# Social discount rate: spaces for agreement

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**Abstract**

We study the problem of aggregating discounted utility preferences into a social discounted utility preference model. We use an axiom capturing a social responsibility of individuals' attitudes to time, called consensus Pareto. We show that this axiom can provide consistent foundations for welfare judgments. Moreover, in conjunction with the standard axioms of anonymity and continuity, consensus Pareto can help adjudicate some fundamental issues related to the choice of the social discount rate: the society selects the rate through a generalized median voter scheme.

**Keywords:** consensus Pareto, social aggregation, discounted utility model, expected utility theory, generalized median.

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# 1 Introduction

The discount rate is the rate at which future costs and benefits are discounted relative to current values. This study introduces an axiom capturing the social responsibility of individuals' attitudes to time into discounted utility preferences. This study contributes to the ongoing debate about what discount rate society should use to evaluate public projects with long-term consequences. The choice of discount rate critically determines the outcome of cost–benefit analysis of long-term public projects. In addition, it critically affects the welfare evaluation of macroeconomic policies in representative agent models. Given the importance of this choice, it seems natural and desirable for the society's discount rate to reflect the distribution of its members' views, so that society avoids the paradoxical situation of using a discount rate that nobody wants. However, until now, prescriptions for this value have been based on dictatorial value judgments (Millner and Heal, [24]).

A key problem in avoiding such paradoxical situations is that people have heterogeneous time preferences. To deal with this problem, Millner and Heal ([24]) suggest two alternative solutions. The efficient solution is to use a declining discount rate, although this means accepting time inconsistency. The political solution is to adopt the discount rate of the median voter in a voting mechanism. The latter approach avoids time inconsistency and can be deemed democratic, although it is inefficient. One contribution of this study is that we investigate precisely when such a voting mechanism is normatively grounded.

We assume that the time preference of each member of the society follows the standard stationary (geometric, exponential) time-additive discounted utility model, which is a descriptive assumption.[1] Furthermore, we assume that the society's decision criterion follows the stationary time-additive discounted utility model, which is a normative requirement.[2] Stationarity, introduced by Koopmans ([22]), is an independence property of preferences at each fixed history, and it explicitly requires the evaluation of two intertemporal allocations to be the same as the evaluation of the corresponding intertemporal allocations obtained by delaying each period-$t$ allocation by one period and adding a common first-period allocation. This axiom expresses the notion that the mere passage of time does not affect the evaluation of intertemporal allocations. Our problem consists of aggregating individuals' lifetime discounted utilities into the society's lifetime discounted utility. We study this aggregation problem in a dynamic choice framework.

The common practice requires the aggregation to satisfy the so-called Pareto's principle. This principle requires that when each member of the society prefers one intertemporal allocation to another, the society's ranking must endorse this view.

Zuber ([32]) and Jackson and Yariv ([21]) show the following result when the aggregation is required to satisfy the Pareto's principle: when everyone's

[1]See Koopmans ([22])'s seminal work on the axiomatization of the discounted utility model in a deterministic setting, and Epstein ([13])'s study on its axiomatization in a risk setting.

[2]Following Koopmans' argument, Kenneth Arrow (1999, [1] and [2]) accepted the need for discounting the welfare of future generations. The reason is that given other standard assumptions, a logical inconsistency would be produced without discounting. In a recent paper, Chichilnisky et al. ([10]) show that this inconsistency dissolves when "the extinction discounting rule", advocated by, *inter alias*, Stern review on climate change, is combined with other two assumptions.

preference, as well as the social preference, is stationary,[3] the only possibility is that everybody's preference, as well as the social preference, is represented in the time-additive discounted utility form, the society's period utility function is an additive aggregation of individual utilities, and people have the same rate of impatience; otherwise, the aggregation must be dictatorial—that is, the preference of only one member of the society determines everything. The crux of this result is that a non-dictatorial Paretian social aggregation cannot be stationary if people have heterogeneous discount rates.

Given that people have different attitudes to time, trade-offs must be made between the Pareto's principle and stationarity to avoid dictatorial social preferences. In this study, we retain the stationarity requirement of social preferences and weaken the Pareto's principle.[4] The reasons of this line of research are detailed below.

A non-dictatorial social preference that is non-stationary is necessarily time inconsistent or time variant (Halevy, [17]; Millner and Heal, [25]).[5]

[3] The social preference is not necessarily time-additive. See Koopmans ([22]) and Epstein ([13]) for classes of stationary utilities that are not necessarily time-additive.

[4] An alternative approach is taken by Chambers and Echenique ([8]), who propose a theory of intertemporal choice that is robust to specific assumptions on the discount rate. Robustness is operationalized in three different ways. One class of models relies on a dominance criterion relating pairs of consumption streams. This dominance ranking says that a stream $x$ 'discounting dominates' a stream $y$ if for every possible discount factor, $x$ yields a lifetime discounted utility larger than $y$. In other words, $x$ discounting dominates $y$ if $x$ is unambiguously better than $y$, independently of the discount factor.

