi_i^{hk} = \tilde{\Pi}_i^{hk}$, or:

$$t_i^{Lk} = \pi_i^{Lk} - \tilde{\Pi}_i^{hk} = \pi_i^{Lk} - \left(\tilde{\pi}_i^{Lk} - t_i^{Hk}\right) = \pi_i^{Lk} - \left(\tilde{\pi}_i^{Lk} - \pi_i^{Hk}\right), \tag{20}$$

where the last equality stems from $t_i^{Hk} = \pi_i^{Hk}$. Furthermore, it follows from $\Pi_i^{Hk} = 0$ that $\left(IC_i^{Hk}\right)$ amounts to:

$$0 \geq \tilde{\Pi}_i^{Hk} = \tilde{\pi}_i^{Hk} - t_i^{Lk},$$

which in turn implies:

$$t_i^{Lk} \geq \tilde{\pi}_i^{Hk} = \pi_i\left(b_i^{Lk}, b_j^{kL}, \theta_i^H, \theta_j^k\right) \geq 0.$$

Finally, as $\theta_i^L < \theta_i^H$, we have:

$$\tilde{\pi}_i^{Lk} - \pi_i^{Hk} = \pi_i\left(b_i^{Hk}, b_j^{kH}, \theta_i^L, \theta_j^k\right) - \pi_i\left(b_i^{Hk}, b_j^{kH}, \theta_i^H, \theta_j^k\right) \geq 0;$$

hence, it follows from (20) that $t_i^{Lk} \leq \pi_i^{Lk}$.

To conclude the proof, it suffices to note that $\beta_i^{hk} \leq \gamma_i^{hk}$ implies $\pi_i^{hk} = 0$; hence, we have $0 \leq t_i^{hk} \leq \pi_i^{hk} = 0$, implying $t_i^{hk} = 0$. Conversely, if $\beta_i^{Hk} > \gamma_i^{Hk}$, then $t_i^{Hk} = \pi_i^{Hk} > 0$. Furthermore, (20) and Lemma 3(ii) together yield:

$$t_i^{Lk} = \pi_i^{Lk} - \left(\tilde{\pi}_i^{Lk} - \pi_i^{Hk}\right) = \pi_i^{Lk} - \tilde{\pi}_i^{Lk} + \pi_i^{Hk} \geq \tilde{\pi}_i^{Hk}.$$

It follows that $\beta_i^{Lk} > \gamma_i^{Hk}$ implies $t_i^{Lk} \geq \tilde{\pi}_i^{Hk} > 0$. ■

The next lemma shows that if a firm loses when it has a low handicap, it *a fortiori* loses when it has a higher handicap:

Lemma 6 (revealed preferences) *For any $i \in \mathcal{I}$ and any $k \in \mathcal{T}$:*

(i) if $\beta_i^{Lk} \geq \gamma_i^{Lk}$ and $\beta_i^{Hk} \geq \gamma_i^{Hk}$, then $\beta_i^{Hk} \geq \beta_i^{Lk}$;

(ii) if $\beta_i^{Lk} \leq \gamma_i^{Lk}$, then $\beta_i^{Hk} \leq \gamma_i^{Lk} \left(< \gamma_i^{Hk}\right)$.

Proof. Part (i). Fix $i \in \mathcal{I}$ and $k \in \mathcal{T}$, and suppose that $\beta_i^{Lk} \geq \gamma_i^{Lk}$ and $\beta_i^{Hk} \geq \gamma_i^{Hk} \left(> \gamma_i^{Lk}\right)$. We then have

$$\pi_i^{Lk} = \underline{\pi}_i^{Lk} \geq 0,$$

and

$$\tilde{\pi}_i^{Lk} = \underline{\tilde{\pi}}_i^{Lk} > \underline{\pi}_i^{Hk} = \pi_i^{Hk} \geq 0,$$

where the strict inequality stems from the fact that $\underline{\tilde{\pi}}_i^{Lk}$ involves the same bandwidth allocation as $\underline{\pi}_i^{Hk}$, but a lower cost for firm i. The monotonicity condition $\left(M_i^k\right)$ therefore yields:

$$\underline{\pi}_i^{Lk} + \underline{\pi}_i^{Hk} \geq \left(\underline{\tilde{\pi}}_i^{Lk} + \tilde{\pi}_i^{Hk} \geq\right) \underline{\tilde{\pi}}_i^{Lk} + \underline{\tilde{\pi}}_i^{Hk},$$

or:

$$C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^L\right) + C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^H\right) \geq$$
$$C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right) + C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^H\right),$$

which after simplification amounts to:

$$C\left(b_i^{Lk} - \theta_i^H\right) - C\left(b_i^{Lk} - \theta_i^L\right) \geq C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right)$$
$$\Longleftrightarrow \phi\left(b_i^{Lk}\right) \geq \phi\left(b_i^{Hk}\right),$$

which in turn implies $b_i^{Hk} \geq b_i^{Lk}$ or, given that there is full allocation, $\beta_i^{Hk} \geq \beta_i^{Lk}$.

Part (ii).

Suppose that $\beta_i^{Lk} \leq \gamma_i^{Lk}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$. It follows from Corollary 2 that $t_i^{Lk} = 0$; hence, $\left(IC_i^{Lk}\right)$ amounts to:

$$0 \geq \tilde{\pi}_i^{Lk} - t_i^{Hk} = \tilde{\pi}_i^{Lk} - \pi_i^{Hk} \geq 0,$$

where the equality stems from Corollary 2 and the last inequality from $\theta_i^L < \theta_i^H$. We therefore have $\tilde{\pi}_i^{Lk} = \pi_i^{Hk}$, or:

$$\max\left\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right), 0\right\} = \max\left\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^H\right), 0\right\},$$

which in turn implies $\beta_i^{Hk} \leq \gamma_i^{Lk}$.[62] ■

Building on Lemma 6 yields:

Lemma 6 can alternatively be stated as follows:

Corollary 3 (revealed preferences) *For any $i \in \mathcal{I}$ and any $k \in \mathcal{T}$:*

(i) if $\beta_j^{kL} \leq \gamma_j^{kL}$ and $\beta_j^{kH} \leq \gamma_j^{kH}$, then $\beta_j^{kH} \leq \beta_j^{kL}$;

(ii) if $\beta_j^{kL} \geq \gamma_j^{kL}$, then $\beta_j^{kH} \geq \gamma_j^{kL}\left(> \gamma_j^{kH}\right)$.

Proof. This follows directly from Lemma 6, rewriting $\beta_i^{hk} \geq \gamma_i^{\hat{h}k}$ as $\beta_j^{kh} \leq \gamma_j^{k\hat{h}}$, for $\mathcal{I}$ and $\left(h, \hat{h}, k\right) \in \mathcal{T}^3$. ■

Building on Lemma 6 yields:

Lemma 6 can alternatively be stated as follows:

Corollary 4 (revealed preferences) *For any $i \in \mathcal{I}$ and any $k \in \mathcal{T}$:*

(i) if $\beta_j^{kL} \leq \gamma_j^{kL}$ and $\beta_j^{kH} \leq \gamma_j^{kH}$, then $\beta_j^{kH} \leq \beta_j^{kL}$;

(ii) if $\beta_j^{kL} \geq \gamma_j^{kL}$, then $\beta_j^{kH} \geq \gamma_j^{kL}\left(> \gamma_j^{kH}\right)$.

Proof. This follows directly from Lemma 6, rewriting $\beta_i^{hk} \geq \gamma_i^{\hat{h}k}$ as $\beta_j^{kh} \leq \gamma_j^{k\hat{h}}$, for $\mathcal{I}$ and $\left(h, \hat{h}, k\right) \in \mathcal{T}^3$. ■

B.0.1 Monotonicity conditions

From revealed preferences, the following monotonicity condition must be satisfied for every $i \in \mathcal{I}$ and every $k \in \mathcal{T}$:

$$\pi_i^{Lk} + \pi_i^{Hk} \geq \tilde{\pi}_i^{Lk} + \tilde{\pi}_i^{Hk}. \qquad (M_i^L)$$

[62] If $\beta_i^{Hk} > \gamma_i^{Lk}$, then the left-hand side is equal to $C(b_j^{kH} - \theta_j^k) - C\left(b_i^{Hk} - \theta_i^L\right) > 0$, and thus strictly exceeds the right-hand side.

Furthermore, the four monotonicity constraints, namely:

$$\pi_i^{LL} - \tilde{\pi}_i^{HL} \geq \tilde{\pi}_i^{LL} - \pi_i^{HL}, \tag{M_i^L}$$

$$\pi_i^{LH} - \tilde{\pi}_i^{HH} \geq \tilde{\pi}_i^{LH} - \pi_i^{HH}, \tag{M_i^H}$$

$$\pi_j^{LL} - \tilde{\pi}_j^{HL} \geq \tilde{\pi}_j^{LL} - \pi_j^{HL}, \tag{M_j^L}$$

$$\pi_j^{LH} - \tilde{\pi}_j^{HH} \geq \tilde{\pi}_j^{LH} - \pi_j^{HH}, \tag{M_j^H}$$

together prevent the regulator from achieving full cost equalization:

Lemma 7 (cost inequalization) *For any $i \in \mathcal{I}$ and any $k \in \mathcal{T}$, there exists $h \in \mathcal{T}$ such that $\beta_i^{hk} \neq \gamma_i^{hk}$.*

Proof. Fix $i \in \mathcal{I}$ and $k \in \mathcal{T}$, and suppose that $\beta_i^{Lk} - \gamma_i^{Lk} = \beta_i^{Hk} - \gamma_i^{Hk} = 0$. From (M_i^k), we have:

$$0 = \pi_i^{Lk} + \pi_i^{Hk} \geq \tilde{\pi}_i^{LH} + \tilde{\pi}_i^{HH} \geq 0,$$

where the first equality stems from $\beta_i^{Lk} - \gamma_i^{Lk} = \beta_i^{Hk} - \gamma_i^{Hk} = 0$ and the last inequality from the definitions of $\tilde{\pi}_i^{LH}$ and $\tilde{\pi}_i^{HH}$, which ensure that they are both non-negative. This, in turn, implies that $\tilde{\pi}_i^{LH} = \tilde{\pi}_i^{HH} = 0$. But the working assumption $\beta_i^{Hk} = \gamma_i^{Hk} = \gamma_i^{Lk} + \delta$ implies $\beta_i^{Hk} > \gamma_i^{Lk}$; hence, $\tilde{\pi}_i^{LH} > 0$, a contradiction. ■

It follows from Lemma 4 that $\beta_i^{hk} < \gamma_i^{hk}$ for some $i \in \mathcal{T}, h, k \in \mathcal{T}^2$ and that some monotonicity conditions are binding (otherwise, the regulator could achieve full cost equalization). Building on this leads to:

Lemma 8 (monotonicity conditions) *For any $i \neq j \in \mathcal{I}$:*

(i) if $\beta_i^{LL} < \gamma_i^{LL}$, then $\beta_i^{LL} \geq \gamma_i^{LH}$ and (M_j^L) is binding;

(ii) if $\beta_i^{LH} < \gamma_i^{LH}$, then (M_j^L) is binding;

(iii) if $\beta_i^{HH} < \gamma_i^{HH}$, then either (M_j^H) binds, or $\beta_i^{HH} \geq \gamma_i^{LH}$ and (M_i^H) binds (or both of these conditions hold);

(iv) if $\beta_i^{HL} < \gamma_i^{HL}$, then $\beta_i^{HL} \geq \gamma_i^{LL}$ and either (M_i^L) or (M_j^H) is binding (or both are binding).

Proof. We consider the various parts of the lemma in turn.

• *Part* (*i*). Suppose that $\beta_i^{LL} < \gamma_i^{LL}$, and consider a change in (b_i^{LL}, b_j^{LL}) to $(b_i^{LL}+\eta, b_j^{LL}-\eta)$, for some positive but small η. These changes can only affect $\pi_i^{LL}, \tilde{\pi}_i^{HL}, \pi_j^{LL}, \tilde{\pi}_j^{HL}$, and can therefore potentially alter $\left(M_i^L\right)$, through $\Delta_i^{LL}(\eta) \equiv \pi_i^{LL}(\eta) - \tilde{\pi}_i^{HL}(\eta)$, and $\left(M_j^L\right)$, through $\Delta_j^{LL}(\eta) \equiv \pi_j^{LL}(\eta) - \tilde{\pi}_j^{HL}(\eta)$. Furthermore, as $\beta_i^{LL} < \gamma_i^{LL} < \gamma_i^{HL}$, for η is sufficiently small:

- $\pi_i^{LL}(\eta)$ and $\tilde{\pi}_i^{HL}(\eta)$ remain equal to zero; hence, $\left(M_i^L\right)$ is unaffected.
- $\pi_j^{LL}(\eta)$ remains positive but is reduced, whereas $\tilde{\pi}_j^{HL}$ either remains equal to zero (if $\beta_j^{LL} \leq \gamma_j^{HL}$ or, equivalently, $\beta_i^{LL} \geq \gamma_i^{LH}$) or is also reduced (if instead $\beta_j^{LL} > \gamma_j^{HL}$ or, equivalently, $\beta_i^{LL} < \gamma_i^{LH}$). In the former case, the difference $\Delta_j^{LL}(\eta)$ is equal to $\underline{\pi}_j^{LL}(\eta)$ and is therefore reduced; in the latter case, it is instead equal to $\underline{\pi}_j^{LL}(\eta) - \underline{\tilde{\pi}}_i^{Hk}(\eta) = \phi\left(b_i^{Lk} - \eta\right)$ and is therefore increased.

It follows that the change is feasible whenever $\beta_i^{LL} < \gamma_i^{LH}$; if instead $\beta_i^{LL} \geq \gamma_i^{LH}$, then the change can cause $\left(M_j^L\right)$ to fail, but has no adverse impact on the other constraints. Hence, if $\beta_i^{LL} < \gamma_i^{LL}$, then $\beta_i^{LL} \geq \gamma_i^{LH}$ and $\left(M_j^L\right)$ is binding.

• *Part* (*ii*). Suppose that $\beta_i^{LH} < \gamma_i^{LH}$, and consider a change in (b_i^{LH}, b_j^{HL}) to $(b_i^{LH} + \eta, b_j^{HL} - \eta)$ for some positive but small η. These changes can only affect $\pi_i^{LH}, \tilde{\pi}_i^{HH}, \pi_j^{HL}, \tilde{\pi}_j^{LL}$, and can therefore potentially alter $\left(M_i^H\right)$, through $\Delta_i^{LL}(\eta) \equiv \pi_i^{LH}(\eta) - \tilde{\pi}_i^{HH}(\eta)$, and $\left(M_j^L\right)$, through $\Delta_j^{HL}(\eta) \equiv \pi_j^{HL}(\eta) - \tilde{\pi}_j^{LL}(\eta)$. Furthermore, as $\beta_i^{LH} < \gamma_i^{LH} < \gamma_i^{HH}$ (or, equivalently, $\beta_j^{HL} > \gamma_j^{HL} > \gamma_j^{HH} = \gamma_j^{LL}$), for η is sufficiently small:

- $\pi_i^{LH}(\eta)$ and $\tilde{\pi}_i^{HH}(\eta)$ remain both equal to zero; hence, $\left(M_i^H\right)$ is unaffected.
- $\pi_j^{HL}(\eta)$ and $\tilde{\pi}_j^{LL}(\eta)$ are both positive; hence, the difference $\Delta_j^{HL}(\eta)$ is equal to $\underline{\pi}_j^{HL}(\eta) - \underline{\tilde{\pi}}_j^{LL}(\eta) = -\phi\left(b_j^{HL} - \eta\right)$ and is therefore also reduced.

It follows that the change can cause $\left(M_j^L\right)$ to fail, but has no adverse impact on the other monotonicity constraints. Hence, if $\beta_i^{LH} < \gamma_i^{LH}$, then $\left(M_j^L\right)$ is binding.

