Merger Analysis with IIA Demand and Type Aggregation *

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Abstract

We propose a framework for merger analysis with multiproduct firms under *generalized* CES (GCES) and *generalized* multinomial logit (GMNL) demand. Despite allowing for arbitrary firm and product heterogeneity, we obtain the *type aggregation property*: All relevant information about a firm's product portfolio can be summarized in a uni-dimensional sufficient statistic, the firm's type. Indeed, GCES and GMNL are shown to be the only IIA demand systems giving rise to that property. Relative to standard CES and MNL demand, our generalization implies that prices, locally, can be strategic complements or substitutes, depending on the local behavior of the curvature of indirect utility. In turn, this distinction is shown to determine whether competition authorities should be more or less strict on mergers in more competitive industries. We obtain further results on the static and dynamic consumer surplus effects of mergers, the aggregate surplus and external effects of mergers, and on the relationship between the market power effect of a merger and the merger-induced change in the Herfindahl index.

Keywords: Multiproduct firms, aggregative game, oligopoly pricing, IIA demand, type aggregation, horizontal merger, Herfindahl index, generalized CES demand, generalized multinomial logit demand.

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1 Introduction

Although almost all mergers involve multiproduct firms selling differentiated products, much of the IO literature on horizontal merger analysis has focused on the homogeneous-goods Cournot model (Farrell and Shapiro, 1990; McAfee and Williams, 1992; Nocke and Whinston, 2010, 2013). We propose a framework for horizontal merger analysis with multiproducts firms under generalized CES (GCES) and generalized multinomial logit (GMNL) demand. Using this framework, we revisit classic questions such as the consumer surplus, aggregate surplus and external effects of mergers. We also relate the harm inflicted on consumers (and society at large) by a merger without synergies to the induced change in the Herfindahl index.

In Section 2, we introduce the oligopoly model with multiproduct firms and GCES/GMNL demand. In Section 3, we show that the multiproduct-firm pricing game is aggregative in that each firm's profit depends on the prices of its rivals only through a uni-dimensional aggregator, which is also a sufficient statistic for consumer surplus. Moreover, we also show that the type aggregation property holds: All relevant information on a firm's product portfolio (number of products, qualities, marginal costs) can be summarized by a uni-dimensional statistic, the firm's type. Relative to standard CES/MNL demand, our generalization to GCES/GMNL demand implies that prices, locally, can be strategic complements or substitutes, depending on the local behavior of the curvature of indirect utility. The equilibrium exists and is unique, and has intuitive comparative statics.

The type aggregation property permits a parsimonious modeling of mergers and merger-induced synergies. In principle, synergies could be reflected not only in changes in the marginal costs of the merger partners' products but also in quality improvements of pre-existing products and a change in the set of products offered by the merged firm. The type aggregation property allows us to summarize such multi-dimensional synergies in a uni-dimensional sufficient statistic, which corresponds to the ratio of the merged firm's type to the sum of the merger partners' pre-merger types.

In Section 4, we study the consumer surplus effects of mergers. We show that for a merger not to harm consumers requires synergies. Moreover, the required synergies are smaller in more competitive industries if prices are local strategic complements but larger if prices are local strategic substitutes. Next, we turn to a dynamic environment in which merger opportunities arise stochastically over time and merger proposals are endogenous. We show that a myopic merger approval policy (which approves a merger whenever it statically raises consumer surplus) is dynamically optimal, provided prices are (global) strategic complements. Instead, under local strategic substitutability, we show that a competition authority adopting a myopic merger approval policy may well have ex-post regret about previously approved mergers. Our results on the dynamic (non-)optimality of a myopic merger approval policy both extend and qualify the earlier results by Nocke and Whinston (2010), obtained in a homogeneous-goods Cournot model, and Nocke and Schutz (2019), obtained in a model with (nested) CES/MNL demand in which prices are always strategic complements.

In Section 5, we show that for a merger to raise aggregate surplus, the type of the merged firm has to be above a unique threshold. The required synergies, if any, are lower than for the merger not to harm consumers. Extending Farrell and Shapiro (1990), we also relate the external effect of a merger (on consumers and the non-merging outsiders) to the level and dispersion of pre-merger market shares.

In Section 6, we study the market power effect of a merger, defined as the impact on consumer (or aggregate) surplus in the absence of synergies. Using Taylor approximations, both around small market shares and around monopolistic competition conduct, we show that a merger's market power effect is positively related to the merger-induced change in the Herfindahl index. This complements Nocke and Whinston (2022) who relate the level of synergies required for a merger not to harm consumers to the change in the Herfindahl index.

In Section 7, we provide an if-and-only-if characterization of the class of IIA demand systems giving rise to the type aggregation property. We show that, essentially, only GCES and GMNL demand permit that property, justifying our focus on those demand systems.

Our paper contributes to two literatures. First, the paper advances the literature on oligopoly with multiproduct firms (e.g., Anderson and de Palma, 1992, 2006; Konovalov and Sándor, 2010; Gallego and Wang, 2014; Nocke and Schutz, 2018, 2023; Garrido, forthcoming) by providing necessary and sufficient conditions for the type aggregation property to obtain. We show that the demand systems consistent with type aggregation are generalizations of standard CES and MNL demand, which—through the local behavior of the curvature of the indirect utility function—allow prices, locally, to be strategic complements or substitutes.

Second, the paper advances the theoretical literature on the unilateral price effects of horizontal mergers (e.g., Williamson, 1968; Farrell and Shapiro, 1990; McAfee and Williams, 1992; Werden, 1996; Froeb and Werden, 1998; Goppelsroeder, Schinkel, and Tuinstra, 2008; Nocke and Whinston, 2010, 2013, 2022; Jaffe and Weyl, 2013; Johnson and Rhodes, 2021). We study the price and welfare effects of mergers in a framework with multiproduct firms, allowing for arbitrary firm and product heterogeneity, and rich forms of merger-induced synergies. This paper complements our earlier work, Nocke and Schutz (2019), by studying the role of strategic complementarity and substitutability for merger analysis in multiproduct-firm oligopoly with *generalized* CES/MNL demand.

2 The Multiproduct-Firm Oligopoly Model

2.1 Demand Side

Consider an industry with a finite set $\mathcal{N}$ of imperfectly substitutable products. We assume throughout that preferences are smooth and quasi-linear, and that the demand system has the IIA property. As shown in Anderson, Erkal, and Piccinin (2020) (see also Goldman and Uzawa, 1964), given the assumption on preferences, the IIA property is equivalent to consumer surplus taking the form

$$V(p) = \Psi\left(H^0 + \sum_{j\in\mathcal{N}} h_j(p_j)\right) \tag{1}$$

for some $H^0 \geq 0$ and some smooth, univariate functions Ψ and h_j ($j \in \mathcal{N}$). Applying Roy's identity, we obtain the demand for product $i \in \mathcal{N}$, $D_i(p) = -h_i'(p_i)\Psi'(H(p))$, where

$$H(p) \equiv H^0 + \sum_{j\in\mathcal{N}} h_j(p_j)$$

is the industry aggregator.

In Section VII in the Online Appendix to Nocke and Schutz (2018), we showed that the following conditions are necessary and sufficient for the above demand system to be derivable from multistage discrete/continuous choice:

Assumption 1. *The function $H \mapsto H\Psi'(H)$ is strictly positive and non-decreasing.*

Assumption 2. *Each function h_j is strictly positive, strictly decreasing, and log-convex.*

In the multistage discrete/continuous choice micro-foundation, a continuum of consumers are engaged in a sequential choice process. In the first stage, each consumer observes the vector of prices p and her idiosyncratic valuation for the stage-1 outside option, and decides whether to take up that outside option. In stage 2, consumers that turned down the outside option observe a vector of idiosyncratic, product-specific taste shocks, drawn i.i.d. from a type-I extreme value distribution, and chooses one of the products or the stage-2 outside option. In stage 3, consumers that chose product i ($i \in \mathcal{N}$) decide how many units of that product to purchase.

With this micro-foundation, $\log h_j(p_j)$ and $\log H^0$ are the mean utilities of product j and the stage-2 outside option, respectively, whereas the function Ψ embodies the distribution of the value of the stage-1 outside option in the population. Under optimal consumer behavior, a mass $H(p)\Psi'(H(p))$ of consumers turn down the stage-1 outside option, each consumer that remains chooses product i with probability $h_i(p_i)/H(p)$ in stage 2, and the stage-3, conditional demand for product i is $-h_i'(p_i)/h_i(p_i)$ by Roy's identity. Taking the product of these three terms yields the total demand for product i, $-h_i'(p_i)\Psi'(H(p))$, which coincides with the above expression for $D_i(p)$.

We assume throughout that Assumption 1 holds. Moreover, we significantly strengthen Assumption 2 as follows:

Assumption 3. *Either (i) for every j, $h_j(p_j) = a_j p_j^{1-\sigma}$ for some parameters $a_j > 0$ and $\sigma > 1$; or (ii) for every j, $h_j(p_j) = \exp \frac{a_j - p_j}{\lambda}$ for some parameters a_j and $\lambda > 0$.*

As we show in Section 7, this stronger version of the assumption is necessary for the type-aggregation property (see Section 3.1) to obtain. Note that if Ψ is (a positive affine transformation of) the logarithm, the demand system is of the CES type when part (i) of Assumption 3 holds and of the MNL type when part (ii) of Assumption 3 holds. Thus, we will henceforth say that demand is *generalized* CES (GCES) when part (i) holds and *generalized* MNL (GMNL) when part (ii) holds.

For every H, let $\epsilon(H) \equiv H(-\Psi''(H))/\Psi'(H)$ denote the curvature of Ψ. We make the following additional assumption:

Assumption 4. *The function ϵ is strictly positive and non-decreasing.*

The first part of the assumption (which boils down to $\Psi'' < 0$) says that products are always substitutes.[1] The second part of the assumption is a technical requirement that, together with the other assumptions, implies that any multiproduct-firm pricing game has a unique Nash equilibrium by Theorem III in the Online Appendix to Nocke and Schutz (2018). Note that Assumptions 1 and 4 imply that $\epsilon(H) \in (0, 1]$ for every H.

A reformulation of Assumption 4. Consider the multistage discrete/continuous choice micro-foundation sketched above. To simplify the exposition, assume that there is a unit mass of consumers and that each consumer's valuation for the stage-1 outside option is zero. At the beginning of stage 1, each consumer observes the realization of an idiosyncratic taste shock ξ, drawn from some smooth cumulative distribution function F. A consumer that turns down the stage-1 outside option expect to receive a utility of ξ plus the expected value of behaving optimally in stage 2. The properties of the type-I extreme value distribution imply that the latter is equal to $\log H$. Thus, a consumer optimally turns down the stage-1 outside option if and only if $\log H + \xi \geq 0$, which happens with probability

$$1 - F(-\log H) = H\Psi'(H).$$

Taking the logarithmic derivative yields:

$$\frac{F'(-\log H)}{1 - F(-\log H)} = 1 - \epsilon(H). \tag{2}$$

[1]To see this, note that, for $i \neq j$, $\partial D_i/\partial p_j = -h_i' h_j' \Psi''$, which has the same sign as $-\Psi''$.

In words, $1-\epsilon$ corresponds to the hazard rate of F in the discrete/continuous choice micro-foundation. Assumption 4 can therefore be rephrased as follows: The hazard rate of F is non-decreasing (so that ϵ is non-decreasing) and strictly less than 1 (so that $\epsilon > 0$).

2.2 Supply Side

We partition the set of products, $\mathcal{N}$, into a set of firms, $\mathcal{F}$. That is, each firm f has exclusive property rights over the production of all products in set f. We assume throughout that there are at least two firms, $|\mathcal{F}| \geq 2$. Each product j has a constant unit cost of production, $c_j > 0$. The profit of firm $f \in \mathcal{F}$ is thus equal to

$$\Pi^f(p) = \sum_{j \in f} (p_j - c_j)(-h_j'(p_j))\Psi'(H(p)). \tag{3}$$

Firms compete by simultaneously setting the prices of all of their products; the equilibrium concept is Nash equilibrium. As welfare measures, we will use both consumer surplus (CS $= \Psi(H)$) and aggregate surplus (AS $= \Psi(H) + \sum_{f \in \mathcal{F}} \Pi^f$).

Firm-level shares will play an important role in our analysis. In terms of the multistage discrete/continuous choice micro-foundation outlined in the previous subsection, the relevant concept for us will be the market share of firm f among the consumers who did not take up the stage-1 outside option:

$$s^f = \sum_{j \in f} \frac{h_j(p_j)}{H}.$$

Thus, s^f corresponds to the probability that a consumer who turned down the stage-1 outside option chooses one of firm f's products. Moreover, it is easily verified that s^f is firm f's output share under GMNL demand and firm f's revenue share under GCES demand—again, this does not account for consumers who took the stage-1 outside option. The market share of the stage-2 outside option is H^0/H.

3 Equilibrium Analysis of the Oligopoly Model

Given Assumptions 1, 3, and 4, Theorem III in the Online Appendix to Nocke and Schutz (2018) implies that first-order conditions are necessary and sufficient for optimality and that the multiproduct-firm pricing game has a unique equilibrium, which can be characterized using an aggregative-games approach. In this section, we establish the type-aggregation property, derive the firm's fitting-in functions, and characterize the unique equilibrium (Section 3.1), establish the key properties of firms' fitting-in functions (Section 3.2), and use them to perform comparative statics that will be useful for the merger analysis (Section 3.3).

3.1 Fitting-in Functions, Type Aggregation, and Equilibrium Characterization

The first-order condition for product i sold by firm f can be written as:

$$-h_i'(p_i)\Psi'(H)\left[1-(p_i-c_i)\frac{h_i''(p_i)}{-h_i'(p_i)}-\frac{\Psi''(H)}{\Psi'(H)}\sum_{j\in f}(p_j-c_j)(-h_j'(p_j))\right]=0,$$

which simplifies to

$$(p_i-c_i)\frac{h_i''(p_i)}{-h_i'(p_i)}=1+\frac{\epsilon(H)}{H}\sum_{j\in f}(p_j-c_j)(-h_j'(p_j)). \tag{4}$$

Nocke and Schutz (2018) refer to the left-hand side of equation (4) as the ι-markup on product i. As the right-hand side of that equation is independent of the identity of i, it follows that firm f charges the same ι-markup, call it μ^f, on all of the products in its portfolio—the common ι-markup property. Under GCES demand, the left-hand side of equation (4) simplifies to $\sigma(p_i-c_i)/p_i$, and so the common ι-markup property means that firm f charges the same Lerner index on all its products. Under GMNL demand, the left-hand side is $(p_i-c_i)/\lambda$ and firm f thus charges the same absolute markup on all its products.

Let us now use the common ι-markup property to simplify the second term on the right-hand side of equation (4). Setting α equal to $(\sigma-1)/\sigma$ under GCES demand and to 1 under GMNL demand, we have:

$$\begin{aligned}\frac{1}{H}\sum_{j\in f}(p_j-c_j)(-h_j'(p_j)) &= \frac{1}{H}\sum_{j\in f}\overbrace{(p_j-c_j)\frac{h_j''(p_j)}{-h_j'(p_j)}}^{=\mu^f}\overbrace{\frac{h_j'(p_j)^2}{h_j''(p_j)}}^{=\alpha h_j(p_j)}\\ &= \alpha\mu^f\frac{1}{H}\sum_{j\in f}h_j(p_j)\\ &= \alpha\mu^f s^f,\end{aligned} \tag{5}$$

Thus, equation (4) simplifies to

$$\mu^f=\frac{1}{1-\alpha\epsilon(H)s^f}. \tag{6}$$

Moreover, equation (5) also implies that, at firm f's optimum:

$$\Pi^f=\alpha\mu^f s^f H\Psi'(H). \tag{7}$$

Next, we use the common ι-markup property to express s^f as a function of μ^f. Under GCES demand, we have:

$$s^f=\frac{1}{H}\sum_{j\in f}h_j(p_j)=\frac{1}{H}\sum_{j\in f}a_j\left(\frac{\sigma}{\sigma-\mu^f}c_j\right)^{1-\sigma}$$

$$= \frac{1}{H} \underbrace{\left(1-(1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}}}_{\equiv \upsilon(\mu^f)} \underbrace{\sum_{j \in f} a_j c_j^{1-\sigma}}_{\equiv T^f} . \quad (8)$$

Similarly, under GMNL demand we obtain:

$$s^f = \frac{1}{H} \sum_{j \in f} h_j(p_j) = \frac{1}{H} \sum_{j \in f} \exp \frac{a_j - c_j - (p_j - c_j)}{\lambda}$$
$$= \frac{1}{H} \underbrace{e^{-\mu^f}}_{\equiv \upsilon(\mu^f)} \underbrace{\sum_{j \in f} \exp \frac{a_j - c_j}{\lambda}}_{\equiv T^f} . \quad (9)$$

Thus, regardless of whether demand is of the GCES or GMNL type, the market share of firm f can be written as

$$s^f = \frac{T^f}{H} \upsilon(\mu^f), \quad (10)$$

where υ is a strictly decreasing function and T^f, firm f's type, is a sufficient statistic that summarizes all the relevant information about firm f's product portfolio—the *type-aggregation property.* That sufficient statistic has the same interpretation (and, in fact, the same expression) as in Nocke and Schutz (2018, 2019): T^f will be higher if firm f supplies more products, delivers higher quality, or has lower costs; it is thus a measure of how good firm f is.

The above analysis implies that the pair (μ^f, s^f) must jointly solve equations (6) and (10). It is easily seen that this system has a unique solution:

Lemma 1. *The system of equations* (6) *and* (10) *has a unique solution in* (μ^f, s^f) *in the domain* $\left(1, \frac{1}{1-\alpha}\right) \times \left(0, \frac{1}{\epsilon(H)}\right)$.[2] *The solution,* $\left(m(H,T^f), S(H,T^f)\right)$*, is smooth. The function* S *is strictly decreasing in* H*, and there exists a cutoff* $\underline{H}(T^f) \geq 0$ *such that* $S(H,T^f) < 1$ *if and only if* $H > \underline{H}(T^f)$*. Moreover,* $\lim_{H \to \underline{H}(T^f)} S(H,T^f) = 1$ *and* $\lim_{H \to \infty} S(H,T^f) = 0$.

Proof. Existence, uniqueness, and monotonicity were established in the Online Appendix to Nocke and Schutz (2018). The result that $\lim_{H \to \infty} S(H,T^f) = 0$ follows immediately as $S(H,T^f) \leq \upsilon(1) T^f / H$. This, together with the monotonicity of $S(\cdot, T^f)$, implies the existence of the cutoff $\underline{H}(T^f)$.

If $\underline{H}(T^f) > 0$, then the result that $\lim_{H \to \underline{H}(T^f)} S(H,T^f) = 1$ follows immediately by the definition of $\underline{H}(T^f)$ and the continuity of $S(\cdot, T^f)$. Suppose instead that $\underline{H}(T^f) = 0$. As $S(\cdot, T^f)$ is bounded, we can take limits in equation (10) to obtain $\lim_{H \to 0} \upsilon(m(H,T^f)) = 0$, and thus $\lim_{H \to 0} m(H,T^f) = 1/(1-\alpha)$. Rearranging terms in equation (6) yields:

$$S(H,T^f) = \left[1 - \frac{1}{m(H,T^f)}\right] \frac{1}{\alpha \epsilon(H)} \underset{H \to 0}{\longrightarrow} \frac{1}{\epsilon(0)} \geq 1.$$

[2] We adopt the convention that $1/(1-\alpha) = +\infty$ when $\alpha = 1$.

Since $S(H,T^f) < 1$ for every H, it follows that $\lim_{H\to 0} S(H,T^f) = 1$. □

In the following, we refer to $m(H,T^f)$ and $S(H,T^f)$ as the markup and market-share fitting-in functions. The interpretation is that $S(H,T^f)$ is the share of the aggregator that is optimally produced by firm f when the aggregator level is H; $m(H,T^f)$ is the associated optimal ι-markup. Using equation (7), we can also define the profit fitting-in function:

$$\begin{aligned}\pi(H,T^f) &= \alpha m(H,T^f)S(H,T^f)H\Psi'(H)\\ &= \frac{\alpha S(H,T^f)}{1-\alpha\epsilon(H)S(H,T^f)}H\Psi'(H),\end{aligned} \tag{11}$$

where we have used equation (6) to obtain the second line. It is also sometimes convenient to work with the profit-per-active-consumer fitting-in function:

$$\tilde{\pi}(H,T^f) \equiv \frac{\pi(H,T^f)}{H\Psi'(H)} = \frac{\alpha S(H,T^f)}{1-\alpha\epsilon(H)S(H,T^f)}, \tag{12}$$

which, in the multistage discrete/continuous choice micro-foundation, corresponds to firm f's profit normalized by the number of consumers who turned down the stage-1 outside option.

Finally, the equilibrium value of the aggregator is pinned down by the condition

$$\frac{H^0}{H} + \sum_{f\in\mathcal{F}} S(H,T^f) = 1, \tag{13}$$

i.e., market shares must add up to unity. Lemma 1 implies that the left-hand side of the above condition is continuous, strictly decreasing, strictly greater than 1 when H is small enough, and strictly lower than 1 when H is high enough. Hence, equation (13) has a unique solution and the pricing game has a unique equilibrium. Summarizing:

Proposition 1. *The multiproduct-firm pricing game has a unique equilibrium. The equilibrium aggregator level H^* is the unique solution of equation* (13). *In equilibrium, firm f sets a markup of $m(H^*,T^f)$, commands a market share of $S(H^*,T^f)$, and earns a profit of $\pi(H^*,T^f)$.*

3.2 Properties of Fitting-in Functions

In this section, we study the properties of fitting-in functions, which will later be useful for deriving comparative statics in the oligopoly model and studying the welfare effects of mergers. For the analysis that follows, it is useful to define

$$\zeta(H) \equiv 1 - \frac{H\epsilon'(H)}{\epsilon(H)}.$$

By Assumption 4, $\zeta(H) \leq 1$ for every H.

We have:

Lemma 2. *The following holds:*

(i) Fitting-in functions are increasing in type: The partial derivatives $\partial m/\partial T$, $\partial S/\partial T$, and $\partial \pi/\partial T$ are all strictly positive.

(ii) The market-share fitting-in function is decreasing in the aggregator level: $\partial S/\partial H < 0$. The profit fitting-in function has the same property when evaluated at an economically relevant aggregator-type pair: $\partial \pi(H,T)/\partial H < 0$ whenever $S(H,T) < 1$.

(iii) Whether the markup fitting-in function is locally increasing or decreasing in H depends on $\zeta(H)$: The partial derivative $\partial m/\partial H$ has the same sign as $-\zeta(H)$.

Proof. See Appendix A. □

Part (i) of the lemma says that, holding fixed the intensity of competition, a firm that has a higher type (i.e., a "better firm") sets a higher markup, commands a higher market share, and makes larger profits. Part (ii) of the lemma says that, holding fixed the type, a firm that operates in a more competitive environment (higher H) commands a lower market share and makes less profits.

Part (iii) is perhaps more surprising: A firm may respond to an increase in the intensity of competition by *raising* its markup; this arises whenever $\zeta(H) < 0$. To see the intuition, note that the absolute value of the price elasticity of demand for product i is given by

$$\left| \frac{d \log(-h_i'(p_i))}{d \log p_i} + p_i h_i'(p_i) \frac{\Psi''(H(p))}{\Psi'(H(p))} \right| = \left| \frac{d \log(-h_i'(p_i))}{d \log p_i} \right| - |p_i h_i'(p_i)| \frac{\epsilon(H(p))}{H(p)}.$$

Thus, whether the demand for product i becomes more or less elastic as p_j rises ($j \neq i$) depends on whether $p_j \mapsto \epsilon(H(p))/H(p)$ is locally decreasing or increasing. This means that prices are local strategic complements (resp. substitutes) if $H \mapsto \epsilon(H)/H$ is locally decreasing (resp. increasing), which holds whenever $\zeta(H) > 0$ (resp. $\zeta(H) < 0$).