[5] In contrast to stationarity, which is a property of a single preference, time consistency is a consistency relationship between preferences at different histories. Stationarity and time consistency are equivalent when the relationship is time invariant; that is, when the ranking of future consumption streams is not affected by the shifting of preferences

Time inconsistency is unacceptable because it prejudices the credibility of the social decision. Thus, the social decision must be time variant. However, this is not desirable either.

The reason is that a time variant social decision imposes no restriction on the dynamic process of the social welfare ranking, which makes the dynamic problem trivial. Indeed, time consistency is trivially satisfied by selecting any social ranking over consumption streams in day 1 and by committing to it from day 2 onwards. Moreover, a time variant social decision may cause moral tensions.[6]

To illustrate this point, consider a two-individual society who must decide on how to divide resources between individuals. Suppose that individual A is patient and that individual B is totally myopic.[7] In this situation, every social decision process that respects the Pareto's principle must have the following form:[8]

- *Day 1*: Everyone obtains an amount of resources.[9] Individual B will

backwards or forwards in time.

[6]These tensions could be dissolved by imposing some restrictions on the social decision process. Hayashi ([19]) proposes a variant of such dynamic constraints.

[7]Formally speaking, the case of total myopia is not in the class of stationary utilities. However, it can be taken to be as the limit of the class of stationary utilities. Alternatively, one can think of the discount factor as being arbitrarily close to zero.

[8]In an infinite-horizon economy, an even sharper implication is known: all but the most patient households "perish" in the long-run, in the sense that their levels of consumption and wealth converge to zero. Becker ([5]) and Bewley ([6]) formally prove this result for a competitive equilibrium, confirming the conjecture of Ramsey ([28]). The same holds for general Pareto-efficient allocations, as this result is independent of initial wealth distribution and earnings.

[9]The planner's ethics may dictate to allocate more consumption to individual B because

receive nothing tomorrow.

- *From day 2 onwards*: Because day 1 is over, individual B receives nothing.

Observe that the just described social decision process is dynamically consistent. Can the planner commit to such a time consistent social decision process? We suspect not, as it causes a significant moral tension of whether to allocate other resources to individual B from day 2 onwards. Time invariance can be viewed as a suitable restriction because it eases this tension.

We propose a weakening of the Pareto's principle, which we call Consensus Pareto, and it requires the social preference to agree with the views of its members only when an intertemporal allocation gives a larger lifetime discounted utility than another for each of its members, according to every member's discount rate. This axiom is motivated by the fact that time discounting is not purely a matter of taste. Indeed, in calculating the lifetime discounted utility of each member of the society, the society should listen to the discounting opinions of all its members. While no single individual is responsible for his or her time preference, the society as a whole is collectively responsible for its time preference.

The idea of collective responsibility has been proposed by Gayer et al. ([15]) in the context of belief heterogeneity, according to which an uncertain prospect Pareto–dominates another if the former gives a higher expected utility than the latter, for everyone and for all individuals' beliefs.[10]

individual B cares only about today consumption.

[10] One drawback of the Consensus Pareto argument, however, is that it takes preference representations as given, and consider utilities and beliefs separately as if they were

We view that the idea of collective responsibility is equally cogent in the context of discount rate heterogeneity. The reason is that the formation of time preferences is affected by sociological aspects that go beyond the control of any individual.

One can argue against the extension of the idea of collective responsibility to the context of discount rate heterogeneity, by pointing out that discount rate heterogeneity is fundamentally distinct from belief heterogeneity, in the sense that when people have different beliefs one of them must have a “wrong” belief, while there is no “wrong” discount rate because it is just a matter of taste. However, there is no “wrong” belief either. The reason is that we can never deduce whether a subjective probability distribution is wrong, but we can assess whether a bet is wrong. Furthermore, in Savage’s subjective expected utility theorem (Savage, [30]), an individual’s belief is derived as part of the representation of his/her preference over acts. Therefore, if we take the Savage theory literally, we should conclude that belief heterogeneity is rather "a matter of taste," which is indeed what the idea of collective responsibility of beliefs is questioning about. We do not want to say that the problem of discount rate heterogeneity is the same as the problem of belief heterogeneity. We simply want to say that they are equally significant in different ways and that individual’s responsibility for his/her own discount rate is at least worth questioning, while its final judgement is an ideological issue. As for belief heterogeneity, Consensus Pareto allows that an individual

primitive objects. Billot and Qu ([7]) provide a set of Consensus Pareto axioms expressed in terms of preference relations. Although we consider discount rates and period utility functions as if they were separate objects, the technique developed by Billot and Qu ([7]) can be used to restate our Consensus Pareto in terms of preference relations.

is held not responsible for his/her own belief because his/her belief formation is affected by his/her educational/informational environment. As for discount rate heterogeneity, Consensus Pareto allows that an individual is held not responsible for his/her own discount rate because his/her patience formation is affected by his/her educational/disciplinary environment.