• *Part* (*iii*). Suppose that $\beta_i^{HH} < \gamma_i^{HH}$, and consider a change in (b_i^{HH}, b_j^{HH}) to $(b_i^{HH} + \eta, b_j^{HH} - \eta)$, for some positive but small η. These changes can only affect $\pi_i^{HH}, \tilde{\pi}_i^{LH}, \pi_j^{HH}, \tilde{\pi}_j^{LH}$, and can therefore potentially alter $\left(M_i^H\right)$, through the impact on $\Delta_i^{HH}(\eta) \equiv \pi_i^{HH}(\eta) - \tilde{\pi}_i^{LH}(\eta)$, and $\left(M_j^H\right)$, through the impact on $\Delta_j^{HH}(\eta) \equiv \pi_j^{HH}(\eta) - \tilde{\pi}_j^{LH}(\eta)$. Furthermore, as $\beta_i^{HH} < \gamma_i^{HH} < \gamma_i^{HL}$, for η is sufficiently small:

- $\pi_i^{HH}(\eta)$ remains equal to zero, whereas $\tilde{\pi}_i^{LH}(\eta)$ either remains equal to zero (if $\beta_i^{HH} < \gamma_i^{LH}$) or is increased (if instead $\beta_i^{HH} \geq \gamma_i^{LH}$). In the former case, $\left(M_i^H\right)$ is unaffected; in the latter case, the change can cause $\left(M_i^H\right)$ to fail.

- $\pi_j^{HH}(\eta)$ and $\tilde{\pi}_j^{LH}(\eta)$ remain positive; hence, the difference $\Delta_j^{HH}(\eta)$ is equal to $\underline{\pi}_j^{HH}(\eta) - \underline{\tilde{\pi}}_j^{LH}(\eta) = -\phi\left(b_j^{HH} - \eta\right)$ and is therefore also reduced.

It follows that the change: (i) can cause $\left(M_j^H\right)$ to fail (through a reduction in $\pi_j^{HH} - \tilde{\pi}_j^{LH}$); (ii) can cause $\left(M_i^H\right)$ provided that $\beta_i^{HH} \geq \gamma_i^{LH}$; (iii) has no other adverse impact on the monotonicity constraints. Hence, if $\beta_i^{HH} < \gamma_i^{HH}$, then either $\left(M_j^H\right)$ binds, or $\beta_i^{HH} \geq \gamma_i^{LH}$ and $\left(M_i^H\right)$ binds (or both of these conditions hold).

- *Part (iv).* Suppose that $\beta_i^{HL} < \gamma_i^{HL}$, and consider a change in (b_i^{HL}, b_j^{LH}) to $(b_i^{HL} + \eta, b_j^{LH} - \eta)$ for some positive but small η. These changes can only affect $\pi_i^{HL}, \tilde{\pi}_i^{LL}, \pi_j^{LH}, \tilde{\pi}_j^{HH}$, and can therefore potentially alter $\left(M_i^L\right)$, through the impact on $\Delta_i^{HL}(\eta) \equiv \pi_i^{HL}(\eta) - \tilde{\pi}_i^{LL}(\eta)$, and $\left(M_j^H\right)$, through the impact on $\Delta_j^{LH}(\eta) \equiv \pi_j^{LH}(\eta) - \tilde{\pi}_j^{HH}(\eta)$. Furthermore, as $\beta_i^{HL} < \gamma_i^{HL}$, for η is sufficiently small:

- π_i^{HL} remains equal to zero, whereas $\tilde{\pi}_i^{LL}$ either remains equal to zero (if $\beta_i^{HL} < \gamma_i^{LL}$) or is increased (if instead $\beta_i^{HL} \geq \gamma_i^{LL}$). In the former case, $\left(M_i^L\right)$ is unaffected; in the latter case, the change can cause $\left(M_i^L\right)$ to fail.

- π_j^{LH} is positive but reduced, whereas $\tilde{\pi}_j^{HH}$ either remains equal to zero (if $\beta_j^{LH} \leq \gamma_j^{HH}$ or, equivalently, $\beta_i^{HL} \geq \gamma_i^{HH}$) or is also reduced (if instead $\beta_j^{LH} > \gamma_j^{HH}$ or, equivalently, $\beta_i^{HL} < \gamma_i^{HH}$). In the former case, the difference $\Delta_j^{LH}(\eta)$ is equal to $\pi_j^{LH}(\eta)$ and is therefore reduced; in the latter case, it is instead equal to $\underline{\pi}_j^{LH}(\eta) - \underline{\tilde{\pi}}_j^{HH}(\eta) = \phi\left(b_j^{LH} - \eta\right)$ and is therefore increased.

It follows that the change is feasible if $\beta_i^{HL} < \gamma_i^{HH}\left(= \gamma_i^{LL}\right)$; if instead $\beta_i^{HL} \geq \gamma_i^{HH}\left(= \gamma_i^{LL}\right)$, then the change can cause either $\left(M_i^L\right)$ or $\left(M_j^H\right)$ (or both) to fail. Hence, if $\beta_i^{HL} < \gamma_i^{HL}$, then $\beta_i^{HL} \geq \gamma_i^{HH}$ and either $\left(M_i^L\right)$ or $\left(M_j^H\right)$ is binding (or both are binding). ■

Lemma 8 can alternatively be stated as follows:

Corollary 5 (monotonicity conditions) *For any $i \neq j \in \mathcal{I}$:*

(i) if $\beta_i^{LL} > \gamma_i^{LL}$, then $\beta_i^{LL} \leq \gamma_i^{HL}$ and $\left(M_i^L\right)$ is binding;

(ii) if $\beta_i^{HL} > \gamma_i^{HL}$, then $\left(M_i^L\right)$ is binding;

(iii) if $\beta_i^{HH} > \gamma_i^{HH}$, then either $\left(M_i^H\right)$ binds, or $\beta_i^{HH} \leq \gamma_i^{HL}$ and $\left(M_j^H\right)$ binds (or both of these conditions hold);

(iv) if $\beta_i^{LH} > \gamma_i^{LH}$, then $\beta_i^{LH} \leq \gamma_i^{HH}$ and either $\left(M_i^H\right)$ or $\left(M_j^L\right)$ is binding (or both are binding).

Proof. This follows directly from Lemma 8, swapping the roles of the two firms. ■

C Proof of Lemma 4

Lemma 9 *The optimal DICM is such that, if $\beta_i^{Lk} \leq \gamma_i^{Lk}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$, then $\beta_i^{Hk} \leq \gamma_i^{Lk} \left(< \gamma_i^{Hk}\right)$.*

Proof. Suppose that $\beta_i^{Lk} \leq \gamma_i^{Lk}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$. It follows from Corollary 2 that $t_i^{Lk} = 0$; hence, $\left(IC_i^{Lk}\right)$ amounts to:

$$0 \geq \tilde{\pi}_i^{Lk} - t_i^{Hk} = \tilde{\pi}_i^{Lk} - \pi_i^{Hk} \geq 0,$$

where the equality stems from Corollary 2 and the last inequality from $\theta_i^L < \theta_i^H$. We therefore have $\tilde{\pi}_i^{Lk} = \pi_i^{Hk}$, or:

$$\max\left\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{HK} - \theta_i^L\right), 0\right\} = \max\left\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{HK} - \theta_i^H\right), 0\right\},$$

which in turn implies $\beta_i^{Hk} \leq \gamma_i^{Lk}$.[63] ■

Corollary 6 *The optimal DICM is such that, if $\beta_i^{hH} \leq \gamma_i^{hH}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$, then $\beta_i^{hL} < \gamma_i^{hL}$.*

[63] If $\beta_i^{Hk} > \gamma_i^{Lk}$, then the left-hand side is equal to $C(b_j^{kH} - \theta_j^k) - C\left(b_i^{HK} - \theta_i^L\right) > 0$, and thus strictly exceeds the right-hand side.

Proof. Suppose that $\beta_i^{hH} \leq \gamma_i^{hH}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$. From Lemma 9, $\beta_j^{Lh} \leq \gamma_j^{Lh}$ would imply $\beta_j^{Hh} \leq \gamma_j^{Lh} < \gamma_j^{Hh}$, contradicting the working assumption $\beta_i^{hH} \leq \gamma_i^{hH}$ (which is equivalent to $\beta_j^{Hh} \geq \gamma_j^{Hh}$). Hence, $\beta_j^{Lh} \leq \gamma_j^{Lh}$, which is equivalent to $\beta_i^{hL} \geq \gamma_i^{hL}$. ■

Throughout the proof, we let $i \in \mathcal{I}$ denote the firm losing the market competition when they are both of type H (in case of a tie, select either one), and $j \in \mathcal{I} \setminus \{i\}$ denote its rival. We thus have $\beta_i^{HH} \leq \gamma_i^{HH}$, or equivalently, $\beta_j^{HH} \geq \gamma_j^{HH}$, and it follows from Corollary 6 that $\beta_i^{HL} < \gamma_i^{HL}$. In other words, θ_i^H loses at least weakly against θ_j^H and does so strictly against θ_j^L.

It follows from Corollary 6 that three configurations may potentially be optimal:

- If $\beta_i^{LH} \leq \gamma_i^{LH}$, then, from Corollary 6, $\beta_i^{LL} < \gamma_i^{LL}$; hence, in that case both θ_i^L and θ_i^H lose at least weakly against θ_j^H, and strictly against θ_j^L (case a hereafter).
- If instead $\beta_i^{LH} > \gamma_i^{LH}$, implying that θ_i^L strictly wins against θ_j^H, then either θ_i^L loses at least weakly against θ_j^L (case b hereafter), or it strictly wins against θ_j^L (case c hereafter).

We will consider these three configurations in turn.

C.1 Preliminaries

It will be convenient to use the following notation, for $(h,k) \in \mathcal{T}^2$:

$$B_i^{hk} \equiv \frac{\left(b_i^{hk} - \theta_i^h\right) + \left(b_j^{kh} - \theta_j^k\right)}{2} = \frac{\Delta}{2} - \frac{\theta_i^h + \theta_j^k}{2},$$
$$\eta_i^{hk} \equiv \frac{\left(b_i^{hk} - \theta_i^h\right) - \left(b_j^{kh} - \theta_j^k\right)}{2} = \frac{\beta_i^{hk} - \gamma_i^{hk}}{2},$$

where, in the first line, the last equality uses $b_i^{hk} + b_j^{kh} = \Delta$; we then have:

$$b_i^{hk} - \theta_i^h = B_i^{hk} + \eta_i^{hk},$$
$$b_j^{kh} - \theta_j^k = B_i^{hk} - \eta_i^{hk}.$$

The welfare for $\theta_i = \theta_i^h$ and $\theta_j = \theta_j^k$ can be expressed as:

$$W_i^{hk} = -p_i^{hk} + \lambda\left(t_i^{hk} + t_j^{kh}\right),$$

where p_i^{hk} is the market price. Furthermore:

- If the market winner is θ_i^H, then the market price corresponds to firm j's cost; hence, letting $k \in \mathcal{T}$ denote the type of firm j,

$$p_i^{Hk} = C\left(b_j^{Hk} - \theta_j^k\right) = C\left(B_i^{Hk} - \eta_i^{Hk}\right),$$

and, from Corollary 2, $t_i^{Hk} + t_j^{kH} = \pi_i^{Hk} = \underline{\pi}_i^{Hk}$, where:

$$\underline{\pi}_i^{hk} = \pi_i^{hk} = C\left(b_j^{kh} - \theta_j^k\right) - C\left(b_i^{hk} - \theta_i^h\right) = C\left(B_i^{hk} - \eta_i^{hk}\right) - C\left(B_i^{hk} + \eta_i^{hk}\right).$$

Expected welfare is thus given by:

$$\begin{aligned} W_i^{Hk} &= -p_i^{Hk} + \lambda \pi_i^{Hk} = -\left(1-\lambda\right) C\left(B_i^{Hk} - \eta_i^{Hk}\right) - \lambda C\left(B_i^{Hk} + \eta_i^{Hk}\right) \\ &= -\hat{c}_i^{Hk}\left(\eta_i^{Hk}\right), \end{aligned} \tag{21}$$

where

$$\hat{c}_i^{hk}\left(\eta\right) \equiv \left(1-\lambda\right) C\left(B_i^{hk} - \eta\right) + \lambda C\left(B_i^{hk} + \eta\right),$$

which satisfies:

$$\frac{d\hat{c}_i^{hk}}{d\eta}\left(0\right) = -\left(1-2\lambda\right) C'\left(B_i^{hk}\right), \tag{22}$$

$$\frac{d^2\hat{c}_i^{hk}}{d\eta^2}\left(\eta\right) = \left(1-\lambda\right) C''\left(B_i^{hk} - \eta\right) + \lambda C''\left(B_i^{hk} + \eta\right) > 0, \tag{23}$$

where the inequality stems from $C''\left(\cdot\right) > 0$.

- Similarly, if the market winner is θ_j^H, then, letting $h \in \mathcal{T}$ denote the type of firm i,

$$p_i^{hH} = C\left(b_i^{hH} - \theta_i^h\right) = C\left(B_i^{hH} + \eta_i^{hH}\right)$$

and $t_i^{hH} + t_j^{Hh} = \pi_j^{Hh} = \underline{\pi}_j^{Hh}$, where

$$\underline{\pi}_j^{Hh} = C\left(b_i^{hH} - \theta_i^h\right) - C\left(b_j^{Hh} - \theta_j^H\right) = C\left(B_i^{hH} + \eta_i^{hH}\right) - C\left(B_i^{hH} - \eta_i^{hH}\right),$$

leading to:

$$W_i^{hH} = -p_i^{hH} + \lambda \pi_j^{Hh} = -\hat{c}_j^{Hh}\left(\eta_i^{hH}\right), \tag{24}$$

where

$$\hat{c}_j^{kh}(\eta) \equiv (1-\lambda) C\left(B_i^{hk} + \eta\right) + \lambda C\left(B_i^{hk} - \eta\right),$$

which satisfies:

$$\frac{d\hat{c}_j^{kh}}{d\eta}(0) = (1-2\lambda) C'\left(B_i^{hk}\right), \tag{25}$$

$$\frac{d^2\hat{c}_j^{kh}}{d\eta^2}(\eta) = (1-\lambda) C''\left(B_i^{hk} + \eta\right) + \lambda C''\left(B_i^{hk} - \eta\right) > 0, \tag{26}$$

- If instead the market winner is θ_i^L, then, letting again $k \in \mathcal{T}$ denote the type of firm j,

$$p_i^{Lk} = C\left(b_j^{kL} - \theta_j^k\right) = C\left(B_i^{Lk} - \eta_i^{Lk}\right),$$

  and, from Corollary 2, the total transfer is now given by:

$$t_i^{Lk} + t_j^{kL} = \pi_i^{Lk} - r_i^{Lk},$$

  where firm i's profit is here equal to

$$\pi_i^{Lk} = C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^L\right) = C\left(B_i^{Lk} - \eta_i^{Lk}\right) - C\left(B_i^{Lk} + \eta_i^{Lk}\right),$$

  and $r_i^{Lk} = \tilde{\pi}_i^{Lk} - \pi_i^{Hk}$ denotes firm i's informational rent, which can be expressed as follows:

  - if $\eta_i^{Hk} > 0$, implying $\beta_i^{Hk} > \gamma_i^{Hk} > \gamma_i^{Lk}$, then $\tilde{\pi}_i^{Lk} = \underline{\tilde{\pi}}_i^{Lk} > \pi_i^{Hk} = \underline{\pi}_i^{Hk} > 0$ and

$$r_i^{Lk} = \underline{\tilde{\pi}}_i^{Lk} - \underline{\pi}_i^{Hk} = C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right) = \hat{r}_i^k\left(\eta_i^{Hk}\right),$$

    where

$$\hat{r}_i^k(\eta) \equiv C\left(B_i^{Hk} + \eta\right) - C\left(B_i^{Hk} + \eta + \delta\right),$$

which satisfies:

$$\frac{d\hat{r}_i^k}{d\eta}(0) = C'\left(B_i^{Hk}\right) - C'\left(B_i^{Hk} + \delta\right) = C'\left(B_i^{Hk}\right)\left(1 - \rho_i^{Hk}\right), \quad (27)$$

where[64]

$$\rho_i^{Hk} \equiv \frac{C'\left(B_i^{Hk} + \delta\right)}{C'\left(B_i^{Hk}\right)} \in (0,1),$$

and

$$\frac{d^2\hat{r}_i^k}{d\eta^2}(\eta) = C''\left(B_i^{Hk} + \eta\right) - C''\left(B_i^{Hk} + \eta + \delta\right) \geq 0, \quad (28)$$

where the inequality stems from $C'''(\cdot) \leq 0$ and $\delta > 0$. Expected welfare is thus given by:

$$W_i^{Lk} = -p_i^{Lk} + \lambda\left(\pi_i^{Lk} - r_i^{Lk}\right) = -\hat{c}_i^{Lk}\left(\eta_i^{Lk}\right) - \lambda\hat{r}_i^k\left(\eta_i^{Hk}\right). \quad (29)$$

– if instead $\eta_i^{Hk} < 0$, implying $\beta_i^{Hk} < \gamma_i^{Hk}$, then $\pi_i^{Hk} = 0$ and

$$\begin{aligned} r_i^{Lk} = \tilde{\pi}_i^{Lk} = & \max\left\{0, \underline{\pi}_i^{Lk}\right\} = \max\left\{0, C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right)\right\} \\ = & \max\left\{0, \hat{\pi}_i^k\left(\eta_i^{Hk}\right)\right\}, \end{aligned}$$

where

$$\hat{\pi}_i^k(\eta) \equiv C\left(B_i^{Hk} - \eta\right) - C\left(B_i^{Hk} + \eta + \delta\right),$$

which is positive for $\eta > -\delta/2$ and satisfies:

$$\begin{aligned} \frac{d\hat{\pi}_i^k}{d\eta}(0) &= -C'\left(B_i^{Hk}\right)\left(1 + \rho_i^{Hk}\right), \quad (30) \\ \frac{d^2\hat{\pi}_i^k}{d\eta^2}(\eta) &= C''\left(B_i^{Hk} - \eta\right) - C''\left(B_i^{Hk} + \eta + \delta\right), \end{aligned}$$

implying (using $C'''(\cdot) \leq 0$ and $\hat{\pi}_i^k(\eta) > 0 \Longleftrightarrow \eta + \delta > -\eta$)

$$\frac{d^2\hat{\pi}_i^k}{d\eta^2}(\eta) \geq 0 \text{ whenever } \hat{\pi}_i^k(\eta) \geq 0. \quad (31)$$

[64]That ρ_i^{hk} lies between 0 and 1 follows from $C'\left(B_i^{hk}\right) < C'\left(B_i^{hk} + \delta\right) < 0$, where the first inequality stems from $C''(\cdot) > 0$ and $\delta > 0$, and the second from $C'(\cdot) < 0$.

