Competition in Two-Sided Markets: An Aggregative-Games Approach*

Preliminary draft: comments welcome

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This article develops an aggregative-games framework for studying asymmetric platform oligopoly in two-sided markets. Using a model of platform choice that has a unique stable consumption equilibrium, I derive an IIA demand system for two-sided platforms that generalizes multinomial logit models. Exploiting the IIA property of the demand, I represent platform competition as an aggregative game and apply it to three competition analyses: platform dominance, platform mergers, and long-run equilibrium with free entry. The dominance of a large platform is associated with a higher consumer surplus on one side only when the consumers benefit from both network effects and two-sided pricing. The merger analysis demonstrates that network effects serve as a synergy but also make large mergers harmful to consumers, and the pre-merger price structure provides useful information on the effects of two-sided pricing. In an equilibrium with free entry, any change in competitive environments that benefits consumers on one side hurts consumers on the other side.

Keywords: Aggregative games; network effects; two-sided markets; dominance; mergers; free entry

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1. Introduction

Recent decades have seen a massive growth of large digital platforms such as Alphabet, Amazon, Apple, Meta, and Microsoft, among others. Competition concerns about these platforms abound, including the potential harms of acquisitions of competitors, predatory behaviors, and other practices such as self-preferencing (e.g., Australian Competition and Consumer Commission, 2018; Khan, 2017; US House of Representatives Subcommittee on Antitrust, 2020).

A common issue in these competition concerns is the role that network effects and two-sidedness play in making these practices anti- or procompetitive. For example, in the merger cases, some scholars discuss the possibility that network effects would make mergers desirable because it allows a beneficial network expansion (Evans and Schmalensee, 2016), whereas others raise the concern that network effects make mergers anticompetitive by amplifying market power of merged platforms (Ocello and Sjödin, 2018). The theoretical assessment of such competition concerns requires a framework for analyzing oligopolistic platform competition that allows for the coexistence of large dominant platforms and small fringe platforms. Yet, as Jullien, Pavan and Rysman (2021) note, "the literature still lacks a tractable model of platform competition in asymmetric [...] markets."

To overcome such a difficulty, I develop a tractable yet flexible model of asymmetric oligopolistic platform competition in two-sided markets. Specifically, I represent price competition between two-sided platforms as an aggregative game so that an arbitrary platform asymmetry can be incorporated in a tractable manner. The framework consists of a demand model with network effects that derives an IIA demand system as a unique stable consumption equilibrium, and an oligopolistic platform competition represented as an aggregative game.

To derive a well-behaved demand for platforms, I consider a class of discrete-continuous choice models of demand for platforms in two-sided markets that combinines a discrete-continuous choice model of IIA demand (Nocke and Schutz, 2018b) and a discrete-choice model of the demand for multi-sided platforms (Tan and Zhou, 2021). Imposing logarithmic form on network effects and the type-I extreme value distribution of taste shocks, both of which are widely adopted in the empirical literature (Ohashi, 2003; Rysman, 2004, 2007), I obtain a unique stable consumption equilibrium despite the presence of network effects. Furthermore, the derived demand function generalizes canonical IIA demand systems, such as the nested-logit demand system, to two-sided markets. The IIA property allows each platform's profit to depend only on its own strategy and the "aggregators" that summarize the intensities of the competition on the two sides of the market.

Using the demand function obtained from the demand model, I represent platform competition as an aggregative game by extending the techniques of Nocke and Schutz (2018b) and Anderson, Erkal and Piccinin (2020). Specifically, the strategic interaction between platforms are determined solely by the aggregators that summarize the intensities of competition on the two sides. Therefore, each platform's pricing can be represented as a best-response to the aggregators, and the equilibrium value of the aggregators are determined in the way that is consistent with the platform's pricing. I provide sufficient conditions under which the unique equilibrium aggregators exist regardless of the platform's characteristics, which enables analyzing an arbitrary heterogeneity between platforms.

As is common in the literature of two-sided markets, the prices set by each platform on each side involves a discount to attract consumers on the same or another side, a subsidization triggered by the within- and cross-group network effects. The tractable characterization of the asymmetric platform competition allows us to examine how the sizes of platforms affect these subsidization incentives along with their market power, which in turn allows us to conduct the competition analysis in the presence of network effects. I apply the the equilibrium characterization to three competition analysis: platform asymmetry, horizontal mergers and free entry. In all the analysis, the demand-side scale economy of network effects and two-sided pricing play important roles that qualitatively affects the welfare results.

The analysis of platform dominance compares consumer welfare under market structures with difference asymmetries. With network effects, asymmetric market structure has both positive and negative effects on consumers. Because the asymmetry increases the size of the large platform, it increases the benefits of network effects, which benefits the consumers. On the contrary, the large platform use its dominance to increase the markup, hurting the consumers. Whether an asymmetric market structure is associated with higher consumer surplus depends on the relative sizes of these two effects. I show that, for the consumer on one side to benefit from platform asymmetry, they need to benefit from both network effects and two-sided pricing. This implies that, in one-sided markets, a dominance of large platform is always associated with lower consumer surplus. Even in two-sided markets, at least consumers on one side are worse off with platform dominance.

In the analysis of horizontal mergers, I first examine whether a merger-specific technological synergy, in the form of cost reduction or quality improvement, improves consumer surplus. I find that although synergies always improve consumer surplus in one-sided markets, a synergy on one side may hurt consumers on the other side in two-sided markets. Therefore, competition authorities may need a caution when using synergies as a benefit to consumers. With this cautionary remark in mind, I characterize the level of CS-neutral synergies, the synergies required for a merger to be neutral to the consumer surplus on both sides because it provides a useful information on the potential effects of mergers and can be interpreted as synergies necessary to improve consumer surplus whenever it is increasing in the synergy (e.g., Nocke and Whinston, 2010, 2021).

The characterization of CS-neutral synergies reveals the roles that network effects and two-sidedness play in merger analysis. To separately identify the importance of network effects and two-sidedness, I first consider the mergers in one-sided markets with within-group network effects and then analyze the general case of two-sided markets. The analysis of one-sided markets shows that network effects may serve as a synergy but also amplifies the market power of large platforms. Because a merger allows merging platforms to expand the network size, consumers benefit from larger network effects. Therefore, as long as the merging platforms do not increase their markups too much, the platform merger benefits consumers even without technological synergies. On the contrary, when merging platforms are sufficiently large compared to the industry, network effects enable a merged platform to easily increase the markup, making it more likely to hurt consumers.

In two-sided markets, the pre-merger price structure of merging platforms provides information about the additional effects of two-sided pricing. For example, if merging platforms set negative markups on one side, the CS-neutral synergy on that side is always negative. This is because when merging platforms subsidize consumers on one side through negative markups, an increase in the market power raises the merged platform's incentive to subsidize those consumers further. In this way, the two-sided pricing of platforms changes the direction in which a merger-induced increase in market power affects the prices.

Finally, I analyze long-run equilibria with the free entry of fringe platforms. I use Anderson, Erkal and Piccinin (2013)'s notion of free entry equilibrium with marginal entrants to model the entry of fringe platforms. In the free entry equilibrium of two-sided markets, the aggregate consumer surplus is no longer neutral to competitive environments, which holds in one-sided markets (e.g., Davidson and Mukherjee, 2007; Anderson et al., 2020). Therefore, the strategic behaviors of the incumbent platform, such as cost-reducing or quality-improving investments, affect the long-run consumer surplus in two-sided markets. Furthermore, a strong see-saw property holds in the long run. That is, any change in competitive environments that benefits the consumer on one side inevitably hurts consumers on the other side. Therefore, the effects of the strategic behaviors of the incumbent platforms must always involve the conflict of interest between two groups of consumers.

The remainder of the paper is organized as follows. The rest of this section discusses the contribution to related literature. Section 2, I present and analyze the general framework of oligopolistic platform competition, which is then used in Section 3 to analyze platform dominance, Section 4 to analyze horizontal mergers, and Section 5 to analyze free entry, respectively. Section 6 concludes.

Related literature This study contributes to the literature on platform competition, aggregative game analysis of oligopoly, and discrete choice theory of demand. Below, I discuss the contribution to each liteature.

The literature on platform competition in multi-sided markets has examined the importance of network effects in platform competition (see Belleflamme and Peitz, 2021; Jullien et al., 2021, for the review of the literature). A methodological contribution of this study to this literature is to develop a tractable and general framework to incorporate both two-sided network effects and platform asymmetry in an oligopolistic setting. Some studies have theoretically analyzed mergers and entry in platform markets.[1] Gama, Lahmandi-Ayed and Pereira (2020) and Correia-da-Silva, Jullien, Lefouili and Pinho (2019) use homogeneous-product Cournot model to examine the welfare effects of a change in the number of platforms in one-sided and two-sided markets, respectively, and find that a reduction in the number of platforms may improve consumer surplus due to network effects. Baranes, Cortade and Cosnita-Langlais (2019) analyze 4-platform oligopoly in a circular city and find that synergy on one side may hurt consumers on the other side. This study shares these findings and, on top of that, provides a general and platform-specific prediction on the likely effects of mergers on consumers using the notion of CS-neutral synergies. One novel result compared with the existing literature is that network effects can make large mergers more harmful to consumers, which is informally discussed by scholars but not established in the context of platform competition (e.g. Faulhaber, 2002; Ocello and Sjödin, 2018). Tan and Zhou (2021) examine the welfare property of the number of symmetric platforms in free entry equilibria, whereas this study focuses on the effect of incumbent platform's behavior on long-run consumer surplus. Adachi, Sato and Tremblay (2021) and Liu, Teh, Wright and Zhou (2021) also consider the oligopolistic platform competition, but their focus is on the role of consumer multihoming.

This study is also related to the literature of an aggregative-games analysis of oligopoly that provides a workhorse framework for analyzing mergers and free entry (see Nocke and Schutz, 2018b; Anderson et al., 2020, for general treatments). The methodological contribution of this study is to generalize the technique Nocke and Schutz (2018a,b) to incorporate two-sided network effects. Technically speaking, the generalization to two-sided markets makes the existence of equilibrium nontrivial. Nontheless, the existence and uniquness of the equilibrium are established. Based on this technical contribution, this study examines the role that network effects and two-sidedness play in merger analysis and free entry. For example, in the presence of network effects, merger may improve consumer surplus even without synergies, which is not the case in a standard oligopoly (e.g., Williamson, 1968; Farrell and Shapiro, 1990; Nocke

[1] Some empirical studies also propsed methods to quantify the effects of mergers in two-sided markets, including Jeziorski (2014) and Song (2021), using simulation analyses.

and Schutz, 2018a). Furthermore, under free entry equilibrium, consumer surplus is no longer neutral to welfare in two-sided markets, as opposed to that in one-sided markets (e.g., Davidson and Mukherjee, 2007; Ino and Matsumura, 2012; Anderson et al., 2013, 2020). As an application of an aggregative-games analysis to two-sided markets, Anderson and Peitz (2020b) also use an aggregative-games approach to analyze mergers and entry in media markets. This study differs from Anderson and Peitz (2020a,b) in homing structure: this study considers singlehoming consumers on both sides, whereas they consider multihoming consumers on advertising side.

Lastly, this study contributes to the literature of discrete choice model of demand (Anderson, De Palma and Thisse, 1989, 1992; Armstrong and Vickers, 2015; Nocke and Schutz, 2018b; Tan and Zhou, 2021). The consumer demand in this study is understood as a combination of Tan and Zhou (2021)'s discrete-choice model of platform choices and Nocke and Schutz (2018b)'s discrete-continuous choice model of demand. Furthermore, the demand function derived in this study generalizes nested-logit demand models (Anderson and De Palma, 1992). The novelty of this study in this literature lies in establishing a natural extension of nested-logit demand model that allows complementarity between the products sold by the same platform, whereas a standard discrete choice model requires all products to be substitutes (see Armstrong and Vickers, 2015).

2. Model of Platform Oligopoly

I present a framework of platform oligopoly that is used for analyzing platform asymmetry, mergers, and free entry. Consider a two-sided market served by a finite number of platforms, where the sides of the market and the set of platforms are indexed by $\{A, B\}$ and $\mathcal{F}$, respectively. Each platform $f \in \mathcal{F}$ charges a price $p_f^k \in \mathbb{R}$ on side k.

In the following, I first describe the demand model that forms the basis for the analysis of platform competition. Then, I analyze the price competition between platforms. After completing the equilibrium characterization, I introduce an extension to multiproduct platform oligopoly, which is used for modeling platform asymmetry and platform mergers. Lastly, I disuss the modeling assumptions.

2.1. Consumer Demand

On each side $k \in \{A, B\}$ of the market, a unit mass of consumers choose which platform to join. In the model, the word "consumer" represents the users of the platforms. For example, in online marketplaces, buyers and sellers correspond to respective consumers in the model.

Each consumer's utility from the joining a platform consists of a stand-alone value, network effects, and an indiosyncratic preference for the product. Formally, on each side $k \in \{A, B\}$, consumer z's indirect utility from joining platform f is given by

$$u^k_{fz} = \log h^k_f(p^k_f) + \alpha_k \log n^k_f + \beta_k \log n^l_f + \varepsilon^k_{fz}. \tag{1}$$

The first term $\log h^k_f(p^k_f)$ is the stand-alone indirect subutility from using platform f at price $p^k_f \in \mathbb{R}$, where $h^k_f(p^k_f)$ is assumed to be decreasing and log-convex in p^k_f. The second and third terms, $\alpha_k \log n^k_f$ and $\beta_k \log n^l_f$, are the benefits of within-group and cross-group network effects, where $\alpha_k \in [0, 1)$ and $\beta_k \in [0, 1)$ are the parameters representing the magnitude of within-group and cross-group effects, and n^k_f and n^l_f are the numbers of consumers on side k and $l \neq k$ who join platform f. The last term, ε^k_{fz}, is an idiosyncratic taste shock that follows an i.i.d. type-I extreme value distribution. I assume that network effects are not too strong so that $\alpha_k + \beta_l < 1$ holds for each $k \in \{A, B\}$ and $l \neq k$, which precludes the possibility that some platforms set infinitely negative prices on one side. The logarithm specification of network effects is not only broadly adopted in the empirical literature (e.g., Ohashi, 2003; Rysman, 2004, 2007), but is also crucial for obtaining the tractable demand function, as I discuss after Proposition 1.

Consumers choose which platform to join and the amount of usage based on the prices $p := (p^A_f, p^B_f)_{f \in \mathcal{F}}$ set by platforms and the expectation $n^e = (n^{A,e}_f, n^{B,e}_f)_{f \in \mathcal{F}}$ over the network sizes. I assume that there is no outside option and that consumers single-home, so that all consumers on side $k \in \{A, B\}$ purchase exactly one product from the set $\mathcal{N}^k$. The demand for each product is derived as a rational-expectation equilibrium among consumers. That is, based on the expectation over the network sizes n^e, each consumer chooses the product that maximizes the utility, and the realized network sizes $n = (n^A_f, n^B_f)_{f \in \mathcal{F}}$ are consistent with the original expectation. I call an equilibrium choice of products a *consumption equilibrium* and the equilibrium network sizes *consumption equilibrium network sizes.* In the following, I characterize the consumer demand for each product as an outcome of a consumption equilibrium.

First, consider the usage of a platform conditional on membership. Applying Roy's identity, the conditional demand function for platform f conditional on the membership is given by $-(h^k_f)'(p^k_f)/h^k_f(p^k_f)$. Next, consider each consumer's membership choice given the expectation over the network sizes n^e. Because ε^k_{fz} follows the type I extreme value distribution, a consumer on side k joins platform $f \in \mathcal{N}^k_f$ with probability

$$\begin{aligned} n^k_f(n^e) &= \Pr\left(u^k_{fz} \geq u^k_{f'z} \text{ for all } f' \in \mathcal{F}\right) \\ &= \frac{h^k_f(p^k_f)\left(n^{k,e}_f\right)^{\alpha_k}\left(n^{l,e}_f\right)^{\beta_k}}{\sum_{f' \in \mathcal{F}} h^k_{f'}(p^k_{f'})\left(n^{k,e}_{f'}\right)^{\alpha_k}\left(n^{l,e}_{f'}\right)^{\beta_k}}. \end{aligned} \tag{2}$$

The consumption equilibrium network sizes n satisfy the condition $\widetilde{n}_f^k(n) = n_f^k$ for all $k = A, B$ and $f \in \mathcal{F}$.

Due to complementarity in network choices, there may be multiple consumption equilibrium network sizes when the network effects are too strong relative to product differentiation, an issue pointed out by Anderson et al. (1992) and Tan and Zhou (2021), among others. The possibility of multiple equilibria prevents us from deriving a well-behaved demand function. In the context of the present setting, equation (2) indicates that whenever consumers expect $n_{f,e}^k = 0$, such an expectation will be self-fulfilling, and $\tilde{n}_f^k(n^e) = 0$ holds. Therefore, there are a number of equilibria in which a subset of platforms will never be chosen. To rule out such an extreme outcome, I select an equilibrium based on some criteria. The literature has used Pareto dominance (Katz and Shapiro, 1986; Fudenberg and Tirole, 2000), coalitional rationalizability (Ambrus and Argenziano, 2009), focality advantage (Caillaud and Jullien, 2003; Halaburda, Jullien and Yehezkel, 2020), and potential maximization (Chan, 2021). In this study, I use asymptotic stability derived from the best-response dynamics as an equilibrium selection criterion, which is formally defined below.

Definition 1. *Define the best-response dynamics and asymptotic stability of network sizes as follows:*

1. *A best-response dynamics $\{n^t\}$ from the initial network sizes $n^0 = \left(n_{f,0}^A, n_{f,0}^B\right)_{f \in \mathcal{F}}$ is defined by a sequence of network sizes $n^t = \left(n_{f,t}^A, n_{f,t}^B\right)_{f \in \mathcal{F}}$ such that $n_{f,t}^k = \widetilde{n}_f^k\left(n^{t-1}\right)$ for all $t = 1, 2, \ldots,$ $f \in \mathcal{F}$ and $k = A, B$.*

2. *A network size $n = \left(n_f^A, n_f^B\right)_{f \in \mathcal{F}}$ is the limit of the best-response dynamics $\{n^t\}$ from the initial network size n^0 if $n = \lim_{t \to \infty}\{n^t\}$.*

3. *Network sizes n are asymptotically stable if for any strictly positive n^0, n is the limit of the best-response dynamics from the initial network sizes n^0.*

Definition 1 refines the consumption equilibrium by requiring the asymptotic stability of the network sizes. It rules out unstable equilibria wherein some perturbation of network sizes leads to a different allocation of network sizes as a result of best-response dynamics. Compared to existing selection methods that choose a focal platform (Katz and Shapiro, 1986; Caillaud and Jullien, 2003), the asymptotic stability only requires robustness to perturbation in network sizes, which is similar to the requirement of evolutionary stability (e.g., Weibull, 1997). Therefore, any equilibrium with an asymptotically stable netowrk sizes does not rely on any normative selection criterion other than stability. I call a consumption equilibrium that has asymptotically stable network sizes an *asymptotically stable consumption equilibrium*.

Using the notion of asymptotic stability, I select a plausible consumption equilibrium that derives a well-defined demand function. To this end, consider a consumption equilibrium in which all network sizes are strictly positive. I call such an equilibrium *interior consumption equilibrium*. Provided that $n_f^k > 0$ for all $f \in \mathcal{F}$ and $k = A, B$, equation (2) has a closed-form solution

$$n_f^k(p) = \frac{\left[h_f^k(p_f^k)\right]^{\Gamma_{kk}} \left[h_f^l(p_f^l)\right]^{\Gamma_{kl}}}{H^k(p)} \tag{3}$$

for all $f \in \mathcal{F}$ and $k \in \{A, B\}$, where, H^k is the industry-level aggregator defined by

$$H^k(p) = \sum_{f \in \mathcal{F}} \left[h_f^k(p_f^k)\right]^{\Gamma_{kk}} \left[h_f^l(p_f^l)\right]^{\Gamma_{kl}},$$

and Γ_{kk} and Γ_{kl} are given by

$$\begin{aligned}\Gamma_{kk} &= \frac{1-\alpha_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}, \text{ and} \\ \Gamma_{kl} &= \frac{\beta_k}{(1-\alpha_k)(1-\alpha_l)-\beta_l\beta_k}.\end{aligned}$$

The network sizes in equation (3) have the following interpretation. Due to network effects, the stand-alone values h_f^k and h_f^l are amplified to $(h_f^k)^{\Gamma_{kk}}(h_f^l)^{\Gamma_{kl}}$ as shown at the numerator of the right-hand side of equation (3). The industry-level aggregator H^k is the sum of such amplified values.