Becker and Mulligan ([4]) argue that an individual's time preference is endogenously determined by his/her parents' choice, which the individual cannot be responsible for. From empirical approach, several studies report that the time preference of an individual is related to his or her socioeconomic status (Lawrance, [23]; Barsky et al, [3]; Tanaka et al, [31]; Dohmen et al, [12]). Although the precise causal effects are yet unclear, it is fair to say that acquiring and maintaining patience is not a straightforward process for which individuals can be kept fully accountable.[11]

When individuals have heterogeneous discount factors, the Pareto's principle requires that the society must endorse the view of its members when each of them prefers one intertemporal allocation to another. This endorse-

[11] Allowing an individual to put an effort into his or her time preference can make the disparity even more severe. Hayashi ([20]) studies a simple dynamic general equilibrium model in which a household can make a costly investment into its 'patience capital', and he reports that the interior long-run steady state is unstable. In the two-dimensional space of patience capital and physical capital, there is a downward sloping curve such that the convergence to the steady state happens only when the initial vector falls exactly on it. Households with initial vectors falling in the upper side of the curve invest more into patience capital, and this leads households to save more, and thus, their consumption level grow in the long-run. However, households with initial vectors falling in the lower side do not invest into patience capital, and this leads to a decay of patience level. This means that in this case households save less, and thus, they will perish in the long-run.

ment must be made even when the superiority of one stream over another is merely justified by disparities in individuals' socioeconomic conditions for which individuals cannot be kept fully accountable. However, it seems bizarre to elevate these disparities to a social rank. This has motivated us to refine the Pareto's principle by seeking "robustness" of the social welfare ordering to the distribution of individuals' socioeconomic characteristics, in some fashion.

In a recent paper, Feng and Ke ([14]) propose a Pareto criterion, called intergenerational Pareto. This axiom is based on the view that present and future 'selves' of an individual are distinct individuals. Intergenerational Pareto requires that whenever an intertemporal allocation $x$ is preferred to $y$ by every individual from every generation, then the society prefers $x$ to $y$.

This axiom is weaker than the Pareto's principle, even when everyone has a time consistent preference. To see it, let us go back to the above example. In this case, to have a unanimous discounting opinion, intergenerational Pareto requires that each current individual and his future selves should agree on how to rank two intertemporal allocations. This is one way to weaken the standard requirement that each 'integrated' individual is responsible for his or her time preference. Although there does not exist any logical relationship between consensus Pareto and intergenerational Pareto, consensus Pareto allows the society to listen to the discounting opinions of all its members in calculating its lifetime discounted utility, even to the opinion of each 'integrated' individual.

We study the implications of consensus Pareto in a dynamic choice framework. Specifically, an allocation in each period is a lottery, and the society's decision criterion, as well as the decision criterion of every member of the society, follows not only the stationary discounted utility model but also the expected utility model.[12] In this set-up, we show that consensus Pareto has a sharp implication: the society's period utility function is a weighted average of individual utilities, which is along the lines of Harsanyi ([18]), and the society's discount rate must reflect the view of only one of its members.

Although this result has a flavor of dictatorship, as it rules out any compromise in time discounting by means of averaging, it gives society the freedom to socially evaluate the discount rates of its members and to choose that which responds better to the society's view. Indeed, by positing that the society's decision criterion satisfies two additional axioms—*anonymity* and *continuity*—, we provide the following complete characterization. The society's period utility function is the symmetric additive average of individual utilities, and the selection rule for the social discount rate has the form of the generalized median (Moulin, [27]), which includes maximum, minimum and median as special cases.[13]

The remainder of this paper is organized as follows. Section 2 sets out the theoretical framework and outlines the basic model. Section 3 shows the implications of consensus Pareto for the aggregation of individual preferences.

[12] In the domain of lotteries over sequences, Epstein ([13]) characterizes the class of expected discounted utility preferences. Thus, it is easy to apply the same argument to the domain of sequences of lotteries over period-wise outcomes.

[13] Under some assumptions, Geber ([16]) shows that individual preferences over discount factors are single-peaked. Thus, simple majority voting over the collective discount factor defines a transitive social preference relation on the set of discount factors and the voting rule that assigns to any profile of individual discount factors the unique Condorcet winner is coalitionally strategy-proof.

Section 4 provides the complete characterization. Section 5 concludes.

# 2 The setting

A set of $n$ agents, denoted by $N = \{1, ..., n\}$, must make a collective decision about sequences of lotteries over social outcomes.

The agents are infinitely lived and consume in discrete periods $t \in \{1, 2, ...\}$.

The set of social outcomes is $C$, which is assumed to be finite for expositional simplicity.[14] The set of lotteries over the set $C$ is denoted by $\mathcal{L}$, which is a compact metric space. The set of sequences of lotteries is denoted by $\mathcal{L}^\infty \equiv \mathcal{L} \times \mathcal{L} \times ...$, which is endowed with the product metric.[15] Thus, for the sake of simplicity, we assume that the set $\mathcal{L}^\infty$ is the domain of social objects, with $\ell = (\ell_1, \ell_2, ...)$ as a typical stream.[16]

We assume that agents' preference rankings and social decision criterion, which are defined over the set $\mathcal{L}^\infty$, follow the discounted utility theory. In other words, each agent's preference ranking is represented in the form

$$\sum_t \beta_i^{t-1} u_i(\ell_t),$$

where $\beta_i \in (0, 1)$ is agent $i$'s discount factor and $u_i : \mathcal{L} \to \mathbb{R}$ is given in the

[14] With obvious adaptations, the results easily extend to the case in which $C$ is a compact metric space. Furthermore, the same results apply to situations in which each agent receives his/her own consumption and the society evaluates consumption allocations.