Expected welfare is thus given by:

$$W_i^{Lk} = -p_i^{Lk} + \lambda \left(\pi_i^{Lk} - r_i^{Lk} \right) = -\hat{c}_i^{Lk} \left(\eta_i^{Lk} \right) - \lambda \hat{\pi}_i^k \left(\eta_i^{Hk} \right). \tag{32}$$

– Finally, if $\eta_i^{Hk} = 0$, implying $\beta_i^{Hk} = \gamma_i^{Hk} > \gamma_i^{Lk}$, then $\tilde{\pi}_i^{Lk} = \tilde{\underline{\pi}}_i^{Lk} > \pi_i^{Hk} = \underline{\pi}_i^{Hk} = 0$; we thus have $\hat{\pi}_i^k(0) = \hat{r}_i^k(0)$ and

$$W_i^{Lk} = -\hat{c}_i^{Lk} \left(\eta_i^{Lk} \right) - \lambda \hat{\pi}_i^k (0) = -\hat{c}_i^{Lk} \left(\eta_i^{Lk} \right) - \lambda \hat{r}_i^k (0). \tag{33}$$

- Likewise, if the market winner is θ_j^L, then, letting again $h \in \mathcal{T}$ denote the type of firm i,

$$p_i^{hL} = C \left(b_i^{hL} - \theta_i^h \right) = C \left(B_i^{hL} - \eta_i^{hL} \right),$$

and $t_i^{hL} + t_j^{Lh} = \pi_j^{Lh} - r_j^{Lh}$, where

$$\pi_j^{Lh} = C \left(b_i^{hL} - \theta_i^h \right) - C \left(b_j^{Lh} - \theta_j^L \right) = C \left(B_i^{hL} + \eta_i^{hL} \right) - C \left(B_i^{hL} - \eta_i^{hL} \right),$$

and firm j's informational rent, r_j^{Lh}, can be expressed as follows:

– if $\eta_i^{hH} < 0$, implying $\beta_j^{Hh} > \gamma_j^{Hh} > \gamma_j^{Lh}$, then $\tilde{\pi}_j^{Lh} = \tilde{\underline{\pi}}_j^{Lh} > \pi_j^{Hh} = \underline{\pi}_j^{Hh} > 0$ and

$$r_j^{Lh} = \tilde{\underline{\pi}}_j^{Lh} - \underline{\pi}_j^{Hh} = C \left(b_j^{Hh} - \theta_j^H \right) - C \left(b_j^{Hh} - \theta_j^L \right) = \hat{r}_j^h \left(\eta_i^{hH} \right),$$

where

$$\hat{r}_j^h (\eta) \equiv C \left(B_i^{hH} - \eta \right) - C \left(B_i^{hH} - \eta + \delta \right),$$

which satisfies:

$$\frac{d\hat{r}_j^h}{d\eta}(0) = -C' \left(B_i^{hH} \right) + C' \left(B_i^{hH} + \delta \right) = -C' \left(B_i^{hH} \right) \left(1 - \rho_i^{hH} \right), \tag{34}$$

$$\frac{d^2\hat{r}_j^h}{d\eta^2}(\eta) = C'' \left(B_i^{hH} - \eta \right) - C'' \left(B_i^{hH} - \eta + \delta \right) \geq 0. \tag{35}$$

Expected welfare is thus given by:

$$W_i^{hL} = -p_i^{hL} + \lambda\left(\pi_j^{Lh} - r_j^{Lh}\right) = -\hat{c}_j^{Lh}\left(\eta_i^{hL}\right) - \lambda\hat{r}_j^h\left(\eta_i^{hH}\right). \quad (36)$$

– if instead $\eta_i^{hH} > 0$, implying $\beta_j^{Hh} \geq \gamma_j^{Hh}$, then $\pi_j^{Hh} = 0$ and

$$\begin{aligned} r_j^{Lh} = \tilde{\pi}_j^{Lh} = & \max\left\{0, \underline{\tilde{\pi}}_j^{Lh}\right\} = \max\left\{0, C\left(b_i^{hH} - \theta_i^h\right) - C\left(b_j^{Hh} - \theta_j^L\right)\right\} \\ = & \max\left\{0, \hat{\pi}_j^h\left(\eta_i^{hH}\right)\right\}, \end{aligned}$$

where

$$\hat{\pi}_j^h(\eta) \equiv C\left(B_i^{hH} + \eta\right) - C\left(B_i^{hH} - \eta + \delta\right),$$

which is positive for $\eta < \delta/2$ and satisfies:

$$\begin{aligned} \frac{d\hat{\pi}_j^h}{d\eta}(0) &= C'\left(B_i^{hH}\right)\left(1 + \rho_i^{hH}\right), \quad (37) \\ \frac{d^2\hat{\pi}_j^h}{d\eta^2}(\eta) &= C''\left(B_i^{hH} + \eta\right) - C''\left(B_i^{hH} - \eta + \delta\right), \end{aligned}$$

implying

$$\frac{d^2\hat{\pi}_j^h}{d\eta^2}(\eta) \geq 0 \text{ whenever } \hat{\pi}_j^h(\eta) \geq 0. \quad (38)$$

Expected welfare is thus given by:

$$W_i^{hL} = -p_i^{hL} + \lambda\left(\pi_j^{Lh} - r_j^{Lh}\right) = -\hat{c}_j^{Lh}\left(\eta_i^{hL}\right) - \lambda\max\left\{0, \hat{\pi}_j^h\left(\eta_i^{hH}\right)\right\}. \quad (39)$$

– Finally, if $\eta_i^{hH} = 0$, implying $\beta_j^{Hh} = \gamma_j^{Hh} > \gamma_j^{Lh}$, then $\tilde{\pi}_j^{Lh} = \underline{\tilde{\pi}}_j^{Lh} > \pi_j^{Hh} = \underline{\pi}_j^{Hh} = 0$; we thus have $\hat{r}_j^h(0) = \hat{\pi}_j^h(0)$ and:

$$W_i^{hL} = -\hat{c}_j^{Lh}\left(\eta_i^{hL}\right) - \lambda\hat{r}_j^h(0) = -\hat{c}_j^{Lh}\left(\eta_i^{hL}\right) - \lambda\max\left\{0, \hat{\pi}_j^h(0)\right\}. \quad (40)$$

C.2 Case a

Letting $i \in \mathcal{I}$ denote a firm losing the competition when both firms are of type H (i.e., $\beta_i^{HH} \leq \gamma_i^{HH}$), we consider here the case firm i moreover always loses strictly against θ_j^L and at least weakly against θ_j^H; we thus have $\beta_i^{LL} < \gamma_i^{LL}$, implying $\beta_i^{HL} \leq$

$\gamma_i^{LL}\left(=\gamma_i^{HL}-\delta\right)$ from Lemma 9, and $\beta_i^{LH} \leq \gamma_i^{LH}$, implying $\beta_i^{HH} \leq \gamma_i^{LH}\left(=\gamma_i^{HH}-\delta\right)$. The DICM thus satisfies $\eta_i = \left(\eta_i^{hk}\right)_{(h,k)\in\mathcal{T}^2} \in \mathcal{E}_a$, where:

$$\mathcal{E}_a \equiv \left\{\eta_i \mid \eta_i^{LL} < 0, \eta_i^{LH} \leq 0 \text{ and } \max\left\{\eta_i^{HL}, \eta_i^{HH}\right\} \leq -\frac{\delta}{2}\right\}.$$

It follows that firm j always wins, as $\eta_i^{hk} \leq 0$ for every $(h,k) \in \mathcal{T}^2$, and when of type L, it earns an informational rent is given by $\hat{r}_j^h\left(\eta_i^{hH}\right)$ (with h denoting the type of firm i), as . We thus have:

$$\begin{aligned}
W_i^{HL} &= -\hat{c}_j^{LH}\left(\eta_i^{HL}\right) - \lambda \hat{r}_j^H\left(\eta_i^{HH}\right),\\
W_i^{LL} &= -\hat{c}_j^{LL}\left(\eta_i^{LL}\right) - \lambda \hat{r}_j^L\left(\eta_i^{LH}\right),\\
W_i^{HH} &= -\hat{c}_j^{HH}\left(\eta_i^{HH}\right),\\
W_i^{LH} &= -\hat{c}_j^{HL}\left(\eta_i^{LH}\right).
\end{aligned}$$

Expected welfare is given by

$$W = \mu_i^{HL}W_i^{HL} + \mu_i^{LL}W_i^{LL} + \mu_i^{HH}W_i^{HH} + \mu_i^{LH}W_i^{LH} \tag{41}$$

and can therefore be expressed here as $W = \hat{W}_{ia}(\eta_i)$ (with the subscript a standing for "case a"), where:

$$\hat{W}_{ia}(\eta_i) \equiv \mu_i^{LL}\hat{W}_{ia}^{LL}\left(\eta_i^{LL}\right) + \mu_i^{HL}\hat{W}_{ia}^{HL}\left(\eta_i^{HL}\right) + \mu_i^{LH}\hat{W}_{ia}^{LH}\left(\eta_i^{LH}\right) + \mu_i^{HH}\hat{W}_{ia}^{HH}\left(\eta_i^{HH}\right),$$

where:

$$\begin{aligned}
\hat{W}_{ia}^{HL}(\eta) &\equiv -\hat{c}_j^{LH}(\eta),\\
\hat{W}_{ia}^{LL}(\eta) &\equiv -\hat{c}_j^{LL}(\eta),\\
\hat{W}_{ia}^{HH}(\eta) &\equiv -\hat{c}_j^{HH}(\eta) - \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}\hat{r}_j^H(\eta),\\
\hat{W}_{ia}^{LH}(\eta) &\equiv -\hat{c}_j^{HL}(\eta) - \lambda\frac{\mu_i^{LL}}{\mu_i^{LH}}\hat{r}_j^L(\eta).
\end{aligned}$$

It follows from (26) and (35) that expected welfare is strictly concave in η_i. Furthermore, for $\eta_i = \mathbf{0}$:

- from (25), we have

$$\begin{aligned}\frac{d\hat{W}_{ia}^{HL}}{d\eta}(0) &= -\frac{d\hat{c}_j^{LH}}{d\eta}(0) = -(1-2\lambda)\,C'\left(B_i^{HL}\right),\\ \frac{d\hat{W}_{ia}^{LL}}{d\eta}(0) &= -\frac{d\hat{c}_j^{LL}}{d\eta}(0) = -(1-2\lambda)\,C'\left(B_i^{LL}\right),\end{aligned}$$

which are both negative for $\lambda < 1/2$;

- and from (25) and (34):

$$\begin{aligned}\frac{d\hat{W}_{ia}^{HH}}{d\eta}(0) &= -\frac{d\hat{c}_j^{HH}}{d\eta}(0) - \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}\frac{d\hat{r}_j^{H}}{d\eta}(0)\\ &= -C'\left(B_i^{HH}\right)\left\{1-\left[2+\frac{\mu_i^{HL}}{\mu_i^{HH}}\left(1-\rho_i^{HH}\right)\right]\lambda\right\}n\\ \frac{d\hat{W}_{ia}^{LH}}{d\eta}(0) &= -\frac{d\hat{c}_j^{HL}}{d\eta}(0) - \lambda\frac{\mu_i^{LL}}{\mu_i^{LH}}\frac{d\hat{r}_j^{L}}{d\eta}(0)\\ &= -C'\left(B_i^{LH}\right)\left\{1-\left[2+\frac{\mu_i^{LL}}{\mu_i^{LH}}\left(1-\rho_i^{LH}\right)\right]\lambda\right\}.\end{aligned}$$

To ensure that moving towards cost equalization (namely, towards $\eta_i = \mathbf{0}$) is always desirable in the range $\eta_i \in \mathcal{E}_a$ (implying $\eta_i \leq \mathbf{0}$), a sufficient condition is for these first-order derivatives to be all non-negative, which amounts to $\lambda \leq \hat{\lambda}_a \equiv \min\left\{\hat{\lambda}_{ia}^{HH}, \hat{\lambda}_{ia}^{LH}\right\}$, where

$$\hat{\lambda}_{ia}^{hH} \equiv \frac{1}{2+\frac{\mu_i^{hL}}{\mu_i^{hH}}\left(1-\rho_i^{hH}\right)}\left(<\frac{1}{2}\right).$$

C.3 Case b

Let $i \in \mathcal{I}$ denote again a firm losing the competition when both firms are of type H (i.e., $\beta_i^{HH} \leq \gamma_i^{HH}$), and suppose now that θ_i^L strictly wins against θ_j^H but still loses at least weakly against θ_j^L (i.e., $\beta_i^{LH} > \gamma_i^{LH}$ and $\beta_i^{LL} \leq \gamma_i^{LL}$, implying $\beta_i^{HL} \leq \gamma_i^{LL}\left(= \gamma_i^{HL} - \delta\right)$ from Lemma 9). The DICM thus satisfies $\eta_i \in \mathcal{E}_b$, where:

$$\mathcal{E}_b \equiv \left\{\eta_i \mid \max\left\{\eta_i^{LL}, \eta_i^{HH}\right\} \leq 0 < \eta_i^{LH} \text{ and } \eta_i^{HL} \leq -\frac{\delta}{2}\right\}.$$

It follows that firm j wins unless it is of type H and faces a type L (as $\max\left\{\eta_i^{LL}, \eta_i^{HH}, \eta_i^{HL}\right\} \leq 0 < \eta_i^{LH}$); furthermore, $r_j^{LH} = \hat{r}_j^H\left(\eta_i^{HH}\right)$ (as $\eta_i^{HH} \leq 0$), whereas $r_j^{LL} = \max\left\{0, \hat{\pi}_j^L\left(\eta_i^{LH}\right)\right\}$

(as $\eta_i^{LH} > 0$) and $r_i^{LH} = \max\left\{0, \hat{\pi}_i^H\left(\eta_i^{HH}\right)\right\}$ (as $\eta_i^{HH} \leq 0$). We thus have:

$$\begin{aligned}
W_i^{HL} &= -\hat{c}_j^{LH}\left(\eta_i^{HL}\right) - \lambda \hat{r}_j^H\left(\eta_i^{HH}\right), \\
W_i^{LL} &= -\hat{c}_j^{LL}\left(\eta_i^{LL}\right) - \lambda \max\left\{0, \hat{\pi}_j^L\left(\eta_i^{LH}\right)\right\}, \\
W_i^{HH} &= -\hat{c}_j^{HH}\left(\eta_i^{HH}\right), \\
W_i^{LH} &= -\hat{c}_i^{LH}\left(\eta_i^{LH}\right) - \lambda \max\left\{0, \hat{\pi}_i^H\left(\eta_i^{HH}\right)\right\}.
\end{aligned}$$