Along with its intuitive interpretation, the interior consumption equilibrium characterized by equation (3) turns out to be asymptotically stable. Particularly, Proposition 1 shows that the interior consumption equilibrium with the network sizes given by equation (3) is the unique asymptotically stable consumption equilibrium.

Proposition 1. *There exists a unique asymptotically stable consumption equilibrium. In the asymptotically stable consumption equilibrium, the network sizes are given by equation* (3)*. The demand for each product $i \in \mathcal{N}^k$ under the asymptotically stable consumption equilibrium is given by*

$$\begin{aligned}D_i^k(p) &= \hat{D}_i^k\left[p_i, h_f^k(p_f^k), h_f^l(p_f^l), H^k(p)\right] \\ &:= -\left[h_f^k(p_f^k)\right]^{\Gamma_{kk}-1}\left[h_f^l(p_f^l)\right]^{\Gamma_{kl}} \frac{(h_f^k)'(p_f^k)}{H^k(p)}.\end{aligned} \tag{4}$$

Proposition 1 provides a tractable characterization of the demand system with within-group and cross-group network effects. Unlike Tan and Zhou (2021)'s model with general network effects and distribution of taste shocks, the demand for each product has an explicit expression along with incorporating within-group and cross-group network effects. The reason why the demand function has a tractable closed-form expression even in the presence of network effects is similar to the reason why existing linear demand models with linear network effects obtain closed-form demand functions (e.g., Armstrong, 2006). In a linear demand model with linear network effects, the interdependence between expected and realized network sizes can be written in a linear form, which allows us to use linear algebra to obtain the closed-form solution for network sizes. In the present study's setting with logarithm network effects and type-I extreme value distributions of taste shocks, the interdependence between the expected and realized network sizes can be written in a log-linear form. Then, using a linear algebra to the log of network sizes, we can obtain the closed-form solution. The assumption of the logarithm form of network effects is crucial because it is necessary for obtaining a log-linear relation between the expected and realized network sizes.

The network sizes characterized by equation (3) exhibit an IIA property, which allows us to generalize the aggregative-games analysis of Nocke and Schutz (2018b) and Anderson et al. (2020) to two-sided markets, thereby making it possible to introduce an arbitrary heterogeneity between platforms. In summary, the demand model of this study simplifies Tan and Zhou (2021) in two dimensions, the forms of network effects and taste distributions, and makes it possible to derive a tractable demand function that allows for an arbitrary platform asymmetry in two-sided markets.

Consumer surplus Consumer surplus CS^k on side k is given by the expected indirect utility of consumers, and the aggregate consumer surplus CS is given by the sum of the consumer surplus on both sides:

$$\begin{aligned} CS^k &= \log \left[\sum_{f \in \mathcal{F}} \left(h_f^k\right)^{\Gamma_{kk}} \left(h_f^l\right)^{\Gamma_{kl}} \frac{1}{(H^k)^{\alpha_k} (H^l)^{\beta_k}} \right] \\ &= (1-\alpha_k) \log H^k - \beta_k \log H^l, \end{aligned}$$

and

$$\begin{aligned} CS &= CS^A + CS^B \\ &= (1-\alpha_A - \beta_B) \log H^A + (1-\alpha_B - \beta_A) \log H^B. \end{aligned}$$

Note that CS^k is increasing in H^k but decreasing in H^l. Because the side-k aggregator H^k measures the intensity of competition on side k, it is natural for H^k to have a positive impact on CS^k. On the contrary, a large value of side-l aggregator H^l reduces the network sizes of platforms on side l. This reduces side-k consumer's benefit from cross-group network effects, thereby negatively affecting CS^k. Nontheless, in aggregate, consumer surplus is increasing in both aggregators, which is shown by the fact that the aggregate consumer surplus CS is increasing in both H^A and H^B.

Logit demand specification As Nocke and Schutz (2018b) discuss, the function h_f^k can take a flexible form, including multinomial-logit specification where $h_f^k(p_f^k) = \exp\left(\frac{a_f^k - p_f^k}{\lambda^k}\right)$ with $\lambda^k > 0$ and CES specification where $h_f^k(p_f^k) = a_f^k \left(p_f^k\right)^{1-\sigma^k}$ with $\sigma^k > 1$. In both specifications, a_f^k represents the quality of the platform. More generally, the framework allows for any log-convex function $h_f^k(p_f^k)$. In the following, to simplify the analysis, I assume that h_f^k takes the multinomial-logit form. A similar analysis is applicable to other specifications such as the CES specification.

Assumption 1. *For each $k \in \{A, B\}$ and $f \in \mathcal{F}$, h_f^k is given by*

$$h_f^k(p_f^k) = \exp\left(\frac{a_f^k - p_f^k}{\lambda^k}\right), \tag{5}$$

where $a_f^k \in \mathbb{R}$ and $\lambda^k > 0$.

The demand system that satisfies Assumption 1 has the form

$$D_f^k(p) = \frac{\exp\left[\frac{\Gamma_{kk}(a_f^k - p_f^k)}{\lambda^k} + \frac{\Gamma_{kl}(a_f^l - p_f^l)}{\lambda^l}\right]}{\lambda^k \sum_{f' \in \mathcal{F}} \exp\left[\frac{\Gamma_{kk}(a_{f'}^k - p_{f'}^k)}{\lambda^k} + \frac{\Gamma_{kl}(a_{f'}^l - p_{f'}^l)}{\lambda^l}\right]}. \tag{6}$$

The demand system given by equation (6) is a natural and tractable generalization of logit demand system to two-sided markets. To see this, suppose that $\alpha_k = \beta_k = 0$ for all $k = A, B$. Then, the demand system is a standard multinomial-logit demand system in one-sided markets. The parameterization of network effects and two-sidedness through two parameters α_k and β_k allows us preserve the usefull IIA property of logit demand system, which in turn allows us to employ an aggregative-game approach to platform competition.

2.2. Platform competition

Using the demand system obtained in Proposition 1, I analyze price competition between platforms.

Each platform $f \in \mathcal{F}$ incurs a constant marginal cost $c_f^k \geq 0$ on side k. Given the demand system $\{(D_f^k)_{f \in \mathcal{F}}\}_{k \in \{A,B\}}$, the profit function of each platform $f \in \mathcal{F}$ is written as a function of the profile of the platform's own prices $p_f := (p_f^A, p_f^B)$ and aggregators H^A and H^B:

$$\Pi_f \left[p_f, H^A(p), H^B(p)\right] = \Pi_f^A + \Pi_f^B, \tag{7}$$

where

$$\Pi_f^k = \hat{D}_f^k \left[p_f, h_f^k(p_f^k), h_f^l(p_f^l), H^k(p)\right] (p_f^k - c_f^k). \tag{8}$$

The pricing game consists of a demand system $\{(D_f^k)_{f\in\mathcal{F}}\}_{k\in\{A,B\}}$, a set of platforms $\mathcal{F}$, and a profile of marginal costs $(c_f^A, c_f^B)_{f\in\mathcal{F}}$. In a pricing game, platforms simultaneously set the prices p_f of their products, with the payoff function Π_f defined by equation (7). I call a Nash equilibrium of this pricing game as a *pricing equilibrium*. In the following analysis, I often suppress the arguments of functions for ease of exposition.

Optimal pricing for each platform The first-order condition for the profit-maximizing prices set by each platform f is given by $\partial\Pi_f/\partial p_i = 0$, which can be transformed into the following equation:

$$-\frac{(h_f^k)''}{(h_f^k)'}(p_f^k - c_f^k) = \mu_f^k, \tag{9}$$

where

$$\mu_f^k := 1 - \underbrace{\frac{1}{n_f^k}\left[(\Gamma_{kk}-1)\Pi_f^k + \beta_l\Gamma_{lk}\Pi_f^l\right]}_{\text{network-externality terms}} + \underbrace{\Gamma_{kk}\Pi_f^k + \Gamma_{lk}\frac{n_f^l}{n_f^k}\Pi_f^l}_{\text{cannibalization terms}}. \tag{10}$$

The right-hand side of equation (10) is independent of the index of the product i. Therefore, the optimal pricing of each platform equates the left side of the equation (9) with some common value μ_f^k. Following Nocke and Schutz (2018b), I call μ_f^k as the ι-markup of platform f on side k. The property that all prices of the products sold by a platform on the same side are summarized into a single ι-markup is driven by the property that the network sizes n_f^k have an IIA property.

The ι-markup concisely summarizes the pricing incentive of each platform. The first term, 1, in equation (10) is the baseline ι-markup, which would be set under the monopolistic competition. The second term is the downward-pricing pressure due to the within-group and cross-group network effects. The third and fourth terms are the upward-pricing pressure due to the cannibalization effects under oligopoly. The relative magnitudes of the second and last terms on each side determine the price level and the price structure of each platform.

Under Assumption 1, we have $-(h_f^k)''/(h_f^k)' = 1/\lambda^k$ and thus $p_f^k = c_f^k + \lambda^k\mu_f^k$, implying that all the product of the same platform on one side has the same markup. Then, the profit of platform f on side k can be written as its network size multiplied by the common markup:

$$\Pi_f^k = n_f^k \mu_f^k. \tag{11}$$

Using this relation, the formula for the ι-markup can be simplified to

$$\mu_f^k = \frac{1}{1 - n_f^k} \left(1 - \alpha_k - \beta_l \frac{n_f^l}{n_f^k} \right) \tag{12}$$

Equation (12) provides a simple generalization of the markup set by a firm facing a logit demand system. In a standard logit model with $\alpha_k = \beta_l = 0$, the ι-markup is set to $1/(1 - n_f^k)$. In the presence of within-group network effects $\alpha_k > 0$, the ι-markup is compressed by α_k. This discount is due to the increase in price elasticity of demand, driven by the positive feedback effects. Lastly, in the presence of cross-group network effects $\beta_l > 0$, the ι-markup is further discounted by the amount $\beta_l n_f^l / n_f^k$. This discount is due to the cross-subsidization incentive of the platform to expand the network on side k to attract consumers on side l, which is widely observed in the models of two-sided platforms (Armstrong, 2006; Rochet and Tirole, 2006; Weyl, 2010; Tan and Zhou, 2021).

The stand-alone value $h_f^k(p_f^k)$ also be simplified to

$$h_f^k(p_f^k) = T_f^k \exp(-\mu_f^k), \tag{13}$$

where $T_f^k := \exp\left(\frac{a_f^k - c_f^k}{\lambda^k}\right)$ is the "type" of platform f that equals the stand-alone value of the platform f when it engages in the marginal cost pricing. Solving the system of equations (12) and (13), along with equation (3), the ι-markup μ_f^k and the network size n_f^k consistent with the platform's optimal pricing are obtained as functions of T_f^A, T_f^B, H^A, and H^B, which I write as

$$\mu_f^k = m^k \left(T_f^k, T_f^l, H^k, H^l\right), \tag{14}$$

and

$$n_f^k = N^k \left(T_f^k, T_f^l, H^k, H^l\right). \tag{15}$$

When the system of equations (12) and (13) has multiple solutions, let the profit-maximizing one be m^k and N^k.

Pricing equilibrium Finally, the equilibrium aggregators (H^A, H^B) must satisfy the condition that the network sizes of the platforms add up to 1:

$$\sum_{f \in \mathcal{N}_f^k} N^k \left(T_f^k, T_f^l, H^k, H^l\right) = 1, \quad J \in \{A, B\}, \quad l \neq J. \tag{16}$$

The analysis so far has characterized the necessary conditions that the pricing equilibrium must satisfy. Proposition 2 shows that these necessary conditions are also sufficient and provides several important cases in which the equilibrium is unique regardless of the type profiles. The proof of this proposition is relegated to the Online Appendix.

Proposition 2. *For any pair of aggregators (H^A, H^B), each platform's corresponding optimal pricing is uniquely given by $p_f^k = c_f^k + \lambda^k m^k \left(T_f^k, T_f^l, H^k, H^l\right)$. Furthermore, the following statements hold:*

1. *If only within-group network effects exist ($\beta_A = \beta_B = 0$), then a unique pricing equilibrium exists.*
2. *If only cross-group effects exist ($\alpha_A = \alpha_B = 0$), then a pricing equilibrium exists. Furthermore, there exists $\underline{\beta}$ such that if $\beta_A \leq \underline{\beta}$ or $\beta_B \leq \underline{\beta}$, then the pricing equilibrium is unique.*

Because a pricing equilibrium is characterized by the pair of industry-level aggregators (H^{A*}, H^{B*}) that satisfies the system of equations (16), the characterization of equilibrium is simplified to the characterization of the system of equations (16). Furthermore, because the consumer surplus $CS^k = (1 - \alpha_k) \log H^k - \beta_k \log H^l$ is determined solely by industry-level aggregators (H^A, H^B), the characterization of equilibrium aggregators directly characterizes the equilibrium consumer surplus. These properties are typically obtained in the aggregative-games frameworks of oligopoly (e.g., Anderson et al., 2020; Nocke and Schutz, 2018b) and simplify the comparative statics and welfare analyses in Sections 3, 4, and 5.

2.3. Extension to multiproduct-firm oligopoly

Our baseline analysis can be generalized to the setting where platforms offer multiple services. Specifically, we can consider the situation where each platform $f \in \mathcal{F}$ sells a finite set $\mathcal{N}_f^k$ of products on side $k \in \{A, B\}$, where each product $i \in \mathcal{N}_f^k$ is priced at $p_i \in \mathbb{R}$. Let $\mathcal{N}^k := \bigcup_{f \in \mathcal{F}} \mathcal{N}_f^k$ be the set of all products on side k sold by the platforms. The baseline setting is a special case where $|\mathcal{N}_f^k| = 1$ for all $k = A, B$ and $f \in \mathcal{F}$.

On each side $k \in \{A, B\}$ of the market, a unit mass of consumers choose which product to purchase. Each consumer's utility from the purchase of a product consists of a stand-alone value, platform-level network effects, and an indiosyncratic preference for the product. Formally, on each side $k \in \{A, B\}$, consumer z's indirect utility from the purchase of product $i \in \mathcal{N}_f^k$ is given by

$$u_{iz}^{k,f} = \log h_i^k(p_i) + \alpha_k \log n_f^k + \beta_k \log n_f^l + \varepsilon_{iz}^k. \tag{17}$$

The term $\log h_i^k(p_i)$ now represents the stand-alone indirect subutility from product i at price $p_i \in \mathbb{R}$, where $h_i^k(p_i)$ is assumed to be decreasing and log-convex in p_i.

The assumption of platform-level rather than product-level network effects is motivated by the observation that the products provided by the same platform (e.g., a group of smartphones and tablets that support the same OS or the services offered by Meta, such as Facebook and Instagram) exhibit greater interoperability than those provided by different platforms do. This assumption holds true in some applications but not in others. For example, if a single firm offers multiple different services as independent incompatible platforms, then the network effects would work at the product level. Nonetheless, as far as single-product platforms are concerned, the platform-level network effects are equivalent to the product-level network effects. In this regard, as long as the analysis focuses on single-product platforms, the model also allows for product-level network effects that apply to prominent two-sided markets such as online marketplaces.

Consumers choose which product to purchase and the amount of purchase based on the prices $p := (p_i)_{i \in \mathcal{N}^A \cup \mathcal{N}^B}$ set by platforms and the expectation $n^e = (n_f^{A,e}, n_f^{B,e})_{f \in \mathcal{F}}$ over the network sizes. I assume that there is no outside option and that consumers single-home, so that all consumers on side $k \in \{A, B\}$ purchase exactly one product from the set $\mathcal{N}^k$. The demand for each product is derived as a rational-expectation equilibrium among consumers. That is, based on the expectation over the network sizes n^e, each consumer chooses the product that maximizes the utility, and the realized network sizes $n = (n_f^A, n_f^B)_{f \in \mathcal{F}}$ are consistent with the original expectation. I call an equilibrium choice of products a *consumption equilibrium* and the equilibrium network sizes *consumption equilibrium network sizes*.

In Online Appendix B.1, I show that all the analysis of single-product setting can be generalized to to the multiproduct setting just by defining the platform's type as $T_f^k = \sum_{j \in \mathcal{N}_f^k} h_j^k(c_j)$. Particularly, for any pair of aggregators (H^A, H^B), each platform's corresponding optimal pricing is uniquely given by $p_i = c_i + \lambda^k m^k \left(T_f^k, T_f^l, H^k, H^l\right)$. The property that all the pricing information is summarized by unidimensional type T_f^k is called the type-aggregation property (Nocke and Schutz, 2018b), which simplifies the analysis of platform dominance and platform mergers.

Note that in the multiproduct extension, the demand system that satisfies Assumption 1 has the form

$$D_i^k(p) = \frac{\left[H_f^k(p_f^k)\right]^{\Gamma_{kk}} \left[H_f^l(p_f^l)\right]^{\Gamma_{kl}}}{\sum_{f\prime \in \mathcal{F}} \left[H_{f\prime}^k(p_{f\prime}^k)\right]^{\Gamma_{kk}} \left[H_{f\prime}^l(p_{f\prime}^l)\right]^{\Gamma_{kl}}} \frac{\exp\left(\frac{a_i - p_i}{\lambda^k}\right)}{\lambda^k \sum_{j \in \mathcal{N}_f^k} \exp\left(\frac{a_j - p_j}{\lambda^k}\right)}, \tag{18}$$

where $H_f^k(p_f^k) = \sum_{j \in \mathcal{N}_f^k} \exp\left(\frac{a_j - p_j}{\lambda^k}\right)$ is a platform-level aggregator on side k. The demand system given by equation (18) is a tractable generalization of nested-logit demand system to two-sided markets. One difference between the demand function in equation (18) and the standard nested-logit demand is that the demand function in equation (18) has the nest coefficient $\Gamma_{kk} > 1$, whereas a standard nested-logit demand function based on discrete-choice model must have a nest coefficient less than 1. This is because if $\Gamma_{kk} > 1$, two products in the same nest are complements when $n_f < (\Gamma_{kk} - 1)/\Gamma_{kk}$, whereas standard discrete-choice models require every pair of products to be substitutes (Armstrong and Vickers, 2015). In other words, by incorporating network effects as a microfoundation for nested-logit demand functions with nest coefficients greater than 1, the demand function given by equation (18) allows for the complementarity between products within a logit-demand framework.

The multiproduct extension allows us to introduce an appropriate notion of platform asymmetry and define the platform mergers as the changes in ownership structures.

2.4. Discussion

Before applying the general framework, I discuss the specific assumptions made on the demand and supply models. The discussions include non-price competition, partially covered markets, general form of network effects, multihoming, and other demand specifications such as CES demand.

Non-price competition The main framework assumes that consumers' stant-alone utilities depend only on the prices set by the platforms. The assumption of price competition might be at odds when platforms do not charge prices on one side. For example, social media typically charges advertisers but does not charge users. The present framework can be extended to such cases, by introducing non-negative price constraints (Choi and Jeon, 2021; Kawaguchi, Kuroda and Sato, 2021) to allow for endogenous zero prices or by using a competition-in-utility frameworks (Armstrong and Vickers, 2001; de Corniére and Taylor, 2021) to allow for non-price competition. However, the type-aggregation property may not continue to hold in general in the extended analysis, although platform competition can still be represented as an aggregative game.