[15] The same argument can be extended to a larger domain $\Delta(C^\infty)$, which is the set of lotteries over infinite consumption streams—which may allow correlation across periods.

[16] The results of the paper could be derived by assuming that the social domain is the set of deterministic streams $C^\infty$, though this would require tedious functional equation arguments.

expected utility form[17]

$$u_i(\ell) = \sum_{c \in C} \ell(c) v_i(c),$$

where $v_i : C \to \mathbb{R}$ is agent $i$'s (instantaneous) utility function. Similarly, the social decision criterion is represented in the form

$$\sum_t \beta_0^{t-1} u_0(\ell_t),$$

where $\beta_0 \in (0,1)$ is the social discount factor and $u_0 : \mathcal{L} \to \mathbb{R}$ is given in the expected utility form

$$u_0(\ell) = \sum_{c \in C} \ell(c) v_0(c).$$

Let $\mathcal{V}$ be the domain of decision criteria, with $(v_i, \beta_i)$ as a typical agent $i$'s decision criterion. This domain is specified below.

A *discounted utility aggregation rule*[18] maps a profile of individual decision criteria $(v_i, \beta_i)_{i \in N} \in (\mathcal{V} \times (0,1))^n$ into a social decision criterion $(v_0, \beta_0) \in \mathcal{V} \times (0,1)$.

We present the following normalization condition, which is natural in our setting (see, for instance, Mongin, [26]).

**Minimal agreement and normalization**: $\underline{c} \in C$ exists such that for all $v \in \mathcal{V}$ and all $c \in C$, it holds that

$$v(c) \geq v(\underline{c}), \qquad v(\underline{c}) = 0 \qquad \text{and} \qquad \sum_{c \in C} v(c) = 1.$$

Consequently, $\mathcal{V}$ is a compact and convex subset of $\mathbb{R}^{|C|}$.

[17] When there is no danger of confusion, with some abuse of notation, sometimes we write $\ell$ for a lottery in a given period.

[18] For expositional simplicity, we omit the full functional formulation and leave it implicit.

# 3 Delegation of social discounting

A well-known efficiency requirement for a social utility function is the so-called *Pareto* condition, which requires that if everyone agrees that one stream is (discernibly) superior to another, then the social utility function should exhibit the same preference. It is formally represented as follows.

**Pareto's principle**: For all $\ell, \ell' \in \mathcal{L}^\infty$,

$$\sum_t \beta_i^{t-1} u_i(\ell_t) > \sum_t \beta_i^{t-1} u_i(\ell'_t) \text{ for all } i \in N \implies \sum_t \beta_0^{t-1} u_0(\ell_t) > \sum_t \beta_0^{t-1} u_0(\ell'_t).$$

As mentioned in the introduction, Jackson and Yariv ([21]) study collective decisions by time-discounting individuals who choose a consumption stream from the set $C \times C \times ....$ The authors show that when agents exhibit heterogeneous time preferences, every non-dictatorial method of aggregating discounted utilities satisfying Pareto must be time–inconsistent, in that it must generate present bias.[19] However, in a significantly different setting, Zuber ([32]) presents an earlier result on the necessity for time–inconsistency in aggregating individual time–preferences. Indeed, in a setting in which each agent can have an independent and arbitrary consumption stream, Zuber ([32]) shows that a Paretian, time-consistent, and history independent aggregation of individual preferences is possible when individual utilities are additively separable and the social decision criterion is a linear combination of these utilities. In addition, the social decision criterion is stationary

[19] Present bias occurs when a smaller immediate reward is preferred to a larger later reward, but the ranking of these rewards is reversed when they are equally delayed.

when all agents have the same constant rate of time discounting. All these requirements are unlikely to be met by individual preferences.

Given that our setting is conceptually connected to that of Zuber ([32]) and Jackson and Yariv ([21]), and that we aim to avoid their conclusions, we propose a weaker version of the Pareto condition, which can be stated as follows.

**Consensus Pareto**: For all $\ell, \ell' \in \mathcal{L}^\infty$,

$$\sum_t \beta_j^{t-1} u_i(\ell_t) > \sum_t \beta_j^{t-1} u_i(\ell'_t) \text{ for all } i, j \in N \implies \sum_t \beta_0^{t-1} u_0(\ell_t) > \sum_t \beta_0^{t-1} u_0(\ell'_t).$$

This condition is weaker than the Pareto requirement in that it requires the Pareto argument to follow only when the stream $\ell$ gives a larger lifetime discounted utility than $\ell'$, for each agent, according to each agent's discount factor.

This condition is an adaptation of a Pareto-type condition proposed by Gayer et al. ([15]) in the context of financial markets, according to which an uncertain prospect Pareto–dominates another if the former gives a higher expected utility than the latter, for each individual and for all individuals' beliefs. The idea behind both conditions is the same—to provide collective responsibility for individual beliefs in the context of Gayer et al. ([15]) and collective responsibility for individual discount factors in our setting.