Expected welfare, given by (41), can now be expressed as:

$$\hat{W}_{ib}(\eta_i) \equiv \mu_i^{HL}\hat{W}_i^{HL}\left(\eta_i^{HL}\right) + \mu_i^{LL}\hat{W}_i^{LL}\left(\eta_i^{LL}\right) + \mu_i^{HH}\hat{W}_i^{HH}\left(\eta_i^{HH}\right) + \mu_i^{LH}\hat{W}_i^{LH}\left(\eta_i^{LH}\right),$$

where:

$$\begin{aligned}
\hat{W}_{ib}^{HL}(\eta) &\equiv -\hat{c}_j^{LH}(\eta), \\
\hat{W}_{ib}^{LL}(\eta) &\equiv -\hat{c}_j^{LL}(\eta), \\
\hat{W}_{ib}^{HH}(\eta) &\equiv -\hat{c}_j^{HH}(\eta) - \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}\hat{r}_j^H(\eta) - \lambda\frac{\mu_i^{LH}}{\mu_i^{HH}}\max\left\{0, \hat{\pi}_i^H(\eta)\right\}, \\
\hat{W}_{ib}^{LH}(\eta) &\equiv -\hat{c}_i^{LH}(\eta) - \lambda\frac{\mu_i^{LL}}{\mu_i^{LH}}\max\left\{0, \hat{\pi}_j^L(\eta)\right\}.
\end{aligned}$$

We first establish concavity:

- from (26), $\hat{W}_{ib}^{HL}(\cdot)$ and $\hat{W}_{ib}^{LL}(\cdot)$ are both strictly concave;
- from (26), (31) and (35), $\hat{W}_{ib}^{HH}(\cdot)$ is strictly concave except possibly at $\eta = -\delta/2$.. However, as $\hat{W}_{ib}^{HH}$ is the sum of two strictly concave functions, and the max of a concave function and zero, it is strictly concave. Furthermore, for $\eta_i = \mathbf{0}$:
  - using (25) yields:

$$\begin{aligned}
\frac{d\hat{W}_{ib}^{HL}}{d\eta}(0) &= -\frac{d\hat{c}_j^{HL}}{d\eta}(0) = -(1-2\lambda)\,C'\left(B_i^{HL}\right), \\
\frac{d\hat{W}_{ib}^{LL}}{d\eta}(0) &= -\frac{d\hat{c}_j^{LL}}{d\eta}(0) = -(1-2\lambda)\,C'\left(B_i^{LL}\right).
\end{aligned}$$

– using (25), (34) and (37) yields:

$$\begin{aligned}\frac{d\hat{W}_{ib}^{HH}}{d\eta}(0) &= -\frac{d\hat{c}_j^{HH}}{d\eta}(0) - \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}\frac{d\hat{r}_j^{H}}{d\eta}(0) - \lambda\frac{\mu_i^{LH}}{\mu_i^{HH}}\frac{d\hat{\pi}_i^{H}}{d\eta}(0) \\ &= -C'\left(B_i^{HH}\right)\left\{1-\left[2+\frac{\mu_i^{HL}}{\mu_i^{HH}}\left(1-\rho_i^{HH}\right)+\frac{\mu_i^{LH}}{\mu_i^{HH}}\left(1+\rho_i^{HH}\right)\right]\lambda\right\}.\end{aligned}$$

– using (22) and (37) yields:

$$\begin{aligned}\frac{d\hat{W}_{ib}^{LH}}{d\eta}(0) &= -\frac{d\hat{c}_i^{LH}}{d\eta}(0) - \lambda\frac{\mu_i^{LL}}{\mu_i^{LH}}\frac{d\hat{\pi}_j^{L}}{d\eta}(0) \\ &= C'\left(B_i^{LH}\right)\left\{1-\left[2+\frac{\mu_i^{LL}}{\mu_i^{LH}}\left(1+\rho_i^{LH}\right)\right]\lambda\right\}.\end{aligned}$$

To ensure that moving towards cost equalization (namely, towards $\eta_i = \mathbf{0}$) is always desirable in the range $\eta_i \in \mathcal{E}_b$ (implying $\left(\eta_i^{LL}, \eta_i^{HL}, \eta_i^{HH}\right) \leq \mathbf{0}$ and $\eta_i^{LH} \geq 0$), a sufficient condition is for the first three first-order derivatives to be non-negative and the last one to be non-positive, which amounts to $\lambda \leq \hat{\lambda}_{ib} \equiv \min\left\{\hat{\lambda}_{ib}^{HH}, \hat{\lambda}_b^{LH}\right\}$, where

$$\begin{aligned}\hat{\lambda}_b^{LH} &\equiv \frac{1}{2+\frac{\mu_i^{LL}}{\mu_i^{LH}}\left(1+\rho_i^{LH}\right)}\left(<\frac{1}{2}\right), \\ \hat{\lambda}_{ib}^{HH} &\equiv \frac{1}{2+\frac{\mu_i^{HL}}{\mu_i^{HH}}\left(1-\rho_i^{HH}\right)+\frac{\mu_i^{LH}}{\mu_i^{HH}}\left(1+\rho_i^{HH}\right)}\left(<\frac{1}{2}\right).\end{aligned}$$

C.4 Case c

Finally, letting again $i \in \mathcal{I}$ denote a firm losing the competition when both firms are of type H (i.e., $\beta_i^{HH} \leq \gamma_i^{HH}$), implying from Corollary 6 that θ_i^H loses strictly against θ_j^L (i.e., $\beta_i^{HL} < \gamma_i^{HL}$), suppose now that θ_i^L strictly wins against both types of firm j (i.e., $\beta_i^{LH} > \gamma_i^{LH}$ and $\beta_i^{LL} > \gamma_i^{LL}$). The DICM thus satisfies $\eta_i \in \mathcal{E}_b$, where:

$$\mathcal{E}_c \equiv \left\{\eta_i \mid \eta_i^{HH} \leq 0, \eta_i^{HL} < 0 < \min\left\{\eta_i^{LL}, \eta_i^{LH}\right\}\right\}.$$

(Note too that $\beta_j^{LL} < \gamma_j^{LL}$ implies, by Corollary 2, that $\beta_j^{HL} \leq \gamma_j^{LL}$ and so $\eta_i^{LH} > \frac{\delta}{2}$. We thus have:

$$\begin{aligned} W_i^{LL} &= -\hat{c}_i^{LL}(\eta_i^{LL}) - \lambda\hat{\pi}_i^L(\eta_i^{HL}) \\ W_i^{LH} &= -\hat{c}_i^{LH}(\eta_i^{LH}) - \lambda\max\{0, \hat{\pi}_i^H(\eta_i^{HH})\} \\ W_i^{HL} &= -\hat{c}_j^{HL}(\eta_i^{HL}) - \lambda\hat{r}_j^L(\eta_i^{HH}) \\ W_i^{HH} &= -\hat{c}_j^{HH}(\eta_i^{HH}) \end{aligned}$$

Expected welfare, given by (41), can now be expressed as:

$$\hat{W}_{ic}(\eta_i) \equiv \mu_i^{HL}\hat{W}_{ic}^{HL}\left(\eta_i^{HL}\right) + \mu_i^{LL}\hat{W}_{ic}^{LL}\left(\eta_i^{LL}\right) + \mu_i^{HH}\hat{W}_{ic}^{HH}\left(\eta_i^{HH}\right) + \mu_i^{LH}\hat{W}_{ic}^{LH}\left(\eta_i^{LH}\right),$$

where:

$$\begin{aligned} \hat{W}_{ic}^{LL}(\eta) &= -\hat{c}_i^{LL}(\eta) \\ \hat{W}_{ic}^{LH}(\eta) &= -\hat{c}_i^{LH}(\eta) \\ \hat{W}_{ic}^{HL}(\eta) &= -\hat{c}_j^{HL}(\eta) - \lambda\frac{\mu_i^{LL}}{\mu_i^{HL}}\hat{\pi}_i^L(\eta) \\ \hat{W}_{ic}^{HH}(\eta) &= -\hat{c}_j^{HH}(\eta) - \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}\hat{r}_j^L(\eta) - \lambda\frac{\mu_i^{LH}}{\mu_i^{HH}}\max\{0, \hat{\pi}_i^H(\eta)\} \end{aligned}$$

We first establish concavity:

- From (26), $\hat{W}_{ic}^{LH}(\cdot)$ and $\hat{W}_{ic}^{LL}(\cdot)$ are both strictly concave;
- From (26) and (38), both $-\hat{c}_j^{HL}(\eta)$ and $-\hat{\pi}_i^L(\eta)$ are strictly concave, which implies $\hat{W}_{ic}^{HL}(\eta)$ is strictly concave.
- from (26), (31) and (35), $\hat{W}_{ic}^{HH}(\cdot)$ is strictly concave except possibly at $\eta = -\delta/2$. However, as $\hat{W}_{ic}^{HH}$ is the sum of two strictly concave functions, and the max of a concave function and zero, it follows that $\hat{W}_{ic}^{JHH}(\eta_i)$ is strictly concave in η_i. Furthermore, for $\eta_i = \mathbf{0}$.

* from (25), we have

$$\begin{aligned}\frac{d\hat{W}_{ic}^{LH}}{d\eta}(0) &= -\frac{d\hat{c}_i^{LH}}{d\eta}(0) = (1-2\lambda)\, C'\left(B_i^{LH}\right), \\ \frac{d\hat{W}_{ic}^{LL}}{d\eta}(0) &= -\frac{d\hat{c}_i^{LL}}{d\eta}(0) = (1-2\lambda)\, C'\left(B_i^{LL}\right),\end{aligned}$$

which are both negative for $\lambda < 1/2$;

* From (31) and (25)

$$\begin{aligned}\frac{d\hat{W}_{ic}^{HL}(\eta)}{d\eta} &= \frac{d\hat{c}_i^{HL}(\eta)}{d\eta} - \lambda\frac{\mu_i^{LL}}{\mu_i^{HL}}\frac{d\hat{\pi}_j^{L}(\eta)}{d\eta} \\ &= -(1-2\lambda)C'(B_i^{HL}) + \lambda\frac{\mu_i^{LL}}{\mu_i^{HL}}C'(B_i^{HL})(1+\rho_i^{HL}) \\ &= -C'(B_i^{HL}) + 2\lambda C'(B_i^{HL}) + \lambda C'(B_i^{HL})\frac{\mu_i^{LL}}{\mu_i^{HL}}(1+\rho_i^{HL})\end{aligned}$$

And $\frac{d\hat{W}_{ic}^{HL}(\eta)}{d\eta} > 0$ if

$$\lambda < \hat{\lambda}_{ic}^{HL} \equiv \frac{1}{2+\frac{\mu_i^{LL}}{\mu_i^{HL}}(1+\rho_i^{HL})}(<\frac{1}{2})$$

* From $(30), (25)$ and (27) we have

$$\begin{aligned}\frac{d\hat{W}_{ic}^{HH}(\eta)}{d\eta} &\geq \frac{d\hat{c}_i^{HH}(\eta)}{d\eta} - \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}\frac{d\hat{r}_j^{L}(\eta)}{d\eta} - \lambda\frac{\mu_i^{LH}}{\mu_i^{HH}}\frac{d\hat{\pi}_i^{H}(\eta)}{d\eta} \\ &= -(1-2\lambda)C'(B_i^{HH}) + \lambda\frac{\mu_i^{HL}}{\mu_i^{HH}}C'(B_i^{HH})(1-\rho_i^{HH}) \\ &\quad +\lambda\frac{\mu_i^{LH}}{\mu_i^{HH}}C'(B_i^{HH})(1+\rho_i^{HH}) \\ &= -C'(B_i^{HH}) + 2\lambda C'(B_i^{HH}) + \lambda C'(B_i^{HH})\frac{\mu_i^{HL}}{\mu_i^{HH}}(1-\rho_i^{HH}) \\ &\quad +\lambda C'(B_i^{HH})\frac{\mu_i^{LH}}{\mu_i^{HH}}(1+\rho_i^{HH})\end{aligned}$$

And $\frac{d\hat{W}_{ic}^{HH}(\eta)}{d\eta} > 0$ if

$$\lambda < \hat{\lambda}_{ic}^{HH} \equiv \frac{1}{2+\frac{\mu_i^{HL}}{\mu_i^{HH}}(1-\rho_i^{HH})+\frac{\mu_i^{LH}}{\mu_i^{HH}}(1+\rho_i^{HH})}(<\frac{1}{2})$$

In the range in which $\eta_i^{hk} > 0$, i.e., for $h = L$ and $k = H$ or L, moving to cost equalization increases welfare as $\frac{d\hat{W}_{ic}^{Lk}(\eta)}{d\eta} < 0$, for $\lambda < \frac{1}{2}$. And in the range in which $\eta_i^{hk} \geq 0$, i.e., for $h = H$ and $k = H$ or L, moving to cost equalization increases welfare as $\frac{d\hat{W}_{ic}^{Lk}(\eta)}{d\eta} > 0$, for $\lambda < \hat{\lambda}_c \equiv \min\{\hat{\lambda}_{ic}^{HH}, \hat{\lambda}_{ic}^{HL}\} < \frac{1}{2}$. So, moving to cost equalization improves welfare when $\lambda < \lambda^c$.

C.5 Recap

Summing-up, moving towards cost equalization (i.e., towards $\eta_i = \mathbf{0}$) is always socially desirable whenever $\lambda \leq \hat{\lambda} \equiv \min\left\{\hat{\lambda}_a, \hat{\lambda}_b, \hat{\lambda}_c\right\}$.

D Proof of Lemma 5

Suppose by way of contradiction that $\sigma_i^{hk} < \Delta$ for some $(h,k) \in \mathcal{T}^2$ and $i \in \mathcal{I}$ such that $\beta_i^{hk} \geq \gamma_i^{hk}$ (that is, we label the firm winning the competition as "firm i" – in case of a tie, either firm can be so labelled). To establish the lemma, it suffices to exhibit a change in the DICM that (i) maintains individual rationality and incentive compatibility, (ii) strictly improves welfare, and (iii) increases the total bandwidth allocated to the firms. We do this below by considering a small change in either b_i^{hk} or b_j^{kh}, together with a transfer adjustment dt_i^{hk} designed to maintain i's payoff, keeping unaltered all other bandwidth allocations and transfers. The following lemma provides sufficient conditions for individual rationality and incentive compatibility:

Lemma 10 *Consider a DICM satisfying $\beta_i^{hk} \geq \gamma_i^{hk}$ for some $i \in \mathcal{I}$ and $(h,k) \in \mathcal{T}^2$, and a small change altering only the allocation designed for $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$, namely, altering the bandwidth allocation (b_i^{hk}, b_j^{kh}) by (db_i^{hk}, db_j^{kh}), with the restriction that $db_i^{hk} = db_j^{kh}$ if $\beta_i^{hk} = \gamma_i^{hk}$, and i's transfer t_i^{hk} by*

$$dt_i^{hk} = C'(b_j^{kh} - \theta_j^k)db_j^{kh} - C'\left(b_i^{hk} - \theta_i^h\right)db_i^{hk}. \tag{42}$$

For db_i^{hk} and db_j^{kh} small enough, the modified DRM is individually rational and incentive compatible whenever (letting $\tilde{h} \neq h \in \mathcal{I}$ and $\tilde{k} \neq k \in \mathcal{I}$ respectively denote i's and j's other types):

(i) when the true handicaps are $\theta_i = \theta_i^{\tilde{h}}$ and $\theta_j = \theta_j^k$, i's misreporting payoff, $\tilde{\Pi}_i^{\tilde{h}k}$, is either initially negative or not increased by the change;

(ii) and when the true handicaps are $\theta_i = \theta_i^h$ and $\theta_j = \theta_j^{\tilde{k}}$, j's misreporting payoff, $\tilde{\Pi}_j^{\tilde{k}h}$, is either initially negative or not increased by the change.

Proof. By construction, the only payoffs that are altered are those obtained when reporting $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$. Therefore, without loss of generality we can restrict attention to (i) the firms' individual rationality and incentive compatibility constraints for $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$, $\left\{\left(IR_i^{hk}\right), \left(IR_j^{kh}\right)\right\}$ and $\left\{\left(IC_i^{hk}\right), \left(IC_j^{kh}\right)\right\}$, (ii) i's incentive constraint for $(\theta_i, \theta_j) = (\theta_i^{\tilde{h}}, \theta_j^k)$, $(IC_i^{\tilde{h}k})$, and (iii) j's incentive constraint for $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^{\tilde{k}})$, $(IC_j^{\tilde{k}h})$. We consider these cases in turn.