Partially covered markets The main analysis also assumes that there is no outside option. This assumption can be relaxed in the following manner. Suppose that each consumer z has an outside option with value $\log H_0^k + \varepsilon_{0z}^k$, where ε_{0z}^k follows an i.i.d. type-I extreme-value distribution, and chooses whether to join one of the networks. Then, the probability that a consumer joins some network is characterized by the following equation:

$$n_f^k = \frac{\left[h_f^k(p_f)\right]^{\Gamma_{kk}} \left[h_f^l(p_f)\right]^{\Gamma_{kl}}}{H^k(p) + H_0^k \frac{\left[h_f^k(p_f)\right]^{\Gamma_{kk}-1} \left[h_f^l(p_f)\right]^{\Gamma_{kl}}}{\left(n_f^k\right)^{\alpha_k} \left(n_f^l\right)^{\beta_k}}}$$

for $k = A, B$ and $I \neq J$. Solving for the system of equations, we can express the network sizes as the function $\tilde{n}^k(h_f^k, h_f^l, H^k, H^l)$. Because the network size depends only on (h_f^k, h_f^l, H^k, H^l), the platform competition can be formalized as an aggregative game, so similar analysis can be made with this demand system. The case where $H_0^k = 0$ for $k = A, B$ corresponds with the model of the main analysis, and as α_k and β_k approach 0, the demand system converges to that of Nocke and Schutz (2018b).

General form of network effects The crucial assumption in this study is the logarithm form of the network effects along with the logit taste shocks. This specification can be regarded as a special case of the model Tan and Zhou (2021). Other theoretical studies also use more general forms of network effects (Hagiu, 2009; Weyl, 2010; Belleflamme and Peitz, 2019). As I discussed after Proposition 1, the logarithm form of network effects along with logit taste shocks is necessary for obtaining the demand system that has both two-sided network effects and IIA property and for conducting aggregative-games analysis. Therefore, any generalization beyond the logarithmic network effects would require giving up an aggregative-games analysis, which poses a challenge in terms of tractability.

Multihoming The main also analysis assumes that the consumers on both sides singlehome. This specification is widely adopted by the literature, including Armstrong (2006) and Tan and Zhou (2021). In some environments, however, consumers on one side often multihome as analyzed by Anderson and Peitz (2020b), and consumers on both sides multi-home in another environment, as analyzed by Adachi et al. (2021), Anderson and Peitz (2020a), and Liu et al. (2021). To focus on the roles of network effects and two-sidedness themselves, I stick to the setting of singlehoming consumers.

CES demand Although the main analysis focuses on the multinomial logit demand system, the same analytical technique can be used for the CES demand system wherre $h_f^k(p_i) = a_f^k(p_f^k)^{\sigma^k}$.

In this case, we have $-h_f^{k\prime\prime}(p_f^k)/h_f^{k\prime}(p_f^k) = \sigma^k/p_f^k$, so the equation (9) is written as $p_f^k = c_f^k/(1 - \mu_f^k/\sigma^k)$. Accordingly, h_f^k, Π_f^k, and T_f^k are given by

$$
\begin{aligned}
h_f^k &= T_f^k \left(1 - \frac{\mu_f^k}{\sigma^k}\right)^{\sigma^k - 1}, \\
\Pi_f^k &= \frac{\sigma^k - 1}{\sigma^k} n_f^k \mu_f^k, \text{ and} \\
T_f^k &= a_f^k (c_f^k)^{\sigma^k - 1}.
\end{aligned}
$$

Then, using equations (9) and (3), each platform's optimal prices can be obtained.

In the following, I apply the general framework to analyze the competition issues surrounding large platforms. In Section 3, I analyze the welfare property of platform asymmetry. Subsequently in Section 4, I analyze the welfare effects of horizontal mergers between platforms. Finally, in Section 5, I introduce the free entry of entrant platforms and examine the welfare properties of incumbent platforms' strategic behaviors.

3. Platform Dominance and Consumer Welfare

As a first application, I examine the welfare property of platform dominance by comparing the different market structures with different degree of platform asymmetries. Such a welfare analysis provides an insight on how the dominance of particular platform is associated with consumer welfare, which in turn provides some theoretical guidance on the structural regulation against dominant platforms.

3.1. Modeling platform asymmetry

To simplify the analysis, I focus on the duopoly platform competition. Suppose that the market is duopoly, that is, $\mathcal{F} = \{S, W\}$. The platform S is *strong*, and the platform W is *weak*. We consider a class of type profiles $(T_S^A, T_S^B, T_W^A, T_W^B)$ satisfying

$$
T_S^k + T_S^k = \overline{T}_k, \tag{19}
$$

for $k = A, B$. That is, the aggregate productivity $\sum_{f=S,W} h_f^k(c_f^k)$ is constant. Then, this class of type profiles can be parameterized by *asymmetry parameters* (δ_A, δ_B) such that

$$
\delta_k = \frac{T_S^k}{T_S^k}.
$$

Given the asymmetry parameters (δ_A, δ_B), the type of platform S can be written as $T_S^k = \delta_k \overline{T}_k/(1+\delta_k)$, and that of platform W can be written as $T_W^k = \overline{T}_k/(1+\delta_k)$.

By comparing the equilibrium consumer surplus under different asymmetry parameters, I analyze the welfare properties of platform asymmetry. The comparison across different asymetry parameters can be regarded as a comparison across the different allocation of the production technology available in the economy. As we observed in Section 2.3, in the multiproduct setting, the type $T_f^k = \sum_{j \in \mathcal{N}_f^k} h_j^k(c_j)$ is additive over the products each platform owns. Therefore, given a fixed production technology, the sum of the types is fixed at $\bar{T}_k = \sum_{j \in \mathcal{N}^k} h_j^k(c_j)$. Then, the asymmetry parameter captures the fraction of the production technology that the strong platform owns relative to the weak platform. For example, when an asset is transfered from the weak platform to the strong platform, it does not change the sum of the types, but increases the fraction of the type of strong platform relative to the weak platform, increasing the asymmetry parameter.

To simplify the exposition, instead of directly conducting the comparative statics with respect to asymmetry parameters, I analyze the network sizes induced as an equilibrium under corresponding asymmetry parameters. Consider the network sizes of strong platform (n_S^A, n_S^B). Such network sizes may or may not be realized as an equilibrium outcome. The following lemma show that for any network sizes of strong platform, there exists an asymmetry parameter that induces the network sizes as an equilibrium outcome. This allows us to analyze the platform asymmetry in terms of the equilibrium network sizes (n_S^A, n_S^B), rather than the asymmetry parameters (δ_A, δ_B).

Lemma 1. *For any network sizes $(n_S^A, n_S^B) \in [1/2, 1)^2$ of platform S, there exists the asymmetry parameters (δ_A, δ_B) under which the equilibrium network sizes of platform S are given by (n_S^A, n_S^B).*

Based on Lemma 1, we compare the consumer welfare under different market structures. Generally, there are two-dimensional asymmetry in two-sided markets, which complicates the analysis. Therefore, to further simplify the exposition and the result, I consider the uni-dimensional platform asymmetry by using the notion of “balanced asymmetry.”

Consider the class of market structures in which the equilibrium network size of platform S is given by $(n_S^A, n_S^B) = (n, n)$ for a given $n \in [1/2, 1)$. In this case, platform asymmetry can be captured by a uni-dimensional market structure $n > 1/2$.

Under the balanced asymmetry with asymmetry parameter $n > 1/2$, the type asymmetry consistent with market structure (n, n) is given by

$$\bar{\delta}_k(n) = \underbrace{\left(\frac{n}{1-n}\right)^{1-\alpha_k-\beta_k}}_{\text{size component}} \underbrace{\exp\left[\left(\frac{1}{1-n} - \frac{1}{n}\right)(1-\alpha_k-\beta_l)\right]}_{\text{markup component}} \tag{20}$$

The asymmetry parameter $\bar{\delta}_k$ is composed of two components. First, suppose for a moment that the markups set by the strong and weak set the same markup. The asymmetry parameter needs to be large enough to achieve the network size $n > 1/2$, which is represented by the *size component* in equation (20). Second, consider the true setting where platforms set the different markups. Because the strong platform with $n > 1/2$ sets relatively higher markup than the weak platform, the asymmetry parameter also needs to be large to compensate for the difference in the markups, which is represented by the *markup component* on equation (20). Finally, total value of the asymmetry parameter consistent with the equilibrium network size n is given by $\bar{\delta}_k$.

3.2. Welfare analysis

By comparing the market structures characterized by balanced asymmetries, characterize the welfare properties of asymmetric market structures in two-sided markets.

Given the pricing equations (12) under the balanced asymmetry, the equilibrium markup set by platform f is given by

$$\mu_f^k = \frac{1-\alpha_k-\beta_l}{1-n_f},$$

for $f = S, W,\ k, l = A, B,\ l \neq k$. Under the balanced market structures, the price structure, μ_f^A/μ_f^B is invariant, but price levels changes proportionally. This simplifies the welfare benefits and costs of platform asymmetry. The welfare benefit of platform asymmetry the increase in the benefit of network effects of larger platform as compared to the decrease in the network effects of smaller platform, captured by $\alpha_k + \beta_k$. The welfare cost of platform asymmetry is the increased markups set by the larger platform as compared to the decreased markups set by the smaller platform. On side k, the scale of markup is given by $1 - \alpha_k - \beta_l$.

When platform S has a network size (n, n), the consumer surplus on side k is given by

$$CS_k^*(n) = \log \bar{T}_k + \log\left(\frac{\bar{\delta}_k(n)}{1+\bar{\delta}_k(n)}\right) - \frac{1-\alpha_k-\beta_l}{1-n} - (1-\alpha_k-\beta_k)\log n. \tag{21}$$

The platform asymmetry is beneficial to side-k consumers if $CS_k^*(n)$ increases with n.

The following proposition shows the condition on the network effects under which platform asymmetry is associated with higher consumer surplus.

Proposition 3. *Consider the side-k consumer surplus $CS_k^*(n)$ realized under balanced asymmetry n.*

1. *If $\beta_k \geq \beta_l$ or $\alpha_k + \beta_k = 0$, then $CS_k^*(n)$ decreases with n.*

2. *If $\alpha_K + \beta_k > 0$, then for any $\bar{n} \in (1/2, 1)$, there exists $\underline{\epsilon} > 0$ such that if $\beta_l > 1 - \alpha_k - \underline{\epsilon}$, then $CS_k^*(n)$ increases with $n \in (1/2, \bar{n}]$.*

Proposition 3 implies that there are two necessary conditions for the consumers on one side to benefit from platform asymmetry. First, consumers on one side must benefit from network expansion. Second, the associated increase in the markup is compressed by the two-sided pricing. If one of these conditions fails to be satisfied, then the consumers are worse off with the platform asymmetry.

For the consumer on side k to benefit from platform asymmetry, it is necessary that the benefit of network expansion of large platform outweighs the cost of the associated increase markup. Because the scale of the markup set on side k is given by $1 - \alpha_k - \beta_l$, the cost of increased markup can be small when $\alpha_k + \beta_l$ is large, that is, the markup is compressed bacause of the cross-subsidization incentive.

It turns out that if $\beta_k \geq \beta_l$, that is, side-k consumers benefit more from cross-group network effects than side l agents, then the platform asymmetry is associated with the lower consumer surplus on side k. This implies that at least consumers on one side are worse off with the increase in the platform asymmetry.

On the contrary, when $\alpha_k + \beta_k > 0$, and $\alpha_k + \beta_l$ is large, then an increase in platform asymmetry may benefit consumers on side k. This is because the increase in the markup on side k is sufficiently compressed, and the benefit of network expansion outweighs the loss from increased markups.

To summarize, an increase in the platform asymmetry may benefit consumers on one side because the increased network effects may outweigh the increased markups, but the it is unlikely that consumers on both side benefit from platform dominance. This implies that as long as the aggregate productivity $\bar{T}_k = T_S^k + T_W^k$ is fixed, any platform asymmetry would hurt some consumers. Therefore, for the platform dominance to be beneficial to all consumers, it is necessary that such a dominance accompanies an increase in the aggregate productivity $\bar{T}_k$.

Recent policy discussion, such as the proposal of the Digital Markets Acts (European Commission, 2020), is driven by the presence of platform dominance. Typically, competition and regulation authorities are concerned about the dominance of specific cplatforms such as Alphabet, Amazon, Apple, and Meta. The results found in this section reveal that unless it is induced by an efficiency gain, platform dominance cannot benefit all the consumers.

4. Merger Analysis

As a short-run competition analysis, I analyze horizontal mergers between platforms. To do so, I first model platform mergers and define merger-specific synergies. Then, I discuss the extent to which a merger-specific synergy improves consumer surplus and then characterize the level of synergies required to make a merger neutral to consumer surplus.

Modeling mergers I model a merger between platforms as a transformation from two platforms f and g with the types $\left(T_f^A, T_f^B\right)$ and $\left(T_g^A, T_g^B\right)$ into a new platform M with the type $\left(T_M^A, T_M^B\right)$. The merger may exhibit a technological synergy, which is captured by $\Delta^k := T_M^k - T_f^k - T_g^k$. The source of technological synergies may be cost reduction, quality improvement, or introduction of new products. For this study, I assume that such synergies are exogenous primitives of the merger rather than an endogenous choice of the merged platform.

Because I assume that network effects work at the platform level, the products of merging platforms become compatible after the merger. This might hold true or not in reality. For example, in 2020, Facebook, the acquirer of Instagram, integrated the messaging services of Instagram and Messenger, so that the users of these services can communicate with each other.[2] Also, Farronato, Fong and Fradkin (2020) document that when a pet-sitting platform Rover acquired another pet-sitting platform DogVacay and decided to shut down DogVacay, Rover allowed the DogVacay users to migrate their accounts. Such a migration can be interpreted as another example of compatibility between merging platforms. In other cases where merged platforms remain incompatible, the assumption of compatible platforms may not hold. Such an example includes online travel agencies Booking.com, Kayak, and Priceline that are owned by Booking Holdings but operate as different platforms.

Based on this specification of mergers, I examine the impact of a merger on consumer surplus. This can be done by examining its impact on equilibrium aggregators, because the consumer surplus on each side depends only on H^A and H^B. Specifically, the post-merger consumer surplus is the same as the pre-merger consumer surplus if the post-merger aggregators remain the same. I call such a merger *CS-neutral merger*. Suppose that the pre-merger aggregators are given by (H^{A*}, H^{B*}). Then, using equation (16), the post-merger equilibrium aggregators

[2] “Say 👋 to Messenger: Introducing New Messaging Features for Instagram”: `https://about.fb.com/news/2020/09/new-messaging-features-for-instagram/`

remain the same if and only if the equation

$$N^k\left(T_M^k, T_M^l, H^{k*}, H^{l*}\right) = N^k\left(T_f^k, T_f^l, H^{k*}, H^{l*}\right) + N^k\left(T_g^k, T_g^l, H^{k*}, H^{l*}\right), \tag{22}$$

holds for $k = A, B$ and $I \neq J$. Let $(\hat{T}_M^A, \hat{T}_M^B)$ be the types of merged platform under a CS-neutral merger. I define *CS-neutral technological synergy* as the technological synergy of CS-neutral merger, which is given by the pair $(\hat{\Delta}^A, \hat{\Delta}^B)$ such that $\hat{\Delta}^k = \hat{T}_M^k - T_f^k - T_g^k$ for $k = A, B$.

The next lemma shows that there exists a unique pair of CS-neutral technological synergy.

Lemma 2. *For any merger between platforms f and g, there exists a unique pair of technological synergies $(\hat{\Delta}^A, \hat{\Delta}^B)$ such that the merger is strictly CS-neutral and thus satisfies the condition* (22)*.*

The characterization of CS-neutral synergies informs the size of synergies required to make the merger neutral to competition. Furthermore, if consumer surplus is monotonically increasing in synergies, then the characterization of CS-neutral synergies also provides a characterization of synergies required to improve consumer surplus. One question is whether the consumer surplus is monotonically increasing in synergies. The following proposition shows that in one-sided markets, consumer surplus is always increasing in synergy, but in two-sided markets, consumer surplus on one side may decrease with the synergy on the other side.

Proposition 4. *Regarding the impact of platform's types on consumer surplus, the following statements hold true:*

1. *In one-sided markets ($\beta_A = \beta_B = 0$), the equilibrium consumer surplus on each side is increasing in T_f^A and T_f^B for all $f \in \mathcal{F}$.*

2. *In two-sided markets ($\beta_A > 0$ or $\beta_B > 0$), there exists a parameterization under which the equilibrium consumer surplus on one side decreases with a platform's type on the other side.*

An intuition is in order. First, consider the case of one-sided markets and suppose that the type of platform M increases by either a reduction in costs or an increase in qualities. Fixed the behaviors of other platforms, such an increase in the type increases the consumer surplus because of the direct benefit of lower prices or higher qualities. Because other platforms shrink their network sizes, they react to the increase in platform M's type by lowering prices, further increasing consumer surplus. Because both of the two effects work in the direction of increasing the consumer surplus, a synergy always improves consumer surplus in one-sided markets.

Second, consider the general case of two-sided markets. In this environment, a synergy on one side does not necessarily benefit consumers on both sides. To see its reason, consider the case where platform M's type on side A increases. Such a change increases platform M's network size on side A, and platform will set a lower price on side B to attract more consumers and exploit network effects. Such a change, however, also reduces the network sizes of other platforms on side A, which in turn reduces the incentive to set low prices on side B. If the latter adverse effect dominates the former benefit, consumers on side B would be worse off due to the synergy of platform M on side A.

This proposition provides the following implication on the welfare properties of mergers in two-sided markets. Unlike one-sided markets, merger-specific synergy may not always benefit consumers on every side. Therefore, competition authorities need to be careful about the use of merger-specific synergies as the benefits to consumers. Note that a similar result is obtained by Baranes et al. (2019).

With these cautionary remarks in mind, I characterize CS-neutral mergers, because it still provides useful information about the potential effects of mergers on consumer welfare. In the following analyses, I separately examine mergers with one-sided network effects and mergers with two-sided network effects. In the analysis of mergers with one-sided network effects, I examine the extent to which the network effects serve as a form of merger-specific synergy, thereby improving consumer surplus without technological improvements. Then, in the analysis of mergers in two-sided markets, I examine the additional effects brought about by two-sided pricing.

One-sided markets Consider the case of one-sided markets where $\beta_A = \beta_B = 0$. Because no interaction between the two sides exists, I drop the index for the side of the market. To highlight the impact of network effects, I also assume that α is strictly positive. By examining the impact of network effects on the CS-neutral synergies, I examine the extent to which network effects serve as a synergy or amplify the market power of platforms.

Because the benefit from network effects is large when the size of a platform is large, a merger-induced increase in the size of platform is beneficial to consumers, unless the merged platform raises the markup. Therefore, network effects may serve as a merger-specific synergy. Particularly, when one of the merging platforms is small, the increase in the markup accompanying the merger is small, leading to an increase in consumer surplus. By contrast, when the merging platforms are large, such benefits from network expansions are offset by the accompanying increase in the markup, harming consumers. These jointly lead to the conjecture that the CS-neutral synergies are negative as long as the merging platforms are small, and then become positive when the sizes of the merging platforms exceed a certain threshold. The following proposition formalizes such an intuition.

Proposition 5. *Consider a merger between one-sided platforms f and g with pre-merger network sizes N_f and N_g and pre-merger equilibrium aggregator H^*. There exists a critical value of network size $\bar{N}_f(N_g) > 0$ such that $\hat{\Delta} < 0$ if and only if $N_f < \bar{N}_f(N_g)$.*

Proposition 5 implies that as long as the pre-merger network sizes of the merging platforms are below certain critical values, the merger between them is CS-increasing without technological synergies. This proposition indicates that for small mergers, network effects serve as a synergy, and merger policy should be lenient to small mergers in the presence of network effects. Note that in the absence of network effects (i.e., $\alpha = 0$), CS-neutral synergy is always positive (Nocke and Schutz, 2018a).

As the next question, we consider the impact of network effects on large mergers. To this end, I introduce some terminologies to describe the platforms' strength, based on the relation between the magnitude of network externalities and the pre-merger network shares. To this end, I first establish the following lemma.

Lemma 3. *For any type profile $\{T_f\}_{f\in\mathcal{F}}$ of one-sided platforms and within-group network effect α, there is a threshold value T^* such that the equilibrium network size of platform f increases with a local increase in α if and only if $T_f \geq T^*$.*

This result is an example of positive feedback effects. Network effects expand the market shares of large platforms and shrink those of small platforms. The threshold type T^* stands for the critical value that defines the direction in which the positive feedback effects influence the market shares. I call the platforms with $T_f > T^*$ as *strong platforms* and the platforms with $T_f < T^*$ as *weak platforms*.