Recall from our discussion at the end of Section 2.1 that, in the multistage discrete/continuous choice micro-foundation, $1-\epsilon$ corresponds to the hazard rate of the distribution of the taste shock that a consumer receives when turning down the stage-1 outside option. Denoting that hazard rate function by $z(\cdot)$ and using equation (2), we have:

$$z(-\log H) = 1 - \epsilon(H).$$

Differentiating yields:

$$z'(-\log H) = H\epsilon'(H) = \epsilon(H)\left(1 - \zeta(H)\right),$$

which implies that

$$\zeta(H) = 1 - \frac{z'(-\log H)}{1 - z(-\log H)}.$$

It follows that $\zeta(H)$ is strictly negative if and only if the derivative of the hazard rate is (locally) sufficiently high. Intuitively, when that condition holds, an increase in p_j (and thus an increase in $-\log H$) gives rise to a large increase in the relative density of consumers who are indifferent between taking the stage-1 outside option and moving on to the second stage of the choice process, thus making the demand for product $i \neq j$ more elastic.

As discussed in the Online Appendix to Nocke and Schutz (2018), the assumptions made in Section 2 do not imply that ζ is everywhere non-negative.[3] In fact, it is easily checked that the function $\Psi(H) = \text{arsinh}(H)$ satisfies Assumptions 1 and 4, but the associated ζ is negative on the interval $(0,1)$ and positive on $(1,\infty)$. Thus, the resulting markup fitting-in function $m(\cdot, T)$ is strictly increasing on $(0,1)$ and strictly decreasing on $(1,\infty)$. Put differently, markups are strategic substitutes when prices are high and strategic complements when prices are low.

There are, of course, examples of functions Ψ that give rise to global strategic complementarity. For instance, if Ψ is the logarithm or a power function (i.e., $\Psi(H) = H^\beta$, $\beta \in (0,1)$), then it has constant curvature, and so $\zeta(H) = 1$ for every H.

3.3 Comparative Statics

Armed with the equilibrium characterization of Section 3.1 and the results of Section 3.2, we can study the equilibrium effects of an increase in the type of one of the firms.

The comparative statics of consumer surplus, markups, market shares, and profits are intuitive. As firm f's type increases, the left-hand side of equilibrium condition (13) increases. Since that left-hand side is decreasing in H, the equilibrium aggregator level must rise. Lemma 2–(ii) implies that the profits and market shares of rival firms decrease; whether their markups decrease or increase depends on whether prices are local strategic complements ($\zeta(H^*) > 0$) or substitutes ($\zeta(H^*) < 0$). As the market shares of rival firms and the outside option (H^0/H^*) decrease, firm f's market share must increase. This, combined with the fact that $\epsilon(H^*)$ increases weakly (Assumption 4), implies that firm f's ι-markup rises (see equation (6)). Combining this with equation (7) and the fact that $H\Psi'(H)$ is non-decreasing, we obtain that firm f's profit increases as well.

Summarizing:

Proposition 2. *Let $f, g \in \mathcal{F}$ with $f \neq g$. A small increase in T^f*

- *raises H^* and thus equilibrium consumer surplus;*
- *raises the equilibrium values of μ^f, s^f, and Π^f;*

[3] That being said, these assumptions do imply that ζ cannot be everywhere non-positive, as this would imply that $\epsilon(H) \notin (0,1]$ for H sufficiently high or sufficiently small.

- *lowers the equilibrium values of s^g and Π^g;*

- *lowers the equilibrium value of μ^g if $\zeta(H^*) > 0$, and raises it if the inequality is reversed.*

By contrast, the comparative statics of aggregate surplus are less clear-cut: An increase in T^f may or may not raise equilibrium aggregate surplus. Intuitively, although consumer surplus and firm f's profit increase, the profits of firm f's rivals fall by Proposition 2. Another way of seeing this is that the increase in T^f redistributes market shares towards firm f, which may amplify pre-existing inefficiencies if firm f was initially producing too much (relative to the other firms).[4] The following proposition provides sufficient conditions on the primitives of the demand system (i.e., α and ϵ) that ensure that, regardless of the original vector of types, an increase in one of the firms' types always raises equilibrium aggregate surplus:

Proposition 3. *Suppose that at least one of the following conditions holds:*

(i) $\zeta(H) = 1$ for every H. (That is, Ψ is either the logarithm or a power function.)

(ii) For every H, $\epsilon(H) \leq \frac{1}{1+\alpha}$.

(iii) For every H, $\zeta(H) \geq 0$ (prices are never strategic substitutes) and $\epsilon(H) \leq \frac{1+\alpha}{2\alpha+\alpha^2}$.

Then, equilibrium aggregate surplus is strictly increasing in the firms' types.

Proof. See Appendix B. □

We already know from Proposition 6 in Nocke and Schutz (2018) that, if Ψ is the logarithm, then equilibrium aggregate surplus is increasing in the firms' types. Part (i) of the proposition shows that this result extends to any Ψ with constant curvature. Part (ii) of the proposition says that, even if Ψ does not have constant curvature, aggregate surplus remains increasing in types provided ϵ and α are not too high. The condition on α and ϵ can be made less stringent if one is willing to assume that prices are always strategic complements, as in part (iii) of the proposition. For example, if α is smaller than approximately 0.62 (which corresponds to the GCES case with $\sigma \leq 2.61$), then the condition on ϵ in part (iii) is automatically satisfied since $\epsilon(H)$ is always less than 1 by Assumption 1.

The intuition for conditions (ii) and (iii) can be understood as follows. In Appendix B (see Lemma 5), we show that the derivative of aggregate surplus with respect to firm f's type has the same sign as

$$\tilde{\Gamma} \equiv 1 + (1-\epsilon)\sum_{g\in\mathcal{F}} \frac{\alpha s^g}{1-\alpha\epsilon s^g} + \frac{\alpha}{(1-\alpha\epsilon s^f)^2}\left[1 - \sum_{g\in\mathcal{F}} s^g\right] + (1-\zeta)\epsilon\sum_{g\in\mathcal{F}}\left(\frac{\alpha s^g}{1-\alpha\epsilon s^g}\right)^2$$

[4] For similar reasons, equilibrium aggregate surplus in the homogeneous-goods Cournot model is not a monotonic function of firms' marginal costs—see Lahiri and Ono (1988) and Zhao (2001).

$$- \sum_{g \in \mathcal{F}} \left| \frac{\partial S(H, T^g)}{\partial \log H} \right| \alpha s^g \left(\frac{1}{(1 - \alpha \epsilon s^g)^2} - \frac{1}{(1 - \alpha \epsilon s^f)^2} \right),$$

where $(s^g)_{g \in \mathcal{F}}$ is the equilibrium vector of market shares, and the functions ϵ, ζ, and S are evaluated at the equilibrium aggregator level.

All the terms on the first line are non-negative. The first term corresponds to the rise in consumer surplus triggered by the increase in T^f. The second term captures the fact that, as H increases, fewer consumers take up the stage-1 outside option, which raises producer surplus and thus aggregate surplus. (Recall that the mass of consumers who turn down the stage-1 outside option is $H\Psi'(H)$, which has elasticity $1 - \epsilon$.) The interpretation of the third term is that firm f steals market shares from the stage-2 outside option, which also increases aggregate surplus since firm f was initially producing too little (relative to the outside option). The fourth term says that, as H increases, ϵ rises (with elasticity $1 - \zeta \geq 0$) by Assumption 4, which mechanically raises firms' markups (see equation (6)).[5]

By contrast, the term on the second line in the definition of $\tilde{\Gamma}$ can be positive or negative. That term captures the fact that, as firm f's product portfolio improves, firm f steals market shares from its rivals. This market-share reallocation effect can be positive or negative, depending on whether firm f's markup, $1/(1 - \alpha \epsilon s^f)$, is higher or lower than its competitors'. An immediate observation is that, if firms were initially symmetric or $T^f \geq T^g$ (i.e., $s^f \geq s^g$) for every g, then the reallocation effect is non-negative. Instead, the effect is negative if $s^f < s^g$ for every g, as firm f is initially overproducing relative to its competitors.

Regardless of the sign of the market-share reallocation effect, its magnitude will be small if there is little dispersion in markups or the change in H has little effect on market shares. A sufficient condition for the former is that α and/or ϵ is small, so that ι-markups do not differ much from 1. A sufficient condition for the latter is that ζ is sufficiently high, as, in that case, firms respond to the increase in H by cutting their markups, thereby mitigating the decrease in their market shares. Parts (ii) and (iii) of Proposition 3 confirm that, under those conditions, the overall effect of an increase in T^f is to raise aggregate surplus.

3.4 Modeling Mergers

As in Nocke and Schutz (2019), the type-aggregation property permits a parsimonious modeling of mergers. Consider a merger between the firms $\mathcal{M} \subsetneq \mathcal{F}$, and let $\mathcal{O} \equiv \mathcal{F} \setminus \mathcal{M}$ be the set of non-merging firms—the outsiders. We assume throughout that the merger does not directly affect the outsiders. That is, each firm $f \in \mathcal{O}$ continues to supply the same products, with the same qualities and costs, post-merger, which implies that T^f remains unchanged.

[5] This is only a partial effect, which does not account for the fact that the market share of firm $g \neq f$ decreases, thus potentially resulting in a lower markup—see Proposition 2.

By contrast, the merger may affect the merging firms' set of products: New products may be added; some products may be dropped; the qualities and costs of pre-existing products may change. By virtue of the type-aggregation property, we do not need to keep track of each of those individual changes, as all that matters is the type of the merged firm, T^M.

A special case of interest arises when the merger does not involve any synergies, in the sense that the merged firm supplies the same products, with the same qualities and costs, as the merger partners did pre-merger. In that case, the type of the merged firm is given by $T^M = \sum_{f \in \mathcal{M}} T^f$. We say that the merger involves synergies if $T^M > \sum_{f \in \mathcal{M}} T^f$.

4 Consumer Surplus Effects of Mergers

In this section, we study the consumer surplus effects of mergers. We provide a static analysis in Section 4.1 and a dynamic one with endogenous mergers in Section 4.2.

4.1 Static Analysis

Consider a merger M between the firms in $\mathcal{M} \subsetneq F$, and let T^M be the merged firm's type. Let H^* and $\overline{H}^*$ denote the pre- and post-merger equilibrium aggregator levels, respectively. We say that the merger is CS-neutral if it does not affect the equilibrium aggregator level, $H^* = \overline{H}^*$; it is CS-increasing if $\overline{H}^* > H^*$, and CS-decreasing if the inequality is reversed.

Suppose first that the merger is CS-neutral. Then, the merger affects neither the outsiders' nor the outside option's market share. Since market shares must add up to unity, it follows that the market share of the merged firm must be equal to the sum of the pre-merger market shares of the merger partners:

$$S(H^*, T^M) = \sum_{f \in \mathcal{M}} S(H^*, T^f). \tag{14}$$

Conversely, if condition (14) holds, then H^* continues to be a solution to equation (13) post-merger, and so the merger is CS-neutral.

As $S(H, \cdot)$ is continuous and strictly increasing (see Lemma 2) and has full range (see Lemma 3 in Appendix A), there exists a unique cutoff type $\widehat{T}^M$ such that condition (14) holds if and only if $T^M = \widehat{T}^M$. Moreover, since the post-merger equilibrium value of the aggregator is strictly increasing in T^M (Proposition 2), the merger is CS-increasing if $T^M > \widehat{T}^M$ and CS-decreasing if the inequality is reversed.

Next, we show that a CS-nondecreasing merger must involve synergies. As $S(H, \cdot)$ is strictly sub-additive (see again Lemma 3), we have that, at $T^M = \sum_{f \in \mathcal{M}} T^f$, i.e., if there are no synergies, the right-hand side of equation (14) strictly exceeds the left-hand side. Since $S(H, \cdot)$ is increasing, it follows that $\widehat{T}^M > \sum_{f \in \mathcal{M}} T^f$.

Finally, let us show that a CS-nondecreasing merger is privately profitable, in the sense that it strictly raises the equilibrium joint profits of the merger partners. Suppose first that the merger is CS-neutral, $T^M = \widehat{T}^M$. Then, the market share of the merged firm, s^M, is equal to the sum of the market shares of the merger partners, $\sum_{f \in \mathcal{M}} s^f$. Hence, $s^M > s^f$ for every merger partner f. Combining this with equation (6) and the fact that the merger does not affect the equilibrium H, we obtain that the markup of the merged firm, μ^M, strictly exceeds the markup μ^f of every merger partner f. Therefore, using equation (7),

$$\Pi^M = \alpha \mu^M s^M H^* \Psi'(H^*) = \alpha \mu^M \sum_{f \in \mathcal{M}} s^f H^* \Psi'(H^*) > \alpha \sum_{f \in \mathcal{M}} \mu^f s^f H^* \Psi'(H^*) = \sum_{f \in \mathcal{M}} \Pi^f.$$

Hence, a CS-neutral merger is profitable. Moreover, by Proposition 2, the merged firm makes larger profits when its type is $T^M > \widehat{T}^M$ than when its type is $\widehat{T}^M$. A CS-increasing merger is therefore even more profitable than a CS-neutral one.

Summing up:

Proposition 4. *For a merger among the firms in $\mathcal{M}$, there exists a unique cutoff type $\widehat{T}^M > \sum_{f \in \mathcal{M}} T^f$ such that the merger is CS-neutral if $T^M = \widehat{T}^M$, CS-decreasing if $T^M < \widehat{T}^M$, and CS-increasing if $T^M > \widehat{T}^M$. Moreover, merger M is privately profitable whenever it is CS-nondecreasing.*

Next, we turn to the comparative statics of the cutoff type:

Proposition 5. *Consider a merger M among the firms in $\mathcal{M}$, and let H^* be the pre-merger level of the aggregator. The post-merger cutoff type $\widehat{T}^M$ is locally strictly decreasing in H^* if $\zeta(H^*) > 0$, and locally strictly increasing if the inequality is reversed.*

Proof. Totally differentiating equation (14) yields:

$$\begin{aligned}
\frac{S(H^*, \widehat{T}^M)}{\partial T} d\widehat{T}^M &= \left[-\frac{S(H^*, \widehat{T}^M)}{\partial H} + \sum_{f \in \mathcal{M}} \frac{\partial S(H^*, T^f)}{\partial H} \right] dH^* \\
&= \left[-S(H^*, \widehat{T}^M) \frac{\partial \log S(H^*, \widehat{T}^M)}{\partial H} + \sum_{f \in \mathcal{M}} S(H^*, T^f) \frac{\partial \log S(H^*, T^f)}{\partial H} \right] dH^* \\
&= \sum_{f \in \mathcal{M}} S(H^*, T^f) \left[\frac{\partial \log S(H^*, T^f)}{\partial H} - \frac{\partial \log S(H^*, \widehat{T}^M)}{\partial H} \right] dH^*,
\end{aligned}$$

where the last line follows by the definition of $\widehat{T}^M$. Since $T^f < \widehat{T}^M$ for every f, Lemma 4 in Appendix A implies that the term inside square brackets on the last line has the same sign as $-\zeta(H^*)$. □

Thus, if prices are local strategic complements, then the synergies required to make a merger CS-nondecreasing are smaller the more competitive is the market pre-merger. If instead prices are local strategic substitutes, then we have the surprising result that an antitrust authority with a consumer surplus standard should require *larger* synergies to approve a merger that takes place in a more competitive industry.

To understand the intuition, consider a merger between two identical single-product firms. Let c denote the merger partners' constant unit cost of production, p^* their pre-merger equilibrium price, and d the diversion ratio between the two products (evaluated at pre-merger equilibrium prices). Suppose that the merger-induced synergies only materialize through a symmetric reduction in the unit cost of production. As shown in Werden (1996), the marginal cost reduction that would make this merger CS-neutral satisfies

$$c - \hat{c} = (p^* - c)\frac{d}{1-d}.$$

Intuitively, the merged firm has more of an incentive to raise the price of one its products if most consumers lost in the process will end up purchasing its other product (high d) and/or if the margin on that other product is high (high $p^* - c$)—larger synergies will then be required to make the merger CS-neutral. The question is therefore how an increase in the intensity of competition affects the pre-merger margin and diversion ratio.

By Lemma 2, the pre-merger margin is locally decreasing (resp. increasing) in H^* if $\zeta(H^*) > 0$ (resp. $\zeta(H^*) < 0$). As for the diversion ratio, straightforward calculations show that, in this symmetric setting,

$$d = \frac{\alpha\epsilon(H^*)s^*}{1-\alpha\epsilon(H^*)s^*} = \mu^* - 1,$$

where s^* and μ^* denote each merger partner's market share and ι-markup, respectively, and we have used equation (6) to obtain the second equality. Using again Lemma 2, we obtain that the diversion ratio is locally decreasing in H^* if $\zeta(H^*) > 0$ and locally increasing if $\zeta(H^*) < 0$. Hence, the derivative of $c - \hat{c}$ with respect to H^* has the same sign as $-\zeta(H^*)$. Proposition 5 generalizes this result to mergers between arbitrary sets of firms and arbitrary forms of merger-induced synergies.

4.2 Dynamic Analysis

If merger opportunities are not isolated events, the static analysis undertaken in the previous subsection may be inappropriate: Merger approval decisions in the current period may affect both the consumer-surplus effects of future mergers (and thus whether those mergers will be approved) and their profitability (and thus whether those mergers will be proposed). This raises the question whether a static merger policy, whereby a merger receives approval if and only if it is CS-nondecreasing given the current market structure, may be dynamically sub-optimal.

We study this question in the dynamic stochastic framework laid out in Section 3.2 of Nocke and Schutz (2019). (See Nocke and Whinston (2010) for the Cournot version of that framework.) We now provide a brief, informal description of the framework, and refer the reader to Nocke and Schutz (2019) for details.

Time is discrete and runs for a finite number of periods, and the initial state of the industry is as described in Section 2. The strategic players are the firms, which maximize the expected present discounted value of profits, and the antitrust authority, which maximizes the expected present discounted value of consumer surplus. There is a collection of potential mergers, each corresponding to a set of merger partners. Importantly, these mergers are disjoint, i.e., each firm can be part of at most one potential merger. In every period, one or several potential mergers can become feasible with some probability, in which case the types of the merged firms are drawn from some probability distributions. We assume that a merger that has become feasible but has not yet been consummated remains feasible, with the same merged firm's type, in all subsequent periods.

Within each period, there are three stages. In the first stage, mergers that have become feasible but have not yet been consummated can be proposed to the antitrust authority—we assume that a merger is proposed if and only if it is in the merger partners' joint interest. In the second stage, the antitrust authority decides which mergers to approve. In the third stage, firms compete in prices given the current market structure. All of those decisions are made under complete information. The equilibrium concept is subgame-perfect equilibrium.

We now show that, under the assumption that prices are global strategic complements ($\zeta > 0$), equilibrium behavior is surprisingly simple. Let the myopically CS-maximizing merger policy be the approval rule that, in each period, maximizes consumer surplus in that period, given current market structure and the set of proposed mergers. We have:

Proposition 6. *Assume that $\zeta(H) > 0$ for every H. There exists a subgame-perfect equilibrium in which the antitrust authority adopts the myopically CS-maximizing merger policy and firms propose all feasible mergers in each period after any history. The resulting outcome maximizes the present discounted value of consumer surplus, no matter what the realized sequence of feasible mergers. Moreover, every subgame-perfect equilibrium results in the same optimal level of consumer surplus in each period.*

Proof. Given Propositions 4 and 5, the analysis in Section VI in the Online Appendix to Nocke and Schutz (2019) readily extends to this setting. □

As in Nocke and Whinston (2010) and Nocke and Schutz (2019), the key insight underlying this result is the sign-preserving complementarity in the consumer surplus effects of mergers. Consider two disjoint mergers M_1 and M_2 and suppose that, given the current market structure, each merger is CS-nondecreasing in isolation. Assuming $\zeta > 0$, Proposition 5 implies that merger M_i remains CS-nondecreasing conditional on merger M_j taking place, since implementing merger M_j weakly raises H^* and thus weakly reduces the cutoff type for merger M_i. The same argument implies that if M_1 and M_2 are both CS-decreasing in isolation, then each remains CS-decreasing conditional on the other taking place.

By contrast, if $\zeta(H) < 0$ for some H, then the sign-preserving complementarity property no longer holds in general, and a counterexample can be constructed in which the myopically CS-maximizing merger policy is not dynamically optimal. Suppose indeed that $\zeta(\hat{H}) < 0$ for some $\hat{H}$. Let $\mathcal{F}$ be a finite set containing at least five elements, and fix an arbitrary vector of market shares $(s^f)_{f\in\mathcal{F}} \in (0,1)^{\mathcal{F}}$ such that $\sum_{f\in\mathcal{F}} s^f = 1$. By Lemmas 2 and 3, for every $f \in \mathcal{F}$, there exists a unique $T^f > 0$ such that $S(\hat{H}, T^f) = s^f$. Since the shares s^f add up to one, equilibrium condition (13) holds for the vector of types $(T^f)_{f\in\mathcal{F}}$ and the aggregator level $\hat{H}$, and so $\hat{H}$ is indeed the equilibrium aggregator level when the vector of types is $(T^f)_{f\in\mathcal{F}}$.

Next, consider two disjoint mergers M_1 and M_2, with merger partners $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, such that $\mathcal{M}_1 \cup \mathcal{M}_2 \subsetneq \mathcal{F}$. Choose $T^{M_1} > \widehat{T}^{M_1}(\hat{H})$ such that $\zeta(H) < 0$ for every $H \in [\hat{H}, H^1]$, where $H^1 > \hat{H}$ is the equilibrium aggregator level after merger M_1 has been implemented with type T^{M_1}. Note that such a T^{M_1} exists since ζ is continuous, the equilibrium aggregator level is continuous in the firms' types, and the post-merger equilibrium aggregator level strictly exceeds $\hat{H}$ whenever the merged firm's type is greater than $\widehat{T}^{M_1}(\hat{H})$ by Proposition 4. Next, consider merger M_2. By Proposition 5, we have that $\widehat{T}^{M_2}(H^1) > \widehat{T}^{M_2}(\hat{H})$. We can thus choose a $T^{M_2} \in \left(\widehat{T}^{M_2}(\hat{H}), \widehat{T}^{M_2}(H^1)\right)$ such that $H^2 \in (\hat{H}, H^1)$, where H^2 is the equilibrium aggregator level after merger M_2 has been implemented with type T^{M_2}. Again, the continuity of the equilibrium aggregator level in types and Proposition 4 ensure that such a T^{M_2} exists. It is clear that the sign-preserving complementarity property does not hold for mergers M_1 and M_2: Although each of those mergers is CS-increasing in isolation, merger M_2 is by construction CS-decreasing conditional on merger M_1 taking place.

Moreover, we also have that $H^1 > H^{12}$, where H^{12} is the equilibrium aggregator level after both M_1 and M_2 have been implemented. Hence, if M_1 and M_2 are both proposed to the antitrust authority, then only M_1 will be approved. Let us now embed those mergers into the dynamic framework described at the beginning of this subsection. Suppose that at time $t = 1$, merger M_2, and only this merger, becomes feasible, with probability 1 and type T^{M_2} as defined above. At time $t = 2$, merger M_1, and only this merger, becomes feasible with probability 1 and type T^{M_1}. No further merger becomes feasible thereafter. Assume that the discount factor is sufficiently close to 1 and that there are sufficiently many periods, so that the consumer surplus level from period $t = 3$ onward dwarfs that in periods 1 and

2. To focus on the antitrust authority's behavior, assume also that mergers M_1 and M_2 are always proposed when feasible. Then, since $H^1 > \max(\hat{H}, H^2, H^{12})$, the antitrust authority optimally blocks merger M_2 in period 1 and approves M_1 in period 2 (and keeps blocking M_2). Instead, the myopically CS-maximizing merger policy would approve M_2 in period 1, and approve M_1 in period 2 if and only if $H^{12} \geq H^2$. Thus, the myopically CS-maximizing merger policy is strictly sub-optimal.

5 Aggregate Surplus and External Effects of Mergers

Although most antitrust authorities around the world have adopted (something close to) a consumer-surplus standard, it seems worthwhile to study approval standards that put some weight on firms' profits. We analyze the aggregate-surplus effects of mergers in Section 5.1 and their external effects in Section 5.2.

5.1 Aggregate Surplus Effects

In this subsection, we assume that at least one of the conditions of Proposition 3 is satisfied, so that equilibrium aggregate surplus is a strictly increasing function of the firms' types.