Consider first the case where the true handicaps are $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$. If the firms report their types truthfully, then: if $\beta_i^{hk} > \gamma_i^{hk}$, θ_i^h strictly wins the competition against θ_j^k and thus keeps doing so under the modified DRM for db_i^{hk} and db_j^{kh} small enough; if instead $\beta_i^{hk} = \gamma_i^{hk}$, then the two firms have initially the same cost, and the restriction $db_i^{hk} = db_j^{kh}$ ensures that this remains the case under the modified DRM. In both instances, both firms' payoffs from truthtelling, Π_i^{hk} and Π_j^{kh}, remain the same:

* If $\beta_i^{hk} > \gamma_i^{hk}$, then j keeps losing the competition and faces the same transfer; and for i, condition (42) ensures that the change in transfer, dt_i^{hk}, exactly offsets the net impact on the market price (which is altered by $C'\left(b_j^{kh} - \theta_j^h\right) db_j^{kh}$) and on its cost (which is altered by $C'\left(b_i^{hk} - \theta_i^h\right) db_i^{hk}$).
* If instead $\beta_i^{hk} = \gamma_i^{hk}$, then initially both firms generate zero profit and the restriction $db_i^{hk} = db_j^{kh}$ ensures that this remains the case; in addition condition (42) then implies $dt_i^{hk} = 0$.

It follows that the individual rationality conditions $\left(IR_i^{hk}\right)$ and $\left(IR_j^{kh}\right)$ remain satisfied; furthermore, as the payoffs from misreporting are unaltered, the incentive compatibility constraints $\left(IC_i^{hk}\right)$ and $\left(IC_j^{kh}\right)$ also remain satisfied.

If instead the true handicaps are $(\theta_i, \theta_j) = (\theta_i^{\tilde{h}}, \theta_j^k)$, then condition (i) ensures that i's payoff from unilaterally misreporting its type, $\tilde{\Pi}_i^{\tilde{h}k}$, either does not increase, or is initially negative, in which case it remains negative for db_i^{hk} and db_j^{kh} small enough. As the payoff from truth-telling remains unaltered and non-negative, the individual rationality constraint $(IC_i^{\tilde{h}k})$ remains satisfied.

Finally, if the true handicaps are $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^{\tilde{k}})$, then condition (ii) ensures that j's payoff from unilaterally misreporting its type, $\tilde{\Pi}_i^{\tilde{k}h}$, either does not increase, or is initially negative, in which case it remains negative for db_i^{hk} and db_j^{kh} small enough. As the payoff from truthtelling remains unaltered and non-negative, the incentive compatibility constraint $(IC_j^{h\tilde{k}})$ remains satisfied.

■

The next lemma shows that all the available bandwidth is allocated when the transfer designed for the winning firm is positive:

Lemma 11 *If* $t_i^{hk} > 0$*, then* $\sigma_i^{hk} = \Delta$.

Proof. Thus, suppose that $t_i^{hk} > 0$, which from Corollary 2 implies $\beta_i^{hk} > \gamma_i^{hk}$. We now exhibit small changes satisfying the conditions of Lemma 10 (as $\beta_i^{hk} > \gamma_i^{hk}$, any db_i^{hk} and db_j^{kh} can be contemplated) and increasing both welfare and total allocated bandwidth. Three cases must be distinguished:

Case 1: $h = H$. Consider the following slight change in the DICM for $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^k)$: (i) $db_i^{hk} > db_j^{kh} = 0$, which reduces i's cost by $dC_i^{hk} \equiv C'\left(b_i^{hk} - \theta_i^h\right) db_i^{hk}\,(< 0)$, and (ii) $dt_i^{hk} = -dC_i^{hk}\,(> 0)$. This change increases the bandwidth allocation and improves welfare by increasing the transfer t_i^{hk}. Furthermore, it satisfies (42) and, for db_i^{hk} low enough:

- ∗ When the true handicaps are $(\theta_i, \theta_j) = (\theta_i^{\tilde{h}}, \theta_j^k)$, by unilaterally misreporting θ_i^h instead of $\theta_i^{\tilde{h}}$, i would initially obtain

$$\tilde{\Pi}_i^{\tilde{h}k} = \max\{C\left(b_j^{kh} - \theta_j^k\right) - C(b_i^{hk} - \theta_i^{\tilde{h}}), 0\} - t_i^{hk}.$$

If $\beta_i^{hk} < \gamma_i^{\tilde{h}k}$, then i initially strictly loses the competition and thus keeps losing it; the change $\left\{db_i^{hk}, dt_i^{hk}\right\}$ thus alters the misreporting payoff by

$$d\tilde{\Pi}_i^{\tilde{h}k} = -dt_i^{hk} < 0.$$

If instead $\beta_i^{hk} \geq \gamma_i^{\tilde{h}k}$, then i initially (weakly) wins the competition and thus now strictly wins it; the change $\left\{db_i^{hk}, dt_i^{hk}\right\}$ thus alters the misreporting payoff by

$$d\tilde{\Pi}_i^{\tilde{h}k} = -C'(b_i^{hk} - \theta_i^{\tilde{h}})db_i^{hk} - dt_i^{hk} = [C'(b_i^{hk} - \theta_i^{h}) - C'(b_i^{hk} - \theta_i^{\tilde{h}})]b_i^{hk} < 0,$$

where the second equality stems from the definition of dt_i^{hk} (namely, $dt_i^{hk} = -dC_i^{hk} = C'\left(b_i^{hk} - \theta_i^{h}\right)db_i^{hk}$) and the inequality follows from $\theta_i^{h} = \theta_i^{H} > \theta_i^{L} = \theta_i^{\tilde{h}}$ and $C''(\cdot) > 0$. Hence, in both instances, the misreporting payoff $\tilde{\Pi}_i^{\tilde{h}k}$ is reduced.

* When instead the true handicaps are $(\theta_i, \theta_j) = (\theta_i^{h}, \theta_j^{\tilde{k}})$, by unilaterally misreporting θ_j^{k} instead of $\theta_j^{\tilde{k}}$, j would initially obtain

$$\tilde{\Pi}_j^{\tilde{k}h} = \max\{C(b_i^{hk} - \theta_i^{h}) - C(b_j^{kh} - \theta_j^{\tilde{k}}), 0\} - t_j^{kh}.$$

The proposed change can only reduce this misreporting payoff, as it increases b_i^{hk} (and so the market price is reduced if j wins) while maintaining b_j^{kh} (and so j's cost remain the same if it wins) and the transfer t_j^{kh}.

Conditions (i) and (ii) of Lemma 10 are therefore satisfied, implying that the modified mechanism is individually rational and incentive compatible.

Case 2: $h = L$ and $\beta_i^{hk} \leq \gamma_i^{\tilde{h}k}$. Consider the same slight change in the DICM for $(\theta_i, \theta_j) = (\theta_i^{h}, \theta_j^{k})$ as in case 1, namely: (i) $db_i^{hk} > db_j^{kh} = 0$, which reduces i's cost by $dC_i^{hk} \equiv C'\left(b_i^{hk} - \theta_i^{h}\right)db_i^{hk}\,(< 0)$, and (ii) $dt_i^{hk} = -dC_i^{hk}\,(> 0)$. This change again increases the bandwidth allocation and improves welfare by increasing the transfer t_i^{hk}. Furthermore, it satisfies (42) and, for db_i^{hk} low enough:

* When the true handicaps are $(\theta_i, \theta_j) = (\theta_i^{\tilde{h}}, \theta_j^k)$, the working assumption $\beta_i^{hk} \leq \gamma_i^{\tilde{h}k}$ ensures that, by unilaterally misreporting θ_i^h instead of $\theta_i^{\tilde{h}}$, i would initially lose the competition against θ_j^k and therefore obtain:

$$\tilde{\Pi}_i^{\tilde{h}k} = -t_i^{hk} < 0,$$

where the inequality stems from the working assumption $t_i^{hk} > 0$.

* When instead the true handicaps are $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^{\tilde{k}})$, as in case 1 the proposed change can only reduce j's payoff from misreporting, as it increases b_i^{hk} while maintaining b_j^{kh} and t_j^{kh}.

It follows from Lemma 10 that the modified mechanism is individually rational and incentive compatible.

Case 3: $h = L$ and $\beta_i^{hk} > \gamma_i^{\tilde{h}k}$. Consider the following slight change in the DICM for $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^k)$: (i) $db_j^{kh} > db_i^{hk} = 0$, which reduces j's cost – and, thus, the market price – by $dC_j^{kh} \equiv C'\left(b_j^{kh} - \theta_j^k\right) db_j^{kh}\,(< 0)$, and (ii) $dt_i^{hk} = dC_j^{kh}\,(< 0)$. This change increases the bandwidth allocation and improves welfare for $(\theta_i, \theta_j) = \left(\theta_i^h, \theta_j^k\right)$ by

$$dW = -dp + \lambda dt_i^{hk} = -\left(1 - \lambda\right) dC_j^{kh} > 0.$$

Furthermore, the proposed change satisfies (42) and, for db_j^{kh} low enough:

* When the true handicaps are $(\theta_i, \theta_j) = (\theta_i^{\tilde{h}}, \theta_j^k)$, the working assumption $\beta_i^{hk} > \gamma_i^{\tilde{h}k}$ ensures that, by unilaterally misreporting θ_i^h instead of $\theta_i^{\tilde{h}}$, i would now strictly win the competition against θ_j^k and therefore obtain:

$$\tilde{\Pi}_i^{\tilde{h}k} = C\left(b_j^{kh} - \theta_j^k\right) - C(b_i^{hk} - \theta_i^{\tilde{h}}) - t_i^{hk}.$$

Following the small change, i would still strictly win the competition, and the misreporting payoff $\tilde{\Pi}_i^{\tilde{h}k}$ is therefore altered by:

$$\begin{aligned} d\tilde{\Pi}_i^{\tilde{h}k} &= C'\left(b_j^{kh} - \theta_j^k\right) db_j^{kh} - dt_i^{hk} \\ &= [C'\left(b_j^{kh} - \theta_j^k\right) - C'\left(b_j^{kh} - \theta_j^k\right)] db_j^{kh} \\ &= 0, \end{aligned}$$

where the second equality stems from the definition of dt_i^{hk} (namely, $dt_i^{hk} = C'\left(b_j^{kh} - \theta_j^k\right) db_j^{kh}$); hence, the change does not alter i's misreporting payoff.

* When instead the true handicaps are $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^{\tilde{k}})$, by unilaterally misreporting θ_j^k instead of $\theta_j^{\tilde{k}}$, j would initially strictly lose the competition, as we have:

$$\beta_i^{hk} > \gamma_i^{\tilde{h}k} = \gamma_i^{hk} + \delta \geq \gamma_i^{h\tilde{k}},$$

where the equality stems from the definition of δ (namely, $\delta = \theta_i^H - \theta_i^L = \theta_i^{\tilde{h}} - \theta_i^h$) and the last inequality from $\gamma_i^{h\tilde{k}} - \gamma_i^{hk} = \theta_j^k - \theta_j^{\tilde{k}} \leq |\theta_j^k - \theta_j^{\tilde{k}}| = \delta$. It follows that, following the small change, j would still lose the competition and thus obtain the same payoff, as the transfer $t_j^{\tilde{k}h}$ is unaltered.

It follows from Lemma 10 that the modified mechanism is individually rational and incentive compatible. ■

The next lemma extends the full allocation result to the case of a tie.

Lemma 12 *The optimal DICM is such that, if $\beta_i^{hk} = \gamma_i^{hk}$ for some $i \in \mathcal{I}$ and $(h,k) \in \mathcal{T}^2$, then $\sigma_i^{hk} = \Delta$.*

Proof. Suppose that $\beta_i^{hk} = \gamma_i^h$ and $\sigma_i^{hk} < \Delta$ for some $i \in \mathcal{I}$ and $(h,k) \in \mathcal{T}^2$, and let $j \neq i \in \mathcal{I}$ denote i's rival, $\tilde{h} \neq h \in \mathcal{I}$ denote i's other type, and $\tilde{k} \neq k \in \mathcal{I}$ denote j's other type. To establish the lemma, we exhibit a slight change in the DICM for $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^k)$ that satisfies the sufficient feasibility conditions provided by Lemma 10, increases the allocated bandwidth and improves welfare. Before proceeding, the following observations will be useful. When the true handicaps are $(\theta_i, \theta_j) = (\theta_i^{\tilde{h}}, \theta_j^k)$, by unilaterally misreporting

θ_i^h instead of $\theta_i^{\tilde{h}}$, i would initially obtain $\tilde{\Pi}_i^{\tilde{h}k} = \tilde{\pi}_i^{\tilde{h}k} - t_i^{hk}$, where

$$\begin{aligned} \tilde{\pi}_i^{\tilde{h}k} &= \max\{C\left(b_j^{kh} - \theta_j^k\right) - C(b_i^{hk} - \theta_i^{\tilde{h}}), 0\} \\ &= \max\{C\left(b_i^{hk} - \theta_i^h\right) - C(b_i^{hk} - \theta_i^{\tilde{h}}), 0\}, \end{aligned}$$

where the second equality stems from the working assumption $\beta_i^{hk} = \gamma_i^{hk}$. It follows that $\tilde{\pi}_i^{\tilde{h}k} > 0$ if $h = H$, and $\tilde{\pi}_i^{\tilde{h}k} < 0$ if $h = L$. Likewise, when the true handicaps are $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^{\tilde{k}})$, by unilaterally misreporting θ_j^k instead of $\theta_j^{\tilde{k}}$, j would initially obtain $\tilde{\Pi}_j^{\tilde{k}h} = \tilde{\pi}_j^{\tilde{k}h} - t_j^{kh}$, where

$$\begin{aligned} \tilde{\pi}_j^{\tilde{k}h} &= \max\{C\left(b_i^{hk} - \theta_i^h\right) - C(b_j^{kh} - \theta_j^{\tilde{k}}), 0\} \\ &= \max\{C\left(b_j^{kh} - \theta_j^k\right) - C(b_j^{kh} - \theta_j^{\tilde{k}}), 0\}, \end{aligned}$$

where the second equality stems again from the working assumption $\beta_i^{hk} = \gamma_i^{hk}$. It follows that $\tilde{\pi}_j^{\tilde{k}h} > 0$ if $k = H$, and $\tilde{\pi}_j^{\tilde{k}h} < 0$ if $k = L$. Four cases can therefore be distinguished.

Case 1: $(h, k) = (L, L)$, implying $\tilde{\pi}_i^{\tilde{h}k} < 0$ and $\tilde{\pi}_j^{\tilde{k}h} < 0$. Consider the following slight change: $db_i^{hk} = db_j^{kh} = db > 0$, which reduces both the market price and the winner's cost by $dp = C'\left(b_i^{hk} - \theta_i^h\right) db = C'\left(b_j^{kh} - \theta_j^k\right) db\,(< 0)$, leaving unchanged the transfers t_i^{hk} and t_j^{kh}. This change increases the bandwidth allocation, improves welfare by decreasing the market price, and satisfies (42). Furthermore, for db_i^{hk} low enough, $\tilde{\pi}_i^{\tilde{h}k}$ and $\tilde{\pi}_j^{\tilde{k}h}$ remain negative, implying that $\tilde{\Pi}_i^{\tilde{h}k}$ and $\tilde{\Pi}_j^{\tilde{k}h}$ remain equal to 0. Hence, from Lemma 10, the modified mechanism is individually rational and incentive compatible.

Case 2: $(h, k) = (H, L)$, implying $\tilde{\pi}_i^{\tilde{h}k} > 0 > \tilde{\pi}_j^{\tilde{k}h}$. Consider instead the following slight change: (i) $db_j^{kh} > 0$, which reduces j's cost by $dC_j^{kh} \equiv C'\left(b_j^{kh} - \theta_j^k\right) db_j^{kh}\,(< 0)$, and (ii) $dt_j^{kh} = -dC_j^{kh}\,(> 0)$, so as to appropriate the benefit of this cost reduction and maintain j's payoff. This change increases again the bandwidth allocation, improves welfare by increasing the transfer t_j^{kh}, and satisfies (42). Furthermore, for db_j^{kh} low enough, $\tilde{\pi}_j^{\tilde{k}h}$ remains negative, implying that $\tilde{\Pi}_j^{\tilde{k}h}$ remains equal to 0, whereas $\tilde{\Pi}_i^{\tilde{h}k}$ is reduced, because i, while winning the competition, now faces a rival with a lower cost and, thus, a lower market price:

$$d\tilde{\Pi}_i^{\tilde{h}k} = d\tilde{\pi}_i^{\tilde{h}k} = dC_j^{kh} < 0.$$

Hence, from Lemma 10, the modified mechanism is individually rational and incentive compatible.