Based on the terminology of strong and weak platforms, I analyze the impact of network effects on CS-neutral technological synergies. The following proposition characterizes how the impact of network effects on CS-neutral technological synergies varies with the sizes of merging platforms.

Proposition 6. *Consider a one-sided market with a type profile $\{T_f\}_{f\in\mathcal{F}}$ and a merger between platforms f and g with pre-merger equilibrium network sizes N_f and N_g.*

1. *If both f and g are weak, then $\hat{\Delta}$ decreases with α.*
2. *If f is strong and g is weak, then there exists $\hat{N} \in (0,1)$ such that if $N_f + N_g < \hat{N}$, then $\hat{\Delta}$ decreases with α.*
3. *If both f and g are strong and $N_f + N_g$ is close to 1. Then $\hat{\Delta}$ increases with α.*

The underlying intuition of the first part of Proposition 6 is similar to that of Proposition 5. Weak platforms suffer from the inability to exploit network effects due to their small network sizes. The stronger the network effects are, the more serious the weak platforms' inability to exploit network effects is. In such cases, mergers involving small platforms soften this problem. Therefore, mergers between weak platforms become more desirable as the magnitude of network effects increases. A similar argument partially extends to the case where one platform in the merging party is strong and the other is weak, leading to the second part of Proposition 6.

The impact of network effects on mergers between strong platforms is ambiguous. When both of the merging parties are strong, the demand-side scale economy may not be enough to compensate for an increase in the markups caused by a greater concentration. When the joint market share of the two platforms is too large, the merged platform will have an extremely strong market power. An increase in the consumer gain from a further increase in the network effects is offset by an increase in the markup of the merged platform. Therefore, when the joint market share of merging platforms is sufficiently large, a greater magnitude network effects would require greater synergy to improve consumer welfare. Note that the role of network effects as synergy is also discussed in existing studies such as Gama et al. (2020). By contrast, the formal result that network effects may make large mergers more likely to harm consumers is a novel result of this study.

In summary, network effects make mergers more likely to improve consumer surplus when the merging parties are small. However, as the sizes of the merging platforms become large relative to the size of the industry, network effects make mergers more likely to harm consumers. Therefore, competition authorities should take the merging platform's sizes into account when evaluating the mergers involvin digital platforms characterized by network effects.

Two-sided markets I now consider the general setting of two-sided markets to highlight the additional effects of two-sided pricing.

Recall that the formula for the markup set by each platform f is given by

$$\mu_f^k = \frac{1}{1 - n_f^k} \left(1 - \alpha_k - \beta_l \frac{n_f^l}{n_f^k} \right).$$

The denominator of the right hand side, $1 - n_f^k$, represents the platform's market power: the ability to set prices away from marginal costs. The numerator, $1 - \alpha_k - \beta_l n_f^l / n_f^k$, represents the direction in which prices diverge from marginal costs. In one-sided markets, the direction in which prices diverge from marginal costs is always positive. By contrast, in two-sided markets, the platform may have an incentive to set prices below marginal costs to expand the network and attract consumers on the other side. When this subsidization effect is so strong that the markup on one side is negative, an increase in the market power allows the platform to *lower* the markup. Consequently, a merger may also decrease the prices set by merging platforms.

To illustrate this point, consider a merger between two platforms that have exactly the same network sizes before the merger. In a CS-neutral merger, the merger doubles the sizes of the merging platforms. Such a change in network size has two effects. First, an increase in network size directly benefits consumers. This makes mergers require a smaller synergy. Second, an increase in network size allows the merged platform to set prices further away from marginal costs. When the merging platforms subsidize consumers on one side, the merged platform uses its increased market power to lower prices. This makes mergers beneficial to consumers on the subsidized side. The following proposition summarizes these two effects.

Proposition 7. *Consider a merger between platforms with the same pre-merger network shares n^A and n^B. Then, $\tilde{\Delta}^k > 0$ if and only if*

$$\left(\frac{1}{1-2n^k} - \frac{1}{1-n^k}\right)\left(1-\alpha_k-\beta_l\frac{n^l}{n^k}\right) - (\alpha_k+\beta_k)\log 2 > 0$$

for $k = A, B$ and $I \neq J$. Particularly, the following holds true.

1. *If $1-\alpha_k-\beta_l\frac{n^l}{n^k} \leq 0$, then $\tilde{\Delta}^k \leq 0$.*
2. *If $1-\alpha_k-\beta_l\frac{n^l}{n^k} > 0$ and $\alpha_k+\beta_k > 0$, then there exists an increasing function $\hat{n}^k(n^l) > 0$ such that $\tilde{\Delta}^k \leq 0$ if and only if $n^k < \hat{n}^k(n^l)$.*
3. *If $1-\alpha_k-\beta_l\frac{n^l}{n^k} > 0$ and $\alpha_k+\beta_k = 0$, then $\tilde{\Delta}^k > 0$.*

Note that in this setting, the condition $1-\alpha_k-\beta_l\frac{n^l}{n^k} \leq 0$ is equivalent to the condition that merging platforms set negative markups on side k. Therefore, the CS-neutral technological synergy on one side is negative if the merging platforms set negative markups on that side before the merger. This provides a practical test for evaluating the impact of two-sidedness on platform mergers, that is, consumers who are subsidized through negative markups are likely to benefit from mergers. Note that a similar result is obtained by Correia-da-Silva et al. (2019)'s analysis of Cournot competition between homogeneous platforms. The difference is that the prediction in this study is at the platform level, whereas the prediction in Correia-da-Silva et al. (2019) is at the market level. Furthermore, the characterization of Proposition 7 shows that merging platform's pre-merger network sizes provide useful information about the competitive effects of mergers. Song (2021) also shows a similar result by using a simulation analysis based on an estimated structural model. Proposition 7 can be considered as a theoretical counterpart of the result of Song (2021).

Overall, in two-sided markets, mergers affect not only the network sizes but also the price structures, by affecting two-sided pricing. When a group of consumers are subsidized before the merger, such consumers are likely to benefit from platform mergers through increased network sizes and lower prices.

5. Long-Run Equilibrium with Free Entry

As a long-run competition, I analyze a platform competition with free entry of fringe platforms. To this end, I extend the baseline framework by incorporating Anderson et al. (2013)'s notion of symmetric marginal entrants.

Suppose that, along with the set of incumbents $\mathcal{I}$, the set $\mathcal{E}$ of marginal entrants choose whether to enter the market where each platform $e \in \mathcal{E}$ has type $\left(T_E^A, T_E^B\right)$ and incurs fixed cost $K > 0$ for entry. Let $\mathcal{A}$ be the set of active entrants. Each incumbent platform $f \in \mathcal{I}$ may have different types (T_f^A, T_f^B). The set of active platforms in the market is given by $\mathcal{F} = \mathcal{I} \cup \mathcal{A}$.

Let $\pi_E\left(H^A, H^B\right)$ be the post-entry profit of a marginal entrant when the values of the aggregators are given by H^A and H^B. Specifically, the post-entry profit $\pi_E\left(H^A, H^B\right)$ is given by

$$\pi_E(H^A, H^B) = \sum_{k=A,B} \mu_E^k \frac{\left(T_E^k\right)^{\Gamma_{kk}} \left(T_E^k\right)^{\Gamma_{kl}} \exp\left(-\Gamma_{kl}\mu_E^k - \Gamma_{kl}\mu_E^l\right)}{H^k}, \tag{23}$$

where $\mu_E^k = m_k(T_E^k, T_E^l, H^k, H^l)$. Using this notation, I define the free-entry equilibrium as follows.

Definition 2. *The set of active marginal entrants $\mathcal{A}$ constitutes a free-entry equilibrium if the triplet $\left(H^A, H^B, |\mathcal{A}|\right)$ satisfies the following conditions:*

$$\pi_E\left(H^A, H^B\right) - K = 0, \tag{24}$$

$$\sum_{f\in\mathcal{I}} N^A\left(T_f^A, T_f^B, H^A, H^B\right) + |\mathcal{A}| N^A\left(T_E^A, T_E^B, H^A, H^B\right) = 1, \tag{25}$$

$$\sum_{f\in\mathcal{I}} N^B\left(T_f^B, T_f^A, H^B, H^A\right) + |\mathcal{A}| N^B\left(T_E^B, T_E^A, H^B, H^A\right) = 1. \tag{26}$$

The definition of free-entry equilibrium endogenizes the number of active marginal entrants $|\mathcal{A}|$ by requiring zero-profit condition (24). Marginal entrants enter the market as long as the post-entry profit exceeds the entry cost, and the entry stops once the additional entry becomes unprofitable. Using Definition 2, I examine the welfare effects of the incumbent platforms' strategic investments, which is captured by a change in (T_f^A, T_f^B) for $f \in \mathcal{I}$.

In the aggregative-games analysis of oligopoly in one-sided markets, the zero-profit condition of marginal entrants uniquely pins down the value of the aggregator (e.g., Davidson and Mukherjee, 2007; Anderson et al., 2013; Ino and Matsumura, 2012; Anderson et al., 2020). Because consumer surplus is determined solely by the value of the aggregator in one-sided markets, any change in competitive environment, such as incumbents' entry and mergers, does not affect consumer surplus. By contrast, in two-sided markets, zero profit condition (24) only pins down the relation between two aggregators (H^A, H^B). Therefore, competitive environments are no longer neutral to the consumer surplus on each side and the aggregate consumer surplus. As an example illustrating this property, I show *strong see-saw property*: the property that any change in competitive environments that improves consumer surplus on one side hurts consumer surplus on the other side.

To see this, I reformulate the profit of marginal entrants and consumer surplus as a function of the network sizes of marginal entrants. Using equations (3) and (12), the profit of marginal entrants can be written as

$$\Pi_E = \frac{n_E^A}{1-n_E^A}\left(1-\alpha_A-\beta_B\frac{n_E^B}{n_E^A}\right)+\frac{n_E^B}{1-n_E^B}\left(1-\alpha_B-\beta_A\frac{n_E^A}{n_E^B}\right)-K. \tag{27}$$

The free-entry condition can be written as $\Pi_e = 0$. The equilibrium consumer surplus can be written as a function of the marginal entrant's type T_e^k and its equilibrium network sizes:

$$CS_k = \log T_E^k + \beta_k \log n_E^l - (1-\alpha_k)\log n_E^k - \frac{1-\alpha_k-\beta_l\frac{n_E^l}{n_E^k}}{1-n_E^k}. \tag{28}$$

Therefore, a change in competitive environments affects the consumer surplus on each side through changes in the relation between (n_E^A, n_E^B). Note that as long as two-sided network effects (β_A, β_B) are not too strong, an increase in network sizes is associated with low consumer surplus, because a large network size of marginal entrants implies a small value of industry-level aggregators.

To see the idea behind the strong see-saw property, consider the case where an incumbent invests in side A so that the marginal entrant's network size on side A decreases. In a standard scenario, competition on side A becomes intense due to the incumbent's investment. At the same time, the entry of marginal entrants decreases, so the competition on side B becomes weaker. In two-sided markets, a subtler strategic interaction may exist due to network effects and changes in two-sided pricing. Nontheless, the following proposition shows that the reasoning of standard scenario still holds in general.

Proposition 8. *Consider an equilibrium with the set of active marginal entrants that constitutes a free entry equilibrium. Any change in competitive environments that increases consumer* *surplus on one side decreases consumer surplus on the other side. Formally, holding the parameters* $(\alpha_A, \alpha_B, \beta_A, \beta_B, T_E^A, T_E^B, K)$ *fixed, compare two long-run equilibria that differ in other parameters. Letting the equilibrium consumer surplus under two settings denoted by* (CS_{A0}, CS_{B0}) *and* (CS_{A1}, CS_{B1})*, we have*

$$(CS_{A0} - CS_{A1})(CS_{B0} - CS_{B1}) < 0.$$

The strong see-saw property poses a challenge to competition authorities facing the strategic behavior of large incumbent platforms in the face of fringe platforms. Because the incumbent platform's strategic behavior generically benefits consumers on one side at the expense of those on the other side, the competition authority must decide which consumers to protect. In some markets, such as on-line commerce, the current antitrust policy is mainly based on "consumer welfare standard," which is focused on the consumer surplus on a particular side. Khan (2017) argues that such an approach fails to recognize other harms of the dominant platform's practices, including the harm to third-party sellers. Proposition 8 indicates that such a conflict is inevitable in two-sided markets with free entry.

6. Conclusion

In this study, I developed an aggregative-game framework for studying oligopolistic platform competition, allowing for an arbitrary asymmetry between platforms. By establishing the existence and uniqueness of equilibrium, I apply the framework to analyze horizontal mergers and free entry of platforms in a unified manner. The framework's ability to incorporate platform asymmetry provides rich implications for the role of network effects and two-sidedness in an asymmetric platform competition.

This study abstracts several aspects of platform competition, which leaves the avenue for future research. First, the analytical framework is static and does not consider dynamics such as R&D competition. Incorporating such dynamics would enrich policy prescriptions. One way would be to use Motta and Tarantino (2021)'s approach to analyze competition in prices and investments in the framework of this study. Second, the framework of this study focuses on the case where consumers on both sides single-home. However, in various online services, consumers often multihome. The possibility of multihoming affects platform competition and the potential effects of mergers on competition in important ways, as discussed by Anderson, Foros and Kind (2019). Unfortunately, the discrete-choice framework of this study does not fit well to address this issue. Therefore, I leave these issues for future research.

Finally, although I only analyzed horizontal mergers and free entry, the framework of this study can be applied to other analyses of oligopolistic platform competition. For example, Sato (2021) applies the framework of this study to analyze the relation between equilibrium market shares and the profits of platforms. Also, the framework of this study can be used as a structural model to estimate using observed data on platform markets, as a multinomial-logit model used as a tractable model for estimating the demand for differentiated products.

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A. Proofs

Proof of Lemma 1 Fix the equilibrium network size of platform S at (n_S^A, n_S^B). To show the existence of asymmetry parameters (δ_A, δ_B) that induces (n_S^A, n_S^B).

Using equation (12), the ι-markup set by platform S and W on side k are given by

$$\mu_S^k = \frac{1}{1-n_S^k}\left(1-\alpha_k-\beta_l\frac{n_S^l}{n_S^k}\right),$$
$$\mu_W^k = \frac{1}{n_S^k}\left(1-\alpha_k-\beta_l\frac{1-n_S^l}{1-n_S^k}\right),$$

where I used the fact that $n_W^k = 1-n_S^k$ for $k = A, B$. Finally, the market share should satisfy the equation

$$n_S^k = \frac{V_S^k/V_W^k}{V_S^k/V_W^k+m},$$

where

$$V_f^k = \left(T_f^k\right)^{\Gamma_{kk}}\left(T_f^l\right)^{\Gamma_{kl}}\exp\left(-\mu_f^k\Gamma_{kk}-\mu_f^l\Gamma_{kl}\right)$$

A calculation shows that the unique value of δ_k consistent with market shares (n_S^A, n_S^B) is given by

$$\delta_k = \left(\frac{n_S^k}{1-n_S^k}\right)^{1-\alpha_k}\left(\frac{n_S^l}{1-n_S^l}\right)^{-\beta_k} m^{-\alpha_k-\beta_k}\exp\left[\mu_S^k-\mu_W^k\right],$$

which completes the proof. □

Proof of Proposition 3 When $(n_S^A, n_S^B) = (n, n)$, by the proof of Lemma 1, the asymmetry parameters consistent with the network sizes are given by $\bar{\delta}_k(n)$ for $k = A, B$, where $\bar{\delta}_k(n)$ is given by equation (20).

Next, we establish that if the platform f of type (T_f^A, T_f^B) has network sizes (n_f^A, n_f^B) in the equilibrium, then the side-k consumer surplus is given by

$$CS_k = \log T_f^k - \mu_f^k - (1-\alpha_k)\log n_f^k + \beta_k \log n_f^l. \tag{29}$$

To see this, note that the value of side-k aggregate H_k must satisfy the relation

$$n_f^k = \frac{\left(T_f^k\right)^{\Gamma_{kk}}\left(T_f^l\right)^{\Gamma_{kl}}\exp\left(-\mu_k\Gamma_{kk}-\mu_l\Gamma_{kl}\right)}{H_k}.$$

Using this relation and the fact that $CS_k = (1-\alpha_k)\log H_k - \beta_k \log H_l$, we obtain the equation (29). Therefore, at the balanced asymmetry, we have

$$CS_k = CS_k^*(n) = \log \bar{T}_k + \log\left(\frac{\bar{\delta}_k(n)}{1+\bar{\delta}_k(n)}\right) - (1-\alpha_k-\beta_k)\log n - \frac{1-\alpha_k-\beta_l}{1-n} \tag{30}$$

Taking the derivative of $CS_k^*(n)$, we obtain

$$CS_k^{*\prime}(n) = \frac{\mathcal{X}_k(n)}{[1+\bar{\delta}_k(n)]n^2(1-n)^2},$$

where

$$\mathcal{X}_k(n) = \left\{(1-\alpha_k-\beta_k)n(1-n)\left[n-(1-n)\bar{\delta}_k\right] - (1-\alpha_k-\beta_l)\left[n^2\delta_k-(1-n)^2\right]\right\}.$$

Note that because $\delta_k \geq 1$, $n^2\delta_k - (1-n)^2 > 0$ for any $n > 1/2$.

Now we proof the statements of Proposition 3.

1. When $\beta_l \leq \beta_k$,

$$\begin{aligned}\mathcal{X}_k(n) &\leq (1-\alpha_k-\beta_k)(1-n+n^2)[(1-n)-n\delta_k] \\ &\leq \frac{(1-\alpha_k-\beta_k)}{4}[1-\delta_k] \\ &\leq 0,\end{aligned}$$

   for all $n \geq 1/2$ with equality only if $n = 1/2$.

2. Let $\beta_l = 1-\alpha_k-\epsilon$, where $\epsilon > 0$. Then,

$$\begin{aligned}\mathcal{X}_k(n) =&(1-\alpha_k-\beta_k)n^2(1-n)\left\{1-\left(\frac{n}{1-n}\right)^{-\alpha_k-\beta_k}\exp\left[\epsilon\left(\frac{1}{1-n}-\frac{1}{n}\right)\right]\right\} \\ &-\epsilon[n^2\bar{\delta}_k-(1-n)^2].\end{aligned}$$

   Because

$$\lim_{\epsilon\to 0}\mathcal{X}_k(n) = (1-\alpha_k-\beta_k)n^2(1-n)\left[1-\left(\frac{n}{1-n}\right)^{-\alpha_k-\beta_k}\right] \geq 0$$

   for all $n \geq 1/2$ with equality only if $n = 1/2$, there exists $\underline{\epsilon} > 0$ such that $\mathcal{X}_k(n) > 0$ for all $\epsilon < \underline{\epsilon}$.

Proof of Lemma 2 First, consider the type (T^A, T^B) of a platform that is consistent with the network sizes (N^A, N^B) and aggregators (H^A, H^B). Explicitly solving for (T^A, T^B) that is consistent with (N^A, N^B) and (H^A, H^B), we obtain the solution

$$T^k = \tau_k(N^A, N^B, H^A, H^B) = \frac{\left(N^kH^k\right)^{1-\alpha_k}}{\left(N^lH^l\right)^{\beta_k}}\exp\left[\frac{1}{1-N^k}\left(1-\alpha_k-\beta_l\frac{N^l}{N^k}\right)\right].$$

Next, I compute CS-neutral synergy given pre-merger network sizes. Let N_a^k be the equilibrium market share of platform $a \in \{f,g\}$ on side $J \in \{A,B\}$. Let $N_M^k = N_f^k + N_g^k$ for each $k \in \{A,B\}$. Consider the merging entity's type (T_M^A, T_M^B) such that

$$\begin{aligned}N^A(T_M^A, T_M^B, H^A, H^B) = N_M^A \\ N^B(T_M^A, T_M^B, H^A, H^B) = N_M^B\end{aligned}$$

Then, we must have $T_M^k = \tau_k(N_M^A, N_M^B, H^A, H^B)$, and the CS-neutral type is given by

$$\widetilde{\Delta} = \tau_k(N_M^A, N_M^B, H^A, H^B) - \tau_k(N_f^A, N_f^B, H^A, H^B) - \tau_k(N_g^A, N_g^B, H^A, H^B)$$

for $k = A, B$. $\square$

Proof of Proposition 4 First, I show that in the case where $\beta_A = \beta_B = 0$, consumer surplus on each side is monotonically increasing in the types of the platforms. Because two sides are independent, I drop the indices of the sides.