Consider a merger M between the firms in $\mathcal{M} \subsetneq F$, and let T^M be the merged firm's type. If the merger is CS-neutral (i.e., $T^M = \widehat{T}^M$), then it is AS-increasing since it raises the profits of the merger partners (Proposition 4) while leaving consumer surplus and the outsiders' profits unchanged. If, instead, T^M is equal to zero, then the merger is equivalent to setting the types of all merger partners equal to zero, which must reduce aggregate surplus by Proposition 3. By continuity of equilibrium aggregate surplus in types, there exists a cutoff $\widetilde{T}^M \in (0, \widehat{T}^M)$ such that the merger is AS-neutral if $T^M = \widetilde{T}^M$. Proposition 3 implies that the merger is AS-increasing if $T^M > \widetilde{T}^M$ and AS-decreasing if the inequality is reversed.

Summing up:

Proposition 7. *Suppose that at least one of the conditions of Proposition 3 holds. For a merger among the firms in $\mathcal{M}$, there exists a unique $\widetilde{T}^M < \widehat{T}^M$ such that the merger is AS-neutral if $T^M = \widetilde{T}^M$, AS-decreasing if $T^M < \widetilde{T}^M$, and AS-increasing if $T^M > \widetilde{T}^M$.*

As discussed in Nocke and Schutz (2019), there is no counterpart to this result in Farrell and Shapiro (1990)'s classical analysis. The reason is that in the homogeneous-goods Cournot model, equilibrium aggregate surplus is never globally monotonic in the firms' marginal costs (Lahiri and Ono, 1988; Zhao, 2001).

The fact that $\widetilde{T}^M < \widehat{T}^M$ means that for a merger to be AS-increasing requires fewer synergies than for it to be CS-increasing. In fact, no synergies may be required at all—we refer the reader to Section 4.1 in Nocke and Schutz (2019) for simple examples where $\widetilde{T}^M$ is greater or lower than $\sum_{f\in\mathcal{F}} T^f$.

5.2 External Effects

Consider a merger M among the firms in $\mathcal{M}$, and let $\mathcal{O} = \mathcal{F} \setminus \mathcal{M}$ be the set of non-merging firms. The merger's external effect is defined as its impact on the sum of consumer surplus and the outsiders' profits. One reason for studying external effects is that, to the extent that a merger will be proposed only if it is in the merger partners' joint interest to do so, a positive external effect is a sufficient condition for a positive impact on aggregate surplus.

Define

$$\mathcal{E}(H) = \Psi(H) + \sum_{f\in\mathcal{O}} \pi(H, T^f).$$

Letting H^* and $\overline{H}^*$ denote the pre- and post-merger equilibrium value of the aggregator, the merger's external effect is thus

$$\Delta\mathcal{E} = \mathcal{E}(\overline{H}^*) - \mathcal{E}(H^*) = \int_{H^*}^{\overline{H}^*} \mathcal{E}'(H)dH.$$

Thus, $\mathcal{E}'(H)dH$ can be viewed as the external effect of an infinitesimal merger. In the following, we focus on CS-decreasing mergers ($dH < 0$, $\overline{H}^* < H^*$) to fix ideas, and let $\eta(H) \equiv -\mathcal{E}'(H)/\Psi'(H)$. An infinitesimal CS-decreasing merger has a positive external effect if and only if $\eta(H) > 0$. In Appendix C, we show that

$$\eta(H) = -1 + \sum_{g\in\mathcal{O}} \phi(s^g, \epsilon(H), \alpha) + (1 - \zeta(H)) \sum_{g\in\mathcal{O}} \chi(s^g, \epsilon(H), \alpha), \tag{15}$$

where $s^g = S(H, T^g)$,

$$\phi(s, \epsilon, \alpha) \equiv \frac{\alpha s}{1 - \alpha\epsilon s}\left[\frac{1 - \epsilon s}{1 - \epsilon s + \alpha(\epsilon s)^2} - (1 - \epsilon)\right],$$

$$\text{and } \chi(s, \epsilon, \alpha) \equiv \frac{(\alpha\epsilon s)^2 s}{(1 - \alpha\epsilon s)(1 - \epsilon s + \alpha(\epsilon s)^2)}.$$

The two sums on the right-hand side of equation (15) represent the increase in the outsiders' profits triggered by the infinitesimal decrease in H. The -1 term represents the decrease in consumer surplus. The external effect of the merger is positive if and only if the former outweighs the latter.

We now argue that, under certain conditions, an infinitesimal CS-decreasing merger is more likely to have a positive external effect if the outsiders' market shares are larger or more concentrated. We formalize the notions of "larger" and "more concentrated" market shares as in Nocke and Schutz (2019). A pre-merger industry structure among outsiders is a vector $(s^f)_{f\in\mathcal{O}}$, where $\mathcal{O}$ is a finite set, $s^f \in (0,1)$ for every $f \in \mathcal{O}$, and $\sum_{f\in\mathcal{O}} s^f < 1$. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ be two such outsider industry structures. Outsiders have *larger market shares* under s than under s', denoted $s \geq_1 s'$, if there exists a one-to-one mapping $\vartheta : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $s^{\vartheta(f)} \geq s'^f$ for every $f \in \mathcal{O}'$. Outsiders' market shares are *more concentrated* under outsider industry structure s than under s', denoted $s \geq_2 s'$, if s and s' have the same length and the distribution of s is a mean-preserving spread of the distribution of s' (Rothschild and Stiglitz, 1970).

Exploiting the convexity and monotonicity properties of the functions $\phi(\cdot, \epsilon, \alpha)$ and $\chi(\cdot, \epsilon, \alpha)$, we prove the following proposition:

Proposition 8. *Consider two infinitesimal CS-decreasing mergers, M and M', with pre-merger industry structures $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$, and pre-merger aggregators H and H'. Suppose that $\epsilon(H) = \epsilon(H')$, $\zeta(H) = \zeta(H')$, and at least one of the following conditions holds:*

(i) $s \geq_1 s'$ and $s^f \leq 3/5$ for every $f \in \mathcal{O}$.

(ii) $s \geq_2 s'$, $s^f \leq 1/5$ for every $f \in \mathcal{O}$, and $s'^f \leq 1/5$ for every $f \in \mathcal{O}'$.

If merger M' has a positive external effect, then so does merger M.

Proof. See Appendix C. □

Note that one of the assumptions of Proposition 8 is that $\epsilon(H) = \epsilon(H')$ and $\zeta(H) = \zeta(H')$. This assumption is automatically satisfied if Ψ is the logarithm or a power function, as ϵ and ζ are then constant functions. If ϵ is strictly increasing, then the assumption reduces to $H = H'$.

To understand the intuition behind Proposition 8, let us study how an infinitesimal decrease in H affects an outsider's profit fitting-in function. Combining equations (7) and (12) with equation (10) yields

$$\pi(H, T^f) = \alpha T^f m(H, T^f) \upsilon(m(H, T^f)) \Psi'(H)$$
$$\text{and } \tilde{\pi}(H, T^f) = \alpha \frac{T^f}{H} m(H, T^f) \upsilon(m(H, T^f)),$$

where the function υ was defined in Section 3.1 (see equations (8) and (9)). Differentiating the profit fitting-in function with respect to H and normalizing by the induced change in consumer surplus, we obtain:

$$\frac{d\pi^f}{\Psi'(H)} = \epsilon(H)\tilde{\pi}^f |dH| - \tilde{\pi}^f \frac{\partial \log m(H, T^f)}{\partial \log H} \left. \frac{\partial \log(\mu \upsilon(\mu))}{\partial \log \mu} \right|_{\mu = m(H, T^f)} |dH|. \tag{16}$$

The positive effect of the infinitesimal CS-decreasing merger on the outsiders' profits can therefore be decomposed as follows. First, holding fixed the outsiders' markups, the merger raises the profit of each outsider f by $\epsilon(H)\tilde{\pi}^f|dH|$. Intuitively, as H decreases, the merger partners lose some consumers. A share $1-\epsilon$ of those consumers is drawn towards the stage-1 outside option, while the remaining share ϵ starts purchasing from the outsiders. Due to the IIA property, those latter consumers are allocated to the firms in proportion to their market shares. Hence, the "direct" effect on outsiders' joint profit is proportional to their joint profit per active consumer. Second, outsiders respond by increasing their markups.

As the outsiders' aggregate profit per active consumer is increasing and convex in outsiders' market shares (see equation (12)), the first, direct effect is larger when those market shares are larger or more concentrated. How the distribution of outsiders' market shares affects the second, indirect effect is less clear. The proposition shows that the intuition that can be gleaned by focusing on the direct effect is valid provided the outsiders' market shares are not too large.

We now turn to the external effects of non-infinitesimal mergers. We begin by deriving conditions under which any CS-decreasing merger has a negative external effect:

Proposition 9. *Suppose at least one of the following holds:*

(i) Ψ is the logarithm or a power function, and α and/or ϵ is lower than 0.8.[6]

(ii) Prices are always weakly strategic complements ($\zeta \geq 0$) and $\alpha\epsilon(H) \leq 1/2$ for every H.

Then, any CS-decreasing merger has a negative external effect.

Proof. See Appendix C. □

A CS-decreasing merger has a negative external effect if the loss in consumer surplus exceeds the gain in outsiders' profits. To understand the intuition for the sufficient conditions in Proposition 9, let us thus return to equation (16), which decomposes the effect of an infinitesimal CS-decreasing merger on outsider f's profit into a direct effect (firm f's market share increases) and an indirect effect (firm f adjusts its markup). Under strategic complementarity, the indirect effect is negative, as $\partial m/\partial H < 0$ and $\partial(\mu\upsilon(\mu))/\partial\mu < 0$.[7] Moreover, the direct effect is proportional to $\epsilon(H)\tilde{\pi}^f$. It will therefore be small if ϵ is small (so that many consumers are lost to the stage-1 outside option when H falls) and/or profit per active consumer is small (so that firm f makes little profit on the additional consumers it gains).

[6] See Appendix C for tighter upper bounds on α and ϵ.

[7] This holds since oligopolistic markups are always above those of monopolistically competitive firms (that perceive H as fixed, and thus set an ι-markup of 1), so any further increase must reduce profit for a fixed H. See Section 6.2 for more details on the monopolistic competition version of the model.

A sufficient condition for the latter is that α and/or ϵ is small, as firms then tend to set low ι-markups.

We close this subsection by providing a sufficient condition for a CS-decreasing merger to have a positive external effect in the case where Ψ has constant curvature. Suppose indeed that Ψ is the logarithm or a power function. Assume that, at the pre-merger market-share vector,

$$-1+\sum \frac{\alpha s^f}{1-\alpha\epsilon s^f}\left[\frac{1-\epsilon s^f}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}-(1-\epsilon)\right]>0, \tag{17}$$

i.e., an infinitesimal CS-decreasing merger at H^* has a positive external effect. Then, along the sequence of infinitesimal mergers from H^* to $\overline{H}^*$, the market shares of all outsiders increase. Therefore, by Proposition 8 (and as long as no outsider reaches a market share larger than 60%), the external effect of each of those infinitesimal mergers is positive. It follows that the external effect of the merger is positive. Importantly, checking whether condition (17) holds only requires knowledge of the outsiders' *pre-merger* market shares.

6 The Herfindahl Index and the Welfare Effects of Mergers

In this section, we show that the merger-induced, naively-computed change in the Herfindahl index provides a suitable approximation to the market power effect of a merger, defined as the impact that the merger would have on consumer surplus (resp. aggregate surplus) if it involved no synergies. We provide approximation results around small market shares in Section 6.1 and around monopolistic competition conduct in Section 6.2.

6.1 Approximation around Small Market Shares

We assume throughout this subsection that consumers have access to a stage-2 outside option, i.e., $H^0>0$. Suppose the equilibrium vector of market shares, $s=(s^f)_{f\in\mathcal{F}}$, is known. As the firms' market shares and the market share of the outside option must add up to unity, the equilibrium aggregator level can be written as a function of s, as

$$H^*(s)=\frac{H^0}{1-\sum_{f\in\mathcal{F}} s^f}.$$

Thus, equilibrium consumer surplus can also be written as a function of s, as $\text{CS}(s)=\Psi(H^*(s))$. The analysis in Section 3.1 implies that producer surplus is given by

$$\text{PS}(s)=\sum_{f\in\mathcal{F}}\frac{\alpha s^f}{1-\alpha\epsilon(H^*(s))s^f}H^*(s)\Psi'\left(H^*(s)\right).$$

We also obtain aggregate surplus: $\mathrm{AS}(s) = \mathrm{CS}(s) + \mathrm{PS}(s)$.

For any vector of market shares s (between 0 and 1, adding up to strictly less than 1), there exists a unique vector of types, $T = \left(T^f(s)\right)_{f\in\mathcal{F}}$ such that the resulting equilibrium market-share vector is s. Firm f's type, $T^f(s)$, is the unique solution of equation

$$s^f = S\left(H^*(s), T^f(s)\right). \tag{18}$$

(Existence and uniqueness of the solution follow immediately by Lemmas 2 and 3).

Now, consider a merger M among the firms in $\mathcal{M} \subsetneq \mathcal{F}$. The set of firms post-merger is $\overline{\mathcal{F}} = \mathcal{O} \cup M$. We define the market power effect of merger M as the impact it would have on equilibrium consumer surplus (resp. aggregate surplus) if there were no synergies. Without synergies, the post-merger vector of types is $\bar{T}(s) = (\bar{T}^f(s))_{f\in\overline{\mathcal{F}}}$, where $\bar{T}^f(s) = T^f(s)$ if $f \in \mathcal{O}$ and $\bar{T}^f(s) = \sum_{g\in\mathcal{M}} T^g(s)$ if $f = M$. This vector of types $\bar{T}(s)$ gives rise to a post-merger equilibrium—the aggregator and the market-share vector are denoted $\overline{H}^*(s)$ and $\bar{s}(s)$, respectively.

The objects of interest are $\Delta\,\mathrm{CS}(s) \equiv \mathrm{CS}(\bar{s}(s)) - \mathrm{CS}(s)$ and $\Delta\,\mathrm{AS}(s) \equiv \mathrm{AS}(\bar{s}(s)) - \mathrm{AS}(s)$, which we seek to approximate at the second order in the neighborhood of $s = 0$. The merger-induced, naively computed change in the Herfindahl index is:

$$\Delta\,\mathrm{HHI}(s) \equiv \left(\sum_{f\in\mathcal{M}} s^f\right)^2 - \sum_{f\in\mathcal{M}} (s^f)^2 = \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g.$$

We have:

Proposition 10. *At the second order in the neighborhood of $s = 0$, the market power effect of the merger is:*[8]

$$\Delta\,CS(s) = -H^0\Psi'(H^0)\alpha\epsilon(H^0)\Delta\,HHI(s) + o(\|s\|^2),$$
$$\Delta\,AS(s) = -H^0\Psi'(H^0)\alpha\epsilon(H^0)\Delta\,HHI(s) + o(\|s\|^2).$$

Proof. See Appendix D. □

Thus, regardless of whether the metric used is consumer surplus or aggregate surplus, the market power effect of the merger is proportional to the merger-induced, naively-computed change in the Herfindahl index. The proportionality coefficient is equal to market size (i.e., $H^0\Psi'(H^0)$, the mass of consumers who turn down the stage-1 outside option) times the product of the elasticity measures α and $\epsilon(H^0)$.

[8] $o(\cdot)$ is Landau's little-o notation: $f(x) = o(g(x))$ in the neighborhood of $x = x^0$ if $f(x)/g(x) \underset{x\to x^0}{\longrightarrow} 0$.

6.2 Approximation around Monopolistic Competition Conduct

Throughout the paper, we have assumed that each firm fully internalizes its impact on the industry aggregator—"Bertrand-Nash" conduct. Under monopolistic competition, instead, firms make their pricing decisions taking the aggregator as given. Given this behavioral assumption, the first-order condition for product i sold by firm f (equation (4)) becomes

$$(p_i - c_i)\frac{h_i''(p_i)}{-h_i'(p_i)} = 1.$$

That is, each firm sets an ι-markup of 1 on all of its products. Under GCES demand, we obtain the familiar pricing formula $(p_i - c_i)/p_i = 1/\sigma$; under GMNL demand, we have $p_i - c_i = \lambda$. An immediate observation is that the market structure (i.e., the firm partition $\mathcal{F}$) does not affect equilibrium prices. This implies in particular that, under monopolistic competition, a merger without synergies affects neither consumer surplus nor aggregate surplus.

Let us now bridge the gap between monopolistic-competition and Bertrand-Nash conduct by introducing a conduct parameter, $\theta \in [0, 1]$, into the oligopoly model. Specifically, we assume that each firm believes that the impact of a small increase in p_i (for every $i \in \mathcal{N}$) on the aggregator is $\theta \partial H/\partial p_i$ instead of $\partial H/\partial p_i$. That is, firms internalize their impact on the aggregator only to some extent.[9] The special cases $\theta = 1$ and $\theta = 0$ correspond to Bertrand-Nash and monopolistic-competition conduct, respectively.

In Appendix E.1, we show that the oligopoly model augmented with a conduct parameter can still be solved using an aggregative games approach and that the type-aggregation property continues to hold, with the definition of types being independent of θ. The only difference is that the markup equation (equation (6)) becomes

$$\mu^f = \frac{1}{1 - \alpha\theta\epsilon(H)s^f}.$$

That is, firms set lower markups when they do not fully internalize their impact on the aggregator. This implies that all the fitting-in functions now depend on θ. The equilibrium aggregator level, $H^*(\theta)$, is still pinned down by the condition that market shares add up to unity. Applying the market-share fitting-in function gives firm f's equilibrium market share: $s^f(\theta) = S(H^*(\theta), T^f; \theta)$. Consumer surplus and aggregate surplus are then given by (see Appendix E.1):

$$\text{CS}(\theta) = \Psi(H^*(\theta)) \quad \text{and} \quad \text{AS}(\theta) = \Psi(H^*(\theta)) + \sum_{f \in \mathcal{F}} \frac{\alpha s^f(\theta)}{1 - \alpha\theta\epsilon(H^*(\theta))s^f(\theta)} H^*(\theta)\Psi'(H^*(\theta)).$$

[9]This way of modeling firm conduct is closely related to the classical approach with quantity setting and homogeneous products as surveyed in Bresnahan (1989).

Consider now a merger M between the firms in $\mathcal{M}$. Suppose the merger involves no synergies, $T^M = \sum_{f\in\mathcal{F}} T^f$, and let $\overline{\text{CS}}(\theta)$ and $\overline{\text{AS}}(\theta)$ denote consumer and aggregate surplus post-merger. The merger-induced, naively computed change in the Herfindahl index is denoted $\Delta\,\text{HHI}(\theta)$. We provide a linear approximation of the market power effect of the merger around monopolistic competition conduct:

Proposition 11. *In the neighborhood of $\theta = 0$, the market power effect of the merger is:*[10]

$$\overline{CS}(\theta) - CS(\theta) = -\alpha\epsilon(H^*)H^*\Psi'(H^*)\Delta\, HHI \times\theta + o(\theta),$$
$$\overline{AS}(\theta) - AS(\theta) = -\alpha\epsilon(H^*)H^*\Psi'(H^*)\Delta\, HHI\left(1 - \alpha\sum_{f\in\mathcal{F}} s^f\right)\theta + o(\theta).$$

Proof. See Appendix E.2. □

Thus, the market power effect of the merger is again proportional to the merger-induced, naively computed change in the Herfindahl index. When consumer surplus is used as measure of market performance, the proportionality coefficient is the same as in Proposition 10 (times the conduct parameter). When instead aggregate surplus is used, there is an additional term in the proportionality coefficient: $1 - \alpha\sum_f s^f$. That term corresponds to the increase in industry profit resulting from the merger. The reason why it is not present in Proposition 10 is that it would be third order.

7 Type Aggregation: A Complete Characterization

Two features of our class of multiproduct-firm pricing games have allowed us to provide a rich and yet tractable merger analysis. First, the pricing game is aggregative and the industry aggregator also pins down consumer surplus; thus, the pre- (and post-) merger state of the industry can be summarized into a uni-dimensional sufficient statistic. Second, the type-aggregation property holds; thus, merger-specific synergies can also be summarized into a uni-dimensional sufficient statistic. Anderson, Erkal, and Piccinin (2020) showed that the first property is equivalent to the indirect utility function being additively separable across products, i.e., as in equation (1). To obtain the type aggregation property, however, we strengthened this requirement by imposing that the h_j-functions take either the GCES or the GMNL form (Assumption 3). In this section, we show that this stronger requirement is (almost) necessary for type aggregation.

The section is organized as follows. We begin by reviewing Nocke and Schutz (2018)'s aggregative-games approach with arbitrary functions Ψ and h_j. Building on this, we then

[10] The functions $H^*(\cdot)$, $\Delta\,\text{HHI}(\cdot)$, and $s^f(\cdot)$ are to be evaluated at either 0 or θ.

propose several formal definitions of the type-aggregation property. Finally, we show that all those definitions are equivalent and provide a complete characterization of the set of h-functions satisfying type aggregation.

Review of Nocke and Schutz (2018). Let Ψ be a function satisfying Assumptions 1 and 4 and $\mathcal{N}$ a finite set. For every $j \in \mathcal{N}$, let h_j be a function satisfying Assumption 2. Fix some $H^0 \geq 0$. The analysis in the Online Appendix to Nocke and Schutz (2018) implies that the demand system with indirect utility function $\Psi\left[H^0 + \sum_{j\in\mathcal{N}} h_j(p_j)\right]$ can be derived from multistage discrete/continuous choice, as described in Section 2.1. For every $j \in \mathcal{N}$ and $p_j > 0$, let $\iota_j(p_j) \equiv -p_j h_j''(p_j)/h_j'(p_j)$ denote the curvature of h_j. To ensure that the resulting multiproduct-firm pricing game is well behaved, we assume that, for every j and p_j, $\iota_j'(p_j) \geq 0$ whenever $\iota_j(p_j) > 1$.[11]

For every j, let $c_j > 0$. Partitioning the set of product $\mathcal{N}$ into a set of firms $\mathcal{F}$, we obtain a multiproduct-firm pricing game with payoff functions

$$\Pi^f(p) = \sum_{\substack{j\in f \\ p_j<\infty}} (p_j - c_j)(-h_j'(p_j))\Psi'(H(p)) \quad \forall p \in (0,\infty]^{\mathcal{N}},\ \forall f \in \mathcal{F}, \tag{19}$$

where $H(p) \equiv H^0 + \sum_{k\in\mathcal{N}} h_k(p_k)$ and $h_k(\infty) \equiv \lim_{p_k'\to\infty} h_k(p_k')$. Note that firm f can, in principle, set infinite prices on some (or all) of its products. Equation (19) implies that firm f makes no profit on products priced at infinity—thus, it is as if firm f were withdrawing those products from the market.[12]

Having set up the oligopoly model, we can now describe how the markup and market-share fitting-in functions of firm f are derived. With arbitrary h_j-functions and potentially infinite prices, a version of the common ι-markup property still holds; formally stating it requires additional notation. Our assumptions on h_j imply the existence of a $\underline{p}_j \geq 0$ such that: For every $p_j > \underline{p}_j$, $\iota_j(p_j) > 1$ and $\iota_j'(p_j) \geq 0$; whereas $\iota_j(p_j) \leq 1$ whenever $p_j \leq \underline{p}_j$.[13] Hence, the function

$$\nu_j : p_j \in \left(\max(\underline{p}_j, c_j), \infty\right) \mapsto \frac{p_j - c_j}{p_j}\iota_j(p_j),$$

[11]This corresponds to Assumption 1 in Nocke and Schutz (2018). When Ψ is the logarithm, it is the weakest assumption that ensures that, in the multiproduct-firm pricing game, first-order conditions are sufficient for optimality (see Section III in the Online Appendix to Nocke and Schutz, 2018).

[12]From a technical viewpoint, allowing firms to set infinite prices compactifies the action sets, thus guaranteeing that a firm's profit-maximization problem has a solution. With arbitrary h_j-functions, it is easy to construct examples in which firms find it optimal to withdraw some of their products in equilibrium—see Section 3.2 in Nocke and Schutz (2018). See also Section II.3 in the Online Appendix to Nocke and Schutz (2018) for an in-depth discussion on infinite prices.