Case 3: $(h,k) = (L,H)$, implying $\tilde{\pi}_i^{\tilde{h}k} < 0 < \tilde{\pi}_j^{\tilde{k}h}$. Consider the mirror change, namely: $db_i^{hk} > 0$, which reduces i's cost by $dC_i^{hk} \equiv C'\left(b_i^{hk} - \theta_i^h\right) db_i^{hk} \, (< 0)$, and (ii) $dt_i^{hk} = -dC_i^{hk} \, (> 0)$, leaving unchanged b_j^{kh} and t_j^{kh}. As in case 2, the proposed change increases the bandwidth allocation, improves welfare and satisfies (42). Furthermore, for db_i^{hk} low enough, $\tilde{\pi}_i^{\tilde{h}k}$ remains negative, implying that $\tilde{\Pi}_i^{\tilde{h}k}$ remains equal to 0, whereas $\tilde{\Pi}_j^{\tilde{k}h}$ is altered by $d\tilde{\Pi}_j^{\tilde{k}h} = d\tilde{\pi}_j^{\tilde{k}h} = dC_i^{hk} < 0$. The modified mechanism thus remains individually rational and incentive compatible.

Case 4: $(h,k) = (H,H)$, implying $\tilde{\pi}_i^{\tilde{h}k} > 0$ and $\tilde{\pi}_j^{\tilde{k}h} > 0$. Consider the same change as in case 1, namely: $db_i^{hk} = db_j^{kh} = db > 0$, leaving unchanged t_i^{hk} and t_j^{kh}. As in case 1, this change increases the bandwidth allocation, improves welfare by reducing the market price, and satisfies (42). Furthermore, it alters $\tilde{\Pi}_i^{\tilde{h}k} = \tilde{\Pi}_i^{LH}$ by

$$\begin{aligned} d\tilde{\Pi}_i^{LH} &= C'\left(b_j^{HH} - \theta_j^H\right) db - C'(b_i^{HH} - \theta_i^L)db \\ &= [C'\left(b_i^{HH} - \theta_i^H\right) - C'(b_i^{HH} - \theta_i^L)]db \\ &< 0, \end{aligned}$$

where the second inequality stems from the working assumption $\beta_i^{HH} = \gamma_i^{HH}$ and the inequality from and $\theta_i^H > \theta_i^L$ and $C''(\cdot) > 0$. Likewise, $\tilde{\Pi}_j^{LH}$ is altered by

$$d\tilde{\Pi}_j^{LH} = [C'\left(b_j^{HH} - \theta_j^H\right) - C'(b_j^{HH} - \theta_j^L)]db < 0.$$

Hence, from Lemma 10, the modified mechanism is individually rational and incentive compatible. ■

Finally, the next lemma extends the full bandwidth allocation result to the remaining potential case:

Lemma 13 *The optimal DICM is such that, if* $t_i^{hk} = 0$ *and* $\beta_i^{hk} > \gamma_i^{hk}$ *for some* $i \in \mathcal{I}$ *and* $(h,k) \in \mathcal{T}^2$*, then* $\sigma_i^{hk} = \Delta$*.*

Proof. Suppose that $\beta_i^{hk} > \gamma_i^{hk}$ for some $i \in \mathcal{I}$ and $(h,k) \in \mathcal{T}^2$, and let $j \neq i \in \mathcal{I}$ denote i's rival, $\left(\tilde{h} \neq h, \tilde{k} \neq k\right) \in \mathcal{I}^2$ respectively denote i's and j's other types. We first note that i's handicap must be the lower one (i.e., $h = L$):

Claim 1 *The optimal DICM is such that, if* $t_i^{hk} = 0$ *and* $\beta_i^{hk} > \gamma_i^{hk}$ *for some* $i \in \mathcal{I}$ *and* $(h,k) \in \mathcal{T}^2$*, then* $h = L$*.*

Proof. Suppose by way of contradiction that the above conditions hold for $h = H$, that is: $t_i^{Hk} = 0$ and $\beta_i^{Hk} > \gamma_i^{Hk}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$. From Lemma **??**, $\left(IR_i^{Hk}\right)$ is binding, leading to:

$$t_i^{Hk} = \pi_i\left(b_i^{Hk}, b_j^{kH}, \theta_i^H, \theta_j^k\right) = \max\left\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^H\right), 0\right\}.$$

However, the working assumptions $t_i^{Hk} = 0$ and $\beta_i^{Hk} > \gamma_i^{Hk}$ imply that the left-hand side is zero whereas the right-hand is positive, a contradiction. ■

We can therefore restrict attention to the case $h = L$, for which the working assumptions become

$$t_i^{Lk} = 0 \text{ and } \beta_i^{Lk} > \gamma_i^{Lk}, \tag{43}$$

which together imply:

$$\Pi_i^{Lk} = C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^L\right) > 0. \tag{44}$$

Furthermore, from Lemma 3, $\left(IC_i^{Lk}\right)$ is binding; together with the above inequality, this yields:

$$0 < \Pi_i^{Lk} = \pi_i\left(b_i^{Hk}, b_j^{kH}, \theta_i^L, \theta_j^k\right) - t_i^{Hk} \leq \pi_i\left(b_i^{Hk}, b_j^{kH}, \theta_i^L, \theta_j^k\right),$$

where the last inequality stems from Corollary 2. Using the definition of profits, this implies $C\left(b_j^{kH} - \theta_j^k\right) > C\left(b_i^{Hk} - \theta_i^L\right)$, or:

$$\beta_i^{Hk} > \gamma_i^{Lk}. \tag{45}$$

The incentive compatibility condition $\left(IC_i^{Lk}\right)$ thus amounts to

$$\Pi_i^{Lk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right) - t_i^{Hk}. \tag{46}$$

Using $\Pi_i^{Hk} = 0$ (from Lemma 3) and the working assumption $t_i^{Lk} = 0$, the incentive compatibility condition $\left(IC_i^{Hk}\right)$ boils down to:

$$0 \geq \pi_i\left(b_i^{Lk}, b_j^{kL}, \theta_i^H, \theta_j^k\right) = \max\{C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^H\right), 0\},$$

implying $C\left(b_j^{kL} - \theta_j^k\right) \leq C\left(b_i^{Lk} - \theta_i^H\right)$, or:

$$\gamma_i^{Hk} \geq \beta_i^{Lk}.$$

The following claim addresses the case in which the above inequality is strict:

Claim 2 *The optimal DICM is such that, if* $t_i^{Lk} = 0$ *and*

$$\gamma_i^{Hk} > \beta_i^{Lk} > \gamma_i^{Lk},$$

for some $i \in \mathcal{I}$ *and* $k \in \mathcal{T}$, *then* $\sigma_i^{Lk} = \Delta$.

Proof. Suppose that $\sigma_i^{Lk} < \Delta$ and consider the following slight change in the DICM for $(\theta_i, \theta_j) = (\theta_i^h, \theta_j^k)$: (i) $db_i^{Lk} > 0$, which reduces i's cost by $dC_i^{Lk} \equiv C'\left(b_i^{Lk} - \theta_i^L\right) db_i^{Lk} \,(< 0)$, and (ii) $dt_i^{Lk} = -dC_i^{Lk} \,(> 0)$, so as to appropriate the benefit of this cost reduction and maintain i's payoff. This change increases the bandwidth allocation, improves welfare by increasing the transfer t_i^{Lk}, and satisfies (42). Furthermore, $(IC_j^{\tilde{k}H})$ remains satisfied because j's rival now benefits from a lower cost, and for db_j^{kh} low enough the strict inequality $\gamma_i^{Hk} > \beta_i^{Lk}$ continues to hold, implying that $\left(IC_i^{Hk}\right)$, too, remains satisfied. It follows that the modified mechanism is individually rational and incentive compatible. ■

In the remainder of the proof, we therefore concentrate on the case where:

$$\beta_i^{Lk} = \gamma_i^{Hk} \left(> \gamma_i^{Lk}\right).$$

Together with $t_i^{Lk} = 0$, this yields:

$$\Pi_i^{Lk} = C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^L\right) = C\left(b_i^{Lk} - \theta_i^H\right) - C\left(b_i^{Lk} - \theta_i^L\right). \quad (47)$$

We next show that $t_i^{Hk} = 0$:

Claim 3 *The optimal DICM is such that, if $t_i^{Lk} = 0$ and $\beta_i^{Lk} = \gamma_i^{Hk}$, for some $i \in \mathcal{I}$ and $k \in \mathcal{T}$, then $t_i^{Hk} = 0$.*

Proof. Suppose by way of contradiction that $t_i^{Hk} > 0$. From Lemma 3, $\left(IR_i^{Hk}\right)$ is binding, implying $\Pi_i^{Hk} = 0$, or:

$$\max\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^H\right), 0\} = t_i^{Hk} > 0,$$

which in turn implies

$$t_i^{Hk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^H\right).$$

The binding incentive compatibility condition $\left(IC_i^{Lk}\right)$ thus amounts to:

$$\Pi_i^{Lk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right) - t_i^{Hk} = C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right).$$

Combining this equality with (47) then implies:

$$\begin{aligned}
0 &= \left[C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right)\right] - \left[C\left(b_i^{Lk} - \theta_i^H\right) - C\left(b_i^{Lk} - \theta_i^L\right)\right] \\
&= \int_{\theta_i^L}^{\theta_i^H} \left[C'\left(b_i^{Lk} - \theta\right) d\theta - C'\left(b_i^{Hk} - \theta\right)\right] d\theta \\
&= \int_{\theta_i^L}^{\theta_i^H} \int_{b_i^{Hk}}^{b_i^{Lk}} C''\left(b - \theta\right) db d\theta.
\end{aligned}$$

It then follows from $\theta_i^H > \theta_i^L$ and $C''(\cdot) > 0$ that

$$b_i^{Hk} = b_i^{Lk} \equiv b_i^k.$$

In addition, we have:

$$C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^k - \theta_i^L\right) = \Pi_i^{Lk} > C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^k - \theta_i^L\right)$$

where the first equality stems from (47) and the inequality follows from $\left(IC_i^{Lk}\right)$, (45) and the working assumption $t_i^{Hk} > 0$; we thus have:

$$b_j^{kH} < b_j^{kL},$$

leading to:

$$\sigma_i^{Hk} = b_i^{Hk} + b_j^{kH} < b_i^{Lk} + b_j^{kL} \leq \Delta.$$

It then follows from Lemma 11 that $t_i^{Hk} = 0$, a contradiction. ■

In the remainder of the proof, we therefore concentrate on the case where:

$$t_i^{Hk} = 0,$$

implying that the incentive condition $\left(IC_i^{Lk}\right)$ boils down to (setting $t_i^{Hk} = 0$ in (46)):

$$\Pi_i^{Lk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right). \tag{48}$$

Furthermore, because $\left(IR_i^{Hk}\right)$ is binding (from Lemma 3), it also implies:

$$0 = \Pi_i^{Hk} = \pi_i^{Hk} = \max\{C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^H\right), 0\},$$

leading to $C\left(b_j^{kH} - \theta_j^k\right) \leq C\left(b_i^{Hk} - \theta_i^H\right)$, or:

$$\beta_i^{Hk} \leq \gamma_i^{Hk}.$$

The following claim addresses the case in which the above condition is satisfied with equality:

Claim 4 *The optimal DICM is such that, if* $t_i^{Lk} = t_i^{Hk} = 0$ *and* $\beta_i^{Lk} = \beta_i^{Hk} = \gamma_i^{Hk}$*, for some* $i \in \mathcal{I}$ *and* $k \in \mathcal{T}$*, then* $\sigma_i^{Lk} = \Delta$*.*

Proof. From (48), we have:

$$\Pi_i^{Lk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right) = C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right),$$

where the second equality stems from $\beta_i^{Hk} = \gamma_i^{Hk}$. Combined with (47), this again implies $b_i^{Hk} = b_i^{Lk}$ and, together with $\beta_i^{Lk} = \beta_i^{Hk}$, it now also leads to $b_j^{kH} = b_j^{kL}$. It follows that the options designed for $(\theta_i, \theta_j) = (\theta_i^L, \theta_j^k)$ and $(\theta_i, \theta_j) = (\theta_i^H, \theta_j^k)$ coincide:

$$b_i^{Hk} = b_i^{Lk} \equiv b_i^k,\ b_j^{kH} = b_j^{kL} \equiv b_j^k, \text{ and } t_j^{kH} = t_j^{kL} \equiv t_j^k.$$

Suppose by way of contradiction that $\sigma_i^{Lk} = b_i^{Lk} + b_j^{kL} < \Delta$, and consider the following uniform change in the DICM for *both* $(\theta_i, \theta_j) = (\theta_i^L, \theta_j^k)$ and $(\theta_i, \theta_j) = (\theta_i^H, \theta_j^k)$: (i) $db_i^{Lk} = db_i^{Hk} \equiv db_i^k > 0$, and (ii) $dt_i^{Lk} = dt_i^{Hk} \equiv dt_i^k = -C'\left(b_i^k - \theta_i^L\right) db_i^k\ (> 0)$, leaving unchanged $b_j^{kH} = b_j^{kL} = b_j^k$ and the transfers t_j^{kH} and t_j^{kL}. This change increases the bandwidth allocation and improves welfare by increasing the transfers t_i^{Lk} and t_i^{Hk}. Furthermore, by construction it maintains Π_i^{Lk}, and it improves Π_i^{Hk} by

$$d\Pi_i^{Hk} = \left[C'\left(b_i^k - \theta_i^H\right) - C'\left(b_i^k - \theta_i^L\right)\right] db_i^k > 0,$$

where the inequality stems from $\theta_i^L > \theta_i^H$ and $C''(\cdot) > 0$; hence, $\left(IR_i^{Lk}\right)$ and $\left(IR_i^{Hk}\right)$ remain satisfied. In addition, (IC_i^{Lk}) and (IC_i^{Hk}) remain trivially satisfied, because the options designed for $(\theta_i, \theta_j) = (\theta_i^L, \theta_j^k)$ and $(\theta_i, \theta_j) = (\theta_i^H, \theta_j^k)$ still coincide, and $(IC_j^{\tilde{k}H})$ and $(IC_j^{\tilde{k}L})$ remain also satisfied, because j's rival now benefits from a lower cost. ■

In the remainder of the proof, we therefore concentrate on the case where $t_i^{Lk} = t_i^{Hk} = 0$ and:

$$\beta_i^{Lk} = \gamma_i^{Hk} > \beta_i^{Hk},$$

which in particular implies that, under truthtelling, θ_j^k wins strictly against

θ_i^H:

$$\pi_j^{kH} > 0. \tag{49}$$

From (48), we have:

$$\Pi_i^{Lk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right) < C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right),$$

where the second equality stems from $\beta_i^{Hk} < \gamma_i^{Hk}$. Combining this with (47) then implies:

$$\begin{aligned} 0 &< \left[C\left(b_i^{Hk} - \theta_i^H\right) - C\left(b_i^{Hk} - \theta_i^L\right)\right] - \left[C\left(b_i^{Lk} - \theta_i^H\right) - C\left(b_i^{Lk} - \theta_i^L\right)\right] \\ &= \int_{\theta_i^L}^{\theta_i^H} \int_{b_i^{Hk}}^{b_i^{Lk}} C''\left(b - \theta\right) db d\theta. \end{aligned}$$