In the case of one-sided markets, the condition for the optimal pricing (9) can be rewritten as

$$\mu_f = \frac{1-\alpha}{1-n_f}. \tag{31}$$

Inserting equations (31) and (13) into equation (3), we obtain the condition for the network size of platform f when it sets the optimal prices as below:

$$n_f = \frac{\gamma(T_f)}{H} \exp\left(-\frac{1}{1-n_f}\right), \tag{32}$$

where $\gamma(x) = x^{\frac{1}{1-\alpha}}$. Let $N_0(\gamma(T_f)/H)$ be the solution to equation (32). As $N_0(\cdot)$ turns out to be an increasing function, platform f has a large network size $n_f = N_0(\gamma(T_f)/H)$ either when it has a high type T_f or the value of industry-level aggregator H is small.

The equilibrium condition for the industry-level aggregator H is given by the condition

$$\sum_{f\in\mathcal{F}} N_0\left(\frac{\gamma(T_f)}{H}\right) = 1. \tag{33}$$

Solving this equation for the industry-level aggregator H, the equilibrium industry-level aggregator, H^*, is obtained. Applying the Implicit Function Theorem, we have

$$\frac{dH^*}{dT_f} = \frac{\frac{\gamma(T_f)}{H^2} N_0'\left(\frac{\gamma(T_f)}{H}\right)}{\sum_{f'\in\mathcal{F}} \frac{\gamma(T_{f'})}{H^2} N_0'\left(\frac{\gamma(T_{f'})}{H}\right)} > 0,$$

which proves that the equilibrium consumer surplus is increasing in the type of platform, because consumer surplus in monotonically increasing in H^*.

Next, I prove the second part of the proposition. To show the existence of the case where an increase in the type of one platform lowers the consumer surplus on the other side, I consider the case where $\alpha_A = \alpha_B = \beta_B = 0$ and $\beta_A = \beta > 0$.

As a preliminary, I consider the effects of the type T_f^A on ι-markups and network sizes. To see the effects of T_f^A on n_f^A and n_f^B note that

$$\begin{aligned}
\frac{\partial m^A}{\partial T_f^A} &= \frac{n_f^A}{T_f^A} \frac{1}{\det(G_f)} \left[\mu_f^A \left(n_f^B \mu_f^B + 1 - n_f^B + \beta \frac{n_f^A}{n_f^B}\right) + \mu_f^A \beta^2 \frac{n_f^A}{n_f^B}\right] > 0 \\
\frac{\partial m^B}{\partial T_f^A} &= -\frac{n_f^A}{T_f^A} \frac{1}{\det(G_f)} \left(\beta \frac{n_f^A}{n_f^B} \mu_f^A (1-\beta) + \beta \frac{1-n_f^A}{n_f^B}\right) < 0,
\end{aligned}$$

and thus

$$\frac{\partial N^A}{\partial T_f^A} = \frac{n_f^A}{T_f^A}\frac{1}{\det(G_f)}(1-n_f^A)\left(n_f^B\mu_f^B + 1 - n_f^B + \beta(1+\beta)\frac{n_f^A}{n_f^B}\right) < 0$$
$$\frac{\partial N^B}{\partial T_f^A} = \frac{n_f^A}{T_f^A}\frac{1}{\det(G_f)}\left(\beta\frac{n_f^A}{n_f^B}\mu_f^A(1-\beta) + \beta\frac{1-n_f^A}{n_f^B}\right)n_f^B < 0.$$

Using the above result, consider the effect of T_f^A on H^A. By the Implicit Function Theorem, we have

$$\begin{pmatrix} \sum \frac{\partial N_f^A}{\partial H^A} & \sum \frac{\partial N_f^A}{\partial H^B} \\ \sum \frac{\partial N_f^B}{\partial H^A} & \sum \frac{\partial N_f^B}{\partial H^B} \end{pmatrix} \begin{pmatrix} \frac{dH^A}{dT_f^A} \\ \frac{dH^B}{dT_f^A} \end{pmatrix} = -\begin{pmatrix} \frac{\partial N_f^A}{\partial T_f^A} \\ \frac{\partial N_f^B}{\partial T_f^A} \end{pmatrix},$$

for T_f^A. Using Cramer's rule, we obtain

$$\operatorname{sign}\left(\frac{\partial H^A}{\partial T_f^A}\right) = \operatorname{sign}\left[-\frac{\partial N_f^A}{\partial T_f^A}\left(\sum_{f'\in\mathcal{F}}\frac{\partial N_f^B}{\partial H^B}\right) + \frac{\partial N_f^B}{\partial x_f}\left(\sum_{f'\in\mathcal{F}}\frac{\partial N_f^A}{\partial H^B}\right)\right]$$

Since $\partial N_f^A/\partial H^B > 0$ for all $f \in \mathcal{F}$ and $\partial N_f^k/\partial T_f^A > 0$ for any $k \in \{A, B\}$, we have $\partial H^A/\partial T_f^A > 0$.

Next, consider the effects of T_f^A on H^B. Using the Cramer's rule, we have

$$\operatorname{sign}\left(\frac{\partial H^B}{\partial T_f^A}\right) = \operatorname{sign}\left[-\frac{\partial N_f^B}{\partial T_f^A}\left(\sum_{f'\in\mathcal{F}}\frac{\partial N_f^A}{\partial H^A}\right) + \frac{\partial N_f^A}{\partial x_f}\left(\sum_{f'\in\mathcal{F}}\frac{\partial N_f^B}{\partial H^A}\right)\right].$$

A calculation shows that

$$\begin{aligned}&\operatorname{sign}\left(\frac{dH^B}{dT_f^A}\right)\\ =&\operatorname{sign}\left[\frac{n_f^A\mu_f^A(1-\beta)+1-n_f^A}{(1-n_f^A)\left(n_f^B\mu_f^B+1-n_f^B+\beta(1+\beta)\frac{n_f^A}{n_f^B}\right)} - \frac{\sum_{f'\in\mathcal{F}}\left(n_{f'}^A\mu_{f'}^A(1-\beta)+1-n_{f'}^A\right)}{\sum_{f'\in\mathcal{F}}\left[(1-n_{f'}^A)\left(n_{f'}^B\mu_{f'}^B+1-n_{f'}^B+\beta(1+\beta)\frac{n_{f'}^A}{n_{f'}^B}\right)\right]}\right]\end{aligned}$$

I provide an example where $dH^B/dT_f^A < 0$. For simplicity, suppose that $\beta \simeq 0$. Then, we have

$$\operatorname{sign}\left(\frac{dH^B}{dT_f^A}\right) \simeq \operatorname{sign}\left[\frac{n_f^A\mu_f^A+1-n_f^A}{(1-n_f^A)\left(n_f^B\mu_f^B+1-n_f^B\right)} - \frac{\sum_{f'\in\mathcal{F}}\left(n_{f'}^A\mu_{f'}^A+1-n_{f'}^A\right)}{\sum_{f'\in\mathcal{F}}\left[(1-n_{f'}^A)\left(n_{f'}^B\mu_{f'}^B+1-n_{f'}^B\right)\right]}\right] \tag{34}$$

Consider further the case where all platform but platform f are symmetric, and platform f's shares are given by $n_f^A \simeq 0$ and $n_f^B \simeq 0$. Then, the terms in the brackets in the second line of equation (34) can be rewritten as

$$1 - \frac{1 + \frac{|\mathcal{F}|^2-|\mathcal{F}|+1}{|\mathcal{F}|-1}}{1 + \frac{|\mathcal{F}|^2-|\mathcal{F}|+1}{|\mathcal{F}|}} < 0,$$

which shows that $dH^B/dT_f^A < 0$. Because the consumer surplus on side B is given by $CS_B = \log H^B - \beta \log H^A$, CS_B decreases with T_f^A in such a case, which completes the proof. $\square$

Proof of Proposition 5 When $\beta_A = \beta_B = 0$, we have

$$\tau_k(N^A, N^B, H^A, H^B) = \tilde{\tau}(N^k, H^k) := (H^k)^{1-\alpha_k} \left(N^k\right)^{1-\alpha_k} \exp\left(\frac{1-\alpha_k}{1-N^k}\right).$$

Therefore, the CS-neutral synergy in one-sided market is computed by

$$\tilde{\tau}(N_f + N_g, H) - \tilde{\tau}(N_f, H) - \tilde{\tau}(N_g, H),$$

which is positive if and only if

$$\begin{aligned}\omega(N_f) = (N_f + N_g)^{1-\alpha} \exp\left(\frac{1-\alpha}{1-N_f-N_g}\right) - (N_f)^{1-\alpha} \exp\left(\frac{1-\alpha}{1-N_f}\right) - (N_g)^{1-\alpha} \exp\left(\frac{1-\alpha}{1-N_g}\right) \\ \geq 0.\end{aligned}$$

Note that we have $\omega(0) = 0$, $\lim_{N_f \to 1-N_g} \omega(N_f) = \infty$. Furthermore, the derivative of $\omega(N)$ is given by

$$\begin{aligned}\omega'(N_f) = (1-\alpha)\Bigg[(N_f + N_g)^{-\alpha} \left(\frac{1}{(1-N_f-N_g)^2} + N_f + N_g\right) \exp\left(\frac{1-\alpha}{1-N_f-N_g}\right) \\ - (N_f)^{-\alpha} \left(\frac{1}{(1-N_f)^2} + N_f\right) \exp\left(\frac{1-\alpha}{1-N_f-N_g}\right), \Bigg]\end{aligned}$$

which is positive if and only if

$$\begin{aligned}\rho(N_f) := \left(1 + \frac{N_g}{N_f}\right)^{-\alpha} \left(\frac{1}{(1-N_f-N_g)^2} + N_f + N_g\right) \exp\left(\frac{1-\alpha}{1-N_f-N_g}\right) \\ - \left(\frac{1}{(1-N_f)^2} + N_f\right) \exp\left(\frac{1-\alpha}{1-N_f-N_g}\right) \geq 0\end{aligned}$$

We have $\rho(0) < 0$, and $\rho'(N) > 0$, implying that there exits $\hat{N} > 0$ such that $\omega'(N_f) < 0$ for all $N_f < \hat{N}$ and $\omega'(N_f) > 0$ for all $N_f > \hat{N}$. Consequently, there exists a critical value $\bar{N}_f(N_g) \in (0, \infty)$ such that $\omega(N_f) < 0$ for all $N_f \in (0, \bar{N}_f(N_g))$ and $\omega(N_f) > 0$ for all $N_f > \bar{N}_f(N_g)$, which completes the proof. $\square$

Proof of Lemma 3 Using the Implicit Function Theorem, we have the following relation:

$$\frac{d}{d\alpha}\left(\frac{\gamma(T_f)}{H^*}\right) = \frac{1}{(1-\alpha)^2} \frac{\gamma(T_f)}{H^*} \frac{\sum_{f' \in \mathcal{F}} (\log T_f - \log T_{f'}) N_0' \left(\frac{\gamma(T_{f'})}{H^*}\right)}{\sum_{f' \in \mathcal{F}} N_0' \left(\frac{\gamma(T_{f'})}{H^*}\right)} \tag{35}$$

From this equation, the lemma is directly obtained.

Proof of Proposition 6 Using the Implicit Function Theorem, we have

$$\frac{d\hat{\Delta}}{d\alpha} = \left\{ \frac{d}{d\Delta} N_0 \left(\frac{\gamma(\hat{T}_M)}{H^*} \right) \right\}^{-1} \left\{ \frac{d}{d\alpha} N_0 \left(\frac{\gamma(T_f)}{H^*} \right) + \frac{d}{d\alpha} N_0 \left(\frac{\gamma(T_g)}{H^*} \right) - \frac{d}{d\alpha} N_0 \left(\frac{\gamma(\hat{T}_M)}{H^*} \right) \right\}.$$

Therefore, $d\hat{\Delta}/d\alpha < 0$ if and only if

$$A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \geq A(T_f)B(T_f) + A(T_g)B(T_g) \tag{36}$$

where

$$A(T_f) = N_0' \left(\frac{\gamma(T)}{H^*} \right) \frac{\gamma(T)}{H^*}$$

and

$$B(T) = \frac{H^*}{\gamma(T)} \frac{d}{d\alpha} \left(\frac{\gamma(T)}{H^*} \right) = \frac{1}{(1-\alpha)^2} \frac{\sum_{f' \in \mathcal{F}} (\log T - \log T_{f'}) N_0' \left(\frac{\gamma(T_{f'})}{H^*} \right)}{\sum_{f' \in \mathcal{F}} N_0' \left(\frac{\gamma(T_{f'})}{H^*} \right)}.$$

Note that $B(T)$ is an increasing function.

I first analyze the concavity and the convexity of the function $\tilde{N}'(x)x$ with respect to the network size, and then use this analysis to prove Proposition 6.1 and Proposition 6.2.

Note that

$$N_0'(x)x \bigg|_{N_0(x)=N} = \frac{N^3 - 2N^2 + N}{N^2 - N + 1} =: \phi(N).$$

We have

$$\phi'(N) = \frac{(1-N)(N^3 - N^2 + 3N - 1)}{(N^2 - N + 1)^2},$$

which is nonnegative in $N \in [0, \hat{N}]$ and negative in $(\hat{N}, 1]$ for some critical value $\hat{N} \in (0,1)$ that satisfies

$$\hat{N}^3 - \hat{N}^2 + 3\hat{N} - 1 = 0.$$

Furthermore, Nocke and Schutz (2018a) show that $\phi(N)$ is concave in N. Therefore, $\phi(N)$ is increasing in $N \in [0, \hat{N})$, nonincreasing in $N \in [\hat{N}, 1]$, and concave in $N \in [0,1]$

Using this result, I prove Proposition 6.1 and Proposition 6.2.

Suppose that two merging platforms f and g are weak. If the merged entity is strong, the left-hand side of equation (36) is positive while the right-hand side of equation (36) is negative. Next, suppose that the merged entity is weak. We have $A(T_f + T_g + \hat{\Delta}) \leq A(T_f) + A(T_g)$ by the concavity of $N_0'(x)x$ in $N \in [0,1]$. Finally, we have the following inequality

$$\begin{aligned} A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) &\geq (A(T_f) + A(T_g))B(T_f + T_g + \hat{\Delta}) \\ &\geq A(T_f)B(T_f) + A(T_g)B(T_g), \end{aligned}$$

where the last inequality follows from the fact that $B(T)$ is increasing in T and $T_f + T_g + \hat{\Delta} \geq \max\{T_f, T_g\}$. Thus, $\hat{\Delta}$ decreases with α.

Next, I show that $\hat{\Delta}$ for the merger between platform f and g decreases with α if platform f is weak and the platform g is strong and if $N_f + N_g < \hat{N}$. This can be observed by

$$A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \geq A(T_f)B(T_f)$$

and

$$A(T_g)B(T_g) \leq 0.$$

Next, I prove Proposition 6.3. When $N_f + N_g \simeq 1$, then $A(T_f + T_g + \hat{\Delta}) \simeq 0$.

$$T_f + T_g + \hat{\Delta} = \left[H^*(N_f + N_g) \exp\left(\frac{1}{1 - N_f - N_g} \right) \right]^{1-\alpha}.$$

Thus, we have

$$\log(T_f + T_g + \hat{\Delta}) = (1 - \alpha)\left[\log H^* + \log(N_f + N_g) + \frac{1}{1 - N_f - N_g} \right]$$

As a result,

$$N'(x)x \log(T_f + T_g + \hat{\Delta}) \to 0 \text{ as } N_f + N_g \to 1.$$

Thus, $A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \to 0$ as $N_f + N_g \to 1$. Thus, if two merging platforms are strong, the LHS of the equation (36) is zero, while the RHS of the equation (36) is positive. Thus, $\hat{\Delta}$ increases with α. □

Proof of Proposition 7 First, for the fixed network sizes n^A and n^B, compute the types that are consistent with network sizes (n^A, n^B):

$$\tau_k(n^A, n^B, H^A, H^B) = \frac{\left(n^k H^k\right)^{1-\alpha_k}}{\left(n^l H^l\right)^{\beta_k}} \exp\left[\frac{1}{1 - n^k} \left(1 - \alpha_k - \beta_l \frac{n^l}{n^k} \right) \right].$$

$$\tau_k(2n^A, 2n^B, H^A, H^B) = 2^{1-\alpha_k-\beta_k} \frac{\left(n^k H^k\right)^{1-\alpha_k}}{\left(n^l H^l\right)^{\beta_k}} \exp\left[\frac{1}{1 - 2n^k} \left(1 - \alpha_k - \beta_l \frac{n^l}{n^k} \right) \right].$$

The CS-neutral technological synergy on side k is positive if and only if

$$\frac{\tau^k(2n^A, 2n^B, H^A, H^B)}{2\tau^k(n^A, n^B, H^A, H^B)} > 1,$$

which is equivalent to the condition

$$\frac{n^k}{(1 - n^k)(1 - 2n^k)} \left(1 - \alpha_k - \beta_l \frac{n^l}{n^k} \right) - (\alpha_k + \beta_k) \log 2 > 0 \tag{37}$$

for $k = A, B$. Based on this analysis, I prove the proposition.

1. If $1 - \alpha_k - \beta_l n^l / n^k \leq 0$, then condition (37) never holds, implying that $\hat{\Delta} < 0$.
2. If $1 - \alpha_k - \beta_l n^l / n^k > 0$ and $\alpha_k + \beta_k = 0$, then condition (37) always holds, implying that $\hat{\Delta} > 0$.
3. If $1 - \alpha_k - \beta_l n^l / n^k > 0$ and $\alpha_k + \beta_k > 0$, then there exists a critical value $\hat{n}^k$ such that condition (37) always holds if and only if $n^k < \hat{n}^k$.

Proof of Proposition 8 Define Π_E by

$$\Pi_E = \frac{n_E^A}{1-n_E^A}\left(1-\alpha_A-\beta_B\frac{n_E^B}{n_E^A}\right)+\frac{n_E^B}{1-n_E^B}\left(1-\alpha_B-\beta_A\frac{n_E^A}{n_E^B}\right)-K. \tag{38}$$

The free entry condition can be written as $\Pi_E = 0$.

The equilibrium consumer surplus can be written as the function of the marginal entrant's type T_e^k and their equilibrium network sizes:

$$CS_k = \log T_E^k + \beta_k \log n_E^l - (1-\alpha_k)\log n_E^k - \frac{1-\alpha_k-\beta_l\frac{n_E^l}{n_E^k}}{1-n_E^k}. \tag{39}$$

Suppose that $n_E^A \geq n_E^B$ and that let n_E^A and n_E^B be written as $n_E^B = n$ and $n_E^A = n\theta$ using $\theta \geq 1$ and $n \leq 1/\theta$. Then, we have

$$\Pi_E = n\left(\frac{\theta-\alpha_A\theta-\beta_B}{1-\theta n}+\frac{1-\alpha_B-\beta_A\theta}{1-n}\right)-K, \tag{40}$$

which has a unique solution in $n(\theta) \in (0, 1/\theta)$ given θ.