[13]See Lemma A in Nocke and Schutz (2018).

which, to every price p_j, associates the ι-markup on product j, is continuous and strictly increasing. It therefore establishes a bijection from its domain to $\Big(\nu_j(\max(\underline{p}_j, c_j)), \iota_j(\infty)\Big)$, where $\iota_j(\infty) = \lim_{p_j \to \infty} \iota_j(p_j)$. Let r_j denote its inverse function. We extend the domain of r_j to $\Big(\nu_j(\max(\underline{p}_j, c_j)), \infty\Big)$ by setting $r_j(\mu) = \infty$ whenever $\mu \geq \iota_j(\infty)$.

If the vector of prices $(p_j)_{j\in f}$ is optimal for firm f (given the prices set by its rivals), then it satisfies the common ι-markup property. That is, there exists an ι-markup $\mu^f \in (1, \max_{k\in f} \iota_k(\infty))$ such that $p_j = r_j(\mu^f)$ for every $j \in f$; moreover, μ^f satisfies

$$\mu^f = 1 + \frac{\epsilon(H(p))}{H(p)} \mu^f \sum_{j\in f} \gamma_j(p_j),$$

where $\gamma_j \equiv (h'_j)^2/h''_j$.[14,15]

For a given H, the value of firm f's markup fitting-in function, $m^f(H)$, is therefore the unique solution in μ^f of the following equation:[16]

$$\mu^f = 1 + \frac{\epsilon(H)}{H} \mu^f \sum_{j\in f} \gamma_j(r_j(\mu^f)). \tag{20}$$

Firm f's market-share fitting-in function is defined as follows:

$$S^f(H) = \frac{1}{H} \sum_{j\in f} h_j(r_j(m^f(H))), \tag{21}$$

where "market share" should be understood as choice probability in the discrete/continuous choice micro-foundation. We also obtain firm f's profit fitting-in function:

$$\begin{aligned}
\pi^f(H) &= H\Psi'(H) \frac{1}{H} \sum_{\substack{j\in f \\ p_j<\infty}} (p_j - c_j)(-h'_j(p_j)) \\
&= H\Psi'(H) \frac{1}{H} \sum_{\substack{j\in f \\ p_j<\infty}} \frac{p_j - c_j}{p_j} \frac{p_j h''_j(p_j)}{-h'_j(p_j)} \frac{h'_j(p_j)^2}{h''_j(p_j)} \\
&= \frac{H\Psi'(H)}{\epsilon(H)} \frac{\epsilon(H)}{H} \mu^f \sum_{j\in f} \gamma_j(p_j)
\end{aligned}$$

[14]See Lemma XIX in the Online Appendix to Nocke and Schutz (2018).

[15]If product j has an infinite price, we have $\gamma_j(\infty) = \lim_{p_j \to \infty} \gamma_j(p_j) = 0$; see Lemma A in Nocke and Schutz (2018).

[16]To see why the solution exists and is unique, note that the equation can be rewritten as

$$\frac{\mu^f - 1}{\mu^f} = \frac{\epsilon(H)}{H} \sum_{j\in f} \gamma_j(r_j(\mu^f)).$$

As μ^f increases from 1 to ∞, the left-hand side increases from 0 to 1, whereas the right-hand side decreases from something strictly positive to 0; see Lemmas A and E in Nocke and Schutz (2018).

$$= \frac{H\Psi'(H)}{\epsilon(H)} \left(m^f(H) - 1\right), \tag{22}$$

where we have used the common ι-markup property to obtain the third line and equation (20) to obtain the fourth line.

The equilibrium aggregator level is then pinned down by the condition that market shares add up to unity:[17]

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}} S^f(H) = 1.$$

Type aggregation: Formal definition(s) and complete characterization. Let $\mathcal{H}$ be a set of smooth functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$ satisfying Assumption 2. We impose the following additional requirements on $\mathcal{H}$. First, $\mathcal{H}$ permits arbitrary quality shifters; that is, for every $h \in \mathcal{H}$ and $a > 0$, we have $ah \in \mathcal{H}$. Second, for every $h \in \mathcal{H}$ and $p > 0$, $d\left(\iota[h](p)\right)/dp \geq 0$ whenever $\iota[h](p) > 1$, where $\iota[h](p) \equiv ph''(p)/(-h'(p))$ is the curvature of h. The first requirement seems desirable for a merger analysis, as mergers can lead to quality improvements (or deterioration). We already discussed the second requirement.

A multiproduct-firm is a tuple $((h_j)_{j\in f}, (c_j)_{j\in f})$, where f is a finite set, and $h_j \in \mathcal{H}$ and $c_j \in \mathbb{R}_{++}$ for every $j \in f$. Let $\Omega(\mathcal{H})$ be the set of all such tuples. For every $\omega = ((h_j)_{j\in f}, (c_j)_{j\in f}) \in \Omega(\mathcal{H})$, we let firm ω's markup and market-share fitting-in functions, $m[\omega](\cdot)$ and $S[\omega](\cdot)$, be as defined in equations (20) and (21). (This definition is unambiguous since neither equation (20) nor equation (21) depends on the characteristics of the other firms in the multiproduct-firm pricing game under consideration.)

We are now in a position to propose several formal definitions for the concept of type aggregation. In a nutshell, the set $\mathcal{H}$ satisfies type aggregation if: (a) There exists a real-valued mapping that assigns a type to each firm $\omega \in \Omega(\mathcal{H})$, such that two firms that have the same type have the same markup and market-share fitting-in functions;[18] and (b) the type mapping is economically meaningful. Note that, for our merger analysis, requirement (b) is essential, as we need the post-merger type to provide information as to whether merger-specific synergies are involved, and, if so, how strong those synergies are.[19] We propose three versions of requirement (b).

[17]Under an additional technical assumption, Nocke and Schutz (2018) show that this equation has a unique solution and, in the firm's profit maximization problem, first-order conditions are necessary and sufficient for global optimality. Hence, the multiproduct-firm pricing game has a unique equilibrium. See Assumption iii-(g), Lemmas XXI and XXIV, and Theorem III in the Online Appendix to Nocke and Schutz (2018).

[18]Equation (22) implies that two such firms also have the same profit fitting-in function.

[19]Without requirement (b), the question of whether $\mathcal{H}$ satisfies type aggregation boils down to whether $\Omega(\mathcal{H})$ can be indexed by the reals, i.e., whether there is an injection from $\Omega(\mathcal{H})$ to $\mathbb{R}$. A sufficient condition for this is that $\mathcal{H}$ has the cardinality of the continuum—see Appendix F.1 for details.

In Section 3.1, we defined firm f's type as $T^f = \sum_{j \in f} h_j(c_j)$. Thus, in the discrete/continuous choice micro-foundation, $\log T^f$ corresponds to the consumer surplus (per stage-2 consumer) that firm f would deliver if there were no other products and firm f were pricing all of its products at marginal cost. This motivates the concept of surplus-based type aggregation:

Definition 1. *The set $\mathcal{H}$ satisfies* surplus-based type-aggregation *if there exist two functions $m : \mathbb{R}^2_{++} \longrightarrow (1, \infty)$ and $S : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_{++}$ such that for every $\omega = ((h_j)_{j \in f}, (c_j)_{j \in f}) \in \Omega(\mathcal{H})$, $m[\omega](\cdot) = m(\cdot, \tau(\omega))$ and $S[\omega](\cdot) = S(\cdot, \tau(\omega))$, where*

$$\tau\left((h_j)_{j \in f}, (c_j)_{j \in f}\right) \equiv \sum_{j \in f} h_j(c_j). \tag{23}$$

When studying the properties of fitting-in functions in Section 3.2, we found that a firm that has a higher type systematically commands a higher market share. This motivates the concept of monotonic type aggregation:

Definition 2. *The set $\mathcal{H}$ satisfies* monotonic type aggregation *if there exist a function $\tau : \Omega(\mathcal{H}) \longrightarrow \mathbb{R}$ with range $\mathcal{T}$ and two functions $m : \mathbb{R}_{++} \times \mathcal{T} \longrightarrow (1, \infty)$ and $S : \mathbb{R}_{++} \times \mathcal{T} \longrightarrow \mathbb{R}_{++}$ such that:*

(a) For every $\omega \in \Omega(\mathcal{H})$, $m[\omega](\cdot) = m(\cdot, \tau(\omega))$ and $S[\omega](\cdot) = S(\cdot, \tau(\omega))$;

(b) for every $H > 0$, $S(H, T)$ is strictly increasing in T.

Finally, observe that the type mapping defined in Section 3.1 is continuous in cost and quality shifters. This motivates the concept of continuous type aggregation:

Definition 3. *The set $\mathcal{H}$ satisfies* continuous type aggregation *if there exist a function $\tau : \Omega(\mathcal{H}) \longrightarrow \mathbb{R}$ with range $\mathcal{T}$ and two functions $m : \mathbb{R}_{++} \times \mathcal{T} \longrightarrow (1, \infty)$ and $S : \mathbb{R}_{++} \times \mathcal{T} \longrightarrow \mathbb{R}_{++}$ such that:*

(a) For every $\omega \in \Omega(\mathcal{H})$, $m[\omega](\cdot) = m(\cdot, \tau(\omega))$ and $S[\omega](\cdot) = S(\cdot, \tau(\omega))$;

(b) for every $n \in \mathbb{N} \setminus \{0\}$ and $(h_j)_{1 \leq j \leq n} \in \mathcal{H}^n$, the mapping

$$(a_j, c_j)_{1 \leq j \leq n} \in \mathbb{R}^{2n}_{++} \mapsto \tau\left((a_j h_j)_{1 \leq j \leq n}, (c_j)_{1 \leq j \leq n}\right)$$

is continuous.

The following proposition provides a complete characterization of the sets $\mathcal{H}$ satisfying type aggregation:

Proposition 12. *Let $\mathcal{H}$ be a set of functions satisfying the assumptions made at the beginning of this section. The following statements are equivalent:*

(i) $\mathcal{H}$ *satisfies surplus-based type-aggregation.*

(ii) $\mathcal{H}$ *satisfies monotonic type-aggregation.*

(iii) $\mathcal{H}$ *satisfies continuous type-aggregation.*

(iv) One of the following assertions holds true:

(1) There exists $\sigma > 1$ *such that for every* $h \in \mathcal{H}$*, there exist* $a > 0$ *and* $\beta \geq 0$ *such that* $h(p) = a(p+\beta)^{1-\sigma}$ *for every* $p > \beta/(\sigma - 1)$*.*[20]

(2) For every $h \in \mathcal{H}$*, there exist* $a \in \mathbb{R}$ *and* $\lambda > 0$ *such that* $h(p) = e^{\frac{a-p}{\lambda}}$ *for every* $p > \lambda$*.*[21]

Proof. See Appendix F.2. □

Thus, our three notions of type aggregation are equivalent. Moreover, the set $\mathcal{H}$ satisfies type aggregation if and only if either (1) all of its elements are power functions with a common exponent $1 - \sigma$, or (2) all of its elements are exponential functions.

The demand systems characterized in part (iv) of the proposition are slightly more general than those considered in the previous sections in that, in case (1) the parameter β can be strictly positive and vary across products, and in case (2) the price sensitivity parameter λ can vary across products. However, once types have been defined using equation (23), it is straightforward to show that a firm's markup and market-share fitting-in functions continue to be pinned down by equations (6) and (10) (see Appendix F.2). This implies that all the results of the paper continue to hold with those slightly more general demand systems. Note however that, if $\beta_j \neq 0$ for some product j (resp. λ_j varies across products), then market shares can no longer be interpreted as revenue shares (resp. output shares).

We close this section by discussing the key steps of the proof of Proposition 12. Showing that (iv) implies surplus-based, monotonic, and continuous type aggregation is straightforward. Moreover, it is clear that surplus-based type aggregation implies continuous type aggregation, as the type mapping in Definition 1 satisfies the continuity requirement of Definition 3. Similarly, we show that monotonic type aggregation also implies continuous type aggregation. Intuitively, if $\mathcal{H}$ satisfies monotonic type aggregation with some type mapping τ, then due to condition (b) in Definition 2 it also satisfies monotonic type aggregation with

[20]Note that, whenever β is strictly positive, the proposition does not pin down h on $(0, \beta/(\sigma-1))$. However, such prices are irrelevant for the following reason. Routine calculations show that $\iota[h](p) = \sigma p/(p+\beta)$ for every $p > \beta/(\sigma-1)$. By continuity of $\iota[h](\cdot)$, and since $\iota[h]'(p) \geq 0$ whenever $\iota[h](p) > 1$, it follows that $\iota[h](p) \leq 1$ for every p in $(0, \beta/(\sigma-1))$. Thus, for any $c > 0$, if the firm were pricing the product at $p \in (0, \beta/(\sigma-1))$, then the ι-markup on the product, $(p-c)/p\iota[h](p)$, would be strictly less than 1. Yet, we see from equation (20) that a firm's optimal choice of ι-markup necessarily exceeds unity.

[21]See footnote 20: Prices below λ are irrelevant, as they give rise to ι-markups that are less than 1.

type mapping $\hat{\tau}(\omega) = S[\omega](1)$; and that latter type mapping is easily shown to satisfy the continuity requirement of Definition 3.

The proof that (iii) implies (iv) is more involved. The first step is to show that, if $\mathcal{H}$ satisfies continuous type aggregation, then, for every $h \in \mathcal{H}$, the function $hh''/(h')^2$ is constant (see Lemma 21 in the Appendix). We do so by contradiction: If this condition is violated for some h, then we can apply the implicit-function theorem and the inverse-function theorem to equations (20) and (21) to establish the existence of an open set $B \subseteq \mathbb{R}^2_{++}$ such that the mapping $(a, c) \in B \mapsto \tau(ah, c) \in \mathbb{R}$ is one-to-one. However, that mapping must also be continuous by Definition 3. Since there is no continuous injection from an open subset of $\mathbb{R}^2$ to $\mathbb{R}$ (see footnote 24 in the Appendix), we have a contradiction.

The next step is to show that the constant $hh''/(h')^2$ is the same for all $h \in \mathcal{H}$ (see Lemma 22 in the Appendix). The proof is again by contradiction: If $h_1 h_1''/(h_1')^2 \neq h_2 h_2''/(h_2')^2$ for some $h_1, h_2 \in \mathcal{H}$, then we can again apply the implicit-function theorem and the inverse-function theorem to equations (20) and (21) to establish the existence of an open set $B \subseteq \mathbb{R}^2_{++}$ such that the mapping $(c_1, c_2) \in B \mapsto \tau((h_1, h_2), (c_1, c_2)) \in \mathbb{R}$ is one-to-one. Since that mapping is also continuous, we obtain the same contradiction as in the previous paragraph. This establishes the existence of a scalar ρ such that $h(p)h''(p)/h'(p)^2 = \rho$ for every $h \in \mathcal{H}$ and p. Integrating this differential equation (see Lemma 20), we obtain that $\mathcal{H}$ must be as described in assertion (iv) of Proposition 12.

8 Conclusion

We have developed a framework of multiproduct-firm oligopoly that is amenable for studying the unilateral price effects of horizontal mergers. The framework allows for arbitrary firm and product heterogeneity and is based on the largest class of demand systems giving rise to the type aggregation property. Through the local behavior of the curvature of indirect utility, the GCES/GMNL class of demand systems allows prices, locally, to be strategic complements or substitutes.

The type aggregation property proves useful for merger analysis, as it permits rich forms of merger-induced synergies—not only through changes in the marginal costs of pre-existing products but also through quality improvements and changes in the set of products. We have derived results on the static and dynamic consumer surplus effects of mergers, the aggregate surplus and external effects of mergers, and on the relationship between the market power effect of a merger and the merger-induced change in the Herfindahl index. Importantly, several results hinge critically on whether prices are (local) strategic complements or substitutes. For instance, a competition authority with a consumer surplus standard should, surprisingly, require *larger* synergies for approving a merger in a *more* competitive industry—if prices are strategic substitutes. This highlights the value of a general framework that flexibly allows for different types of strategic interaction.

Appendix

A Technical Results on Fitting-in Functions

In this section, we derives formulas for the partial derivatives of fitting-in functions, prove Lemma 2, and derive two additional intermediate results.

Totally differentiating equations (6) and (10) yields (firm superscript dropped to ease notation):

$$d\log\mu = \alpha\mu\left[\epsilon'(H)sdH + \epsilon(H)ds\right] = \alpha\mu\epsilon s\left(\frac{\epsilon'}{\epsilon}dH + \frac{ds}{s}\right) = (\mu-1)\left(\frac{\epsilon'}{\epsilon}dH + d\log s\right)$$

and

$$d\log s = d\log(T/H) + \frac{\mu\upsilon'(\mu)}{\upsilon(\mu)}d\log\mu = d\log(T/H) - \frac{\alpha\mu}{1-(1-\alpha)\mu}d\log\mu.$$

Rewriting this in matrix form as

$$\begin{pmatrix} 1 & -(\mu-1) \\ \frac{\alpha\mu}{1-(1-\alpha)\mu} & 1 \end{pmatrix}\begin{pmatrix} d\log\mu \\ d\log s \end{pmatrix} = \begin{pmatrix} (\mu-1)(1-\zeta(H))d\log H \\ d\log(T/H) \end{pmatrix},$$

and applying Cramer's formula yields:

$$\begin{pmatrix} d\log\mu \\ d\log s \end{pmatrix} = \frac{1}{1+\frac{\alpha\mu(\mu-1)}{1-(1-\alpha)\mu}}\begin{pmatrix} 1 & (\mu-1) \\ -\frac{\alpha\mu}{1-(1-\alpha)\mu} & 1 \end{pmatrix}\begin{pmatrix} (\mu-1)(1-\zeta(H))d\log H \\ d\log(T/H) \end{pmatrix}$$
$$= \frac{1}{1+\frac{\alpha\mu(\mu-1)}{1-(1-\alpha)\mu}}\begin{pmatrix} (\mu-1)\left[-\zeta(H)d\log H + d\log T\right] \\ -\left[\frac{\alpha\mu(\mu-1)}{1-(1-\alpha)\mu}(1-\zeta(H))+1\right]d\log H + d\log T \end{pmatrix}.$$

We have thus obtained the partial derivatives of the markup and market-share fitting-in functions:

$$\frac{\partial\log m}{\partial\log H} = -\zeta\frac{m-1}{1+\frac{\alpha m(m-1)}{1-(1-\alpha)m}}, \tag{24}$$

$$\frac{\partial\log m}{\partial\log T} = \frac{m-1}{1+\frac{\alpha m(m-1)}{1-(1-\alpha)m}}, \tag{25}$$

$$\frac{\partial\log S}{\partial\log H} = -1 + \frac{\frac{\alpha m(m-1)}{1-(1-\alpha)m}}{1+\frac{\alpha m(m-1)}{1-(1-\alpha)m}}\zeta, \tag{26}$$

$$\frac{\partial\log S}{\partial\log T} = \frac{1}{1+\frac{\alpha m(m-1)}{1-(1-\alpha)m}}. \tag{27}$$

For our comparative statics on aggregate surplus, our analysis of the external effects of mergers, and our results on the Herfindahl index as a measure of the market-power effect of mergers, it is also useful to express the partial derivatives of S as a function of S (instead of m). Plugging equation (6) into equations (26) and (27) yields:

$$\frac{\partial \log S}{\partial \log H} = -1 + \frac{\alpha S}{1 - \epsilon S + \alpha(\epsilon S)^2}\epsilon\zeta, \tag{28}$$

$$\frac{\partial \log S}{\partial \log T} = \frac{(1-\epsilon S)(1-\alpha\epsilon S)}{1 - \epsilon S + \alpha(\epsilon S)^2}. \tag{29}$$

We can now prove Lemma 2:

Proof. The results for the partial derivatives of S and m and for $\partial\pi/\partial T$ follow readily from equations (24)–(27) and (11). (Recall that $\zeta \leq 1$.)

Next, suppose that $S(H,T) < 1$, and consider the partial derivative $\partial\pi/\partial H$. Consider a small open interval I containing H and such that $S(H',T) < 1$ for every $H' \in I$. (The continuity of S imply that I exists.) For every $H' \in I$, let $\widehat{H}(H') \equiv H' - H'S(H',T) > 0$, and note that $\widehat{H}$ has a strictly positive derivative. Consider the maximization problem

$$\max_{\mu} \alpha\mu T\upsilon(\mu)\Psi'\left[\widehat{H}(H') + T\upsilon(\mu)\right].$$

The objective function corresponds to the profit of a firm with type T when it sets an ι-markup of μ and its rivals contribute $\widehat{H}(H')$ to the industry aggregator (see equations (7) and (10)). The analysis in the Online Appendix to Nocke and Schutz (2018) implies that, for every $H' \in I$, this maximization problem has a unique solution, $\mu = m(H',T)$, and the maximized value of the objective function is $\pi(H',T)$. It follows that

$$\begin{aligned}\left.\frac{\partial\pi(H',T)}{\partial H'}\right|_{H'=H} &= \left.\frac{\partial}{\partial H'}\left(\max_{\mu}\alpha\mu T\upsilon(\mu)\Psi'\left[\widehat{H}(H') + T\upsilon(\mu)\right]\right)\right|_{H'=H} \\ &= \alpha T m(H,T)\upsilon(m(H,T))\widehat{H}'(H)\Psi''\left[\widehat{H}(H) + T\upsilon(\mu)\right] < 0,\end{aligned}$$

where we have used the envelope theorem. □

The following technical lemmas will be useful when studying the consumer-surplus effects of mergers:

Lemma 3. *For every H, $\lim_{T\to 0} S(H,T) = 0$ and $\lim_{T\to\infty} S(H,T) \geq 1$. Moreover, $S(H,\cdot)$ is strictly concave and thus strictly sub-additive.*

Proof. The result that $\lim_{T\to 0} S(H,T) = 0$ follows immediately as

$$S(H,T) \leq \frac{T}{H}\upsilon(1) \underset{T\to 0}{\longrightarrow} 0.$$

(See equation (10).) Assume for a contradiction that $l \equiv \lim_{T\to\infty} S(H,T)$, which exists by monotonicity, is strictly less than 1. Taking limits in equation (6), this implies that

$$\lim_{T\to\infty} m(H,T) = \frac{1}{1 - \alpha\epsilon(\infty)l} < \frac{1}{1-\alpha}.$$

Using this to take limits in equation (10), we obtain the following contradiction:

$$\lim_{T\to\infty} S(H,T) = \underbrace{\lim_{T\to\infty}\frac{T}{H}}_{=\infty}\underbrace{\lim_{T\to\infty}\upsilon(m(H,T))}_{>0} = \infty.$$

Next, we prove the strict concavity of $S(H,\cdot)$. Using equation (27), we have that

$$\frac{\partial S}{\partial T} = \frac{S}{T}\frac{1}{1+\frac{\alpha m(m-1)}{1-(1-\alpha)m}} = \frac{1}{H}\frac{\upsilon(m)}{1+\frac{\alpha m(m-1)}{1-(1-\alpha)m}},$$

where we have used equation (10) to obtain the second equality. As the right-hand side is strictly decreasing in m (holding H fixed) and m is strictly increasing in T (see Lemma 1), it follows that $\partial S/\partial T$ is strictly decreasing in T. Hence, $S(H,\cdot)$ is strictly concave; since $S(T,0)=0$, it is also strictly sub-additive. □

Lemma 4. *The partial derivative $\partial^2 \log S/\partial H\partial T$ has the same sign as $\zeta(H)$.*

Proof. This follows immediately from equation (26):

$$\frac{\partial^2 \log S}{\partial H\partial T} = \frac{\zeta(H)}{H}\underbrace{\frac{\partial}{\partial T}\frac{1}{1+\frac{1-(1-\alpha)m}{\alpha m(m-1)}}}_{>0}.$$ □

B Proof of Proposition 3

For every $T^f > 0$, let $H^*(T^f)$ (resp. $W^*(T^f)$) be the equilibrium aggregator level (resp. aggregate surplus) when firm f's type is T^f. (Since the other firms' types are held constant, we do not include them as arguments.) We have:

Lemma 5. *The derivative $W^{*\prime}(T^f)$ has the same sign as*

$$\tilde{\Gamma} \equiv 1 + \frac{\alpha}{(1-\alpha\epsilon s^f)^2}\left[1-\sum_g s^g\right] + \sum_g\left[\frac{\alpha s^g}{1-\alpha\epsilon s^g}(1-\epsilon) + \left(\frac{\alpha s^g}{1-\alpha\epsilon s^g}\right)^2(1-\zeta)\epsilon \right.$$
$$\left. -\alpha s^g\left(1-\frac{\alpha s^g}{1-\epsilon s^g+\alpha(\epsilon s^g)^2}\epsilon\zeta\right)\left(\frac{1}{(1-\alpha\epsilon s^g)^2}-\frac{1}{(1-\alpha\epsilon s^f)^2}\right)\right],$$

where $(s^g)_{g\in\mathcal{F}}$ is the equilibrium market share vector and ϵ and ζ are evaluated at $H^*(T^f)$.