We now state our first main result. If agents hold heterogeneous discount factors and their (instantaneous) utility functions are linearly independent, then any social decision criterion that respects consensus Pareto is dictatorial in the discount factor component: the social discount factor coincides with the discount factor of one of the agents. Moreover, the social utility can take only a limited form, namely, it is a linear combination of individual utility functions. The requirement that agents' utility functions are linear independent means that their tastes over temporal outcomes are sufficiently diverse. This result is formally represented as follows.

**Theorem 1** Let $((v_i, \beta_i)_{i \in N}, (v_0, \beta_0)) \in (\mathcal{V} \times (0,1))^n \times (\mathcal{V} \times (0,1))$ be a profile of decision criteria for individuals and society, in which each criterion satisfies the minimal agreement and normalization. Suppose that $\beta_1, \cdots, \beta_n$ are all distinct and that $v_1, \cdots, v_n$ are linearly independent. Then, the social decision criterion $(v_0, \beta_0)$ satisfies consensus Pareto if and only if there is a non-zero vector $a \in \mathbb{R}^n_+$, with $\sum_{i \in N} a_i = 1$, and an agent $s \in N$ such that

$$v_0 = \sum_{i \in N} a_i v_i \qquad \text{and} \qquad \beta_0 = \beta_s.$$

**Proof.** Let the premises hold. Since the proof of the "if" part of the statement is obvious, we prove the "only if" part. Suppose that $(v_0, \beta_0)$ satisfies consensus Pareto. Then, the Harsanyi–theorem (Harsanyi, [18]) shows that there is a non-zero matrix $\Gamma = (\gamma_{ij})_{i,j \in N} \in \mathbb{R}^{n^2}_+$ such that for all $\ell \in \mathcal{L}^\infty$,

$$\sum_t \beta_0^{t-1} u_0(\ell_t) = \sum_{i \in N} \sum_{j \in N} \gamma_{ij} \sum_t \beta_j^{t-1} u_i(\ell_t)$$

Restricting attention to sequences of the form $(\ell, \underline{c}, \underline{c}, \underline{c}, \cdots) \in \mathcal{L} \times C \times C \times ...$ yields

$$u_0(\ell) = \sum_{i \in N} \sum_{j \in N} \gamma_{ij} u_i(\ell). \tag{1}$$

By restricting attention to sequences in which an arbitrary $\ell \in \mathcal{L}$ appears in period $t$ and in which agents consume $\underline{c}$ in all other periods $t' \neq t$, we obtain

$$\beta_0^{t-1} u_0(\ell) = \sum_{i \in N} \sum_{j \in N} \gamma_{ij} \beta_j^{t-1} u_i(\ell). \tag{2}$$

Using (1) in (2), we obtain

$$\beta_0^{t-1} \sum_{i \in N} \sum_{j \in N} \gamma_{ij} u_i(\ell) = \sum_{i \in N} \sum_{j \in N} \gamma_{ij} \beta_j^{t-1} u_i(\ell) \tag{3}$$

for any arbitrary $\ell \in \mathcal{L}$. Since the individual instantaneous utility functions $v_1, \cdots, v_n$ are linearly independent, (3) yields

$$\beta_0^{t-1} \sum_{j \in N} \gamma_{ij} = \sum_{j \in N} \gamma_{ij} \beta_j^{t-1} \tag{4}$$

for all $i \in N$.

For $t = 2$, (4) becomes

$$\beta_0 \sum_{j \in N} \gamma_{ij} = \sum_{j \in N} \gamma_{ij} \beta_j \tag{5}$$

for all $i \in N$. By plugging (5) into (4), we obtain

$$\left( \sum_{j \in N} \gamma_{ij} \beta_j \right)^{t-1} = \left( \sum_{j \in N} \gamma_{ij} \right)^{t-2} \sum_{j \in N} \gamma_{ij} \beta_j^{t-1} \tag{6}$$

for all $i \in N$.

For $t = 3$, (6) becomes

$$\left( \sum_{j \in N} \gamma_{ij} \beta_j \right)^2 = \left( \sum_{j \in N} \gamma_{ij} \right) \sum_{j \in N} \gamma_{ij} \beta_j^2 \tag{7}$$

for all $i \in N$. Simplifying (7) yields

$$\sum_{j \in N} \sum_{k \in N - \{j\}} \gamma_{ij} \gamma_{ik} (\beta_j - \beta_k)^2 = 0$$

for all $i \in N$. Since $(\beta_1, ..., \beta_n)$ are all distinct, we obtain

$$\gamma_{ij}\gamma_{ik} = 0$$

for all $i, j \in N$ and $k \in N - \{j\}$.