Using the Implicit Function Theorem, we have

$$n'(\theta) = -\frac{W(\theta)}{V(\theta)}, \tag{41}$$

where

$$W(\theta) = \frac{n}{(1-\theta n)^2}(1-\alpha_A-n\beta_B)-\frac{\beta_A}{1-n}.$$

and

$$V(\theta) = \frac{\theta}{1-\theta n}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right)+\frac{1-\alpha_B-\beta_A\theta}{1-n}+\frac{\theta}{(1-\theta n)^2}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right)+\frac{1-\alpha_B-\beta_A\theta}{(1-n)^2}$$

I show that $W(\theta) > 0$ for all $\theta \geq 1$. Note that at θ such that $W(\theta) = 0$, or $n'(\theta) = 0$, the derivative of $W(\theta)$ with respect to θ is given by

$$W'(\theta)\frac{2n}{1-\theta n}\frac{\beta_A}{1-n} > 0.$$

Provided that $V(\theta)$ is positive, there exists $\hat{\theta}$ such that $W(\theta) \geq 0$ if and only if $\theta \geq \hat{\theta}$. Note that at $\theta = 1$, we have

$$n(1) = \frac{K}{2-\alpha_A-\alpha_B-\beta_A-\beta_B+K},$$

and

$$
\begin{aligned}
W(1) &= \frac{1}{1-n(1)}\left[\frac{n(1)}{1-n(1)}(1-\alpha_A-n(1)\beta_B)-\beta_A\right] \\
&= \frac{1}{1-n(1)}\frac{(2-\alpha_A-\alpha_B-\beta_A-\beta_B+K)^2(1-\alpha_A)-(4-2\alpha_A-2\alpha_B-2\beta_A-2\beta_B+K)K\beta_B}{(2-\alpha_A-\alpha_B-\beta_A-\beta_B+K)^2} \\
&\geq \frac{1}{1-n(1)}\frac{(2-\alpha_A-\alpha_B-\beta_A-\beta_B)^2(1-\alpha_A)}{(2-\alpha_A-\alpha_B-\beta_A-\beta_B+K)^2} > 0,
\end{aligned} \tag{42}
$$

which implies that $\hat{\theta} < 1$, implying that $W(\theta)$ is always positive, provided that the denominator is positive, which I show below.

To show that $V(\theta) > 0$, note that $V(\theta)$ is written as

$$
\begin{aligned}
V(\theta) =& \frac{\theta}{1-\theta n}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right)+\frac{1-\alpha_B-\beta_A\theta}{1-n}+\frac{\theta}{(1-\theta n)^2}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right)+\frac{1-\alpha_B-\beta_A\theta}{(1-n)^2} \\
=& \frac{\theta}{n}\left[\frac{n}{(1-\theta n)^2}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right)-\frac{n\beta_A}{1-n}\right]+\frac{\theta}{1-\theta n}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right)+\frac{1-\alpha_B}{1-n}+\frac{1-\alpha_B-\beta_A\theta}{(1-n)^2} \\
=& \frac{\theta}{n}W+X,
\end{aligned} \tag{43}
$$

where

$$
\begin{aligned}
X &= \frac{\beta_B(-1-n^2+2\theta n)}{(1-\theta n)^2}+\frac{\theta}{n}\beta_A+\frac{\theta(1-\alpha_A)}{1-\theta n}+\frac{(2-n)(1-\alpha_B)-\beta_A\theta}{(1-n)^2} \\
&= -\beta_B-\frac{(1-\theta^2)n^2}{(1-\theta n)^2}\beta_B+\frac{\theta}{n}\beta_A+\frac{\theta(1-\alpha_A)}{1-\theta n}+\frac{(2-n)(1-\alpha_B)-\beta_A\theta}{(1-n)^2}
\end{aligned} \tag{44}
$$

I show that $X > 0$. Because $\theta \geq 1$, we have

$$
\begin{aligned}
X \geq& \frac{\theta(1-\alpha_A)-(1-\theta n)\beta_B}{1-\theta n}+\frac{n(2-n)(1-\alpha_B)-[3n-1-n^2]\theta\beta_A}{n(1-n^2)} \\
\geq& \frac{(1+n)\theta-1}{1-\theta n}(1-\alpha_A)+\frac{n(2-n)(1-\alpha_B)-[3n-1-n^2]\theta\beta_A}{n(1-n^2)}.
\end{aligned} \tag{45}
$$

If $3n-1-n^2 < 0$, $X > 0$. If $3n-1-n^2 \geq 0$, we have

$$
X \geq \frac{(1+n)\theta-1}{1-\theta n}(1-\alpha_A)+\frac{3}{n(1+n)}(1-\alpha_B) > 0,
$$

Therefore, $X > 0$ always holds, and we have

$$
n'(\theta) = -\frac{W}{\frac{\theta}{n}W+X} \in \left(-\frac{n}{\theta},0\right).
$$

Using this result, I show that $dCS_A/d\theta < 0$ and $dCS_B/d\theta > 0$. To see that $dCS_A/d\theta < 0$, note that

$$\frac{dCS_A}{d\theta} = \frac{\partial CS_A}{\partial \theta} + \frac{\partial n}{\partial \theta}\frac{\partial CS_A}{\partial n}. \tag{46}$$

We have

$$\begin{aligned}\frac{\partial CS_A}{\partial \theta} &= -\frac{1-\alpha_A}{\theta} - \frac{\beta_B}{1-\theta n}\frac{1}{\theta^2} - \frac{n}{(1-\theta n)^2}\left(1-\alpha_A - \frac{\beta_B}{\theta}\right)\\ &\leq -\frac{1-\alpha_A}{\theta} - \frac{\beta_B}{(1-\theta n)^2\theta^2}(1-2\theta n) - \frac{n}{(1-\theta n)^2}(1-\alpha_A) < 0,\end{aligned}$$

and

$$\frac{\partial CS_A}{\partial n} = -\frac{1-\alpha_A-\beta_A}{n} - \frac{\theta}{(1-\theta n)^2}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right) \tag{47}$$

Because $\partial CS_A/\partial\theta < 0$ and $\partial n/\partial\theta < 0$, if $\partial CS_A/\partial n > 0$, $dCS_A/d\theta$ always holds. Consider the case where $\partial CS_A/\partial n < 0$. In this case, we have

$$\begin{aligned}\frac{dCS_A}{d\theta} &\leq -\frac{1-\alpha_A}{\theta} - \frac{\beta_B}{1-\theta n}\frac{1}{\theta^2} - \left(1+\frac{\theta}{n}\frac{\partial n}{\partial\theta}\right)\frac{n}{(1-\theta n)^2}\left(1-\alpha_A-\frac{\beta_B}{\theta}\right) + \frac{1-\alpha_A-\beta_A}{\theta}\\ &\leq -\frac{\beta_A}{\theta} - \frac{\beta_B}{(1-\theta n)^2}(1-2\theta n) < 0.\end{aligned} \tag{48}$$

Next, I show that $dCS_B/d\theta > 0$. To see this, note that

$$\frac{\partial CS_B}{\partial\theta} = \frac{\beta_B}{\theta} + \frac{\beta_A}{1-n} > 0, \tag{49}$$

and

$$\frac{\partial CS_B}{\partial n} = -\frac{1-\alpha_B-\beta_B}{n} - \frac{1-\alpha_B-\beta_A\theta}{(1-n)^2}. \tag{50}$$

If $\partial CS_B/\partial n < 0$, $dCS_B/d\theta > 0$. If $\partial CS_B/\partial n > 0$, we have

$$\frac{dCS_B}{d\theta} \geq \frac{1-\alpha_B}{\theta} + \frac{(1-2n)\beta_A}{(1-n)^2} + \frac{n(1-\alpha_B)}{\theta(1-n)^2} > 0. \tag{51}$$

Therefore, along the $\Pi_E = 0$ curve with $n_E^A/n_E^B \geq 1$, CS_B is increasing in (n_E^A/n_E^B) and CS_A is decreasing in (n_E^A/n_E^B).

Next, consider the case where $n_E^B > n_E^A$. Then, we can write (n_E^A, n_E^B) as $(n_E^A, n_E^B) = (n, \theta' n)$, where $\theta' > 1$. Using the same analysis, we can show that CS_A increases with θ' and CS_B decreases with θ'. Therefore, along the $\Pi_E = 0$ curve with $n_E^A/n_E^B \leq 1$, again CS_B isincreasing in (n_E^A/n_E^B) and CS_A is decreasing in (n_E^A/n_E^B).

Finally, compare the long-run equilibria with the same parameters $(\alpha_A, \alpha_B, \beta_A, \beta_B, T_E^A, T_E^B, K)$ but different parameters other than that. Then, the two equilibria are characterized by the network sizes of marginal entrants (n_0^A, n_0^B) and (n_1^A, n_1^B) along the $\Pi_E = 0$ curve. Let (CS_{A0}, CS_{B0}) and (CS_{A1}, CS_{B1}) be the consumer surplus on two sides under respective equilibria. Suppose without loss of generality that $n_0^A/n_0^B \geq n_1^A/n_1^B$. Then, we have $CS_{A0} \leq CS_{A1}$ and $CS_{B0} \geq CS_{B1}$ with equality if and only if $(n_0^A, n_0^B) = (n_1^A, n_1^B)$. Therefore, we have

$$(CS_{A0} - CS_{A1})(CS_{B0} - CS_{B1}) < 0,$$

which completes the proof. $\square$

B. Online Appendix (Not for Publication)

B.1. Full analysis of multiproduct-firm oligopoly

I present a multiproduct version of the platform oligopoly presented in Section 2. Consider a two-sided market served by a finite number of platforms, where the sides of the market and the set of platforms are indexed by $\{A, B\}$ and $\mathcal{F}$, respectively. Each platform $f \in \mathcal{F}$ sells a finite set $\mathcal{N}_f^k$ of products on side $k \in \{A, B\}$, where each product $i \in \mathcal{N}_f^k$ is priced at $p_i \in \mathbb{R}$. Let $\mathcal{N}^k := \bigcup_{f \in \mathcal{F}} \mathcal{N}_f^k$ be the set of all products on side k sold by the platforms.

In the following, I first describe the demand model and derive the demand function as an equilibrium outcome of the demand model, which forms the basis for the analysis of platform competition. Then, I analyze the price competition between platforms using the derived demand function.

B.1.1. Consumer Demand

On each side $k \in \{A, B\}$ of the market, a unit mass of consumers choose which product to purchase. In the model, the word "consumer" represents the users of the platforms. For example, in online marketplaces, buyers and sellers correspond to respective consumers in the model.

Each consumer's utility from the purchase of a product consists of a stand-alone value, platform-level network effects, and an indiosyncratic preference for the product. Formally, on each side $k \in \{A, B\}$, consumer z's indirect utility from the purchase of product $i \in \mathcal{N}_f^k$ is given by

$$u_{iz}^{k,f} = \log h_i^k(p_i) + \alpha_k \log n_f^k + \beta_k \log n_f^l + \varepsilon_{iz}^k. \tag{B.1}$$

The first term $\log h_i^k(p_i)$ is the stand-alone indirect subutility from product i at price $p_i \in \mathbb{R}$, where $h_i^k(p_i)$ is assumed to be decreasing and log-convex in p_i. The second and third terms, $\alpha_k \log n_f^k$ and $\beta_k \log n_f^l$, are the benefits of within-group and cross-group network effects, where $\alpha_k \in [0, 1)$ and $\beta_k \in [0, 1)$ are the parameters representing the magnitude of within-group and cross-group effects, and n_f^k and n_f^l are the numbers of consumers on side k and $l \neq k$ who purchase products provided by platform f. The last term, ε_{iz}^k, is an idiosyncratic taste shock that follows an i.i.d. type-I extreme value distribution. I assume that network effects are not too strong so that $\alpha_k + \beta_l < 1$ holds for each $k \in \{A, B\}$ and $l \neq k$, which precludes the possibility that some platforms set infinitely negative prices on one side.

Consumers choose which product to purchase and the amount of purchase based on the prices $p := (p_i)_{i \in \mathcal{N}^A \cup \mathcal{N}^B}$ set by platforms and the expectation $n^e = (n_f^{A,e}, n_f^{B,e})_{f \in \mathcal{F}}$ over the network sizes. I assume that there is no outside option and that consumers single-home, so that all consumers on side $k \in \{A, B\}$ purchase exactly one product from the set $\mathcal{N}^k$. The demand for each product is derived as a rational-expectation equilibrium among consumers. That is, based on the expectation over the network sizes n^e, each consumer chooses the product that maximizes the utility, and the realized network sizes $n = (n_f^A, n_f^B)_{f \in \mathcal{F}}$ are consistent with the original expectation. I call an equilibrium choice of products a *consumption equilibrium* and the equilibrium network sizes *consumption equilibrium network sizes*. In the following, I characterize the consumer demand for each product as an outcome of a consumption equilibrium.

First, consider the demand for each product conditional on purchase. Applying Roy's identity, the conditional demand function for product i conditional on the purchase is given by $-(h_i^k)'(p_i)/h_i^k(p_i)$. Next, consider each consumer's choice of products given the expectation over the network sizes n^e.

Because ε_{iz}^{k} follows the type I extreme value distribution, a consumer on side k chooses product $i \in \mathcal{N}_f^k$ with probability

$$s_i^k(n^e) = \Pr\left(u_{iz}^{k,f} \geq u_{jz}^{k,f} \text{ for all } j \in \mathcal{N}^k\right)$$
$$= \frac{h_i^k(p_i)\left(n_f^{k,e}\right)^{\alpha_k}\left(n_f^{l,e}\right)^{\beta_k}}{\sum_{f' \in \mathcal{F}} \sum_{j \in \mathcal{N}_{f'}^k} h_j^k(p_j)\left(n_{f'}^{k,e}\right)^{\alpha_k}\left(n_{f'}^{l,e}\right)^{\beta_k}}. \tag{B.2}$$

The realized network size of each platform f is given by the sum of the probability that a consumer chooses the product provided by platform f:

$$\widetilde{n}_f^k(n^e) := \sum_{i \in \mathcal{N}_f^k} s_i^k(n^e). \tag{B.3}$$

The consumption equilibrium network sizes n satisfy the condition $\widetilde{n}_f^k(n) = n_f^k$ for all $k = A, B$ and $f \in \mathcal{F}$.

Due to complementarity in network choices, there may be multiple consumption equilibrium network sizes when the network effects are too strong relative to product differentiation. The possibility of multiple equilibria prevents us from deriving a well-behaved demand function. In the context of the present setting, equation (B.2) indicates that whenever consumers expect $n_{f,e}^k = 0$, such an expectation will be self-fulfilling, and $\tilde{n}_f^k(n^e) = 0$ holds. Therefore, there are a number of equilibria in which a subset of platforms will never be chosen. To rule out such an extreme outcome, I use asymptotic stability derived from the best-response dynamics as an equilibrium selection criterion, which is formally defined below.

Definition B.1. *Define the best-response dynamics and asymptotic stability of network sizes as follows:*

1. *A best-response dynamics $\{n^t\}$ from the initial network sizes $n^0 = \left(n_{f,0}^A, n_{f,0}^B\right)_{f \in \mathcal{F}}$ is defined by a sequence of network sizes $n^t = \left(n_{f,t}^A, n_{f,t}^B\right)_{f \in \mathcal{F}}$ such that $n_{f,t}^k = \widetilde{n}_f^k\left(n^{t-1}\right)$ for all $t = 1, 2, \ldots$, $f \in \mathcal{F}$ and $k = A, B$.*
2. *A network size $n = \left(n_f^A, n_f^B\right)_{f \in \mathcal{F}}$ is the limit of the best-response dynamics $\{n^t\}$ from the initial network size n^0 if $n = \lim_{t \to \infty}\{n^t\}$.*
3. *Network sizes n are asymptotically stable if for any strictly positive n^0, n is the limit of the best-response dynamics from the initial network sizes n^0.*

I call a consumption equilibrium that has asymptotically stable network sizes an *asymptotically stable consumption equilibrium.*

Using the notion of asymptotic stability, I select a plausible consumption equilibrium that derives a well-defined demand function. To this end, consider a consumption equilibrium in which all network sizes are strictly positive. I call such an equilibrium *interior consumption equilibrium.* Provided that $n_f^k > 0$ for all $f \in \mathcal{F}$ and $k = A, B$, equation (B.3) has a closed-form solution

$$n_f^k(p) = \frac{\left[(H_f^k(p_f^k)\right]^{\Gamma_{kk}}\left[H_f^l(p_f^l)\right]^{\Gamma_{kl}}}{H^k(p)} \tag{B.4}$$

for all $f \in \mathcal{F}$ and $k \in \{A, B\}$, where, H_f^k and H^k are the platform-level and industry-level aggregators defined by

$$
\begin{aligned}
H_f^k(p_f^k) &= \sum_{i \in \mathcal{N}_f^k} h_i^k(p_i), \\
H^k(p) &= \sum_{f \in \mathcal{F}} \left[H_f^k(p_f^k)\right]^{\Gamma_{kk}} \left[H_f^l(p_f^l)\right]^{\Gamma_{kl}},
\end{aligned}
$$

and Γ_{kk} and Γ_{kl} are given by

$$
\begin{aligned}
\Gamma_{kk} &= \frac{1-\alpha_l}{(1-\alpha_k)(1-\alpha_l) - \beta_k \beta_l}, \text{ and} \\
\Gamma_{kl} &= \frac{\beta_k}{(1-\alpha_k)(1-\alpha_l) - \beta_l \beta_k}.
\end{aligned}
$$

The network sizes in equation (B.4) have the following interpretation. The platform-level aggregator H_f^k is the total value provided by platform f on side k. Due to network effects, platform-level aggregators H^k and H^l are amplified to $(H^k)^{\Gamma_{kk}}(H^l)^{\Gamma_{kl}}$ as shown at the numerator of the right-hand side of equation (B.4). The industry-level aggregator H^k is the sum of such amplified values.

Along with its intuitive interpretation, the interior consumption equilibrium characterized by equation (B.4) turns out to be asymptotically stable. Particularly, Proposition B.1 shows that the interior consumption equilibrium with the network sizes given by equation (B.4) is the unique asymptotically stable consumption equilibrium.

Proposition B.1. *There exists a unique asymptotically stable consumption equilibrium. In the asymptotically stable consumption equilibrium, the network sizes are given by equation* (3)*. The demand for each product $i \in \mathcal{N}^k$ under the asymptotically stable consumption equilibrium is given by*

$$
\begin{aligned}
D_i^k(p) &= \hat{D}_i^k\left[p_i, H_f^k(p_f^k), H_f^l(p_f^l), H^k(p)\right] \\
&:= -\left[H_f^k(p_f^k)\right]^{\Gamma_{kk}-1} \left[H_f^l(p_f^l)\right]^{\Gamma_{kl}} \frac{(h_i^k)'(p_i)}{H^k(p)}.
\end{aligned}
\tag{B.5}
$$

Proof. I first derive an interior consumption equilibrium and the demand for each product. Then I show that the best-resopnse dynamics from any starting value of network sizes converges to the interior consumption equilibrium.

Applying Holman and Marley's Theorem, the consumer choice probability s_i^k of product $i \in \mathcal{N}_f^k$ given the expectation over network shares $n^e = (n_{f'}^{A,e}, n_{f'}^{B,e})_{f' \in \mathcal{F}}$ is given by equation (B.2). I require that the network share is consistent with the consumers' behaviors, that is, the network share n_f^k of platform f on side k is given by equation (B.3). From equations (B.2)) and (B.3), the share of product $i \in \mathcal{N}_f^k$ in the set of products sold by platform f is given by

$$
\frac{s_i^k}{n_f^k} = \frac{h_i^k(p_i)}{H_f^k(p_f^k)}. \tag{B.6}
$$

The network share n_f^k of platform f on side k in the interior consumption equilibrium is given by equation (B.4).