Proof. We will prove the result by showing that $W^{*\prime}(T^f)$ is the product of $\tilde{\Gamma}$ and a strictly positive term. Define

$$W(H,T^f) \equiv \Psi(H) + \sum_{g\in\mathcal{F}}\tilde{\pi}(H,T^g)H\Psi'(H),$$

and note that $W^*(T^f) = W(H^*(T^f), T^f)$. The object of interest is therefore

$$W^{*\prime}(T^f) = \frac{\partial W}{\partial T^f} + \frac{d\log H^*}{dT^f}\frac{\partial W}{\partial \log H}. \tag{30}$$

We thus require expressions for each of the three terms on the right-hand side of equation (30).

Clearly,

$$\frac{\partial W}{\partial T^f} = \frac{\partial \tilde{\pi}(H, T^f)}{\partial T^f} H\Psi'(H) = \frac{\partial S(H, T^f)}{\partial T^f}\frac{\alpha}{(1-\alpha\epsilon s^f)^2} H\Psi'(H). \tag{31}$$

Totally differentiating equilibrium condition (13) yields:

$$\frac{d\log H^*}{dT^f} = \frac{\frac{\partial S(H,T^f)}{\partial T^f}}{\frac{H^0}{H} - \sum_g \frac{\partial S(H,T^g)}{\partial \log H}} = \frac{\frac{\partial S(H,T^f)}{\partial T^f}}{1 - \sum_g s^g - \sum_g s^g \frac{\partial \log S(H,T^g)}{\partial \log H}}. \tag{32}$$

Moreover, as, for every g,

$$\begin{aligned}\frac{\partial \tilde{\pi}(H, T^g) H\Psi'(H)}{\partial \log H} &= H\Psi'(H)\left[\tilde{\pi}^g(1-\epsilon) + H\epsilon'(\tilde{\pi}^g)^2 + \frac{\partial S(H,T^g)}{\partial \log H}\frac{\alpha}{(1-\alpha\epsilon s^g)^2}\right] \\ &= H\Psi'(H)\left[\tilde{\pi}^g(1-\epsilon) + \epsilon(1-\zeta)(\tilde{\pi}^g)^2 + \frac{\partial \log S(H,T^g)}{\partial \log H}\frac{(\tilde{\pi}^g)^2}{\alpha s^g}\right],\end{aligned}$$

we have that

$$\frac{\partial W}{\partial \log H} = H\Psi'(H)\left(1 + \sum_g \left[\tilde{\pi}^g(1-\epsilon) + (\tilde{\pi}^g)^2\epsilon(1-\zeta) + \frac{(\tilde{\pi}^g)^2}{\alpha s^g}\frac{\partial \log S(H,T^g)}{\partial \log H}\right]\right). \tag{33}$$

Plugging equations (31)–(33) into equation (30), we obtain:

$$\begin{aligned}W^{*\prime}(T^f) = H\Psi'(H)\frac{d\log H^*}{dT^f}&\left(1 + \sum_g \left[\tilde{\pi}^g(1-\epsilon) + (\tilde{\pi}^g)^2\epsilon(1-\zeta) + \frac{(\tilde{\pi}^g)^2}{\alpha s^g}\frac{\partial \log S(H,T^g)}{\partial \log H}\right]\right. \\ &\left. + \frac{\alpha}{(1-\alpha\epsilon s^f)^2}\left[1 - \sum_g s^g - \sum_g s^g \frac{\partial \log S(H,T^g)}{\partial \log H}\right]\right) \\ = H\Psi'(H)\frac{d\log H^*}{dT^f}&\left(1 + \frac{\alpha}{(1-\alpha\epsilon s^f)^2}\left[1 - \sum_g s^g\right] + \sum_g \left[\tilde{\pi}^g(1-\epsilon) + (\tilde{\pi}^g)^2\epsilon(1-\zeta)\right.\right. \\ &\left.\left. + \alpha s^g \frac{\partial \log S(H,T^g)}{\partial \log H}\left(\frac{1}{(1-\alpha\epsilon s^g)^2} - \frac{1}{(1-\alpha\epsilon s^f)^2}\right)\right]\right).\end{aligned}$$

Using equation (28), we see that the term inside parentheses is indeed equal to $\tilde{\Gamma}$, as defined in the statement of the lemma. $\square$

To prove the proposition, we show that, if the assumptions of Proposition 3 are satisfied, then $\tilde{\Gamma}$ is strictly positive for any vector (s^g) such that $s^g \in [0,1]$ for every g and $\sum_g s^g \leq 1$. Note that $\tilde{\Gamma} > \Gamma$, where

$$\Gamma \equiv 1 + \sum_g \left[\frac{\alpha s^g}{1-\alpha\epsilon s^g}(1-\epsilon) + \left(\frac{\alpha s^g}{1-\alpha\epsilon s^g}\right)^2 (1-\zeta)\epsilon \right.$$
$$\left. -\alpha s^g \left(1 - \frac{\alpha s^g}{1-\epsilon s^g + \alpha(\epsilon s^g)^2}\epsilon\zeta\right)\left(\frac{1}{(1-\alpha\epsilon s^g)^2} - 1\right)\right].$$

It is thus enough to show that, under the assumptions of Proposition 3, Γ is non-negative.

Let us rewrite Γ as an affine function of $(1-\zeta)$:

$$\Gamma = \kappa + (1-\zeta)\epsilon\tau,$$

where

$$\kappa \equiv 1 + \sum_g \left[\frac{\alpha s^g}{1-\alpha\epsilon s^g}(1-\epsilon) - \alpha s^g \left(1 - \frac{\alpha\epsilon s^g}{1-\epsilon s^g + \alpha(\epsilon s^g)^2}\right)\left(\frac{1}{(1-\alpha\epsilon s^g)^2} - 1\right)\right]$$

and

$$\tau \equiv \sum_g (\alpha s^g)^2 \left[\frac{1}{(1-\epsilon\alpha s^g)^2} - \frac{1}{1-\epsilon s^g + \alpha(\epsilon s^g)^2}\left(\frac{1}{(1-\epsilon\alpha s^g)^2} - 1\right)\right]$$
$$= \sum_g \frac{(\alpha s^g)^2 \left(1-(1+\alpha)\epsilon s^g\right)}{(1-\alpha\epsilon s^g)(1-\epsilon s^g + \alpha(\epsilon s^g)^2)}. \tag{34}$$

Let us show that κ is strictly positive:

Lemma 6. *For every $\alpha \in (0,1]$, $\epsilon \in (0,1]$, and for every vector of market shares (s^g), κ is strictly greater than $1-\alpha$.*

Proof. Note that

$$\kappa > 1 + \alpha \sum_g \underbrace{\left[s^g(1-\epsilon) - s^g\left(1 - \frac{\alpha\epsilon s^g}{1-\epsilon s^g + \alpha(\epsilon s^g)^2}\right)\left(\frac{1}{(1-\alpha\epsilon s^g)^2} - 1\right)\right]}_{\equiv \phi(s^g,\epsilon,\alpha)}.$$

It is thus enough to show that, for every α, ϵ, and (s^g), $\sum_g \phi(s^g,\epsilon,\alpha) \geq -1$. Routine calculations show that ϕ is decreasing in α. Hence, it is enough to prove the inequality for $\alpha = 1$. Let $\phi(s,\epsilon) \equiv \phi(s,\epsilon,1)$. Routine calculations show that, if $\epsilon \leq 1/2$, then $\phi(\cdot,\epsilon)$ is strictly concave on $(0,1)$; if instead $\epsilon > 1/2$, then $\phi(\cdot,\epsilon)$ is strictly concave on $(0, 1/(2\epsilon))$.

Suppose first that $\epsilon \leq 1/2$. Since $\phi(0,\epsilon) = 0$ and $\phi(\cdot,\epsilon)$ is concave, $\phi(\cdot,\epsilon)$ is sub-additive. Thus,

$$\sum_g \phi(s^g,\epsilon) \geq \phi\left(\sum_g s^g, \epsilon\right) \geq \min\{\phi(0,\epsilon), \phi(1,\epsilon)\},$$

where the second inequality follows by the concavity of $\phi(\cdot, \epsilon)$. Clearly, $\phi(0, \epsilon) = 0$, whereas

$$\phi(1, \epsilon) = \frac{1 - 4\epsilon + 3\epsilon^2 - \epsilon^3}{1 - \epsilon + \epsilon^2},$$

which is easily shown to be greater than -1 for every $\epsilon \in (0, 1]$.

Suppose next that $\epsilon > 1/2$, and let (s^g) be a vector of market shares. If $\bar{s} \equiv \sum_g s^g \leq 1/(2\epsilon)$, then the above sub-additivity and concavity arguments immediately imply that $\sum_g \phi(s^g, \epsilon) \geq -1$. Suppose instead that $\bar{s} > 1/(2\epsilon)$, and assume without loss of generality that the profile of market shares $s = (s^g)_{1 \leq g \leq N}$ is sorted in increasing order. We argue that there exists $(s_1, s_2) \in [0, 1]^2$ such that $s_1 + s_2 = \bar{s}$, $s_2 \geq 1/(2\epsilon)$, and

$$\sum_g \phi(s^g, \epsilon) \geq \phi(s_1, \epsilon) + \phi(s_2, \epsilon). \tag{35}$$

To this end, let us define the function ξ, which takes as argument a strictly positive vector of market shares $s' = (s'_1, \ldots, s'_k)$ (with $k \geq 2$) adding up to $\bar{s}$ and sorted in increasing order. The function ξ is defined as follows:

- If $s'_2 \geq 1/(2\epsilon)$, then $\xi(s') = s'$.
- If $s'_2 < 1/(2\epsilon)$, then do the following:
  - If $s'_1 + s'_2 \leq 1/(2\epsilon)$, then form the $(k-1)$-dimensional vector with first component $s'_1 + s'_2$ and remaining components $(s'_i)_{3 \leq i \leq k}$, and sort it in increasing order to obtain $\xi(s')$.
  - If instead $s'_1 + s'_2 > 1/(2\epsilon)$, then form the k-dimensional vector with first component $s'_1 + s'_2 - 1/(2\epsilon)$, second component $1/(2\epsilon)$, and remaining components $(s'_i)_{3 \leq i \leq k}$, and sort it in increasing order to obtain $\xi(s')$.

Clearly, $\xi(s')$ is a strictly positive vector of market shares adding up to $\bar{s}$ and sorted in increasing order. Moreover, the concavity and sub-additivity properties of ϕ imply that

$$\sum_{s'} \phi(\cdot, \epsilon) \geq \sum_{\xi(s')} \phi(\cdot, \epsilon),$$

where, for every market-share vector $\hat{s} = (\hat{s}_i)_{1 \leq i \leq k}$,

$$\sum_{\hat{s}} \phi(\cdot, \epsilon) \equiv \sum_{i=1}^{k} \phi(\hat{s}_i, \epsilon).$$

Define the sequence $(s^{(n)})_{n \geq 0}$ by induction as follows: $s^{(0)} = s$; for every $n \geq 1$, $s^{(n)} = \xi(s^{(n-1)})$. The properties of ξ imply that, for every n, $\sum_s \phi(\cdot, \epsilon) \geq \sum_{s^{(n)}} \phi(\cdot, \epsilon)$. For every n, let J_n be the dimensionality of vector $s^{(n)}$ and K_n the number of components of vector $s^{(n)}$ that are strictly less than $1/(2\epsilon)$. By construction, $(J_n)_{n\geq 0}$ and $(K_n)_{n\geq 0}$ are non-increasing sequences of integers. Hence, they are eventually stationary: There exists $N \geq 0$ such that $J_{N+1} = J_N$ and $K_{N+1} = K_N$. The definition of ξ implies that $\xi(s^{(N)}) = s^{(N)}$ and $s_2^{(N)} \geq 1/(2\epsilon)$. Thus, condition (35) holds with $(s_1, s_2) = (s_1^{(N)}, s_2^{(N)})$.

Routine calculations show that $\phi(\cdot, \epsilon)$ is decreasing on $(1/(2\epsilon), 1)$. It follows that

$$\sum_g \phi(s^g, \epsilon) \geq \phi(s_1, \epsilon) + \phi(1 - s_1, \epsilon).$$

Thus, all we need to do is show that $\phi(s, \epsilon) + \phi(1 - s, \epsilon) \geq -1$ for every s and ϵ. Straightforward but tedious calculations show that this condition holds. □

We are now in a position to prove Proposition 3:

Lemma 7. *Under the assumptions of Proposition 3,* Γ *is non-negative for every vector of market shares* (s^g)*.*

Proof. If condition (i) of Proposition 3 holds, then $\Gamma = \kappa$, which is positive by Lemma 6. If condition (ii) of Proposition 3 holds, then τ is strictly positive for every vector of market shares (see equation (34)). Since $1 - \zeta \geq 0$ and $\kappa > 0$, this implies that $\Gamma > 0$.

Finally, suppose that condition (iii) of Proposition 3 holds. We have just shown that Γ is positive when $\zeta = 1$ (for any ϵ, α, and (s^g)). Since Γ is linear in ζ, all we need to do is show that Γ is also positive when $\zeta = 0$. At $\zeta = 0$, we find after some algebra that

$$\Gamma = 1 + \sum_g \underbrace{\frac{\alpha s^g (1 - \epsilon - \alpha \epsilon s^g)}{1 - \alpha \epsilon s^g}}_{\equiv \phi(s, \epsilon, \alpha)}.$$

Routine calculations show that $\phi(\cdot, \epsilon, \alpha)$ is concave and $\phi(0, \epsilon, \alpha) = 0$. Hence, $\phi(\cdot, \epsilon, \alpha)$ is sub-additive. It follows that $\Gamma \geq 1 + \phi(\bar{s}, \epsilon, \alpha)$, where $\bar{s} = \sum_g s^g$. By concavity, $\phi(\bar{s}, \epsilon, \alpha)$ is minimized at either $\bar{s} = 0$ (in which case $\phi(\bar{s}, \epsilon, \alpha) = 0$) or $\bar{s} = 1$, in which case

$$1 + \phi(\bar{s}, \epsilon, \alpha) = \frac{1 + \alpha - 2\alpha\epsilon - \alpha^2 \epsilon}{1 - \alpha \epsilon},$$

which is indeed non-negative given condition (iii). □

C External Effects: Derivations and Proofs

We begin by deriving the formula for $\eta(H)$ stated in the main text (equation (15)). Using equation (33) (and replacing $\mathcal{F}$ by $\mathcal{O}$), we have:

$$\eta(H) = -1 - \sum_{g \in \mathcal{O}} \left[\tilde{\pi}^g (1 - \epsilon) + (\tilde{\pi}^g)^2 \epsilon (1 - \zeta) + \frac{(\tilde{\pi}^g)^2}{\alpha s^g} \frac{\partial \log S(H, T^g)}{\partial \log H} \right]$$

$$= -1 - \sum_{g\in\mathcal{O}} \left[\tilde{\pi}^g(1-\epsilon) + (\tilde{\pi}^g)^2\epsilon(1-\zeta) + \frac{(\tilde{\pi}^g)^2}{\alpha s^g}\left(-1 + \frac{\alpha s^g}{1-\epsilon s^g + \alpha(\epsilon s^g)^2}\epsilon\zeta\right)\right]$$
$$= -1 - \sum_{g\in\mathcal{O}} \left[\tilde{\pi}^g(1-\epsilon) - \frac{(\tilde{\pi}^g)^2}{\alpha s^g}\frac{(1-\alpha\epsilon s^g)(1-\epsilon s^g)}{1-\epsilon s^g + \alpha(\epsilon s^g)^2} - \epsilon(1-\zeta)(\tilde{\pi}^g)^2\frac{\epsilon s^g(1-\alpha\epsilon s^g)}{1-\epsilon s^g + \alpha(\epsilon s^g)^2}\right].$$

Using the definition of $\tilde{\pi}^g$ and simplifying, we obtain equation (15).

We now put on record the following facts on the functions ϕ and χ, which appear in equation (15):

Lemma 8. *The function ϕ has the following properties:*

(a) For every $(\alpha, \epsilon) \in (0,1]^2$, $\phi(0,\epsilon,\alpha) = 0$, and $\phi(s,\epsilon,\alpha) > 0$ for every $s > 0$.

(b) There exist a cutoff $\underline{\epsilon} \simeq 0.81$ and a strictly decreasing function $\bar{\alpha} : [\underline{\epsilon}, 1] \longrightarrow [\underline{\alpha}, 1]$ such that the following assertions are equivalent:

(i) For every $s \in (0,1)$, $\phi(s,\epsilon,\alpha) \leq s$.

(ii) Either $\epsilon \leq \underline{\epsilon}$, or $\epsilon > \underline{\epsilon}$ and $\alpha \leq \bar{\alpha}(\epsilon)$.

Moreover, $\bar{\alpha}(\underline{\epsilon}) = 1$ and $\underline{\alpha} = \bar{\alpha}(1) \simeq 0.82$.

(c) For any $(\alpha, \epsilon) \in (0,1]^2$, $\phi(\cdot,\epsilon,\alpha)$ is strictly increasing on $[0, 3/5]$.

(d) For any $(\alpha, \epsilon) \in (0,1]^2$, $\phi(\cdot,\epsilon,\alpha)$ is strictly convex on $[0, 1/5]$.

Moreover,

(e) For every $(\alpha, \epsilon) \in (0,1]^2$, $\chi(\cdot,\epsilon,\alpha)$ is strictly positive, strictly increasing, and strictly convex on $(0,1)$.

Proof. The lemma is proven analytically using Mathematica. Mathematica files are available upon request. □

Armed with Lemma 8, we can prove Proposition 8:

Proof. Suppose first that condition (i) holds and merger M' has a positive external effect. Then, there exists an injection $\vartheta : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $s^{\vartheta(f)} \geq s'^f$ for every $f \in \mathcal{O}'$. Hence,

$$\begin{aligned} 0 &< -1 + \sum_{f\in\mathcal{O}'} \phi(s'^f, \epsilon(H'), \alpha) + (1-\zeta(H')) \sum_{f\in\mathcal{O}'} \chi(s'^f, \epsilon(H'), \alpha) \\ &\leq -1 + \sum_{f\in\mathcal{O}'} \phi(s^{\vartheta(f)}, \epsilon(H), \alpha) + (1-\zeta(H)) \sum_{f\in\mathcal{O}'} \chi(s^{\vartheta(f)}, \epsilon(H), \alpha) \\ &\leq -1 + \sum_{f\in\mathcal{O}} \phi(s^f, \epsilon(H), \alpha) + (1-\zeta(H)) \sum_{f\in\mathcal{O}} \chi(s^f, \epsilon(H), \alpha), \end{aligned}$$

where the second inequality follows by parts (c) and (e) of Lemma 8, and the third inequality follows by injectivity of ϑ and parts (a) and (e) of Lemma 8. Hence, merger M also has a positive external effect.

Next, suppose that condition (ii) holds and merger M' has a positive external effect. Then,

$$\begin{aligned} 0 &< -1 + \sum_{f \in \mathcal{O}'} \phi(s'^f, \epsilon(H'), \alpha) + (1 - \zeta(H')) \sum_{f \in \mathcal{O}'} \chi(s'^f, \epsilon(H'), \alpha) \\ &\leq -1 + \sum_{f \in \mathcal{O}} \phi(s^f, \epsilon(H), \alpha) + (1 - \zeta(H)) \sum_{f \in \mathcal{O}} \chi(s^f, \epsilon(H), \alpha), \end{aligned}$$

where the second inequality follows as s is a mean-preserving spread of s' and $\phi(\cdot, \epsilon(H), \alpha)$ and $\chi(\cdot, \epsilon(H), \alpha)$ are convex on $[0, 1/5]$ by parts (d) and (e) of Lemma 8. Hence, merger M also has a positive external effect. □

Finally, we prove Proposition 9:

Proof. Consider a CS-decreasing merger M with set of outsiders $\mathcal{O}$ and pre- and post-merger equilibrium aggregator levels H^* and $\overline{H}^*$, respectively.

We begin by stating a weaker version of condition (i) in the proposition: Ψ is the logarithm or a power function; and either $\epsilon \leq \underline{\epsilon}$, or $\epsilon > \underline{\epsilon}$ and $\alpha \leq \bar{\alpha}(\epsilon)$ (where $\underline{\epsilon}$ and $\bar{\alpha}$ were defined in Lemma 8). Note that this is indeed weaker than condition (i) since $\underline{\epsilon}$ and $\bar{\alpha}(1) = \underline{\alpha}$ are both greater than 0.8 and the function $\bar{\alpha}(\cdot)$ is decreasing. Suppose indeed that this weaker condition holds. For every $H \in [\overline{H}^*, H^*]$, we have

$$\begin{aligned} \eta(H) &= -1 + \sum_{f \in \mathcal{O}} \phi\left(S(H, T^f), \epsilon(H), \alpha\right) \quad \text{since } \zeta = 1 \\ &\leq -1 + \sum_{f \in \mathcal{O}} S(H, T^f) \quad \text{by part (b) of Lemma 8} \\ &\leq -1 + \sum_{f \in \mathcal{O}} S(\overline{H}^*, T^f) \quad \text{by monotonicity of } S(\cdot, T^f). \end{aligned}$$

Since the outsiders' market shares add up to less than 1 in the post-merger equilibrium, it follows that $\eta(H) < 0$ for every $H \in [\overline{H}^*, H^*]$, and so $\Delta\mathcal{E} < 0$.

Next, suppose that condition (ii) holds. Let $H \in [\overline{H}^*, H^*]$ and $s^f \equiv S(H, T^f)$ for every $f \in \mathcal{O}$. For every outsider f, define also $\tilde{s}^f \equiv S(\overline{H}^*, T^f)$, and note that $\tilde{s}^f \geq s^f$ since $S(\cdot, T^f)$ is decreasing. As ζ and χ are non-negative, we have that

$$\begin{aligned} \eta(H) &\leq -1 + \sum_{f \in \mathcal{O}} \left[\phi(s^f, \epsilon(H), \alpha) + \chi(s^f, \epsilon(H), \alpha)\right] \\ &= -1 + \sum_{f \in \mathcal{O}} \frac{\alpha\epsilon(H)s^f}{1 - \alpha\epsilon(H)s^f} \end{aligned}$$

$$\le -1 + \sum_{f\in\mathcal{O}} \frac{\alpha\epsilon(H)\tilde{s}^f}{1-\alpha\epsilon(H)\tilde{s}^f} \quad \text{since } \tilde{s}^f \ge s^f$$
$$\le -1 + \frac{\alpha\epsilon(H)\sum_{f\in\mathcal{O}}\tilde{s}^f}{1-\alpha\epsilon(H)\sum_{f\in\mathcal{O}}\tilde{s}^f} \quad \text{by super-additivity of } s \mapsto \frac{\alpha\epsilon(H)s}{1-\alpha\epsilon(H)s}$$
$$< -1 + \frac{\alpha\epsilon(H)}{1-\alpha\epsilon(H)} \quad \text{since } \sum_{f\in\mathcal{O}}\tilde{s}^f < 1$$
$$\le 0 \quad \text{since } \alpha\epsilon(H) \le \frac{1}{2}.$$

Hence, $\eta(H) < 0$ for every $H \in [\overline{H}^*, H^*]$, and so $\Delta\mathcal{E} < 0$. □

D Proof of Proposition 10

We prove a series of lemmas that jointly imply Proposition 10. Throughout this section, $\Sigma(s) \equiv \sum_{f\in\mathcal{F}} s^f$ denotes the aggregate market share of all firms.