It then follows from $C''(\cdot) > 0$ that

$$b_i^{Lk} > b_i^{Hk}. \tag{50}$$

Combining (44) and (48) yields:

$$C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_i^{Lk} - \theta_i^L\right) = \Pi_i^{Lk} = C\left(b_j^{kH} - \theta_j^k\right) - C\left(b_i^{Hk} - \theta_i^L\right),$$

leading to:

$$C\left(b_j^{kL} - \theta_j^k\right) - C\left(b_j^{kH} - \theta_j^k\right) = C\left(b_i^{Lk} - \theta_i^L\right) - C\left(b_i^{Hk} - \theta_i^L\right) < 0,$$

where the inequality stems from (50), implying:

$$b_j^{kL} > b_j^{kH}.$$

We thus have:

$$\sigma_i^{Hk} = b_i^{Hk} + b_j^{kH} < b_i^{Lk} + b_j^{kL} \leq \Delta.$$

It follows from Lemma 11 that

$$t_j^{kH} = 0, \tag{51}$$

which, together with (49), implies:

$$\Pi_j^{kH} > 0,$$

and (from Lemma 2):

$$k = L.$$

Building on these observations, without loss of generality we can now focus on DICMs satisfying (using (51), $\beta_j^{LL} = \beta_i^{LL}$, $\gamma_j^{LH} = \gamma_i^{HL}$, $\beta_j^{LH} = \beta_i^{HL}$ and $\sigma_j^{LH} = \sigma_i^{HL}$):

$$\begin{aligned} t_j^{LH} &= 0, \\ \beta_j^{LL} &= \gamma_j^{LH} < \beta_j^{LH}, \\ \sigma_j^{LH} &< \Delta. \end{aligned}$$

Using $\Pi_j^{HH} = 0$ (from Lemma 2) and $t_j^{LH} = 0$, the incentive compatibility condition $\left(IC_j^{HH}\right)$ boils down to:

$$0 \geq \pi_j \left(b_j^{LH}, b_i^{HL}, \theta_j^H, \theta_i^H\right) = \max\{C\left(b_i^{HL} - \theta_i^H\right) - C\left(b_j^{LH} - \theta_j^H\right), 0\},$$

implying $C(b_i^{HL} - \theta_i^H) \leq C(b_j^{LH} - \theta_j^H)$, or $\gamma_j^{HH} \geq \beta_j^{LH}$. Furthermore, from Claim 2 (applied to "i"$= j$ and "k"$= H$), together with the conditions $t_j^{LH} = 0$, $\beta_j^{LH} > \gamma_j^{LH}$ and $\sigma_j^{LH} < \Delta$, this inequality cannot be strict. Hence, $\beta_j^{LH} = \gamma_j^{HH}$. It then follows from Claim 3 (applied to "i"$= j$ and "k"$= H$, and using the conditions $t_j^{LH} = 0$ and $\beta_j^{LH} = \gamma_j^{LH}$), that $t_j^{HH} = 0$. Furthermore, from Lemma 3, $\Pi_j^{HH} = 0$; together with $t_j^{HH} = 0$, this implies $\gamma_j^{HH} \geq \beta_j^{HH}$. Summing up, we have

$$t_j^{LH} = t_j^{HH} = 0, \ \sigma_j^{LH} < \Delta, \text{ and } \beta_j^{LH} = \gamma_j^{HH} \geq \beta_j^{HH}.$$

To conclude the argument, it suffices to exhibit a contradiction. Two cases can be distinguished.

Case 1: $\gamma_j^{HH} = \beta_j^{HH}$. We would then have $t_j^{LH} = t_j^{HH} = 0$, $\beta_j^{LH} = \beta_j^{HH} = \gamma_j^{HH}$ and $\sigma_j^{LH} < \Delta$, which is ruled out by Claim 4 (applied to "i"$= j$ and "k"$= H$).

Case 2: $\gamma_j^{HH} > \beta_j^{HH}$. Under truthtelling, θ_i^H would then win the competition against θ_i^H and generate a positive profit: $\pi_i^{HH} > 0$. Furthermore, it follows from Claim 3 (applied to "k"$= H$) that $t_i^{HH} = 0$. Together, this would imply

$$\Pi_i^{HH} = \pi_i^{HH} > 0,$$

which is ruled out by Lemma 2. ■

Online Appendix - 3
(Not for Publication)

A An Example Where Bunching Is Not Optimal

We provide here an example where bunching is no longer optimal.

A.1 Setup

A.1.1 Demand and Supply Conditions

We will adopt the following specifications:

- *Linear demand:* Letting p denote the market price (i.e., the lowest of the firms' prices), consumer demand is given by:

$$D(p) = 1 - p.$$

  It follows that consumer surplus is equal to:

$$S(p) = \frac{(1-p)^2}{2},$$

  and the industry monopoly profit, based on a constant marginal cost γ, is equal to:

$$p^m(\gamma) = \frac{1+\gamma}{2}.$$

- *Linear unit cost:* If a firm benefits from a bandwidth $\tilde{B}$, its unit cost is given by:

$$c\left(\tilde{B}\right) = C - \tilde{B}.$$

A.1.2 Two types of entrant

We denote the bandwidth initially available to the incumbent by

$$B_I = B$$

and that initially available to the entrant by

$$B_E = B - \tilde{\theta}.$$

Thus, as before, the parameter $\tilde{\theta}$ reflects the handicap of the entrant. We assume that this handicap can take two values:

- With probability $\rho \in (0,1)$, the handicap is given by $\tilde{\theta} = 0$; that is, the entrant is initially as efficient as the incumbent.
- With probability $\rho' = 1 - \rho$, the handicap is given by $\tilde{\theta} = B$; that is, the entrant has initially no bandwidth.

A.1.3 Calibration

For the sake of exposition, we further assume that:

- The additional bandwidth is large enough to enable both types of entrant to win the market competition; that is:

  $$\Delta = B + 2\varepsilon,$$

  where $\varepsilon > 0$. With this notation, the relevant values of the critical bandwidth threshold,

  $$\hat{b}(\theta) = \frac{\Delta + \theta}{2},$$

  are equal to:

  $$\hat{b} = \hat{b}(0) = \frac{\Delta}{2} = \frac{B}{2} + \varepsilon,$$
  $$\hat{b}' = \hat{b}(B) = \frac{B + \Delta}{2} = B + \varepsilon.$$

- The cost function is normalized such that

  $$C = B + \Delta = 2(B + \varepsilon).$$

  This ensures that all unit costs remain non-negative (the incumbent benefits from a zero unit cost if it obtains all the additional bandwidth, and both unit costs are positive otherwise), and also simplifies some of the exposition.[65]

A.1.4 Prices

When an entrant of type $\tilde{\theta}$ obtains an additional bandwidth $\tilde{b} \geq \hat{b}(\tilde{\theta})$, the market price $p_{\tilde{\theta}}\left(\tilde{b}\right)$ is equal to the cost of the incumbent:

$$c\left(B_I + \Delta - \tilde{b}\right) = C - \left(B + \Delta - \tilde{b}\right).$$

Using $C = B + \Delta$, this simplifies to

$$p_{\tilde{\theta}}\left(\tilde{b}\right) = \tilde{b}.$$

When instead the incumbent wins the competition (i.e., when $\tilde{b} < \hat{b}\left(\theta\right)$), the market price is determined by the cost of the entrant, and is thus equal to:

$$p_{\tilde{\theta}}\left(\tilde{b}\right) \equiv \begin{cases} B + 2\varepsilon - \tilde{b} & \text{if } \tilde{\theta} = \theta = 0, \\ 2\left(B + \varepsilon\right) - \tilde{b} & \text{if } \tilde{\theta} = \theta' = B. \end{cases}$$

A.1.5 Profit

When an entrant of type $\tilde{\theta}$ obtains an additional bandwidth $\tilde{b} \in [\hat{b}(\tilde{\theta}), 1]$, it wins the product-market competition and obtains a profit equal to:

$$\begin{aligned} \pi(\tilde{b}, \tilde{\theta}) &\equiv [c(B + \Delta - \tilde{b}) - c(B - \tilde{\theta} + \tilde{b})]D(c(B + \Delta - \tilde{b})) \\ &= 2[\tilde{b} - \hat{b}(\tilde{\theta})](1 - \tilde{b}). \end{aligned}$$

We have:

$$\frac{\partial \pi}{\partial \tilde{\theta}}(\tilde{b}, \tilde{\theta}) = -(1 - \tilde{b}) \leq 0,$$

[65] See for instance below the derivation of the market price $p_{\tilde{\theta}}\left(\tilde{b}\right)$ for $\tilde{b} \geq \hat{b}\left(\tilde{\theta}\right)$.

with a strict inequality for $\tilde{b} < 1$, and:

$$\frac{\partial^2 \pi}{\partial \tilde{\theta} \partial \tilde{b}} \pi(\tilde{b}, \tilde{\theta}) = 1 > 0.$$

Finally,

$$\partial_{\tilde{b}} \pi(\tilde{b}, \tilde{\theta}) = 2(1 - \tilde{b}) - 2(\tilde{b} - \hat{b}(\tilde{\theta})) = 4 \left[\frac{1 + \hat{b}(\tilde{\theta})}{2} - \tilde{b} \right] = 4 \left[\hat{p}^m(\hat{b}(\tilde{\theta})) - \tilde{b} \right],$$

where

$$\hat{p}^m(\gamma) = \frac{1 + \gamma}{2}$$

denotes the monopoly price based on a unit cost γ. Hence, $\partial_{\tilde{b}} \pi(\tilde{b}, \tilde{\theta})$ is positive as long as:

$$p_{\tilde{\theta}}(\tilde{b}) = \tilde{b} < \hat{p}^m(\hat{b}(\tilde{\theta})) = \frac{1 + \hat{b}(\tilde{\theta})}{2} = \frac{1 + \frac{\Delta + \tilde{\theta}}{2}}{2} = \frac{1}{2} \left(1 + B + \varepsilon + \frac{\tilde{\theta}}{2} \right).$$

In particular, when the entrant has no handicap ($\theta = 0$), in order to ensure that the price ($p_\theta(b) = b$) remains below the monopoly level ($\hat{p}^m(\hat{b})$) in the relevant range ($b \in [\hat{b}, \hat{b}']$), we need:

$$\begin{aligned} \hat{b}' = B + \varepsilon < \hat{p}^m(\hat{b}) &= \frac{1 + \frac{B}{2} + \varepsilon}{2} \\ \iff B < \frac{2}{3}(1 - \varepsilon). \end{aligned} \tag{52}$$

A.2 Bunching Mechanisms

When considering bunching mechanisms, which allocate the same additional bandwidth b to both types of entrant, without loss of generality we can restrict attention to $b \in [\hat{b}, \hat{b}']$, as any lower value ($b < \hat{b}$) is dominated by $b = \hat{b}$, and any higher value ($b > \hat{b}'$) is dominated by $b = \hat{b}'$. For any value b in that range:

- when the entrant has no handicap, the market price is equal to the cost of the incumbent; and
- otherwise, the market price is equal to the cost of the entrant.

Hence, expected consumer surplus is equal to:

$$\begin{aligned}S_B(b) &= \rho S(c(B+\Delta-b)) + \rho' S(c(B-\theta'+b)) \\ &= \rho S(b) + \rho' S(2B+2\varepsilon-b).\end{aligned}$$

This expected surplus is convex in b:

$$\begin{aligned}S'_B(b) &= -\rho D(b) + \rho' D(2B+2\varepsilon-b), \\ S''_B(b) &= \rho+\rho' > 0.\end{aligned}$$

It follows that the best bunching mechanism consists of allocating either $\hat{b}$ or $\hat{b}'$ to the entrant; both options are moreover equivalent when:

$$\begin{aligned}&S_B(\hat{b}') = S_B(\hat{b}) \\ \Longleftrightarrow &\rho S(C-s(B+\Delta-\hat{b}')) + \rho' S(C-s(B-\theta'+\hat{b}')) \\ &\quad = \rho S(C-s(B+\Delta-\hat{b})) + \rho' S(C-s(B-\theta'+\hat{b})) \qquad (53) \\ \Longleftrightarrow &\frac{\rho}{\rho'} = \frac{S(C-(B-\theta'+\hat{b}')) - S(C-(B-\theta'+\hat{b}))}{S(C-(B+\Delta-\hat{b})) - S(C-(B+\Delta-\hat{b}'))} \\ \Longleftrightarrow &\frac{\rho}{\rho'} = \frac{1-\varepsilon-\frac{5B}{4}}{1-\varepsilon-\frac{3B}{4}}. \qquad (54)\end{aligned}$$

A.3 Discriminating mechanisms

We will consider a candidate discriminating mechanism which gives the non-handicapped entrant a bandwidth $b \in [\hat{b}, \hat{b}']$ (in exchange for a transfer t), and the handicapped entrant a higher bandwidth $b' > \hat{b}'$ (in exchange for a transfer t').[66] To be incentive-compatible, the mechanism must satisfy:

$$\begin{aligned}\pi(b,\theta) - t &\geq \pi(b',\theta) - t', \\ \pi(b',\theta') - t' &\geq -t.\end{aligned}$$

[66] By inspecting the incentive constraints for all possible cases (where $b \gtreqless \hat{b}$ and $b' \gtreqless \hat{b}'$), it can be checked that the best discriminating mechanism has indeed these features.

Combining these conditions imposes:

$$\pi\left(b,\theta\right) \geq \pi\left(b',\theta\right) - \pi\left(b',\theta'\right). \tag{55}$$

The right-hand side of this inequality decreases as b' increases:

$$\begin{aligned}\frac{d}{db'}\left(\pi\left(b',\theta\right)-\pi\left(b',\theta'\right)\right) &= \frac{\partial \pi}{\partial \tilde{b}}\left(b',\theta\right) - \frac{\partial \pi}{\partial \tilde{b}}\left(b',\theta'\right) \\ &= -\int_{\theta}^{\theta'} \frac{\partial^2 \pi}{\partial \tilde{\theta}\partial \tilde{b}}\left(b',x\right)dx \\ &< 0.\end{aligned}$$

Hence, for any given bandwidth, $b \in \left[\hat{b},\hat{b}'\right]$, that an entrant with no handicap would receive, the best value for the bandwidth, b', that a handicapped entrant should receive is the lowest one that is compatible with (55); that is, b' should be chosen such that:

$$\begin{aligned}&\pi\left(b,\theta\right) = \pi\left(b',\theta\right) - \pi\left(b',\theta'\right) \\ &\quad\Longleftrightarrow 2\left(b-\hat{b}\right)\left(1-b\right) = 2\left(b'-\hat{b}\right)\left(1-b'\right) - 2\left(b'-\hat{b}'\right)\left(1-b'\right) \\ &\quad\Longleftrightarrow b' = \beta\left(b\right) \equiv \left[1-\frac{b-\hat{b}}{\hat{b}'-\hat{b}}\left(1-b\right)\right] = 2-b-\frac{2}{B}\left(b-\varepsilon\right)\left(1-b\right) = \frac{B\left(2-b\right)-2\left(b-\varepsilon\right)\left(1-b\right)}{B}\end{aligned}$$

This optimal value is such that:

$$\beta\left(\hat{b}'\right) = \frac{\hat{b}'\left(\hat{b}'-\hat{b}\right)}{\hat{b}'-\hat{b}} = \hat{b}',$$

and, for $b \in \left[\hat{b},\hat{b}'\right]$:

$$\begin{aligned}\beta'\left(b\right) &= \frac{d}{db}\left(\frac{B\left(2-b\right)-2\left(b-\varepsilon\right)\left(1-b\right)}{B}\right) \\ &= -\frac{2+B+2\varepsilon-4b}{B} \\ &= -\frac{1+\hat{b}-2b}{\hat{b}'-\hat{b}} = -2\frac{\hat{p}^m\left(\hat{b}\right)-b}{\hat{b}'-\hat{b}},\end{aligned}$$

which is negative as long as

$$b < \hat{p}^m\left(\hat{b}\right) = \frac{1+\hat{b}}{2} = \frac{1}{2}\left(1 + \frac{B}{2} + \varepsilon\right).$$

In particular:

$$\beta'\left(\hat{b}'\right) = -2\frac{\hat{p}^m\left(\hat{b}\right) - \hat{b}'}{\hat{b}' - \hat{b}} = \left[-2\frac{\frac{1}{2} + \frac{\Delta}{4} - \frac{\Delta+B}{2}}{\frac{B}{2}}\right]_{\Delta=B+2\varepsilon} = -\frac{2\left(1-\varepsilon\right) - 3B}{B},$$

which is thus negative as long as B satisfies (52). Hence, as long as B satisfies this condition, $\beta\left(b\right)$ is indeed higher than $\hat{b}'$ for b lower than but close to $\hat{b}'$.