This means that every row of the matrix $\Gamma$ can have at most one non-zero entry. For any $i$ with a non-zero entry (there is at least one such $i$ since $\Gamma$ is a non-zero matrix), let $\gamma_{ij(i)}$ be such a non-zero entry. Then, (5) yields

$$\beta_0\gamma_{ij(i)} = \gamma_{ij(i)}\beta_{j(i)},$$

that is,

$$\beta_0 = \beta_{j(i)}$$

for any such $i \in N$. Since $(\beta_1, \cdots, \beta_n)$ are all distinct, the only possibility is that $j(i)$ is identical for all $i$ who have the non-zero entry in $(\gamma_{ij})_{j \in N}$. Let $s$ be such an index, so that $\beta_0 = \beta_s$. Finally, let us define the non-zero vector $a$ as follows: $a_i = \gamma_{is}$ if $i$ has a non-zero entry in $(\gamma_{ij})_{j \in N}$; otherwise, $a_i = 0$. Thus, we have that $v_0 = \sum_{i \in N} a_i v_i$, where $\sum_{i \in N} a_i = 1$ follows from the normalization condition. This completes the proof. ■

We are able to circumvent the impossibility result mentioned above because consensus Pareto allows us to separate the problem of selecting the collective (instantaneous) utility function from the problem of selecting the social discount factor. The Pareto's principle forces us to tie together the two problems by requiring to match the collective (instantaneous) utility function with the utility function of the agent whose discount factor represents the social discount factor.

Theorem 1 has two main implications in the context of our analysis. First, it shows that when society as a whole is considered responsible for its members' discount factors, as captured by consensus Pareto, and agents have heterogeneous discount factors, the society's period utility funtion can be non-dictatorial. This is particularly interesting, given that the consistency between the Pareto's principle and the discounted utility model is called into question by Zuber ([32]) and Jackson and Yariv ([21]). Indeed, if we were to insist on a social decision criterion that respects the Pareto's principle, this would bring us back to dictatorial social decisions.

**Corollary 1 (Zuber ([32]) and Jackson and Yariv ([21]))** Let $((v_i, \beta_i)_{i\in N}, (v_0, \beta_0)) \in (\mathcal{V}\times (0,1))^n \times (\mathcal{V}\times (0,1))$ be a profile of decision criteria for individuals and society, in which each criterion satisfies the minimal agreement and normalization. Suppose that $\beta_1, \cdots, \beta_n$ are all distinct and that $v_1, \cdots, v_n$ are linearly independent. Then, the social decision criterion $(v_0, \beta_0)$ satisfies Pareto's principle if and only if there exists an agent $s \in N$ such that

$$v_0 = v_s \qquad \text{and} \qquad \beta_0 = \beta_s.$$

**Proof.** There is an immediate violation of the Pareto's principle if $a_i > 0$ for some $i \neq s$. ■

Theorem 1 shows that such a conclusion is partly avoidable.

Second, when dealing with intertemporal allocation problems, the standard approach in macroeconomic theory is to use the discount factor of a fictional representative consumer as a measure of social discounting. For instance, the most common intertemporal social decision criterion, used in seminal models of economic growth ([28]) and of optimal resource allocation ([11]), exhibits exponential discounting and is additively separable. This form of social criterion corresponds to the model of dynamic decision making described by Samuelson ([29]). However, for the discount factor of the representative consumer to have such social significance, it should have a normative foundation—that is, it should be the result of a Paretian aggregation of individual preferences. Theorem 1 may provide such a normative foundation when some other properties are imposed. The next section clarifies this point.

## 4 The social discounting selection rule

According to Theorem 1, the social discount factor can be neither a linear combination of the individual discount factors nor a product of individuals' discount factors. Formally, it cannot take the following forms:

$$\beta_0 = \sum_{i=1}^{n} \eta_i \beta_i$$

and

$$\beta_0 = \prod_{i=1}^{n} \beta_i^{\eta_i}$$

where $\eta_i \in (0, 1)$ for each agent $i \in N$.

However, Theorem 1 allows the social discount factor to be chosen according to one of the following selection criteria:

$$\beta_0 = \text{med}\{\beta_1, \cdots, \beta_n\}$$

$$\beta_0 = \min\{\beta_1, \cdots, \beta_n\}$$

$$\beta_0 = \max\{\beta_1, \cdots, \beta_n\}$$

These criteria belong to a version of the class of generalized median (Moulin, [27]):

$$\beta_0 = \mathrm{med}\{\overbrace{0, \cdots, 0}^{k}, \beta_1, \cdots, \beta_n, \overbrace{1, \cdots, 1}^{l}\},$$

with $k, l \geqq 0$ being integers satisfying $k + l = n - 1$.

We now introduce two natural and well-known axioms that a social discounting selection rule may be required to satisfy—anonymity and continuity.

A social discounting selection rule $f$ for the set $\mathcal{V} \times (0, 1)$ and the domain $(\mathcal{V} \times (0, 1))^n$ is a function from $(\mathcal{V} \times (0, 1))^n$ to the set of social decision criteria $\mathcal{V} \times (0, 1)$.

An anonymous/impartial social discounting selection rule focuses on the individual decision criteria, and not on the identities of people who display particular decision criteria.