We begin by obtaining second-order approximations of pre-merger market-performance measures:

Lemma 9. *At the second order in the neighborhood of $s = 0$,*

$$H^*(s) = H^0\left(1+\Sigma(s)+\Sigma(s)^2\right) + o(\|s\|^2),$$
$$CS(s) = \Psi(H^0) + H^0\Psi'(H^0)\left(\Sigma(s) + \left[1 - \frac{\epsilon(H^0)}{2}\right]\Sigma(s)^2\right) + o(\|s\|^2),$$
$$\textit{and}\ \ PS(s) = H^0\Psi'(H^0)\left(\alpha\Sigma(s) + \alpha(1-\epsilon(H^0))\Sigma(s)^2 + \alpha^2\epsilon(H^0)\sum_{f\in\mathcal{F}}(s^f)^2\right) + o(\|s\|^2).$$

Proof. To ease notation, let $\tilde{s} \equiv \Sigma(s)$. We have:

$$H^*(s) = \frac{H^0}{1-\tilde{s}} = H^0(1+\tilde{s}+\tilde{s}^2) + o(\tilde{s}^2) = H^0(1+\tilde{s}+\tilde{s}^2) + o(\|s\|^2).$$

Since

$$\Psi(H^0(1+x)) = \Psi(H^0) + \Psi'(H^0)H^0x + \frac{1}{2}\Psi''(H^0)(H^0x)^2 + o(x^2)$$
$$= \Psi(H^0) + H^0\Psi'(H^0)\left(x - \frac{\epsilon(H^0)}{2}x^2\right) + o(x^2), \tag{36}$$

it follows that

$$\text{CS}(s) = \Psi(H^*(s)) = \Psi(H^0) + H^0\Psi'(H^0)\left(\tilde{s} + \tilde{s}^2 - \frac{\epsilon(H^0)}{2}\tilde{s}^2\right) + o(\|s\|^2),$$

as stated.

Next, we turn to profits. Clearly,

$$\frac{\alpha s^f}{1-\alpha\epsilon(H^*(s))s^f} = \alpha s^f \left[1+\alpha\epsilon(H^0)s^f\right] + o(\|s\|^2).$$

Moreover, since

$$\begin{aligned} H^*(s)\Psi'(H^*(s)) &= H^0\left(1+\tilde{s}\right)\left[\Psi'(H^0)+H^0\Psi''(H^0)\tilde{s}\right]+o(\|s\|) \\ &= H^0\Psi'(H^0)\left[1+(1-\epsilon(H^0))\tilde{s}\right]+o(\|s\|), \end{aligned}$$

we have that

$$\begin{aligned} \mathrm{PS}(s) &= \sum_{f\in\mathcal{F}} \alpha s^f \left[1+\alpha\epsilon(H^0)s^f\right] H^0\Psi'(H^0)\left[1+(1-\epsilon(H^0))\tilde{s}\right]+o(\|s\|^2) \\ &= H^0\Psi'(H^0)\left[\alpha\tilde{s}+\alpha(1-\epsilon(H^0))\tilde{s}^2+\alpha^2\epsilon(H^0)\sum_{f\in\mathcal{F}}(s^f)^2\right]+o(\|s\|^2). \end{aligned}$$

□

Next, we compute the first and second partial derivatives of $T^f(s)$:

Lemma 10. *The first and second partial derivatives of $T^f(s)$ at $s=0$ are given by*

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} \frac{H^0}{\upsilon(1)} & \text{if } f=f', \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\left.\frac{\partial^2 T^f}{\partial s^{f'} s^{f''}}\right|_{s=0} = \begin{cases} 2\frac{H^0}{\upsilon(1)}\left[1+\alpha\epsilon(H^0)\right] & \text{if } f=f'=f'', \\ 0 & \text{if } f',f''\neq f, \\ \frac{H^0}{\upsilon(1)} & \text{otherwise.} \end{cases}$$

Proof. Totally differentiating equation (18) yields:

$$\begin{aligned} ds^f &= \frac{H^0}{\left(1-\sum_f s^g\right)^2}\frac{\partial S(H^*,T^f)}{\partial H}\sum_g ds^g + \frac{\partial S(H^*,T^f)}{\partial T^f}dT^f \\ &= \frac{H^*}{H^0}s^f\frac{\partial \log S(H^*,T^f)}{\partial \log H^*}\sum_g ds^g + \frac{s^f}{T^f}\frac{\partial \log S(H^*,T^f)}{\partial \log T^f}dT^f. \end{aligned}$$

Hence,

$$\frac{\partial T^f}{\partial s^f} = \frac{1-\frac{H^*}{H^0}s^f\frac{\partial \log S(H^*,T^f)}{\partial \log H^*}}{\frac{s^f}{T^f}\frac{\partial \log S(H^*,T^f)}{\partial \log T^f}}$$

$$= \frac{1 + \frac{H^*}{H^0} s^f \left[1 - \frac{\alpha s^f}{1-\epsilon s^f + \alpha(\epsilon s^f)^2}\epsilon\zeta\right]}{\frac{s^f}{T^f}\frac{(1-\epsilon s^f)(1-\alpha\epsilon s^f)}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}} \quad \text{using equations (28) and (29)}$$

$$= \frac{T^f}{s^f}\frac{1-\epsilon s^f + \alpha(\epsilon s^f)^2}{(1-\epsilon s^f)(1-\alpha\epsilon s^f)}\left(1 + s^f\frac{H^*}{H^0}\left[1 - \frac{\alpha s^f}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}\epsilon\zeta\right]\right), \tag{37}$$

and, for $f' \neq f$,

$$\frac{\partial T^f}{\partial s^{f'}} = \frac{\frac{H^*}{H^0} s^f \left[1 - \frac{\alpha s^f}{1-\epsilon s^f + \alpha(\epsilon s^f)^2}\epsilon\zeta\right]}{\frac{s^f}{T^f}\frac{(1-\epsilon s^f)(1-\alpha\epsilon s^f)}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}}$$

$$= T^f\frac{H^*}{H^0}\frac{1-\epsilon s^f + \alpha(\epsilon s^f)^2}{(1-\epsilon s^f)(1-\alpha\epsilon s^f)}\left[1 - \frac{\alpha s^f}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}\epsilon\zeta\right]. \tag{38}$$

Taking limits in equation (37) and using the fact that

$$\frac{T^f}{s^f} = \frac{H^*(s)}{\upsilon(m(H^*(s), T^f(s)))} \underset{s\to 0}{\longrightarrow} \frac{H^0}{\upsilon(1)}$$

yields

$$\left.\frac{\partial T^f}{\partial s^f}\right|_{s=0} = \frac{H^0}{\upsilon(1)}.$$

Taking limits in equation (38) gives

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = 0.$$

Next, we turn to the second partial derivatives of T^f. Differentiating equation (37) at $s = 0$ yields:[22]

$$\left.\frac{\partial^2 T^f}{\partial (s^f)^2}\right|_{s=0} = \left.\frac{\partial \frac{T^f}{s^f}}{\partial s^f}\right|_{s=0} \times 1 + \left[\lim_{s\to 0}\frac{T^f}{s^f}\right]\alpha\epsilon(H^0) \times 1 + \left[\lim_{s\to 0}\frac{T^f}{s^f}\right] \times 1$$

$$= \frac{H^0}{\upsilon(1)}[1+\alpha\epsilon] + \lim_{s\to 0}\frac{1}{s^f}\left(\frac{\partial T^f}{\partial s^f} - \frac{T^f}{s^f}\right)$$

$$= \frac{H^0}{\upsilon(1)}[1+\alpha\epsilon] + \lim_{s\to 0}\frac{1}{s^f}\frac{T^f}{s^f}\left(\frac{1-\epsilon s^f+\alpha(\epsilon s^f)^2}{(1-\epsilon s^f)(1-\alpha\epsilon s^f)}\right.$$

$$\left.\times\left(1 + s^f\frac{H^*}{H^0}\left[1 - \frac{\alpha s^f}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}\epsilon\zeta\right]\right) - 1\right) \quad \text{using equation (37)}$$

$$= \frac{H^0}{\upsilon(1)}[1+\alpha\epsilon] + \lim_{s\to 0}\frac{1}{s^f}\frac{T^f}{s^f}\left(\frac{\alpha\epsilon s^f}{(1-\epsilon s^f)(1-\alpha\epsilon s^f)}\right.$$

[22]There is no need to differentiate the second fraction on the right-hand side with respect to ϵ because that derivative is equal to s^f (which tends to zero) times a finite number. Likewise, there is no need to differentiate the term inside square brackets because that term is pre-multiplied by s^f, which tends to zero.

$$+s^f \frac{1-\epsilon s^f+\alpha(\epsilon s^f)^2}{(1-\epsilon s^f)(1-\alpha\epsilon s^f)} \frac{H^*}{H^0}\left[1-\frac{\alpha s^f}{1-\epsilon s^f+\alpha(\epsilon s^f)^2}\epsilon\zeta\right]\Bigg)$$
$$= 2\frac{H^0}{\upsilon(1)}\left[1+\alpha\epsilon\right].$$

Differentiating equation (37) with respect to $s^{f'}$ $(f' \neq f)$ yields:

$$\left.\frac{\partial^2 T^f}{\partial s^f s^{f'}}\right|_{s=0} = \left(\lim_{s\to 0}\frac{1}{s^f}\frac{\partial T^f}{\partial s^{f'}}\right)\times 1 = \lim_{s\to 0}\frac{T^f}{s^f} = \frac{H^0}{\upsilon(1)},$$

where we have used equation (38) to obtain the second equality. Finally, differentiating equation (38) with respect to $s^{f''}$ $(f', f'' \neq f)$ yields:

$$\left.\frac{\partial^2 T^f}{\partial s^{f'} s^{f''}}\right|_{s=0} = \left.\frac{\partial T^f}{\partial s^{f''}}\right|_{s=0} = 0. \qquad \square$$

We are now in a position to approximate the post-merger level of the aggregator:

Lemma 11. *At the second order in the neighborhood of $s = 0$,*

$$\overline{H}^*(s) = H^0\left[1+\Sigma(s)+\Sigma(s)^2-\alpha\epsilon(H^0)\Delta\, HHI(s)\right]+o(\|s\|^2).$$

Proof. The post-merger equilibrium aggregator level $\overline{H}^*(s)$ is pinned down by the following condition:

$$\frac{H^0}{\overline{H}^*(s)}+\sum_{g\in\overline{\mathcal{F}}} S\left(\overline{H}^*(s), \bar{T}^g(s)\right) = 1. \tag{39}$$

Differentiating this with respect to s^f yields

$$\left[-\frac{H^0}{\overline{H}^*}+\sum_{g\in\overline{\mathcal{F}}}\bar{s}^g\frac{\partial\log S(\overline{H}^*,\bar{T}^g)}{\partial\log H}\right]\frac{\partial\log\overline{H}^*}{\partial s^f}+\left[\sum_{g\in\overline{\mathcal{F}}}\frac{\partial\bar{T}^g}{\partial s^f}\frac{\bar{s}^g}{\bar{T}^g}\frac{\partial\log S(\overline{H}^*,\bar{T}^g)}{\partial\log T^g}\right]=0.$$

Using equation (39) to eliminate H^0 yields:

$$\left[-1+\sum_{g\in\overline{\mathcal{F}}}\bar{s}^g\left(1+\frac{\partial\log S(\overline{H}^*,\bar{T}^g)}{\partial\log H}\right)\right]\frac{\partial\log\overline{H}^*}{\partial s^f}+\left[\sum_{g\in\overline{\mathcal{F}}}\frac{\partial\bar{T}^g}{\partial s^f}\frac{\bar{s}^g}{\bar{T}^g}\frac{\partial\log S(\overline{H}^*,\bar{T}^g)}{\partial\log T^g}\right]=0.$$

Combining this with equations (28) and (29), we obtain:

$$\frac{\partial\overline{H}^*}{\partial s^f} = \overline{H}^*\frac{\sum_{g\in\overline{\mathcal{F}}}\frac{\bar{s}^g}{\bar{T}^g}\frac{(1-\epsilon(\overline{H}^*)\bar{s}^g)(1-\alpha\epsilon(\overline{H}^*)\bar{s}^g)}{1-\epsilon(\overline{H}^*)\bar{s}^g+\alpha(\epsilon(\overline{H}^*)\bar{s}^g)^2}\frac{\partial\bar{T}^g}{\partial s^f}}{1-\zeta(\overline{H}^*)\sum_{g\in\overline{\mathcal{F}}}\frac{\alpha\epsilon(\overline{H}^*)(\bar{s}^g)^2}{1-\epsilon(\overline{H}^*)\bar{s}^g+\alpha(\epsilon(\overline{H}^*)\bar{s}^g)^2}}. \tag{40}$$

The denominator clearly tends to 1 as s tends to 0. Moreover, by Lemma 10, as s tends to zero each term indexed by g in the numerator tends to 0 if either $f \notin \mathcal{M}$ and $g \neq f$ or $f \in \mathcal{M}$ and $g \neq M$; otherwise, that term tends to

$$\lim_{s\to 0}\frac{\bar{s}^g}{\bar{T}^g}\frac{\partial\bar{T}^g}{\partial s^f} = \lim_{s\to 0}\frac{\upsilon(m(\overline{H}^*,\bar{T}^g))}{\overline{H}^*}\frac{\partial\bar{T}^g}{\partial s^f} = 1.$$

It follows that

$$\left.\frac{\partial \overline{H}^*}{\partial s^f}\right|_{s=0} = H^0.$$

Next, we differentiate equation (40) with respect to $s^{f'}$ (where f' may or may not be equal to f) and take the limit as s tends to zero. Note that the derivative of the denominator tends to zero, so we can ignore that term. Moreover, when differentiating the numerator, we can ignore the terms coming from the derivative of ϵ as all those terms will be pre-multiplied by $\bar{s}^g$, which tends to zero. We have:

$$\begin{aligned}\left.\frac{\partial^2 \overline{H}^*}{\partial s^f \partial s^{f'}}\right|_{s=0} = H^0 + H^0 \sum_{g\in\overline{\mathcal{F}}} \Bigg[& \left.\frac{\partial \bar{s}^g/\bar{T}^g}{\partial s^{f'}}\right|_{s=0} \left.\frac{\partial \bar{T}^g}{\partial s^f}\right|_{s=0} \\ & -\frac{\upsilon(1)}{H^0}\alpha\epsilon(H^0) \left.\frac{\partial \bar{s}^g}{\partial s^{f'}}\right|_{s=0} \left.\frac{\partial \bar{T}^g}{\partial s^f}\right|_{s=0} + \frac{\upsilon(1)}{H^0} \left.\frac{\partial^2 \bar{T}^g}{\partial s^f \partial s^{f'}}\right|_{s=0}\Bigg]\end{aligned} \tag{41}$$

Thus, we require $\left.\frac{\partial \bar{s}^g}{\partial s^{f'}}\right|_{s=0}$ and $\left.\frac{\partial \bar{s}^g/\bar{T}^g}{\partial s^{f'}}\right|_{s=0}$. Since $\bar{s}^g = S(\overline{H}^*, \bar{T}^g)$, we have:

$$\begin{aligned}\frac{\partial \bar{s}^g}{\partial s^{f'}} &= \frac{\partial \overline{H}^*}{\partial s^{f'}} \frac{\bar{s}^g}{\overline{H}^*} \frac{\partial \log S(\overline{H}^*, \bar{T}^g)}{\partial \log H} + \frac{\partial \bar{T}^g}{\partial s^{f'}} \frac{\bar{s}^g}{\bar{T}^g} \frac{\partial \log S(\overline{H}^*, \bar{T}^g)}{\partial \log T^g} \\ &\underset{s\to 0}{\longrightarrow} 0 + \left.\frac{\partial \bar{T}^g}{\partial s^{f'}}\right|_{s=0} \times \frac{\upsilon(1)}{H^0} \times 1 \\ &= \begin{cases} 0 & \text{if either } f' \notin \mathcal{M} \text{ and } g \neq f' \text{ or } f' \in \mathcal{M} \text{ and } g \neq M \\ 1 & \text{otherwise.} \end{cases}\end{aligned} \tag{42}$$

Moreover,

$$\begin{aligned}\left.\frac{\partial \bar{s}^g/\bar{T}^g}{\partial s^{f'}}\right|_{s=0} &= \left.\frac{\bar{s}^g}{\bar{T}^g} \frac{\partial \log \bar{s}^g/\bar{T}^g}{\partial s^{f'}}\right|_{s=0} = \frac{\upsilon(1)}{H^0}\left.\left[\frac{\partial \log \bar{s}^g}{\partial s^{f'}} - \frac{\partial \log \bar{T}^g}{\partial s^{f'}}\right]\right|_{s=0} \\ &= \frac{\upsilon(1)}{H^0}\left.\left[\frac{\partial \overline{H}^*}{\partial s^{f'}} \frac{1}{\overline{H}^*} \frac{\partial \log S(\overline{H}^*, \bar{T}^g)}{\partial \log H} + \frac{\partial \log \bar{T}^g}{\partial s^{f'}} \left(\frac{\partial \log S(\overline{H}^*, \bar{T}^g)}{\partial \log T^g} - 1\right)\right]\right|_{s=0} \\ &= \frac{\upsilon(1)}{H^0}\left.\left[-1 + \frac{\partial \bar{T}^g}{\partial s^{f'}} \frac{1}{\bar{T}^g} \left(\frac{\left(1-\epsilon(\overline{H}^*)\bar{s}^g\right)\left(1-\alpha\epsilon(\overline{H}^*)\bar{s}^g\right)}{1-\epsilon(\overline{H}^*)\bar{s}^g + \alpha(\epsilon(\overline{H}^*)\bar{s}^g)^2} - 1\right)\right]\right|_{s=0} \\ &= \frac{\upsilon(1)}{H^0}\left.\left[-1 - \frac{\partial \bar{T}^g}{\partial s^{f'}} \frac{\bar{s}^g}{\bar{T}^g} \frac{\alpha\epsilon(\overline{H}^*)}{1-\epsilon(\overline{H}^*)\bar{s}^g + \alpha(\epsilon(\overline{H}^*)\bar{s}^g)^2}\right]\right|_{s=0} \\ &= -\frac{\upsilon(1)}{H^0}\left[1 + \alpha\epsilon(H^0)\frac{\upsilon(1)}{H^0} \left.\frac{\partial \bar{T}^g}{\partial s^{f'}}\right|_{s=0}\right] \\ &= -\frac{\upsilon(1)}{H^0} \times \begin{cases} 1 & \text{if either } f' \notin \mathcal{M} \text{ and } g \neq f' \text{ or } f' \in \mathcal{M} \text{ and } g \neq M \\ 1+\alpha\epsilon(H^0) & \text{otherwise.} \end{cases}\end{aligned}$$

Combining these results and Lemma 10, we can now evaluate equation (41). Suppose first that $f, f' \in \mathcal{O}$. If $f' \neq f$, then only the terms $g = f$ and $g = f'$ on the right-hand side of equation (41) can be different from zero:

$$\left. \frac{\partial^2 \overline{H}^*}{\partial s^f \partial s^{f'}} \right|_{s=0} = H^0 \left(1 + [-1+0+1] + [0+0+1]\right) = 2H^0.$$

If instead $f' = f$, then only the term $g = f$ can be different from zero:

$$\left. \frac{\partial^2 \overline{H}^*}{\partial s^f \partial s^{f'}} \right|_{s=0} = H^0 \left(1 + \left[-(1+\alpha\epsilon(H^0)) - \alpha\epsilon(H^0) + 2(1+\alpha\epsilon(H^0))\right]\right) = 2H^0.$$

Next, suppose that one firm in (f, f') belongs to $\mathcal{M}$ while the other one belongs to $\mathcal{O}$; to fix ideas, suppose $f \in \mathcal{M}$ and $f' \in \mathcal{O}$. Then, only the terms $g = M$ and $g = f'$ can be different from zero:

$$\left. \frac{\partial^2 \overline{H}^*}{\partial s^f \partial s^{f'}} \right|_{s=0} = H^0 \left(1 + [-1+0+1] + [0+0+1]\right) = 2H^0.$$

Finally, suppose that $f, f' \in \mathcal{M}$, so that only the term $g = M$ in the sum on the right-hand side of equation (41) can be different from zero. If $f' \neq f$, we have:

$$\left. \frac{\partial^2 \overline{H}^*}{\partial s^f \partial s^{f'}} \right|_{s=0} = H^0 \left(1 + \left[-(1+\alpha\epsilon(H^0)) - \alpha\epsilon(H^0) + 2\right]\right) = 2H^0(1-\alpha\epsilon(H^0)).$$

If instead $f = f'$, then:

$$\left. \frac{\partial^2 \overline{H}^*}{\partial s^f \partial s^{f'}} \right|_{s=0} = H^0 \left(1 + \left[-(1+\alpha\epsilon(H^0)) - \alpha\epsilon(H^0) + 2(1+\alpha\epsilon(H^0))\right]\right) = 2H^0.$$

The Taylor theorem implies that

$$\begin{aligned} \overline{H}^*(s) &= H^0 \left[1 + \sum_{f \in \mathcal{F}} s^f + \frac{1}{2}\left(2 \sum_{f,g \in \mathcal{F}} s^f s^g - 2\alpha\epsilon(H^0) \sum_{\substack{f,g \in \mathcal{M} \\ f \neq g}} s^f s^g\right)\right] + o(\|s\|^2) \\ &= H^0 \left[1 + \Sigma(s) + \Sigma(s)^2 - \alpha\epsilon(H^0)\Delta\, \mathrm{HHI}(s)\right] + o(\|s\|^2). \end{aligned}$$

□

Combining Lemmas 9 and 11, we obtain the merger-induced distortion to consumer surplus, which proves the first part of Proposition 10:

Lemma 12. *At the second order in the neighborhood of $s = 0$,*

$$\Delta\, CS(s) = -H^0 \Psi'(H^0) \alpha\epsilon(H^0) \Delta\, HHI(s) + o(\|s\|^2).$$

Proof. Combining equation (36) and Lemma 11 yields

$$\begin{aligned}
\mathrm{CS}(\bar{s}(s)) &= \Psi(\overline{H}^*(s)) \\
&= \Psi(H^0) + H^0\Psi'(H^0)\left[\Sigma(s) + \Sigma(s)^2 - \alpha\epsilon(H^0)\Delta\,\mathrm{HHI}(s) - \frac{\epsilon(H^0)}{2}\Sigma(s)^2\right] + o(\|s\|^2) \\
&= \mathrm{CS}(s) - H^0\Psi'(H^0)\alpha\epsilon(H^0)\Delta\,\mathrm{HHI}(s) + o(\|s\|^2),
\end{aligned}$$

where the last line follows by Lemma 9. ☐

The next step is to approximate post-merger market shares:

Lemma 13. *At the second order in the neighborhood of $s = 0$,*

$$\bar{s}^f = s^f + o(\|s\|^2) \quad \forall f \in \mathcal{O}$$
$$\textit{and } \bar{s}^M = \sum_{f\in\mathcal{M}} s^f - \alpha\epsilon(H^0)\Delta\,\mathit{HHI}(s) + o(\|s\|^2).$$

Proof. We already showed in the proof of Lemma 11 that

$$\left.\frac{\partial \bar{s}^g}{\partial s^f}\right|_{s=0} = \begin{cases} 0 & \text{if either } f \notin \mathcal{M} \text{ and } g \neq f \text{ or } f \in \mathcal{M} \text{ and } g \neq M \\ 1 & \text{otherwise.} \end{cases}$$

Plugging equations (28) and (29) into equation (42) yields:

$$\frac{\partial \bar{s}^g}{\partial s^f} = \frac{\partial \overline{H}^*}{\partial s^f}\frac{\bar{s}^g}{\overline{H}^*}\left[-1 + \frac{\alpha\bar{s}^g}{1-\epsilon(\overline{H}^*)\bar{s}^g + \alpha(\epsilon(\overline{H}^*)\bar{s}^g)^2}\epsilon(\overline{H}^*)\zeta(\overline{H}^*)\right] + \frac{\partial \bar{T}^g}{\partial s^f}\frac{\bar{s}^g}{\bar{T}^g}\frac{(1-\epsilon(\overline{H}^*)\bar{s}^g)(1-\alpha\epsilon(\overline{H}^*)\bar{s}^g)}{1-\epsilon(\overline{H}^*)\bar{s}^g + \alpha(\epsilon(\overline{H}^*)\bar{s}^g)^2}.$$

Differentiating this with respect to $s^{f'}$ and taking the limit as s tends to zero yields

$$\left.\frac{\partial^2 \bar{s}^g}{\partial s^f \partial s^{f'}}\right|_{s=0} = -\frac{\partial \bar{s}^g}{\partial s^{f'}} + \frac{\partial^2 \bar{T}^g}{\partial s^f \partial s^{f'}}\frac{\upsilon(1)}{H^0} + \frac{\partial \bar{T}^g}{\partial s^f}\left[\frac{\partial \bar{s}^g/\bar{T}^g}{\partial s^{f'}} - \frac{\upsilon(1)}{H^0}\alpha\epsilon(H^0)\right], \tag{43}$$

where all the derivatives on the right hand-side should be understood as being evaluated at $s = 0$. Recall from the proof of Lemma (42) that

$$\left.\frac{\partial \bar{s}^g/\bar{T}^g}{\partial s^{f'}}\right|_{s=0} = -\frac{\upsilon(1)}{H^0} \times \begin{cases} 1 & \text{if either } f' \notin \mathcal{M} \text{ and } g \neq f' \text{ or } f' \in \mathcal{M} \text{ and } g \neq M \\ 1 + \alpha\epsilon(H^0) & \text{otherwise.} \end{cases}$$

Suppose first that $g \in \mathcal{O}$. If $f \neq g$ and $f' \neq g$, then all the terms on the right-hand side of equation (43) are equal to zero by lemma 10, and so

$$\left.\frac{\partial^2 \bar{s}^g}{\partial s^f \partial s^{f'}}\right|_{s=0} = 0.$$

If $f' = g$ and $f \neq g$, then

$$\left.\frac{\partial^2 \bar{s}^g}{\partial s^f \partial s^g}\right|_{s=0} = -1 + 1 + 0 = 0.$$

Finally, if $f' = f = g$, then

$$\left.\frac{\partial^2 \bar{s}^g}{\partial (s^g)^2}\right|_{s=0} = -1 + 2(1 + \alpha\epsilon(H^0)) + \left[-(1 + \alpha\epsilon(H^0)) - \alpha\epsilon(H^0)\right] = 0.$$

Hence, if $g \in \mathcal{O}$, then $\bar{s}^g = s^g + o(\|s\|^2)$ by the Taylor theorem.