Expected consumer surplus is then equal to:

$$\begin{aligned} S_D\left(b\right) &= \rho S\left(c\left(B+\Delta-b\right)\right) + \rho' S\left(c\left(B+\Delta-\beta\left(b\right)\right)\right) \\ &= \rho\frac{\left(1-b\right)^2}{2} + \rho'\frac{\left(1-\beta\left(b\right)\right)^2}{2}. \end{aligned}$$

Therefore:

$$S_D'\left(b\right) = -\rho\left(1-b\right) - \rho'\left(1-\beta\left(b\right)\right)\beta'\left(b\right),$$

Bunching will for instance not be optimal if:

- the probabilities of the two types are such that expected consumer surplus is the same in the situation where both types receive $\hat{b}$ and in the situation where they both receive $\hat{b}'$; and,
- starting from the latter situation, where both types receive $\hat{b}'$, a small reduction in the bandwidth b allocated to the entrant in case of no handicap, together with an increase in the bandwidth b' allocated to the entrant in case of a large handicap, up to $b' = \beta\left(b\right)$, increases expected consumer surplus.

Hence, to exhibit an example where bunching is not optimal, it suffices to find parameters B and ε such that $S_D'\left(\hat{b}'\right) < 0$ for the probabilities ρ and ρ' that satisfy (54). As:

$$S'_D\left(\hat{b}'\right) = -\rho\left(1-\hat{b}'\right) - \rho'\left(1-\hat{b}'\right)\beta'\left(\hat{b}'\right)$$
$$= -\rho\left(1-\hat{b}'\right)\left[1+\frac{\rho'}{\rho}\beta'\left(\hat{b}'\right)\right],$$

this amounts to finding parameters B and ε such that the terms within square brackets is positive, that is:

$$-\beta'\left(\hat{b}'\right) = \frac{2-3B-2\varepsilon}{B} < \frac{\rho}{\rho'} = \frac{1-\varepsilon-\frac{5B}{4}}{1-\varepsilon-\frac{3B}{4}}.$$

This requires:

$$\frac{2-3B-2\varepsilon}{B} < \frac{1-\frac{5B}{4}-\varepsilon}{1-\frac{3B}{4}-\varepsilon}$$
$$\Longleftrightarrow (2-3B-2\varepsilon)\left(1-\frac{3B}{4}-\varepsilon\right) < B\left(1-\frac{5B}{4}-\varepsilon\right)$$
$$\Longleftrightarrow 0 < B\left(1-\frac{5B}{4}-\varepsilon\right) - (2-3B-2\varepsilon)\left(1-\frac{3B}{4}-\varepsilon\right) = -2\left(1-\varepsilon-\frac{7B}{4}\right)(1-\varepsilon-$$

which amounts to:

$$\frac{4}{7}(1-\varepsilon) < B < 1-\varepsilon.$$

Combining these conditions with (52), it suffices to choose B and ε such that:

$$\frac{4}{7}(1-\varepsilon) < B < \frac{2}{3}(1-\varepsilon).$$

A.4 Numerical example

A.4.1 Parameter values

For $\varepsilon = 0$, the above conditions boil down to:

$$\frac{4}{7} = \frac{12}{21} < B < \frac{2}{3} = \frac{14}{21}.$$

We will thus consider the case

$$B = \frac{13}{21},$$

and choose ε "small enough" to satisfy (52), namely, such that:

$$B < \frac{2}{3}(1-\varepsilon) \Longleftrightarrow \varepsilon < 1 - \frac{3B}{2} = \frac{1}{14} \simeq 0.07.$$

We will thus take $\varepsilon = 0.05\,(= 1/20)$. We then have:

$$\begin{aligned}
\hat{b} &= \frac{13}{42} + \frac{1}{20} = \frac{151}{420} \simeq 0.36, \\
\hat{b}' &= \frac{13}{21} + \frac{1}{20} = \frac{281}{420} \simeq 0.67, \\
\Delta &= \frac{13}{21} + \frac{1}{10} = \frac{151}{210} \simeq 0.72, \\
\beta(b) &= \left[\frac{B(2-b) - 2(b-\varepsilon)(1-b)}{B}\right]_{B=\frac{13}{21},\varepsilon=\frac{1}{20}} = \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2, \\
p^m\left(\hat{b}\right) &= \left[\frac{1+\hat{b}}{2}\right]_{\hat{b}=13/42+1/20} = \frac{571}{840} \simeq 0.68, \\
p^m\left(\hat{b}'\right) &= \left[\frac{1+\hat{b}'}{2}\right]_{\hat{b}'=13/21+1/20} = \frac{701}{840} \simeq 0.83, \\
\rho &= \frac{37}{139} \simeq 0.27, \\
\rho' &= \frac{102}{139} \simeq 0.73,
\end{aligned}$$

and:

$$-\beta'\left(\hat{b}'\right) = \frac{9}{130} \simeq 0.06 < \frac{\rho}{\rho'} = \frac{37}{102} \simeq 0.36.$$

The function $\beta(b)$ is depicted by the following figure (for $b \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$):

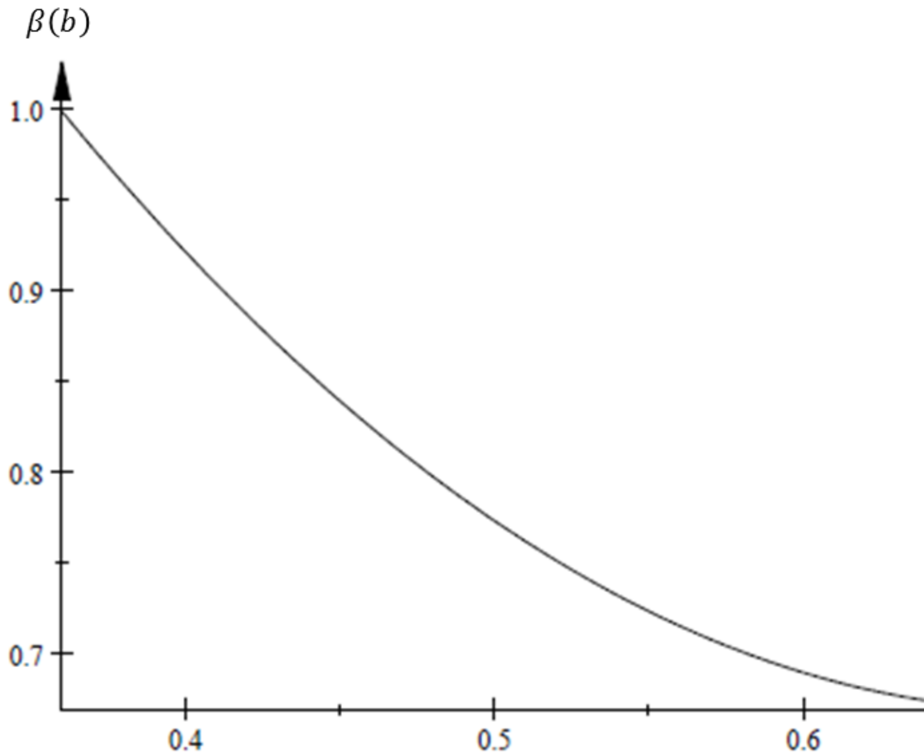

Figure 3: Bandwidth for handicapped entran

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In particular, we have:

$$\begin{aligned}\beta(b) \leq \Delta &\Longleftrightarrow \frac{281}{130} - \frac{571}{130} b + \frac{42}{13} b^2 \leq \frac{151}{210} \\ &\Longleftrightarrow b \geq \bar{b} = \frac{571}{840} - \frac{\sqrt{3}\sqrt{3667}}{840} \simeq 0.55.\end{aligned}$$

It can be checked that the above conditions are satisfied in this example; in particular:

– Demand is positive (i.e., $b < 1$) in the relevant ranges $b \leq \hat{b}'$ (as $\hat{b}' \simeq 0.67 < 1$) and $b' \leq \Delta$ (as $\Delta \simeq 0.72 < 1$). It follows that $\partial_\theta \pi(b, \theta) < 0$ in the relevant ranges.

– We also have $\partial_b \pi(b,\theta) > 0$ (i.e., $b < \hat{p}^m\left(\hat{b}(\theta)\right)$) in these ranges:

$$b \leq \hat{b}' = \frac{281}{420} \simeq 0.67 < p^m\left(\hat{b}\right) = \frac{571}{840} \simeq 0.68,$$
$$b' \leq \Delta = \frac{151}{210} \simeq 0.72 \leq p^m\left(\hat{b}'\right) = \frac{701}{840} \simeq 0.83.$$

A.4.2 Prices

In case of bunching, for $b = b' \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$, the price is equal to b if the entrant faces no handicap, and it is otherwise equal to:

$$\begin{aligned} p_{\theta'}(b') &= [C - (B - \theta' + b)]_{\theta'=B,\Delta=B+2\varepsilon,C=2(B+\varepsilon)} \\ &= \frac{281}{210} - b. \end{aligned}$$

In case of discrimination, for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67]$ and $b' = \beta(b) = \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2$, the price is the same as in the previous scenario (i.e., it is equal to b) if the entrant faces no handicap, and it is otherwise equal to:

$$p_{\theta'}(b') = \beta(b) = \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2.$$

The following figure depicts the price in case of handicap, in the two scenarios: bunching (thin line, for $b \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$) and discrimination (bold curve, for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67])$:

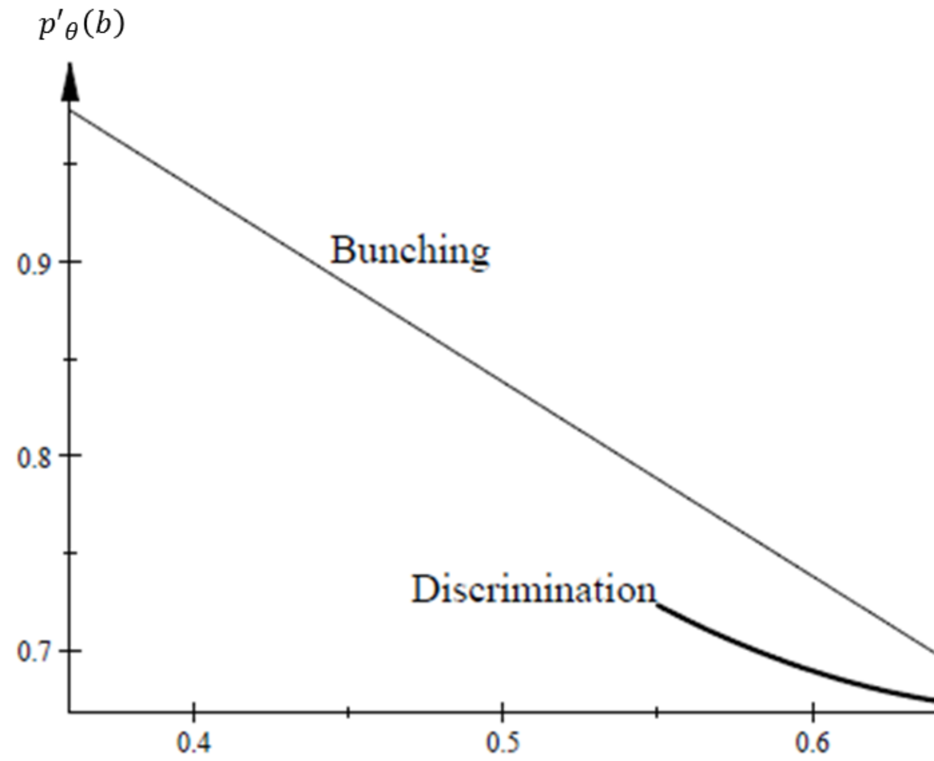

Figure 4: Market price when the entrant is handic

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The figure confirms that the price $p_{\theta'}(b)$ is lower in the discriminating scenario than in the bunching scenario.

A.4.3 Consumer surplus

$$p_{\theta'}(b') = \begin{cases} \frac{281}{210} - b & \text{if} \quad b' = b, \\ \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2 & \text{if} \quad b' = \beta(b). \end{cases}$$

Building on the above analysis, and using

$$\rho = \frac{37}{139} \simeq 0.27 \text{ and } \rho' = \frac{102}{139} \simeq 0.73,$$

consumers' expected surplus is given by:

– If $b' = b$ ("Bunching"), then for $b = b' \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$:

$$\begin{aligned} S_B(b) &= \rho \frac{(1-b)^2}{2} + \rho' \frac{(1 - p_{\theta'}(b))^2}{2} \\ &= \frac{37}{139} \frac{(1-b)^2}{2} + \frac{102}{139} \frac{\left(1 - \left(\frac{281}{210} - b\right)\right)^2}{2} \\ &= \frac{1}{2} b^2 - \frac{18}{35} b + \frac{2573}{14\,700}. \end{aligned}$$

From the above analysis, this expected consumer is maximal for $b = \hat{b}$ and $b = \hat{b}'$, where it is equal to:

$$S_B\left(\hat{b}\right) = S_B\left(\hat{b}'\right) = \frac{19321}{352800} \simeq 0.05.$$

– If $b' = \beta(b)$ ("Discriminating"), then for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67]$ and $b' \in \left[\hat{b}', \Delta\right] \simeq [0.67, 0.72]$):

$$\begin{aligned} S_D(b) &= \rho \frac{(1-b)^2}{2} + \rho' \frac{(1 - \beta(b))^2}{2} \\ &= \frac{37}{139} \frac{(1-b)^2}{2} + \frac{102}{139} \frac{\left(1 - \left(\frac{281}{130} - \frac{571}{130} b + \frac{42}{13} b^2\right)\right)^2}{2} \\ &= \frac{8996400 b^2 - 6468840 b + 1475501}{2349100} (1-b)^2. \end{aligned}$$

The following figure depicts expected consumer surplus in the bunching scenario (thin curve, for $b \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$) and the discriminating scenario (bold curve, for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67]$); it shows that discriminating is indeed optimal:

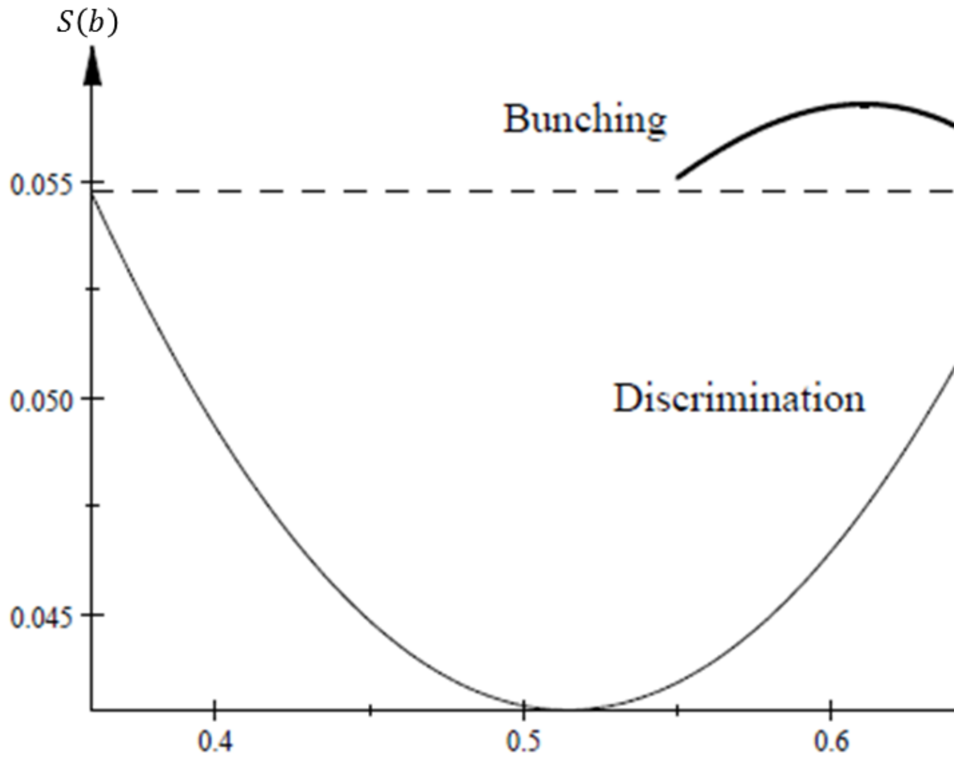

Figure 5: Consumer surplus

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To determine the socially optimal mechanism, it suffices to maximize $S_D(b)$, which yields:

$$S'_S(b) = \frac{359856}{23491}b^3 - \frac{3669246}{117455}b^2 + \frac{1800737}{90350}b - \frac{4709921}{1174550} = 0,$$

leading to

$$b^* = \frac{291}{560} + \frac{\sqrt{51}\sqrt{1189\,211}}{85\,680} \simeq 0.610\,54.$$