Formally, suppose that $\sigma$ is a permutation of $N$. Such a permutation induces a map $\sigma$ on profiles of individual decision criteria:

$$\sigma\left(\left(v_1, \beta_1\right), \left(v_2, \beta_2\right), ..., \left(v_n, \beta_n\right)\right) = \left(\left(v_{\sigma(1)}, \beta_{\sigma(1)}\right), \left(v_{\sigma(2)}, \beta_{\sigma(2)}\right), ..., \left(v_{\sigma(n)}, \beta_{\sigma(n)}\right)\right).$$

**Anonymity**: A social discounting selection rule $f$ is *anonymous* if for every permutation $\sigma$ on $N$ and every profile of individual decision criteria $((v_1, \beta_1),$ $(v_2, \beta_2), ..., (v_n, \beta_n))$ in the domain of $f$, it holds that

$$f\left(\left(v_1, \beta_1\right), \left(v_2, \beta_2\right), ..., \left(v_n, \beta_n\right)\right) = f\left(\left(v_{\sigma(1)}, \beta_{\sigma(1)}\right), \left(v_{\sigma(2)}, \beta_{\sigma(2)}\right), ..., \left(v_{\sigma(n)}, \beta_{\sigma(n)}\right)\right).$$

We now turn to the second axiom. A social discounting selection rule is continuous if changes in the social decision criterion can be bounded to be arbitrarily small by taking sufficiently small changes in individual decision criteria.

Formally, let $f^{-1}(A)$ be the set of all profiles of individual decision criteria for which $f((v_1, \beta_1), (v_2, \beta_2), ..., (v_n, \beta_n)) \in A$.[20]

**Continuity**: A social discounting selection rule $f$ is *continuous* (relative to the Euclidean topology) if for each open subset $A$ of $\mathcal{V} \times (0,1)$, the set $f^{-1}(A)$ is an open subset of $(\mathcal{V} \times (0,1))^n$.

A common objection to discontinuous social selection rules is sensitivity to small changes in individual decision criteria, and thus, to measurement errors. These issues are particularly relevant in empirical applications and policy debates, although they are possibly secondary in theoretical analyses. As Chichilnisky ([9], p. 346) aptly notes,

*Continuity is a natural assumption that is made throughout the body of economic theory, and it is certainly desirable as it permits approximation of social preferences on the basis of a sample of individual preferences, and makes mistakes in identifying preferences less crucial. These are relevant considerations in a world of imperfect information.*

Theorem 2 concerns the implications of impartial approaches for social discounting selection rules that are robust to small changes in individual decision criteria. The first implication is that the social utility that must be used to provide policy guidance and to choose the intertemporal allocation

[20] Note that in our set-up, continuity is equivalent to uniform continuity.

of resources takes the form of a weighted sum of individual utilities, in which each individual's weight equals $\frac{1}{n}$. The second implication is that the choice of the social discount factor that must be used to define optimal policies has to be confined to a version of the class of generalized median (Moulin, [27]). In contrast to Theorem 1, our next result does not require that agents' (instantaneous) utility functions are linearly independent. We do not repeat the statement of consensus Pareto for the selection rule $f$, as it is a fixed-profile axiom and we maintain it in the variable profile setting as well.

**Theorem 2** A social discounting selection rule satisfies consensus Pareto, anonymity and continuity if and only if there are integers $k, l \geqq 0$ with $k + l = n - 1$ such that for all profiles $((v_i, \beta_i)_{i \in N}, (v_0, \beta_0)) \in (\mathcal{V} \times (0,1))^n \times (\mathcal{V} \times (0,1))$, in which each criterion satisfies the minimal agreement and normalization, it holds that

$$v_0 = \frac{1}{n} \sum_{i=1}^{n} v_i$$

and

$$\beta_0 = \mathrm{med}\{\overbrace{0, \cdots, 0}^{k}, \beta_1, \cdots, \beta_n, \overbrace{1, \cdots, 1}^{l}\}. \tag{8}$$

**Proof.** Let the premises hold. Since the proof of the "if" part of the statement is obvious, we prove the "only if" part. Suppose that $f$ satisfies consensus Pareto, anonymity and continuity. Let $((v_i, \beta_i)_{i \in N}, (v_0, \beta_0)) \in (\mathcal{V} \times (0,1))^n \times (\mathcal{V} \times (0,1))$ be a profile of decision criteria for individuals and society, in which each criterion satisfies the minimal agreement and normalization.

By anonymity, we assume without loss of generality that $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \beta_n$. In addition, by continuity, we assume without loss of generality that $\beta_1 < \beta_2 < \cdots < \beta_{n-1} < \beta_n$.

By restricting attention to sequences of the form $(\ell, \underline{c}, \underline{c}, \underline{c}, \cdots)$, where $\ell \in \mathcal{L}$ is an arbitrary lottery, Theorem 1 and anonymity implies that

$$v_0 = \frac{1}{n} \sum_{i=1}^{n} v_i.$$

Now, we need only to show that (8) holds. To this end, we assume that agents' utilities $v_1, ..., v_n$ are linearly independent.

Let us first illustrate the proof for $n = 1, 2, 3$.

When $n = 1$, it is clear that $k = l = 0 = 1 - 1$, and thus, $\beta_0 = \beta_1$.