Combining equations (B.6) and (B.4), the probability that product $i \in \mathcal{N}_f^k$ is purchased by a consumer is given by the equation

$$s_i^k(p) = n_f^k(p) \times \frac{h_i^k(p_i)}{H_f^k(p_f^k)}.$$

Finally, the demand for the product $i \in \mathcal{N}_f^k$ given the profile of prices p has the following form.

$$\begin{aligned}
D_i^k(p) &= \hat{D}_i^k \left(p_i, H_f^k(p_f^k), H_f^l(p_f^l), H^k(p)\right) \\
&= s_i^k(p) \times \frac{-(h_i^k)'(p_i)}{h_i^k(p_i)} \\
&= -\left[H_f^k(p_f^k)\right]^{\Gamma_{kk}-1} \left[H_f^l(p_f^l)\right]^{\Gamma_{kl}} \frac{(h_i^k)'(p_i)}{H^k(p)}
\end{aligned}$$

I next show that the best-resopnse dynamics from any starting value of network sizes converges to the interior consumption equilibrium. First, fix an initial value of the vector of network shares $(n_{f,0}^k)_{f\in\mathcal{F}}$ such that $n_{f,0}^k > 0$ for all $f \in \mathcal{F}$ and $k \in \{A, B\}$. Next, for each $t > 0$, update the network share based on the value of network share in the previous iteration $t-1$. Then, the sequence of network shares $\left\{(n_f^t)_{f\in\mathcal{F}}\right\}_{t=0,...}$ is obtained. Here, for any $t > 0$, we have

$$\frac{n_{f,t}^k}{n_{g,t}^k} = \frac{H_f^k}{H_g^k} \left(\frac{n_{f,t-1}^k}{n_{g,t-1}^k}\right)^{\alpha_k} \left(\frac{n_{f,t-1}^l}{n_{g,t-1}^l}\right)^{\beta_k}$$

By taking the logarithm and letting $x_t^k := \log(n_{f,t}^k/n_{g,t}^k)$ and $\psi^k := \log(H_f^k/H_g^k)$, we have

$$\begin{pmatrix} x_t^A \\ x_t^B \end{pmatrix} = A \begin{pmatrix} x_{t-1}^A \\ x_{t-1}^B \end{pmatrix} + \begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix},$$

where

$$A = \begin{bmatrix} \alpha_A & \beta_A \\ \beta_B & \alpha_B \end{bmatrix}.$$

If any eigenvalue of A has an absolute value less than 1, (x_t^A, x_t^B) converges to a unique value (x^A, x^B) regardless of the initial value (x_0^A, x_0^B) (see Luenberger, 1979). At such value, we must satisfy $x_t^k = x_{t-1}^k = x^k$. Solving for x^k, we have

$$x^k = \frac{(1-\alpha_l)\psi^k + \beta_k\psi^l}{(1-\alpha_k)(1-\alpha_l) - \beta_k\beta_l}.$$

Then, using the relation $(n_f^k/n_g^k) = x^k$, we can solve for the value of n_f^k, which corresponds with that of (B.4). Therefore, from any starting value of positive network shares, the best-response dynamics converges to the interior consumption equilibrium.

Lastly, I show that any eigenvalue of A has an absolute value less than 1. A scalar b is an eigenvalue of A if and only if it is the solution to the quadratic equation

$$\xi(b) = b^2 - (\alpha_A + \alpha_B)b + (\alpha_A\alpha_B - \beta_A\beta_B) = 0.$$

Because

$$\xi(-1) = 1 + \alpha_A + \alpha_B + \alpha_A\alpha_B - \beta_A\beta_B > 0,$$
$$\xi\left(\frac{\alpha_A + \alpha_B}{2}\right) = -\frac{\alpha_A^2 + \alpha_B^2 + 2\beta_A\beta_B}{2} < 0,$$
$$\xi(1) = (1-\alpha_A)(1-\alpha_B) - \beta_A\beta_B > 0,$$

The two solutions to $\xi(b) = 0$ lies in $(-1, 1)$, which completes the proof.

□

The network sizes characterized by equation (B.4) exhibit an IIA property, which allows us to generalize the aggregative-games analysis of Nocke and Schutz (2018b) to two-sided markets, thereby making it possible to introduce an arbitrary heterogeneity between platforms. In summary, the demand model of this study simplifies Tan and Zhou (2021) in two dimensions, the forms of network effects and taste distributions, and makes it possible to derive a tractable demand function that allows for an arbitrary platform asymmetry in two-sided markets.

Consumer surplus Consumer surplus CS^k on side k is given by the expected indirect utility of consumers, and the aggregate consumer surplus CS is given by the sum of the consumer surplus on both sides:

$$\begin{aligned} CS^k &= \log\left[\sum_{f\in\mathcal{F}} \left(H_f^k\right)^{\Gamma_{kk}} \left(H_f^l\right)^{\Gamma_{kl}} \frac{1}{(H^k)^{\alpha_k}(H^l)^{\beta_k}}\right] \\ &= (1-\alpha_k)\log H^k - \beta_k \log H^l, \end{aligned}$$

and

$$\begin{aligned} CS &= CS^A + CS^B \\ &= (1-\alpha_A-\beta_B)\log H^A + (1-\alpha_B-\beta_A)\log H^B. \end{aligned}$$

Logit demand specification The demand system that satisfies Assumption 1 has the form

$$D_i^k(p) = \frac{\left[H_f^k(p_f^k)\right]^{\Gamma_{kk}} \left[H_f^l(p_f^l)\right]^{\Gamma_{kl}}}{\sum_{f'\in\mathcal{F}} \left[H_{f'}^k(p_{f'}^k)\right]^{\Gamma_{kk}} \left[H_{f'}^l(p_{f'}^l)\right]^{\Gamma_{kl}}} \frac{\exp\left(\frac{a_i - p_i}{\lambda^k}\right)}{\lambda^k \sum_{j\in\mathcal{N}_f^k} \exp\left(\frac{a_j - p_j}{\lambda^k}\right)}. \tag{B.7}$$

B.1.2. Platform competition

Using the demand system obtained in Proposition B.1, I analyze price competition between platforms.

Each product $i \in \mathcal{N}^k$ has a constant marginal cost $c_i \geq 0$ of production. Given the demand system $\{(D_i^k)_{i\in\mathcal{N}^k}\}_{k\in\{A,B\}}$, the profit function of each platform $f \in \mathcal{F}$ is written as a function of the profile of the platform's own prices $p_f := (p_i)_{\mathcal{N}_f^A \cup \mathcal{N}_f^B}$ and aggregators H^A and H^B:

$$\Pi_f\left[p_f, H^A(p), H^B(p)\right] = \Pi_f^A + \Pi_f^B, \tag{B.8}$$

where

$$\Pi_f^k = \sum_{i\in\mathcal{N}_f^k} \hat{D}_i^k \left[p_i, H_f^k(p_f^k), H_f^l(p_f^l), H^k(p)\right](p_i - c_i). \tag{B.9}$$

The pricing game consists of a demand system $\{(D_i^k)_{i\in\mathcal{N}^k}\}_{k\in\{A,B\}}$, a set of platforms $\mathcal{F}$, and a profile of marginal costs $(c_i)_{i\in\mathcal{N}_f^A\cup\mathcal{N}_f^B}$. In a pricing game, platforms simultaneously set the prices p_f of their products, with the payoff function Π_f defined by equation (B.8). I call a Nash equilibrium of this pricing game as a *pricing equilibrium*. In the following analysis, I often suppress the arguments of functions for ease of exposition.

Optimal pricing for each platform The first-order condition for the profit-maximizing prices set by each platform f is given by $\partial\Pi_f/\partial p_i = 0$, which can be transformed into the following equation:

$$-\frac{(h_i^k)''}{(h_i^k)'}(p_i - c_i) = \mu_f^k, \tag{B.10}$$

where

$$\mu_f^k := 1 - \underbrace{\frac{1}{n_f^k}\left[(\Gamma_{kk}-1)\Pi_f^k + \beta_l\Gamma_{lk}\Pi_f^l\right]}_{\text{network-externality terms}} + \underbrace{\Gamma_{kk}\Pi_f^k + \Gamma_{lk}\frac{n_f^l}{n_f^k}\Pi_f^l}_{\text{cannibalization terms}}. \tag{B.11}$$

The right-hand side of equation (B.11) is independent of the index of the product i. Therefore, the optimal pricing of each platform equates the left side of the equation (B.10) with some common value μ_f^k. Following Nocke and Schutz (2018b), I call μ_f^k as the ι-markup of platform f on side k. The property that all prices of the products sold by a platform on the same side are summarized into a single ι-markup is driven by the property that the network sizes n_f^k have an IIA property.

Under Assumption 1, we have $-(h_i^k)''/(h_i^k)' = 1/\lambda^k$ and thus $p_i = c_i + \lambda^k\mu_f^k$, implying that all the product of the same platform on one side has the same markup. Then, the profit of platform f on side k can be written as its network size multiplied by the common markup:

$$\Pi_f^k = n_f^k\mu_f^k. \tag{B.12}$$

Using this relation, the formula for the ι-markup can be simplified to

$$\mu_f^k = \frac{1}{1-n_f^k}\left(1 - \alpha_k - \beta_l\frac{n_f^l}{n_f^k}\right) \tag{B.13}$$

The formula for the platform-level aggregator can also be simplified to

$$H_f^k = T_f^k\exp(-\mu_f^k), \tag{B.14}$$

where $T_f^k := \sum_{i\in\mathcal{N}_f^k}\exp\left(\frac{a_i - c_i}{\lambda^k}\right)$ is the "type" of platform f that equals the value of the platform-level aggregator of platform f when it engages in the marginal cost pricing. Solving the system of equations (12) and (13), along with equation (3), the ι-markup μ_f^k and the network size n_f^k consistent with the platform's optimal pricing are obtained as functions of T_f^A, T_f^B, H^A, and H^B, which I write as

$$\mu_f^k = m^k\left(T_f^k, T_f^l, H^k, H^l\right), \tag{B.15}$$

and

$$n_f^k = N^k \left(T_f^k, T_f^l, H^k, H^l\right). \tag{B.16}$$

When the system of equations (B.13) and (B.14) has multiple solutions, let the profit-maximizing one be m^k and N^k.

The property that all the pricing information is summarized by unidimensional type T_f^k is called the type-aggregation property (Nocke and Schutz, 2018b), which simplifies the analysis of horizontal mergers and free entry.

Pricing equilibrium Finally, the equilibrium aggregators (H^A, H^B) must satisfy the condition that the network sizes of the platforms add up to 1:

$$\sum_{f \in \mathcal{N}_f^k} N^k \left(T_f^k, T_f^l, H^k, H^l\right) = 1, \quad J \in \{A, B\}, \quad l \neq J. \tag{B.17}$$

The analysis so far has characterized the necessary conditions that the pricing equilibrium must satisfy. Proposition B.2 shows that these necessary conditions are also sufficient and provides several important cases in which the equilibrium is unique regardless of the type profiles. The proof of this proposition is relegated to the Online Appendix.

Proposition B.2. *For any pair of aggregators (H^A, H^B), each platform's corresponding optimal pricing is uniquely given by $p_i = c_i + \lambda^k m^k \left(T_f^k, T_f^l, H^k, H^l\right)$. Furthermore, the following statements hold:*

1. *If only within-group network effects exist ($\beta_A = \beta_B = 0$), then a unique pricing equilibrium exists.*

2. *If only cross-group effects exist ($\alpha_A = \alpha_B = 0$), then a pricing equilibrium exists. Furthermore, there exists $\underline{\beta}$ such that if $\beta_A \leq \underline{\beta}$ or $\beta_B \leq \underline{\beta}$, then the pricing equilibrium is unique.*

Because a pricing equilibrium is characterized by the pair of industry-level aggregators (H^{A*}, H^{B*}) that satisfies the system of equations (B.17), the characterization of equilibrium is simplified to the characterization of the system of equations (B.17). Furthermore, because the consumer surplus $CS^k = (1 - \alpha_k) \log H^k - \beta_k \log H^l$ is determined solely by industry-level aggregators (H^A, H^B), the characterization of equilibrium aggregators directly characterizes the equilibrium consumer surplus.

B.2. Proof of Proposition B.2

Before proceeding to the proof, I introduce several notations that are used in both proofs. First, let

$$\Omega_f^{kI}(p_f) := \left(H_f^k(p_f^k)\right)^{\frac{1-\alpha_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}} \left(H_f^l(p_f^l)\right)^{\frac{\beta_k}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}}$$

and

$$H_{-f}^k = \sum_{f' \in \mathcal{F} \setminus \{f\}} \Omega_{f'}^{kI}(p_{f'}).$$

Then, the profit-maximization problem of platform f the problem can be rewritten as

$$\max_{p_f \in \{\mathbb{R} \cup \{\infty\}\}^{\mathcal{N}_f^A \cup \mathcal{N}_f^B}} G_f(p_f) := \Pi_f \left(p_f, \Omega_f^{AB}(p_f) + H_{-f}^A, \Omega_f^{BA}(p_f) + H_{-f}^B\right) \tag{B.18}$$

I show that the solution to (B.18) takes unique finite value and given by the first-order condition (B.10). Here, I list the steps of the proof.

1. Fixing p_f^A, p_f^B that maximizes $G_f(p_f^A, p_f^B)$ is finite and unique. Let $\tilde{p}_f^B(p_f^A)$ denote such p_f^B.
2. p_f^A that maximizes $G_f(p_f^A, \tilde{p}_f^B(p_f^A))$ is finite and unique.
3. Setting infinite price for some good is never optimal.
4. The optimal prices should satisfy the first-order condition (B.10).

I first show that all platforms' prices are bounded below. Next, I show that all platform' prices are bounded above.

Fix $p_f^l := (p_i^l)_{i\in\mathcal{N}_f^l}$. Then, I show that the value of $(p_f^k) := (p_i^k)_{i\in\mathcal{N}_f^k}$ that maximize the profit of platform f has finite absolute values. To see this, note that

$$\begin{aligned}
&\text{sign}\left(\frac{\partial G(p_f)}{\partial p_i}\right)\\
=&\text{sign}\left(1-\frac{p_i-c_i}{\lambda}+\frac{(1-\alpha_l)-\{(1-\alpha_l)\alpha_k+\beta_l\beta_k\}\frac{1}{n_f^k}}{(1-\alpha_l)(1-\alpha_k)-\beta_l\beta_k}\Pi_f^k-\frac{\beta_l}{(1-\alpha_l)(1-\alpha_k)-\beta_l\beta_k}\frac{(1-n_f^l)}{n_f^k}\Pi_f^l\right).
\end{aligned}$$

the last term converges to 0 as $p_i \to -\infty$ because $n_f^l \to 1$ as $p_i \to -\infty$. The sum of the second and the third terms is nonnegative as $p_i \to -\infty$ because

$$\begin{aligned}
&-\frac{p_i-c_i}{\lambda}+\frac{(1-\alpha_l)-\{(1-\alpha_l)\alpha_k+\beta_l\beta_k\}\frac{1}{n_f^k}}{(1-\alpha_l)(1-\alpha_k)-\beta_l\beta_k}\Pi_f^k\\
&\geq -\frac{p_i-c_i}{\lambda}+\Pi_f^k\\
&\geq -\frac{p_i-c_i}{\lambda}+n_f^k\frac{p_i-c_i}{\lambda}
\end{aligned}$$

as $p_i \to -\infty$. Thus we have $\partial G/\partial p_i > 0$ for sufficiently small p_i.

The fact that platform never sets infinite price is shown in the same manner as Nocke and Schutz (2018a).

As a result, fixed the values of p_f^l, the platform's optimal pricing p_f^k is restricted to a compact cube $[\underline{B}_f^k, \overline{B}_f^k]^{|\mathcal{N}_f^k|}$. Since the profit function is continuous in p_f^k, Wierstrass' theorem implies that there is optimal price in $[\underline{B}, \overline{B}]^{\mathcal{N}_f^k}$. In particular, since optimal price is interior, p_f^k is given by the common ι-markup pricing, which is given by (B.10). Let $\mu_f^k(p_f^l)$ be the optimal ι-markup on side k given the profile of prices on the other side p_f^l. Then, under MNL demand, we have $p_i = c_i + \lambda\mu_f^k(p_f^l)$ for $i \in \mathcal{N}_f^k$.

Next, I show that the optimal value of p_f^l that maximizes $G(p)$ where $p_f^k = (c_i + \lambda\mu_f^k(p_f^l))_{i\in\mathcal{N}_f^k}$ is finite. To see this, it is sufficient to show that

$$\begin{aligned}
&\text{sign}\left(\frac{\partial G((p_f^l, (c_i+\lambda\mu_f^k(p_f^l))_{i\in\mathcal{N}_f^k})}{\partial p_j}\right)\\
=&\text{sign}\left(1-\frac{p_j-c_j}{\lambda}+\frac{(1-\alpha_k)-\{(1-\alpha_l)\alpha_k+\beta_l\beta_k\}\frac{1}{n_f^l}}{(1-\alpha_l)(1-\alpha_k)-\beta_l\beta_k}\Pi_f^l-\frac{\beta_k}{(1-\alpha_l)(1-\alpha_k)-\beta_l\beta_k}\frac{(1-n_f^k)}{n_f^l}\Pi_f^k\right)
\end{aligned}$$

becomes positive as $p_j \to -\infty$ and negative as $p_j \to \infty$.

Using the first-order condition for μ_f^k, B.10, we have

$$\begin{aligned}&\frac{1-\alpha_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}(1-n_f^k)\mu_f^k\\ =&1-\frac{\beta_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}\frac{n_f^l}{n_f^k}(1-n_f^l)\sum_{j'\in\mathcal{N}_f^l}\frac{\exp\left(\frac{a_{j'}-p_{j'}}{\lambda}\right)}{H_f^l}\frac{p_{j'}-c_{j'}}{\lambda}\end{aligned}$$

By l'Hopital's rule, we have

$$\begin{aligned}&\lim_{p_i\to-\infty}\frac{\beta_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}\frac{n_f^l}{n_f^k}(1-n_f^l)\sum_{j'\in\mathcal{N}_f^l}\frac{\exp\left(\frac{a_{j'}-p_{j'}}{\lambda}\right)}{H_f^l}\frac{p_{j'}-c_{j'}}{\lambda}\\ =&\lim_{p_j\to-\infty}\frac{\beta_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}\frac{\sum_{j'\in\mathcal{N}_f^l\setminus\{i\}}\exp\left(\frac{a_{j'}-p_{j'}}{\lambda}\right)}{\sum_{j'\in\mathcal{N}_f^l\setminus\{i\}}\exp\left(\frac{a_{j'}-p_{j'}}{\lambda}\right)+\exp\left(\frac{a_i-p_i}{\lambda}\right)}\frac{p_i-c_i}{\lambda}\\ =&\lim_{p_j\to-\infty}\frac{\beta_l}{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}\frac{\sum_{j'\in\mathcal{N}_f^l\setminus\{i\}}\exp\left(\frac{a_{j'}-p_{j'}}{\lambda}\right)}{-\lim_{p_i\to\infty}\lambda\exp\left(\frac{a_i-p_i}{\lambda}\right)}\\ =&0.\end{aligned}$$

Thus, we have

$$\lim_{p_j\to-\infty}\frac{(1-n_f^k)}{n_f^l}\Pi_f^k=\frac{(1-\alpha_k)(1-\alpha_l)-\beta_k\beta_l}{1-\alpha_l},$$

and

$$\begin{aligned}&\text{sign}\lim_{p_j\to-\infty}\left(\frac{\partial G((p_f^l,(c_i+\lambda\mu_f^k(p_f^l))_{i\in\mathcal{N}_f^k})}{\partial p_j}\right)\\ =&\text{sign}\lim_{p_j\to-\infty}\left(1-\frac{\beta_k}{1-\alpha_l}-\frac{p_j-c_j}{\lambda}+\frac{(1-\alpha_k)-\{(1-\alpha_l)\alpha_k+\beta_l\beta_k\}\frac{1}{n_f^l}}{(1-\alpha_l)(1-\alpha_k)-\beta_l\beta_k}\Pi_f^l\right)>0.\end{aligned}$$

as $p_j \to -\infty$ for some $\mathcal{N}_f^l$.

Again, the fact that platform never sets infinite price is shown in the same manner as Nocke and Schutz (2018a).

As a result, the platforms' choice of prices p_f^l can be restricted to some compact cube $[\underline{B}_f^l, \overline{B}_f^l]^{|\mathcal{N}_f^l|}$. As a result, an application of Wierstrass' theorem implies that there exists an optimal price p_f^l in $[\underline{B}_f^l, \overline{B}_f^l]^{|\mathcal{N}_f^l|}$. Because the optimal profile of prices is interior, which implies that the optimal prices should satisfy the first-order condition (B.10).

By the facts that (i) there is finite price profile that maximizes the profit and (ii) any optimal price profile should satisfy equation (B.10) jointly show that there is a solution to (B.10) that maximizes the platform's profit. If there is no solution to equation (B.10) that maximize the platform's profit, then there must be some finite price profile that maximizes the platform's profit but does not satisfy equation (B.10), which contradicts the necessity of (B.10).