Suppose instead that $g = M$. If neither f nor f' belongs to $\mathcal{M}$, then we again have that all the terms on the right-hand side of equation (43) are equal to zero, and so

$$\left.\frac{\partial^2 \bar{s}^g}{\partial s^f \partial s^{f'}}\right|_{s=0} = 0.$$

If $f' \in \mathcal{M}$ but $f \notin \mathcal{M}$, then

$$\left.\frac{\partial^2 \bar{s}^g}{\partial s^f \partial s^{f'}}\right|_{s=0} = -1 + 1 + 0 = 0.$$

Next, assume that f and f' both belong to $\mathcal{M}$. If $f \neq f'$, then

$$\left.\frac{\partial^2 \bar{s}^g}{\partial s^f \partial s^{f'}}\right|_{s=0} = -1 + 2 + \left[-(1 + \alpha\epsilon(H^0)) - \alpha\epsilon(H^0)\right] = -2\alpha\epsilon(H^0).$$

Finally, if $f = f'$, then we have again

$$\left.\frac{\partial^2 \bar{s}^g}{\partial (s^f)^2}\right|_{s=0} = -1 + 2(1 + \alpha\epsilon(H^0)) + \left[-(1 + \alpha\epsilon(H^0)) - \alpha\epsilon(H^0)\right] = 0.$$

Thus, by the Taylor theorem,

$$\bar{s}^M = \sum_{f \in \mathcal{M}} s^f - \frac{1}{2}\alpha\epsilon(H^0) \sum_{\substack{f, f' \in \mathcal{M} \\ f \neq f'}} 2 s^f s^{f'} + o(\|s\|^2) = \sum_{f \in \mathcal{M}} s^f - \alpha\epsilon(H^0)\Delta\,\mathrm{HHI}(s) + o(\|s\|^2). \quad \square$$

Combining the above lemma with Lemma 12, we can prove the second part of Proposition 10:

Lemma 14. *At the second order in the neighborhood of $s = 0$,*

$$\Delta\, AS(s) = -H^0 \Psi'(H^0)\alpha\epsilon(H^0)\Delta\, HHI(s) + o(\|s\|^2).$$

Proof. All that is left to do is approximate post-merger profits. For every $f \in \mathcal{O}$, we have that

$$\frac{\alpha \bar{s}^f}{1 - \alpha\epsilon(\overline{H}^*)\bar{s}^f} = \alpha s^f \left[1 + \alpha\epsilon(H^0) s^f\right] + o(\|s\|^2)$$

by Lemma 13. (Note that we can simply take the order-zero approximation of $\epsilon(\overline{H}^*)$ since that term is pre-multiplied by $(s^f)^2$.) Moreover, using again Lemma 13, we have that

$$\begin{aligned}
\frac{\alpha \bar{s}^M}{1-\alpha\epsilon(\overline{H}^*)\bar{s}^M} &= \alpha\left[\sum_{g\in\mathcal{M}} s^g - \alpha\epsilon(H^0)\Delta\,\mathrm{HHI}(s)\right]\left[1+\alpha\epsilon(H^0)\sum_{g\in\mathcal{M}} s^g\right] + o(\|s\|^2) \\
&= \alpha\sum_{g\in\mathcal{M}} s^g - \alpha^2\epsilon(H^0)\Delta\,\mathrm{HHI}(s) + \alpha^2\epsilon(H^0)\left[\sum_{g\in\mathcal{M}} s^g\right]^2 + o(\|s\|^2) \\
&= \sum_{g\in\mathcal{M}} \alpha s^g\left[1+\alpha\epsilon(H^0)s^g\right] + o(\|s\|^2).
\end{aligned}$$

Hence,

$$\sum_{f\in\overline{\mathcal{F}}} \frac{\alpha\bar{s}^f}{1-\alpha\epsilon(\overline{H}^*)\bar{s}^f} = \alpha\sum_{f\in\mathcal{F}} s^f + \alpha^2\epsilon(H^0)\sum_{f\in\mathcal{F}}(s^f)^2 + o(\|s\|^2).$$

Since, by Lemma 11,

$$\overline{H}^*(s)\Psi'(\overline{H}^*(s)) = H^0\Psi'(H^0)\left[1+(1-\epsilon(H^0))\sum_{f\in\mathcal{F}} s^f\right] + o(\|s\|),$$

we have that

$$\begin{aligned}
\mathrm{PS}(\bar{s}(s)) &= H^0\Psi'(H^0)\left[1+(1-\epsilon(H^0))\Sigma(s)\right]\left[\alpha\Sigma(s)+\alpha^2\epsilon(H^0)\sum_{f\in\mathcal{F}}(s^f)^2\right] + o(\|s\|^2) \\
&= H^0\Psi'(H^0)\left[\alpha\sum_{f\in\mathcal{F}} s^f + \alpha(1-\epsilon(H^0))\Sigma(s)^2 + \alpha^2\epsilon(H^0)\sum_{f\in\mathcal{F}}(s^f)^2\right] + o(\|s\|^2) \\
&= \mathrm{PS}(s) + o(\|s\|^2),
\end{aligned}$$

where the last line follows by Lemma 9. Hence, $\Delta\,\mathrm{AS}(s) = \Delta\,\mathrm{CS}(s) + o(\|s\|^2)$. □

E Conduct Approximation

E.1 Conduct in the Oligopoly Model

In this subsection, we provide more details on our treatment of firm conduct. Consider a multiproduct-firm pricing game, as defined in Section 2, and let $\theta \in [0,1]$ be a conduct parameter, as introduced in Section 6.2. The first-order condition for product i sold by firm f is given by:

$$-h_i'(p_i)\Psi'(H)\left[1-(p_i-c_i)\frac{h_i''(p_i)}{-h_i'(p_i)} - \theta\frac{\Psi''(H)}{\Psi'(H)}\sum_{j\in f}(p_j-c_j)(-h_j'(p_j))\right] = 0,$$

which simplifies to

$$(p_i - c_i)\frac{h_i''(p_i)}{-h_i'(p_i)} = 1 + \theta\epsilon(H)\frac{1}{H}\sum_{j\in f}(p_j - c_j)(-h_j'(p_j)). \tag{44}$$

Hence, the common ι-markup property continues to hold—let μ^f denotes firm f's ι-markup. It follows that equation (5) still holds, so that equation (44) simplifies to:

$$\mu^f = \frac{1}{1-\alpha\theta\epsilon(H)s^f}. \tag{45}$$

Moreover, as equations (7)–(10) only relied on the common ι-markup property, those equations continue to hold. That is, firm f's profit can still be written as $\Pi^f = \alpha\mu^f s^f H\Psi'(H)$ and we still have that

$$s^f = \frac{T^f}{H}\upsilon(\mu^f), \tag{46}$$

where the expression for firm f's type (T^f) and the function υ are as in Section 3.1.

The system of equations (45) and (46) has a unique solution in (μ^f, s^f), which defines the markup and market-share fitting-in functions: $m(H, T^f; \theta)$ and $S(H, T^f; \theta)$. It is straightforward to show that S is strictly decreasing in H, and that S is close to 1 or greater than 1 when H is close to zero, and close to 0 when H tends to infinity. The profit fitting-in function is

$$\pi(H, T^f; \theta) = \alpha m(H, T^f; \theta)S(H, T^f; \theta)H\Psi'(H) = \frac{\alpha S(H, T^f; \theta)}{1-\alpha\theta\epsilon(H)S(H, T^f; \theta)}H\Psi'(H).$$

The equilibrium condition is still that market shares add up to unity:

$$\frac{H^0}{H} + \sum_{f\in\mathcal{F}} S\left(H, T^f; \theta\right) = 1. \tag{47}$$

The properties of S imply that this equation has a unique solution, $H^*(\theta)$.

E.2 Proof of Proposition 11

In this subsection, we prove a series of lemmas that jointly imply Proposition 11. We begin by computing the partial derivatives of S with respect to H and θ at $\theta = 0$:

Lemma 15. *We have:*

$$\left.\frac{\partial S}{\partial H}\right|_{(H,T;0)} = -\frac{S(H,T;0)}{H}$$
$$\left.\frac{\partial S}{\partial \theta}\right|_{(H,T;0)} = -\alpha\epsilon(H)S(H,T;0)^2.$$

Proof. Combining equations (45) and (46) and dropping the firm superscript to ease notation yields:

$$s = \frac{T}{H}\upsilon\left(\frac{1}{1-\alpha\theta\epsilon(H)s}\right).$$

Differentiating this expression with respect to s, H, and θ at $\theta = 0$, we obtain:

$$\begin{aligned} ds &= -\frac{T}{H^2}\upsilon(1)dH + \frac{T}{H}\alpha\epsilon(H)s\upsilon'(1)d\theta \\ &= -\frac{s}{H}dH + \alpha\epsilon(H)s^2\frac{\upsilon'(1)}{\upsilon(1)}d\theta = -\frac{s}{H}dH - \alpha\epsilon(H)s^2 d\theta. \end{aligned}$$

□

Next, let us fix a profile of types $(T^f)_{f\in\mathcal{F}}$ and a value of the stage-2 outside option $H^0 \geq 0$. Let $\text{HHI}(\theta) = \sum_{f\in\mathcal{F}} s^f(\theta)^2$ be the equilibrium Herfindahl index when the conduct parameter is θ. We now compute the derivative of the equilibrium aggregator level, $H^*(\theta)$, with respect to θ at $\theta = 0$:

Lemma 16. *We have:*

$$\left.\frac{dH^*}{d\theta}\right|_{\theta=0} = -\alpha\epsilon(H^*(0))\, HHI(0)H^*(0).$$

Hence, in the neighborhood of $\theta = 0$,

$$CS(\theta) = CS(0) - \alpha\epsilon(H^*)H^*\Psi'(H^*)\, HHI\,\theta + o(\theta),$$

where the functions $H^*(\cdot)$ and $HHI(\cdot)$ are evaluated at either 0 or θ.

Proof. Totally differentiating equilibrium condition (47), we obtain:

$$\frac{dH^*}{H^*}\left(-\frac{H^0}{H^*} + \sum_{f\in\mathcal{F}} H^*\frac{\partial S(H^*, T^f;\theta)}{\partial H}\right) + d\theta\sum_{f\in\mathcal{F}}\frac{\partial S(H^*, T^f;\theta)}{\partial\theta} = 0.$$

Evaluating this at $\theta = 0$ and using Lemma 15 and the equilibrium condition yields:

$$-\frac{dH^*}{H^*(0)} - d\theta\sum_{f\in\mathcal{F}}\alpha\epsilon(H^*(0))s^f(0)^2 = 0,$$

which proves the first part of the lemma.

The second part of the lemma follows by Taylor's theorem:

$$\begin{aligned} \text{CS}(\theta) &= \Psi(H^*(0)) + H^{*\prime}(0)\Psi'(H^*(0))\theta + o(\theta) \\ &= \text{CS}(0) - \alpha\epsilon(H^*(0))H^*(0)\Psi'(H^*(0))\,\text{HHI}(0)\theta + o(\theta) \\ &= \text{CS}(0) - \alpha\epsilon(H^*(\theta))H^*(\theta)\Psi'(H^*(\theta))\,\text{HHI}(\theta)\theta + o(\theta), \end{aligned}$$

where the third line follows as

$$\epsilon(H^*(\theta))H^*(\theta)\Psi'(H^*(\theta))\,\text{HHI}(\theta) = \epsilon(H^*(0))H^*(0)\Psi'(H^*(0))\,\text{HHI}(0) + o(1).$$

□

Let $\mathrm{PS}(\theta)$ be equilibrium industry profit when the conduct parameter is θ. We have:

Lemma 17.

$$PS'(0) = \alpha^2 H^* \Psi'(H^*) \epsilon(H^*)^2 \, HHI \sum_{f \in \mathcal{F}} s^f,$$

where the functions $H^*(\cdot)$, $HHI(\cdot)$, and $s^f(\cdot)$ are evaluated at 0.

Proof. Let

$$\pi^f(\theta) = \frac{\alpha s^f(\theta)}{1 - \alpha\theta\epsilon(H^*(\theta)) s^f(\theta)} H^* \Psi'(H^*(\theta))$$

denote firm f's equilibrium profit. We have that

$$\begin{aligned} s^{f\prime}(0) &= H^{*\prime}(0) \left. \frac{\partial S}{\partial H} \right|_{(H^*(0), T^f; 0)} + \left. \frac{\partial S}{\partial \theta} \right|_{(H^*(0), T^f; 0)} \\ &= \alpha\epsilon(H^*(0)) \left[\mathrm{HHI}(0) s^f(0) - s^f(0)^2 \right], \end{aligned}$$

where we have used Lemmas 15 and 16. It follows that

$$\begin{aligned} \pi^{f\prime}(0) &= \left[\alpha s^{f\prime} + \alpha^2 \epsilon(H^*)(s^f)^2 \right] H^* \Psi'(H^*) + \alpha s^f H^{*\prime} \Psi'(H^*) \left[1 - \epsilon(H^*) \right] \\ &= \alpha^2 H^* \Psi'(H^*) \epsilon(H^*)^2 \, \mathrm{HHI} \, s^f, \end{aligned}$$

where all the functions are evaluated at $\theta = 0$. Adding up over all firms proves the lemma. □

Combining Lemmas 16 and 17 yields:

Lemma 18. *In the neighborhood of $\theta = 0$,*

$$AS(\theta) = AS(0) - \alpha\epsilon(H^*) H^* \Psi'(H^*) \, HHI \left[1 - \alpha\epsilon(H^*) \sum_{f \in \mathcal{F}} s^f \right] \theta + o(\theta),$$

where the functions $H^*(\cdot)$, $HHI(\cdot)$, and $s^f(\cdot)$ are evaluated at either 0 or θ.

We are now in a position to prove Proposition 11:

Proof. Consider a merger M between the firms in $\mathcal{M}$, and suppose the merger involves no synergies. For every θ, let $\mathrm{CS}(\theta)$, $\mathrm{AS}(\theta)$, $H^*(\theta)$, $\mathrm{HHI}(\theta)$, and $s^f(\theta)$ denote the pre-merger equilibrium values of consumer surplus, aggregate surplus, the aggregator, the Herfindahl index, and firm f's market share, respectively. We use the same notation with an upper bar to denote the post-merger values of those quantities. The discussion at the beginning of Section 6.2 implies that, under monopolistic competition ($\theta = 0$), $\mathrm{CS}(0) = \overline{\mathrm{CS}}(0)$, $\mathrm{AS}(0) = \overline{\mathrm{AS}}(0)$, $H^*(0) = \overline{H}^*(0)$, and $s^f(0) = \overline{s}^f(0)$ for every $f \in \mathcal{F} \setminus \mathcal{M}$. Moreover, $s^M(0) = \sum_{f \in \mathcal{M}} s^f(0)$ and

$$\overline{\mathrm{HHI}}(0) - \mathrm{HHI}(0) = \sum_{f \in (\mathcal{F} \setminus \mathcal{M}) \cup M} \overline{s}^f(0)^2 - \sum_{f \in \mathcal{F}} s^f(0)^2$$

$$= \left(\sum_{f \in \mathcal{M}} s^f(0)\right)^2 - \sum_{f \in \mathcal{M}} s^f(0)^2$$
$$= \Delta \, \mathrm{HHI}(0).$$

Thus, at $\theta = 0$, the merger-induced, naively computed change in the Herfindahl index coincides with the actual change in the Herfindahl index.

By Lemma 16, we have that

$$\begin{aligned}
\overline{\mathrm{CS}}(\theta) - \mathrm{CS}(\theta) &= \alpha \left[\epsilon(H^*(0))H^*(0)\Psi'(H^*(0)) \, \mathrm{HHI}(0) - \epsilon(\overline{H}^*(0))\overline{H}^*(0)\Psi'(\overline{H}^*(0))\overline{\mathrm{HHI}}(0)\right] \theta + o(\theta) \\
&= \alpha\epsilon(H^*(0))H^*(0)\Psi'(H^*(0)) \left[\mathrm{HHI}(0) - \overline{\mathrm{HHI}}(0)\right] \theta + o(\theta) \\
&= -\alpha\epsilon(H^*(0))H^*(0)\Psi'(H^*(0))\Delta \, \mathrm{HHI}(0)\theta + o(\theta).
\end{aligned}$$

Note that the coefficient on θ can be replaced by $-\alpha\epsilon(H^*(\theta))H^*(\theta)\Psi'(H^*(\theta))\Delta \, \mathrm{HHI}(\theta)$ since all the functions involved are continuous.

Similarly, applying Lemma 18 to $\overline{\mathrm{AS}}(\theta) - \mathrm{AS}(\theta)$, we obtain the second part of the proposition. □

F Technical Results on Type Aggregation

F.1 Type Indexation

In the main text, we proposed three definitions of the concept of type aggregation (Definitions 1–3). In a nutshell, $\mathcal{H}$ satisfies type aggregation if: (a) There exists a mapping $\tau : \Omega(\mathcal{H}) \longrightarrow \mathbb{R}$ such that firms ω and ω' have the same markup and market-share fitting-in functions if and only if $\tau(\omega) = \tau(\omega')$; and (b) the mapping τ is economically meaningful. In the following, we drop requirement (b) and say that $\mathcal{H}$ satisfies *type indexation* if requirement (a) holds. We now argue that a sufficient condition for type indexation is that $\mathcal{H}$ has the cardinality of the continuum.

Suppose indeed that there exists a bijection $H : x \in \mathbb{R} \mapsto H[x] \in \mathcal{H}$. The set of multiproduct firms can be written as $\Omega(\mathcal{H}) = \bigcup_{n \geq 1} \Omega_n(\mathcal{H})$, where

$$\begin{aligned}
\Omega_n(\mathcal{H}) &= \{((h_j)_{1 \leq j \leq n}, (c_j)_{1 \leq j \leq n}) : \ \forall j, \ h_j \in \mathcal{H} \text{ and } c_j > 0\} \\
&= \{((H[x_j])_{1 \leq j \leq n}, (c_j)_{1 \leq j \leq n}) : \ \forall j, \ x_j \in \mathbb{R} \text{ and } c_j \in \mathbb{R}_{++}\}.
\end{aligned}$$

Thus, with a slight abuse of notation we can write $\Omega_n(\mathcal{H}) = \mathbb{R}^n \times \mathbb{R}^n_{++}$ and $\Omega(\mathcal{H}) = \bigcup_{n \geq 1} \mathbb{R}^n \times \mathbb{R}^n_{++}$.

For every $n \geq 1$, let $\tilde{\tau}_n$ be an injection from $\mathbb{R}^n \times \mathbb{R}^n_{++}$ to the interval $(n-1, n)$.[23] For every $\omega \in \Omega(\mathcal{H})$, let $\tilde{\tau}(\omega) \equiv \tilde{\tau}_{n(\omega)}(\omega)$, where $n(\omega)$ is the number of products that firm ω owns (i.e., $\omega \in \Omega_{n(\omega)}(\mathcal{H})$). Then, $\tilde{\tau}$ is an injection from $\Omega(\mathcal{H})$ to $\mathbb{R}_{++}$.

Next, define the equivalence relation $\sim$ as follows: For every $\omega, \omega' \in \Omega(\mathcal{H})$, $\omega \sim \omega'$ if and only if $m[\omega] = m[\omega']$ and $S[\omega] = S[\omega']$. For every y in the quotient set $\Omega(\mathcal{H})/\sim$, let $\phi(y) \in \Omega(\mathcal{H})$ be a representative of the equivalence class y. Finally, define $\tau : \omega \in \Omega(\mathcal{H}) \mapsto \tilde{\tau}(\phi(e(\omega))) \in \mathbb{R}_{++}$, where $e(\omega)$ denotes the equivalence class of ω. Since $\tilde{\tau}$ and ϕ are injective, we have that $\tau(\omega) = \tau(\omega')$ is equivalent to $e(\omega) = e(\omega')$, which is in turn equivalent to $m[\omega] = m[\omega']$ and $S[\omega] = S[\omega']$. It follows that $\Omega(\mathcal{H})$ satisfies type indexation (with type mapping τ).

F.2 Proof of Proposition 12

It is clear that (i) implies (iii), i.e., surplus-based type aggregation implies continuous type aggregation, as the type mapping in Definition 1 (see equation (23)) obviously satisfies the continuity requirement of Definition 3. Next, let us verify that (iv) implies (i) and (ii):

Proof. Suppose first that part (1) of assertion (iv) holds for some $\sigma > 1$, and let the type mapping τ be as defined in equation (23). Consider a multiproduct firm $\omega = ((h_j)_{j \in f}, (c_j)_{j \in f}) \in \Omega(\mathcal{H})$. Then, there exist $(a_j)_{j \in f} \in \mathbb{R}^f_{++}$ and $(\beta_j) \in \mathbb{R}^f_{+}$ such that for every $j \in f$ and $p_j > \beta_j/(\sigma - 1)$, $h_j(p_j) = a_j (p_j + \beta_j)^{1-\sigma}$. Routine calculations show that (using the notation of Section 7):

$$
\begin{aligned}
\iota_j(p_j) &= \sigma \frac{p_j}{p_j + \beta_j} \qquad \forall p_j > \frac{\beta_j}{\sigma - 1}, \\
\underline{p}_j &= \frac{\beta_j}{\sigma - 1}, \\
\gamma_j(p_j) &= \frac{\sigma - 1}{\sigma} h_j(p_j) \qquad \forall p_j > \frac{\beta_j}{\sigma - 1}, && (48) \\
\text{and } \nu_j(p_j) &= \sigma \frac{p_j - c_j}{p_j + \beta_j} \qquad \forall p_j > \max\left(\underline{p}_j, c_j\right). && (49)
\end{aligned}
$$

Plugging equation (48) into equation (20), using equation (21), and rearranging terms, we obtain

$$
m[\omega](H) = \frac{1}{1 - \alpha \epsilon(H) S[\omega](H)},
$$

[23]The classic example of an injection from $(0,1)^2$ to $(0,1)$ is the function χ, which, to every x and y with decimal expansions $x = \sum_{k=1}^{\infty} x_k 10^{-k}$ and $y = \sum_{k=1}^{\infty} y_k 10^{-k}$, associates $\chi(x, y) = \sum_{k=0}^{\infty} \left(x_k 10^{-1} + y_k 10^{-2}\right) 10^{-2k}$. Using the same approach, one can construct an injection from $(0,1)^n$ to $(0,1)$ for any n. Since the interval $(0,1)$ can be put in bijection with any non-degenerate interval, one can then construct an injection from $\mathbb{R}^n \times \mathbb{R}^n_{++}$ to $(n-1, n)$ for any n.

where $\alpha \equiv (\sigma-1)/\sigma$.