Let $n = 2$. Suppose that $\beta_0 = \beta_2$. Then, by continuity, $\beta_0 = \beta_2$ always holds. Thus, $k = 0$ and $l = 1$, so that $\beta_0 = \text{med}\{\beta_1, \beta_2, 1\}$. Suppose that $\beta_0 = \beta_1$. Similarly, by continuity, it must always be the case that $\beta_0 = \beta_1$. Hence, $k = 1$ and $l = 0$, and thus, $\beta_0 = \text{med}\{0, \beta_1, \beta_2\}$.

Let $n = 3$. Assume that $\beta_0 = \beta_3$. Then, continuity assures that $\beta_0 = \beta_3$ always holds. It follows that $k = 0$ and $l = 2$, and thus, $\beta_0 = \text{med}\{\beta_1, \beta_2, \beta_3, 1, 1\}$. Suppose that $\beta_0 = \beta_2$. Again, continuity implies that $\beta_0 = \beta_2$ always holds. Thus, $k = 1$ and $l = 1$, so that $\beta_0 = \text{med}\{0, \beta_1, \beta_2, \beta_3, 1\}$. Finally, let $\beta_0 = \beta_1$. Given that it is always the case that $\beta_0 = \beta_1$, by continuity, it follows that $k = 2$ and $l = 0$, and thus, $\beta_0 = \text{med}\{0, 0, \beta_1, \beta_2, \beta_3\}$.

We now return back to the general $n$-person case.[21] Suppose that $\beta_0 = \beta_s$

[21] One may be puzzled by our continuity argument that only order statistics matter—because, otherwise, the selection rule jumps—and may have the impression that it is an artifact of the assumption that $N$ is finite. When there is a continuum of agents $i \in [0, 1]$ and a profile of discount factors is given by a continuous (and monotone increasing/decreasing) function $\beta : [0, 1] \to (0, 1)$, the analogue of our argument of selecting $\beta_0$ from $\beta([0, 1])$ allows $\beta^{-1}(\beta_0)$ to move continuously even when $\beta$ changes in a way that

for some $s \in N$. By continuity, it always holds that $\beta_0 = \beta_s$. Hence, $k = n - s$ and $l = s - 1$, so that

$$\beta_0 = \mathrm{med}\{\overbrace{0, \cdots, 0}^{k}, \beta_1, \cdots, \beta_n, \overbrace{1, \cdots, 1}^{l}\}.$$

To complete the proof, we need to show that the choice of $k$ and $l$ is independent of the choice of individual utilities $v_1, ...v_n$. Without loss of generality, we know that all $\beta_1, \cdots, \beta_n$ are distinct. If the choice of $k$ and $l$ depended on the profile $(v_i)_{i \in N}$, then there would be a jump of the social discount factor from some $\beta_s$ to another $\beta_{s'}$, which would be a violation of continuity. This completes the proof. ■

# 5 Conclusions

When people have heterogeneous discount rates, choosing a representative agent involves trading off efficiency against stationarity of social preferences. In this study, we retain the assumption of stationarity, and propose a weak

its ordinal nature is unchanged. Note, however, that the space of continuous functions is extremely small compared to the space of Lebesgue measurable functions, though it is the natural choice in the context of a continuum of agents. Moreover, the space of continuous functions does not allow studying anonymous selection rules, because continuity is not preserved under permutations (measure-preserving transformations).

When we consider the domain in which $\beta : [0, 1] \to (0, 1)$ is a Lebesgue measurable function, first we can use our argument in the subspace of simple functions defined over increasing families of subintervals, generated, for example, by binary expansions, and then we can extend it to the whole domain by continuity with respect to $L^\infty([0, 1])$, because the domain consists of bounded functions. In this way, we obtain that the rule selects the social discount factor $\beta_0$ by means of a fixed percentile.

variant of the Pareto's principle, called consensus Pareto. This decision is mainly dictated by the fact that the Pareto's principle can have implications that run counter to our intuition of intergenerational equity (Becker, [5]; Bewley, [6]). This is in line with the political approach advocated by Millner and Heal ([24]) for choosing the appropriate degree of intertemporal social impatience when people have different time preferences.

The Pareto's principle states that if all agents are strictly better off in $\ell$ than in $\ell'$, then $\ell$ should be socially strictly preferred to $\ell'$. Consensus Pareto is weaker in that it requires all agents to be strictly better off in $\ell$ than in $\ell'$ according to each agent's rate of time preference. We view the concept of consensus Pareto as a first step toward considering how to make individuals more socially responsible for their discount rates than that allowed by the Pareto's principle. In other words, we believe that people's attitudes to time are not purely a matter of taste, because they carry a responsibility role for determining the appropriate degree of intertemporal social impatience.

The main message of the study is that consensus Pareto makes it possible to aggregate individuals' lifetime discounted utilities into the society's lifetime discounted utility. The society's period utility function is the weighted average of individual utilities and the social discount rate reflects the opinion of only one member of society. Although this result may seem to have a flavor of dictatorship, we show that it gives to the society the freedom to socially evaluate the discount rates of its members and to choose that which responds better to the society's view. The study shows that, in effect, when the social decision criterion is anonymous and continuous, the society's period utility function is the symmetric additive average of individual utilities, and the selection rule for the social discount rate has the form of the generalized median (Moulin, [27]).

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