B.3. Proof of Proposition B.2.1

Suppose that $\beta_A = \beta_B = 0$. Then, two sides of markets are independent, and thus it suffice to focus on one side of the market. The ι-markup is uniquely given by the equation (31). Solving the equation (31), the ι-markup is obtained as $\mu_f = m(\gamma(T_f)/H, \alpha)$. Using this ι-markup function, we further obtain the network share n_f as

$$n_f = N\left(\frac{\gamma(T_f)}{H}, \alpha\right) := \frac{\gamma(T_f)}{H} \exp\left(-\frac{m\left(\frac{\gamma(T_f)}{H}, \alpha\right)}{1-\alpha}\right), \tag{B.19}$$

Finally, since the market share equation $N(\gamma(T)/H, \alpha)$ defined by equation (B.19) is monotonically decreasing in H, $\lim_{x\to 0} N(x,\alpha) = 0$, and $\lim_{x\to\infty} N(x,\alpha) = 1$, the intermediate value theorem implies that the value of aggregator H that satisfies $\sum_f N(\gamma(T_f)/H, \alpha) = 1$ is unique.

B.4. Proof of Proposition B.2.2

I show that there is the unique pair of ι-markup that satisfies the system of equations (B.10), and the existence of equilibrium aggregators when $\alpha_A = \alpha_B = 0$. Then I show that there exists $\underline{\beta} > 0$ such that there is the unique pair of aggregators that satisfies the system of equation (B.10) when $\beta_k \leq \underline{\beta}$.

First, I show the uniqueness of ι-markups given the value aggregators. Let x_f^k for $k = A, B$ be defined by

$$x_f^k = \frac{\left(T_f^k\right)^{\frac{1}{1-\beta_A\beta_B}} \left(T_f^l\right)^{\frac{\beta_l}{1-\beta_A\beta_B}}}{H^k}$$

After several manipulations, the system of first-order conditions (9) can be rewritten as

$$\begin{aligned} 0 =& g_A(\mu_f^A, \mu_f^B) \\ =& \left[1 - x_f^A \exp\left(-\frac{\mu_f^A + \beta_A \mu_f^B}{1-\beta_A\beta_B}\right)\right]\mu_f^A - 1 + \beta_B \frac{x_f^B}{x_f^A} \exp\left(-\frac{\mu_f^B(1-\beta_B) - \mu_f^A(1-\beta_A)}{1-\beta_A\beta_B}\right), \end{aligned} \tag{B.20}$$

$$\begin{aligned} 0 =& g_B(\mu_f^A, \mu_f^B) \\ =& \left[1 - x_f^B \exp\left(-\frac{\mu_f^B + \beta_B \mu_f^A}{1-\beta_A\beta_B}\right)\right]\mu_f^B - 1 + \beta_A \frac{x_f^A}{x_f^B} \exp\left(-\frac{\mu_f^A(1-\beta_A) - \mu_f^B(1-\beta_B)}{1-\beta_A\beta_B}\right). \end{aligned} \tag{B.21}$$

Let $g(\mu_f^A, \mu_f^B) := \{g_A(\mu_f^A, \mu_f^B, g_B(\mu_f^A, \mu_f^B)\}$. To show that this system of equations has a unique solution, I show that the determinant of the Jacobian of $g(\mu_f^A, \mu_f^B)$ is positive.[3] A calculation leads to

$$\begin{aligned} \det G_f =& \left(1-n_f^A\right)\left(1-n_f^B\right) + \left(1-n_f^A\right)\left(\frac{n_f^B\mu_f^B + \beta_A(1-\beta_B)\frac{n_f^A}{n_f^B}}{1-\beta_A\beta_B}\right) + \left(1-n_f^B\right)\left(\frac{n_f^A\mu_f^A + \beta_B(1-\beta_A)\frac{n_f^B}{n_f^A}}{1-\beta_A\beta_B}\right) \\ &+ \frac{n_f^A\mu_f^A n_f^B \mu_f^B}{1-\beta_A\beta_B} + \frac{\frac{n_f^A}{n_f^B}[\beta_A(1-\beta_B) + \beta_A^2(1-\beta_A)]n_f^A\mu_f^A + \frac{n_f^B}{n_f^A}[\beta_B(1-\beta_A) + \beta_B^2(1-\beta_B)]n_f^B\mu_f^B}{(1-\beta_A\beta_B)^2}, \end{aligned}$$

[3] See chapter 2 of Vives (2001).

where

$$G_f := \begin{pmatrix} \frac{\partial g_A}{\partial \mu_f^A} & \frac{\partial g_A}{\partial \mu_f^B} \\ \frac{\partial g_B}{\partial \mu_f^A} & \frac{\partial g_B}{\partial \mu_f^B} \end{pmatrix}, \tag{B.22}$$

and

$$\begin{aligned} \frac{\partial g^k}{\partial \mu_f^k} &= 1 - n_f^k + \frac{n_f^k \mu_f^k}{1-\beta_A\beta_B} + \frac{\beta_l(1-\beta_k)}{1-\beta_A\beta_B}\frac{n_f^l}{n_f^k}, \\ \frac{\partial g^k}{\partial \mu_f^l} &= \frac{1}{1-\beta_A\beta_B}\left(\beta_k n_f^k \mu_f^k - \beta_l(1-\beta_l)\frac{n_f^l}{n_f^k}\right), \end{aligned}$$

for $J, I \in \{A, B\}$, and $I \neq I$.

I show that $\det G_f > 0$ for all $n_f^A, n_f^B, \beta_A, \beta_B \in (0,1)$. If $\mu_f^k \geq 0$ for $k = A, B$, $\det G_f > 0$ holds. Suppose that $\mu_f^A < 0$. Then, we must have $\beta_B \frac{n^B}{n^A} > 1$ and $\mu_f^B > \frac{1-\beta_A\beta_B}{1-n^A} > 0$. Note that, suppressing the platform index, we have

$$\begin{aligned} \det G_f =& \underbrace{(1-n^A)(1-n^B)}_{>0} + \underbrace{\frac{(1-n^B)\left(n^A\mu^A + n^B\mu^B\right)}{1-\beta_A\beta_B}}_{(X)} + \underbrace{\frac{(n^B-n^A)n^B\mu^B}{1-\beta_A\beta_B}}_{(Y)} \\ &+ \underbrace{\frac{(1-n^A)\beta_A(1-\beta_B)\frac{n^A}{n^B} + (1-n^B)\beta_B(1-\beta_A)\frac{n^B}{n^A}}{1-\beta_A\beta_B}}_{>0} \\ &+ \underbrace{\frac{n^A\mu^A n^B\mu^B}{1-\beta_A\beta_B} + \frac{\beta_A\frac{n^A}{n^B}[1-\beta_B+\beta_A(1-\beta_A)]n^A\mu^A + \beta_B\frac{n^B}{n^A}[1-\beta_A+\beta_B(1-\beta_B)]n^B\mu^B}{(1-\beta_A\beta_B)^2}}_{(Z)} \end{aligned}$$

First, note that

$$n^A\mu^A + n^B\mu^B = \frac{n^A - \beta_B n^B}{1-n^A} + \frac{n^B - \beta_A n^A}{1-n^B} > \frac{n^A(1-\beta_A) + n^B(1-\beta_B)}{1-n^B} > \frac{n^B(1-\beta_B)}{1-n^B},$$

where the first inequality follows from the fact that $n^A - \beta_B n^B < 0$ and $n^A < n^B$. Using this inequality and the fact that $\beta_B \frac{n^B}{n^A} > 1$ we have

$$(X) + (Y) + (Z) > \frac{n^B(1-\beta_B)}{1-\beta_A\beta_B} + \frac{(n^B)^2}{1-n^B}(1-\beta_B) + (Z)$$

Further, by additionally using the fact that $n^A\mu^A = (n^A - \beta_B n^B)/(1-n^A) \geq -\beta_B n^B$, we have

$$(Z) > \frac{1}{(1-\beta_A\beta_B)^2}\frac{n^B}{1-n^B}\left[(1-\beta_A\beta_B)[1-(1-\beta_A\beta_B)\beta_B n^B] - (1-n^B)n^B\beta_A\beta_B^2[1-\beta_B+\beta_A(1-\beta_A)]\right].$$

Putting them together, we have

$$
\begin{aligned}
&(X)+(Y)+(Z)\\
=&\frac{1}{(1-\beta_A\beta_B)^2}\frac{n^B}{1-n^B}\Bigg[(1-n^B)\left[(1-\beta_A\beta_B)-n^B\beta_A^2\beta_B^2(1-\beta_A)\right]\\
&\qquad +n^B(1-\beta_B)\left[(1-\beta_A\beta_B)-(1-n^B)\beta_A\beta_B^2\right]+(1-\beta_A\beta_B)[1-(1-\beta_A\beta_B)\beta_B n^B]\Bigg]\\
>&\frac{1}{(1-\beta_A\beta_B)^2}\frac{n^B}{1-n^B}\Bigg[(1-n^B)\left[(1-\beta_A\beta_B)-n^B\beta_A^2\beta_B^2(1-\beta_A)\right]\\
&\qquad +n^B(1-\beta_B)\left[1+(1-\beta_A\beta_B)(1-\beta_B n^B)-(1-n^B)\beta_A\beta_B^2\right]\Bigg]\\
>&0.
\end{aligned}
$$

As a result, $\det G_f > (X)+(Y)+(Z) > 0$ always holds, and the pair of ι-markups that satisfies the first-order condition (B.10) is unique.

Next, I show the existence of the aggregators that satisfy the equilibrium condition. To do this, note that the network share n_f^k under optimal pricing can be written as

$$
n_f^k = \frac{\left(T_f^k\right)^{\frac{1}{1-\beta_A\beta_B}}\left(T_f^l\right)^{\frac{\beta_l}{1-\beta_A\beta_B}}}{H^k}\exp\left[-\frac{\frac{1}{1-n_f^k}\left(1-\beta_l\frac{n_f^l}{n_f^k}\right)+\beta_k\frac{1}{1-n_f^l}\left(1-\beta_k\frac{n_f^k}{n_f^l}\right)}{1-\beta_A\beta_B}\right] \tag{B.23}
$$

I first show that for any fixed value of H^A, there exists $H^B(H^A)$ such that

$$
\sum_f N^B(T_f^A, T_f^B, H^A, H^B(H^A)) = 1.
$$

We have $\lim_{H^B\to 0} N^B(T_f^A, T_f^B, H^A, H^B) = 1$ because the right-hand side of (B.23) goes to infinity as $H^B \to 0$ if $n_f^k < 1$. We also have $\lim_{H^B\to\infty} N^B(T_f^A, T_f^B, H^A, H^B) = 0$ because the right-hand side of equation (B.23) goes to infinity as $H^B \to 0$. Thus, the intermediate value theorem implies that there exists $\hat{H}^B(H^A)$ such that

$$
\sum_f N^B(T_f^A, T_f^B, H^A, \hat{H}^B(H^A)) = 1.
$$

Similarly, there exists $\hat{H}^A$ such that

$$
\sum_f N^A(T_f^A, T_f^B, \hat{H}^A, \hat{H}^B(\hat{H}^A)) = 1.
$$

To see this, first consider the case where $H^A \to \infty$. I argue that $N^A(T_f^A, T_f^B, H^A, \hat{H}^B(H^A)) \to 0$ for all f. Suppose to the contrary that $n_f^A \to \underline{n}^A > 0$ as $H^A \to \infty$ for some f Then, we must have $n_f^B \to 0$ as $H^A \to \infty$ to satisfy equation (B.23). Next, dividing equations (B.23) for $k = A$ and $k = B$, we have

$$\frac{n_f^B}{n_f^A} = \frac{\left(T_f^B\right)^{\frac{1-\beta_B}{1-\beta_A\beta_B}}}{\left(T_f^A\right)^{\frac{1-\beta_A}{1-\beta_A\beta_B}}} \frac{H^A}{\hat{H}^B} \exp\left[-\frac{\frac{(1-\beta_A)}{1-n_f^B}\left(1-\beta_A\frac{n_f^A}{n_f^B}\right) - \frac{(1-\beta_B)}{1-n_f^A}\left(1-\beta_B\frac{n_f^B}{n_f^A}\right)}{1-\beta_A\beta_B}\right]. \tag{B.24}$$

From this equation, we have that if $\lim_{H^A\to\infty} n_f^B/n_f^A = 0$, then $\lim_{H^A\to\infty}(H^A/\hat{H}^B(H^A)) = 0$. We also have that if $\lim_{H^A\to\infty}(H^A/\hat{H}^B(H^A)) = 0$, then $\lim_{H^A\to\infty} n_f^B/n_f^A = 0$ for all $f \in \mathcal{F}$. This implies that $n_f^B \to 0$ for all f as $H^A \to \infty$, which contradicts the definition of $\hat{H}^B(H^A)$. Thus, we must have $\lim_{H^A\to\infty} N^A(T_f^A, T_f^B, H\ A, \hat{H}^B(H^A)) = 0$ for all $f \in \mathcal{F}$. This implies that

$$\lim_{H^A\to\infty} \sum_{f\in\mathcal{F}} N^A(T_f^A, T_f^B, H\ A, \hat{H}^B(H^A)) = 0.$$

Next, consider the case where $H^A \to 0$. In this case, we must have either $n_f^A \to 1$ or $n_f^B \to 1$ to satisfy equation (B.23). First, by the definition of $\hat{H}^B(H^A)$, there is at most one platform with $n_f^B \to 1$. Thus, for all other platforms $f' \neq f$, we must have $n_{f'}^A \to 1$. Next, I argue that $\lim_{H^A\to 0} n_f^A > 0$ for the platform f such that $n_f^B \to 1$. This is because if $n_f^A \to 0$, we have $\lim_{H^A\to 0} n_f^A/n_f^B = 0$, which implies from equation (B.24) that $\lim_{H^A\to 0}(\hat{H}^B(H^A)/H^A) = 0$ and that $n_{f'}^A \to 0$ for all $f' \in \mathcal{F}$, a contradiction. Consequently, we have $\lim_{H^A\to 0} N^A(T_f^A, T_f^B, H\ A, \hat{H}^B(H^A)) = 1$ for all but one f and $\lim_{H^A\to 0} N^A(T_f^A, T_f^B, H^A, \hat{H}^B(H^A)) > 0$ for one f. This implies that

$$\lim_{H^A\to 0} \sum_{f\in\mathcal{F}} N^A(T_f^A, T_f^B, H\ A, \hat{H}^B(H^A)) > 1.$$

Finally, applying the intermediate value theorem, there exists an aggregator $H^A \in (0, \infty)$ such that $\sum_{f\in\mathcal{F}} N^A(T_f^A, T_f^B, H\ A, \hat{H}^B(H^A)) = 1$, which implise that there exists a pair of aggregators (H^A, H^B) that satisfy the equilibrium condition. This establishes the equilibrium existence.

Finally, I show that there exists $\underline{\beta} > 0$ such that if $\beta_k \leq \underline{\beta}$ for some $k \in \{A, B\}$, then there exists a unique pair of aggregators that satisfy the equilibrium condition. Because the model is continuous at $\beta_k = 0$, it suffices to show that there is a unique pair of aggregators that satisfy the equilibrium condition at $\beta_k = 0$ for some $k \in \{A, B\}$. Then, by continuity we have the above result. Without loss of generality, suppose that $\beta_B = 0$. To show the uniqueness at $\beta_B = 0$, I present several comparative statics of m^k, $k \in \{A, B\}$, with respect to several parameters x. This is given by the Implicit Function Theorem

$$G_f \begin{pmatrix} \frac{\partial m^A}{\partial x} \\ \frac{\partial m^B}{\partial x} \end{pmatrix} = -\begin{pmatrix} \frac{\partial g_A}{\partial x} \\ \frac{\partial g_B}{\partial x} \end{pmatrix}.$$

by Cramer's Rule, I obtain

$$\frac{\partial \mu^A}{\partial x} = \frac{\det\begin{pmatrix} -\frac{\partial g_A}{\partial x} & \frac{\partial g_A}{\partial \mu_B} \\ -\frac{\partial g_B}{\partial x} & \frac{\partial g_B}{\partial \mu_B} \end{pmatrix}}{\det G_f}, \quad \frac{\partial \mu^B}{\partial x} = \frac{\det\begin{pmatrix} \frac{\partial g_A}{\partial \mu_A} & -\frac{\partial g_A}{\partial x} \\ \frac{\partial g_B}{\partial \mu_A} & -\frac{\partial g_B}{\partial x} \end{pmatrix}}{\det G_f}.$$

Using this comparative statics in ι-markups, I conduct a comparative statics in market shares N^A and

N^B:

$$\frac{\partial N^A}{\partial x} = \frac{\partial n^A}{\partial x} - \frac{\partial m^A}{\partial x} n^A - \beta_A \frac{\partial m^B}{\partial x} n^A$$

$$\frac{\partial N^B}{\partial x} = \frac{\partial n^B}{\partial x} - \frac{\partial m^B}{\partial x} n^B$$

Based on this observation, I derive the effects of H^A and H^B on N^A and N^B.

Fist, for H^A, we have

$$\frac{\partial m^A}{\partial H^A} = -\frac{n_f^A}{H^A} \frac{1}{\det(G_f)} \left[\mu_f^A \left(n_f^B \mu_f^B + 1 - n_f^B + \beta_A \frac{n_f^A}{n_f^B} \right) + \mu_f^A \beta_A^2 \frac{n_f^A}{n_f^B} \right] < 0$$

$$\frac{\partial m^B}{\partial H^A} = \frac{n_f^A}{H^A} \frac{1}{\det(G_f)} \left(\beta_A \frac{n_f^A}{n_f^B} \mu_f^A (1-\beta_A) + \beta_A \frac{1-n_f^A}{n_f^B} \right) > 0,$$

and thus

$$\frac{\partial N^A}{\partial H^A} = -\frac{n_f^A}{H^A} \frac{1}{\det(G_f)} (1-n_f^A) \left(n_f^B \mu_f^B + 1 - n_f^B + \beta_A (1+\beta_A) \frac{n_f^A}{n_f^B} \right) < 0$$

$$\frac{\partial N^B}{\partial H^A} = -\frac{n_f^A}{H^A} \frac{1}{\det(G_f)} \left(\beta_A \frac{n_f^A}{n_f^B} \mu_f^A (1-\beta_A) + \beta_A \frac{1-n_f^A}{n_f^B} \right) n_f^B < 0.$$

For H^B, we have

$$\frac{\partial m^A}{\partial H^B} = \frac{n_f^B}{H^B} \frac{1}{\det(G_f)} \left(\mu_f^B + \beta_A \frac{n_f^A}{(n_f^B)^2} \right) \beta_A n_f^A \mu_f^A > 0$$

$$\frac{\partial m^B}{\partial H^B} = -\frac{n_f^B}{H^B} \frac{1}{\det(G_f)} \left(\mu_f^B + \beta_A \frac{n_f^A}{(n_f^B)^2} \right) (n_f^A \mu_f^A + 1 - n_f^A) < 0,$$

and thus

$$\frac{\partial N^A}{\partial H^B} = \beta_A \frac{n_f^A}{n_f^B} \frac{n_f^B}{H^B} \frac{1}{\det(G_f)} \left(n_f^B \mu_f^B + \beta_A \frac{n_f^A}{n_f^B} \right) (1-n_f^A) > 0,$$

$$\frac{\partial N^B}{\partial H^B} = -\frac{n_f^B}{H_f^B} \frac{1}{\det(G_f)} \left((n_f^A \mu_f^A + 1 - n_f^A)(1-n_f^B) + \beta_A^3 \frac{n_f^A}{n_f^B} n_f^A \mu_f^A \right) < 0.$$

As a result, we have

$$\det \begin{pmatrix} \sum \frac{\partial N^A}{\partial H^A} & \sum \frac{\partial N^A}{\partial H^B} \\ \sum \frac{\partial N^B}{\partial H^A} & \sum \frac{\partial N^B}{\partial H^B} \end{pmatrix} > 0,$$

which implies that the pair (H^A, H^B) that satisfies the condition (B.17) is unique when $\beta_B = 0$. By continuity, there exists $\underline{\beta} > 0$ such that if $\beta_k \leq \underline{\beta}$, there exists a unique equilibrium.