Inverting equation (49), we obtain $r_j(\mu) = (\sigma c_j + \mu\beta_j)/(\sigma-\mu)$ for every $\mu \in (1,\sigma)$. Plugging this into equation (21) yields:

$$\begin{aligned} S[\omega](H) &= \frac{1}{H}\sum_{j\in f} a_j \left(\beta_j + \frac{\sigma c_j + m[\omega](H)\beta_j}{\sigma - m[\omega](H)}\right)^{1-\sigma} \\ &= \frac{1}{H}\left(1-\frac{m[\omega](H)}{\sigma}\right)^{\sigma-1}\sum_{j\in f}(c_j+\beta_j)^{1-\sigma} \\ &= \frac{\tau(\omega)}{H}\left(1-(1-\alpha)m[\omega](H)\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Hence, $(m[\omega](H), S[\omega](H))$ is the unique solution (in (μ^f, s^f)) of the system of equations (6) and (10) (with T^f replaced by $\tau(\omega)$). It follows that $\mathcal{H}$ satisfies surplus-based type aggregation. Moreover, part (i) of Lemma 2 implies that $\mathcal{H}$ also satisfies monotonic type aggregation.

Next, suppose that part (2) of assertion (iv) holds, and let the type mapping τ be as defined in equation (23). Consider a multiproduct firm $\omega = ((h_j)_{j\in f}, (c_j)_{j\in f}) \in \Omega(\mathcal{H})$. Then, there exist $(a_j)_{j\in f} \in \mathbb{R}^f$ and $(\lambda_j) \in \mathbb{R}^f_{++}$ such that for every $j \in f$ and $p_j > \lambda_j$, $h_j(p_j) = \exp\frac{a_j - p_j}{\lambda_j}$. Routine calculations show that

$$\begin{aligned} \iota_j(p_j) &= \frac{p_j}{\lambda_j} \qquad \forall p_j > \lambda_j, \\ \underline{p}_j &= \lambda_j, \\ \gamma_j(p_j) &= h_j(p_j) \qquad \forall p_j > \lambda_j, \qquad (50)\\ \text{and } \nu_j(p_j) &= \frac{p_j - c_j}{\lambda_j} \qquad \forall p_j > \max\left(\underline{p}_j, c_j\right). \qquad (51) \end{aligned}$$

Plugging equation (50) into equation (20), using equation (21), and rearranging terms, we obtain

$$m[\omega](H) = \frac{1}{1-\epsilon(H)S[\omega](H)}.$$

Inverting equation (51), we obtain $r_j(\mu) = \lambda_j\mu + c_j$ for every $\mu \in (1,\infty)$. Plugging this into equation (21) yields:

$$S[\omega](H) = \frac{1}{H}\sum_{j\in f} \mathrm{e}^{\frac{a_j - c_j}{\lambda_j}}\,\mathrm{e}^{-m[\omega](H)} = \frac{\tau(\omega)}{H}\,\mathrm{e}^{-m[\omega](H)}.$$

Hence, $(m[\omega](H), S[\omega](H))$ is the unique solution (in (μ^f, s^f)) of the system of equations (6) and (10) (with $\alpha = 1$ and T^f replaced by $\tau(\omega)$). It follows that $\mathcal{H}$ satisfies surplus-based type aggregation. Moreover, part (i) of Lemma 2 implies that $\mathcal{H}$ also satisfies monotonic type aggregation. □

Next, we introduce new notation. For every $h \in \mathcal{H}$ and $p > 0$, let $\iota[h](p) \equiv -ph''(p)/h'(p)$ and $\gamma[h](p) \equiv (h'(p))^2/h''(p)$. For every $c > 0$, let $r[h, c](\cdot)$ be the inverse function of

$$\nu[h,c] : p \in \left(\max(c, \underline{p}[h]), \infty\right) \mapsto \frac{p-c}{p}\iota[h](p) \in \left(\nu[h,c]\left(\max(c,\underline{p}[h])\right), \iota[h](\infty)\right),$$

where $\underline{p}[h] \equiv \inf\{p > 0 : \iota[h](p) > 1\}$. We extend the domain of $r[h, c]$ to $\left(\nu[h,c]\left(\max(c,\underline{p}[h])\right), \infty\right)$ by setting $r[h, c](\mu) = \infty$ for every $\mu \geq \iota[h](\infty)$. An immediate observation, which will be very useful in many of the proofs below, is that, for every $a > 0$, $c > 0$, and $h \in \mathcal{H}$, $\iota[ah] = \iota[h]$, $\gamma[ah] = a\gamma[h]$, and $r[ah, c] = r[h, c]$.

Let us show that (ii) implies (iii):

Lemma 19. *If $\mathcal{H}$ satisfies monotonic type aggregation, then $\mathcal{H}$ satisfies continuous type aggregation.*

Proof. Suppose $\mathcal{H}$ satisfies monotonic type aggregation, and let $\tau(\omega)$, $S(\cdot, \tau(\omega))$, and $m(\cdot, \tau(\omega))$ be as in Definition 2. For every $\omega \in \Omega(\mathcal{H})$, define $\hat{\tau}(\omega) \equiv S[\omega](1)$, and let $\hat{T}$ be the range of $\hat{\tau}$. Requirement (b) of Definition 2 implies that, for every $\omega, \omega' \in \Omega(\mathcal{H})$, $\hat{\tau}(\omega) = \hat{\tau}(\omega')$ if and only if $\tau(\omega) = \tau(\omega')$. For every $t \in \hat{T}$, we can thus define $\hat{m}(\cdot, t)$ and $\hat{S}(\cdot, t)$ by picking an arbitrary ω such that $\hat{\tau}(\omega) = t$ and setting $\hat{m}(\cdot, t) \equiv m(\cdot, \tau(\omega))$ and $\hat{S}(\cdot, t) \equiv S(\cdot, \tau(\omega))$. The triple $(\hat{\tau}, \hat{m}, \hat{S})$ then satisfies requirement (a) of Definition 3.

Next, let us show that $\hat{\tau}$ satisfies the continuity requirement of Definition 3. Fix some finite set f and some $(h_j)_{j\in f}$ in $\mathcal{H}^f$. For every $(a, c) = ((a_j)_{j\in f}, (c_j)_{j\in f}) \in \mathbb{R}^f_{++} \times \mathbb{R}^f_{++}$, let $\omega(a, c) = ((a_j h_j)_{j\in f}, (c_j)_{j\in f}) \in \Omega(\mathcal{H})$. For every (a, c), $m[\omega(a, c)](1)$ is the unique solution in μ of equation

$$\mu = 1 + \epsilon(1)\mu \sum_{j\in f} \gamma[a_j h_j]\left(r[a_j h_j, c_j](\mu)\right),$$

which can be rewritten as

$$\mu = 1 + \epsilon(1)\mu \sum_{j\in f} a_j \gamma[h_j]\left(r[h_j, c_j](\mu)\right).$$

As all the functions on the right-hand side are continuous (see Lemma E in Nocke and Schutz, 2018), standard arguments imply that $m[\omega(a, c)](1)$ is continuous. It follows that

$$\hat{\tau}(\omega(a,c)) = S[\omega(a,c)](1) = \sum_{j\in f} a_j h_j\left(r[h_j, c_j]\left(m[\omega(a,c)](1)\right)\right)$$

is continuous in (a, c). □

We now turn our attention to the more challenging part of the proof, i.e., showing that (iii) implies (iv). The following lemma reformulates assertion (iv):

Lemma 20. *The following statements are equivalent:*

(i) Assertion (iv) in Proposition 12 holds.

(ii) There exists $\rho \geq 1$ *such that, for every* $h \in \mathcal{H}$ *and* $p > \underline{p}[h]$, $h(p)h''(p)/h'(p)^2 = \rho$.

Proof. It is straightforward to check that (i) implies (ii) (with $\rho = \sigma/(\sigma-1)$ in case (1) and $\rho = 1$ in case (2)). Conversely, suppose that (ii) holds for some $\rho \geq 1$, and let $h \in \mathcal{H}$. Then, on $(\underline{p}[h], \infty)$, we have

$$\frac{h''}{h'} = \rho \frac{h'}{h}.$$

Integrating yields:

$$\log(-h') = \rho \log h + K,$$

where K is a constant of integration. Taking exponentials yields $-h' = \mathrm{e}^K h^\rho$, i.e., $-h'/h^\rho = \mathrm{e}^K$. If $\rho = 1$, then this implies that $\log h(p) = -\mathrm{e}^K p + L$ for some constant of integration L, and thus

$$h(p) = \exp\left(L - \mathrm{e}^K p\right) \equiv \exp \frac{a-p}{\lambda}$$

for every $p > \underline{p}[h]$. It follows that $\iota[h](p) = p/\lambda$ for every $p > \underline{p}[h]$, and so $\underline{p}[h] = \lambda$. Hence, $h(p) = \exp \frac{a-p}{\lambda}$ for every $p > \lambda$, as stated.

If, instead, $\rho > 1$, then we obtain

$$-\frac{1}{1-\rho} h^{1-\rho}(p) = \mathrm{e}^K p + L, \quad \text{i.e.,} \quad h^{1-\rho}(p) = (\rho - 1)\,\mathrm{e}^K p + L'.$$

Taking exponent $1/(1-\rho)$ in the above equation and setting $\sigma = \rho/(\rho-1)$, we obtain that

$$h(p) = \underbrace{\left[(\rho-1)\,\mathrm{e}^K\right]^{1-\sigma}}_{\equiv a} \left[p + \underbrace{\frac{L'}{(\rho-1)\,\mathrm{e}^K}}_{\equiv \beta} \right]^{1-\sigma}$$

for every $p > \underline{p}[h]$. Moreover, as $\iota[h](p) = \sigma p/(p+\beta)$ and $\iota[h](\cdot)$ must be non-decreasing on $(\underline{p}[h], \infty)$, β must be non-negative. Moreover, $\underline{p}[h] = \beta/(\sigma-1)$. Hence, $h(p) = a(p+\beta)^{1-\sigma}$ for every $p > \beta/(\sigma-1)$, as stated. □

Thus, we need to show that continuous type aggregation implies assertion (ii) in Lemma 20. We do so in two steps:

Lemma 21. *If* $\mathcal{H}$ *satisfies continuous type aggregation, then for every* $h \in \mathcal{H}$, $hh''/(h')^2$ *is constant on* $(\underline{p}[h], \infty)$.

Proof. Assume for a contradiction that $\mathcal{H}$ satisfies continuous type aggregation and there exists $h \in \mathcal{H}$ such that $hh''/(h')^2 = h/\gamma[h]$ is not constant on $(\underline{p}[h], \infty)$. To ease notation, let $\iota \equiv \iota[h]$, $\underline{p} \equiv \underline{p}[h]$, $\gamma \equiv \gamma[h]$, and $r(\cdot, c) \equiv r[h, c](\cdot)$.

As h/γ is non-constant, there exists $p^0 > \underline{p}$ such that $\frac{d}{dp}\frac{h(p)}{\gamma(p)}\Big|_{p^0} \neq 0$. Choose $\mu^0 \in (1, \iota(\infty))$ and $c^0 > 0$ such that $r(\mu^0, c^0) = p^0$; such a pair (μ^0, c^0) exists since $r(\mu, c)$ is continuous and $\lim_{c\to 0} r(1, c) = \underline{p} < p^0 < \lim_{c\to\infty} r(1, c) = \infty$. Next, let a^0 be such that $\mu^0 = 1 + \epsilon(1)\mu^0 a^0 \gamma(p^0)$. Finally, let $s^0 \equiv a^0 h(p^0)$.

For every $(a, c) \in \mathbb{R}^2_{++}$, let $\tilde{m}(a, c)$ be the unique solution in μ of equation

$$1 = \frac{1}{\mu} + \epsilon(1) a \gamma\left(r(\mu, c)\right). \tag{52}$$

Define also $\tilde{S}(a, c) \equiv ah\left(r(\tilde{m}(a, c), c)\right)$ and $\xi(a, c) \equiv \left(\tilde{m}(a, c), \tilde{S}(a, c)\right)$. Clearly, $\xi(a^0, c^0) = (\mu^0, s^0)$ and $\xi(a, c) = (m[(ah, c)](1), S[(ah, c)](1))$ for every (a, c).

Note that: The right-hand side of equation (52) is continuously differentiable in (μ, a, c); the partial derivative with respect to μ is equal to $-\frac{1}{\mu^2} + a\epsilon(1)\frac{\partial r}{\partial \mu}\gamma'(r(\mu, c))$, which is different from zero since $\gamma' < 0$ and $\partial r/\partial \mu > 0$ by Lemmas A and E in Nocke and Schutz (2018); the partial derivative with respect to c is also different from 0 (see again Lemmas A and E in Nocke and Schutz, 2018). The implicit function theorem implies that $\tilde{m}$ is continuously differentiable and $\partial \tilde{m}(a, c)/\partial c$ is different from 0 for every (a, c). Moreover,

$$\frac{\partial \tilde{m}}{\partial a} = \frac{\gamma}{a\gamma' \partial r/\partial c}\frac{\partial \tilde{m}}{\partial c},$$

where γ and γ' are evaluated at $r(\tilde{m}(a, c), c)$ and $\partial r/\partial c$ is evaluated at $(\tilde{m}(a, c), c)$. As $\tilde{m}$ is continuously differentiable, so are $\tilde{S}$ and ξ.

Let us now show that the Jacobian matrix of ξ at (a^0, c^0) is non-singular. Differentiating $\tilde{S}(a, c)$ yields

$$\frac{\partial S}{\partial a} = h + ah'\frac{\partial r}{\partial \mu}\frac{\partial \tilde{m}}{\partial a} \quad \text{and} \quad \frac{\partial S}{\partial c} = ah'\left[\frac{\partial r}{\partial \mu}\frac{\partial \tilde{m}}{\partial c} + \frac{\partial r}{\partial c}\right],$$

where h and h' are evaluated at $r(\tilde{m}(a, c), c)$ and the partial derivatives of r are evaluated at $(\tilde{m}(a, c), c)$. The Jacobian determinant of ξ is

$$\begin{aligned}
\frac{\partial \tilde{m}}{\partial a}\frac{\partial \tilde{S}}{\partial c} - \frac{\partial \tilde{S}}{\partial a}\frac{\partial \tilde{m}}{\partial c} &= \frac{\partial \tilde{m}}{\partial a} ah'\left[\frac{\partial r}{\partial \mu}\frac{\partial \tilde{m}}{\partial c} + \frac{\partial r}{\partial c}\right] - \frac{\partial \tilde{m}}{\partial c}\left[h + ah'\frac{\partial r}{\partial \mu}\frac{\partial \tilde{m}}{\partial a}\right] \\
&= \frac{\partial \tilde{m}}{\partial a} ah'\frac{\partial r}{\partial c} - \frac{\partial \tilde{m}}{\partial c} h \\
&= \frac{\partial \tilde{m}}{\partial c}\left[h'\frac{\gamma}{\gamma'} - h\right],
\end{aligned}$$

where h and γ and their derivatives are evaluated at $r(\tilde{m}(a, c), c)$. Since $\partial \tilde{m}/\partial c \neq 0$, the Jacobian of ξ is non-singular at (a, c) if and only if the term inside square brackets is different from zero, which is equivalent to $(h(p)/\gamma(p))'$ being different from zero at $p = r(\tilde{m}(a,c),c)$. At (a^0, c^0), $r(\tilde{m}(a^0,c^0),c^0) = p^0$, and so the derivative is indeed different from zero.

Thus, ξ is continuously differentiable and it has a non-singular Jacobian matrix at (a^0, c^0). The inverse function theorem implies the existence of an open ball $B \subset \mathbb{R}^2_{++}$ containing (a^0, c^0) such that the restriction of ξ to B is one-to-one. Hence, for every (a, c) and (a', c') in B such that $(a, c) \neq (a', c')$, we have $\xi(a, c) \neq \xi(a', c')$. This means that, for every such (a, c) and (a', c'), $m[(ah, c)](1) \neq m[(a'h, c')](1)$ and/or $S[(ah, c)](1) \neq S[(a'h, c')](1)$, and thus $\tau((ah, c)) \neq \tau((a'h, c'))$. It follows that the function $\tilde{\tau} : (a, c) \in B \mapsto \tau((ah, c)) \in \mathbb{R}$ is one-to-one. Moreover, $\tilde{\tau}$ is continuous since $\mathcal{H}$ satisfies continuous type aggregation. We have thus found a continuous injection from an open subset of $\mathbb{R}^2$ to $\mathbb{R}$. As is well known, such a function does not exist; we therefore have a contradiction.[24] $\square$

Lemma 22. *If $\mathcal{H}$ satisfies continuous type aggregation, then there exists $\rho \geq 1$ such that, for every $h \in \mathcal{H}$ and $p > \underline{p}[h]$, $h(p)h''(p)/h'(p)^2 = \rho$.*

Proof. Suppose that $\mathcal{H}$ satisfies continuous type aggregation. By Lemma 21, for every $h \in \mathcal{H}$, there exists a scalar $\rho[h]$ such that $h(p)h''(p)/h'(p)^2 = \rho[h]$ for every $p > \underline{p}[h]$. Moreover, by log-convexity, $\rho[h] \geq 1$ for every $h \in \mathcal{H}$. Assume for a contradiction that there exist $h_1, h_2 \in \mathcal{H}$ such that $\rho[h_1] \neq \rho[h_2]$. To ease notation, let $\iota_i \equiv \iota[h_i]$, $\underline{p}_i \equiv \underline{p}[h_i]$, $\rho_i \equiv \rho[h_i]$, $\gamma_i \equiv \gamma[h_i]$, and $r_i(\cdot, c) \equiv r[h_i, c](\cdot)$ for $i = 1, 2$. Note that $h_i(p_i) = \rho_i \gamma_i(p_i)$ for every $p_i > \underline{p}_i$ $(i = 1, 2)$.

Let $\mu^0 \in (1, \min(\iota_1(\infty), \iota_2(\infty)))$ and $c^0 = (c_1^0, c_2^0) \in \mathbb{R}^2_{++}$. Let $a > 0$ be such that

$$\mu^0 = 1 + a\epsilon(1)\mu^0 \left[\gamma_1(r_1(\mu^0, c_1^0)) + \gamma_2(r_2(\mu^0, c_2^0))\right].$$

For every $c = (c_1, c_2) \in \mathbb{R}^2_{++}$, let $\tilde{m}(c)$ be the unique solution in μ of equation

$$1 = \frac{1}{\mu} + a\epsilon(1) \left[\gamma_1(r_1(\mu, c_1)) + \gamma_2(r_2(\mu, c_2))\right]. \tag{53}$$

Clearly, $\tilde{m}(c^0) = \mu^0$, and so $r_i(\tilde{m}(c^0), c_i) < \infty$ for $i = 1, 2$. Define also

$$\tilde{S}(c) \equiv a\left[h_1(r_1(\tilde{m}(c), c_1)) + h_2(r_2(\tilde{m}(c), c_2))\right]$$

and $\xi(c) \equiv (\tilde{m}(c), \tilde{S}(c))$. Note that

$$\tilde{m}(c) = m[(ah_1, ah_2), (c_1, c_2)](1) \quad \text{and} \quad \tilde{S}(c) = S[(ah_1, ah_2), (c_1, c_2)](1).$$

[24] The reason why such a function does not exist is the following. As $\tilde{\tau}$ is continuous and B is connected, $\tilde{\tau}(B)$ is a connected subset of $\mathbb{R}$; it is therefore an interval. Let t be a point in the interior of that interval, and let $(\hat{a}, \hat{c})$ be the unique element of B such that $\tilde{\tau}(\hat{a}, \hat{c}) = t$; $(\hat{a}, \hat{c})$ is indeed unique, as $\tilde{\tau}$ is one-to-one. Then, despite $\tilde{\tau}$ being continuous and $B \setminus \{(\hat{a}, \hat{c})\}$ being connected, $\tilde{\tau}(B \setminus \{(\hat{a}, \hat{c})\}) = \tilde{\tau}(B) \setminus \{t\}$ is not connected, a contradiction.

Applying the implicit function theorem to equation (53), we obtain that $\tilde{m}$ is continuously differentiable in a neighborhood of c^0 and that its partial derivatives are different from zero. (The argument is the same as in the proof of lemma 21.) Moreover,

$$\frac{\partial \tilde{m}}{\partial c_2} = \frac{\gamma_2' \frac{\partial r_2}{\partial c_2}}{\gamma_1' \frac{\partial r_1}{\partial c_1}} \frac{\partial \tilde{m}}{\partial c_1}, \tag{54}$$

where γ_i is evaluated at $r_i(\tilde{m}(c), c_i)$ and $\partial r_i / \partial c_i$ is evaluated at $(\tilde{m}(c), c_i)$, $i = 1, 2$. Since $\tilde{m}$ is $\mathcal{C}^1$, so are $\tilde{S}$ and ξ. The partial derivative of $\tilde{S}$ with respect to c_i is:

$$\frac{\partial \tilde{S}}{\partial c_i} = a \left[h_i' \frac{\partial r_i}{\partial c_i} + \frac{\partial \tilde{m}}{\partial c_i} \sum_{j=1}^{2} h_j' \frac{\partial r_j}{\partial \mu} \right],$$

where h_k' is evaluated at $r_k(\tilde{m}(c), c_k)$ and the partial derivatives of r_k are evaluated at $(\tilde{m}(c), c_k)$.

Let us now show that the Jacobian matrix of ξ is non-singular at c^0. The Jacobian determinant is:

$$\begin{aligned}
\frac{\partial \tilde{m}}{\partial c_1} \frac{\partial \tilde{S}}{\partial c_2} - \frac{\partial \tilde{m}}{\partial c_2} \frac{\partial \tilde{S}}{\partial c_1} &= a \frac{\partial \tilde{m}}{\partial c_1} \left[h_2' \frac{\partial r_2}{\partial c_2} + \frac{\partial \tilde{m}}{\partial c_2} \sum_{j=1}^{2} h_j' \frac{\partial r_j}{\partial \mu} \right] - a \frac{\partial \tilde{m}}{\partial c_2} \left[h_1' \frac{\partial r_1}{\partial c_1} + \frac{\partial \tilde{m}}{\partial c_1} \sum_{j=1}^{2} h_j' \frac{\partial r_j}{\partial \mu} \right] \\
&= a \left[\frac{\partial \tilde{m}}{\partial c_1} h_2' \frac{\partial r_2}{\partial c_2} - \frac{\partial \tilde{m}}{\partial c_2} h_1' \frac{\partial r_1}{\partial c_1} \right] \\
&= a \frac{\partial \tilde{m}}{\partial c_1} \frac{\partial r_2}{\partial c_2} \gamma_2' \left[\frac{h_2'}{\gamma_2'} - \frac{h_1'}{\gamma_1'} \right] \quad \text{using equation (54)} \\
&= a \frac{\partial \tilde{m}}{\partial c_1} \frac{\partial r_2}{\partial c_2} \gamma_2' (\rho_2 - \rho_1) \neq 0.
\end{aligned}$$

We can conclude as in the proof of Lemma 22. As the $\mathcal{C}^1$ function ξ has a non-singular Jacobian matrix at c^0, the inverse function theorem implies that it is locally one-to-one. Hence, there is an open ball B containing c^0 such that, for every $c, c' \in B$ such that $c \neq c'$, we have that $m[(ah_1, ah_2), (c_1, c_2)](1) \neq m[(ah_1, ah_2), (c_1', c_2')](1)$ and/or $S[(ah_1, ah_2), (c_1, c_2)](1) \neq S[(ah_1, ah_2), (c_1', c_2')](1)$. It follows that the function $\tilde{\tau} : c \in B \mapsto \tau((ah_1, ah_2), (c_1, c_2)) \in \mathbb{R}$, which is continuous since $\mathcal{H}$ satisfies continuous type aggregation, is one-to-one. Since there is no continuous injection from an open subset of $\mathbb{R}^2$ to $\mathbb{R}$ (see footnote 24), we have a contradiction. □

Combining Lemmas 20 and 22, we obtain that (iii) implies (iv), which concludes the proof of the proposition.

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