# Arrovian Social Choice with Non -Welfare Attributes*

Ryo-Ichi Nagahisa† Kansai University
Koichi Suga‡ Waseda University

November 14, 2017

**Abstract**

A social state is assumed to be chracterized by welfare and non-welfare attributes. Welfare attributes are preference profiles and all the relevant information. Non-welfare attributes are intrinsic to social states irrespective of preferences. Converting all of the Arrow's axioms, full rationality, Pareto, and Arrow's IIA, into this setting, we show that dictatorship still holds (Theorem 1). We also show that non-welfare information is used only if the dictater is indifferent beween two social states (Theorem 2).

## 1 Introduction

## 2 Notation and Definitions

Let $N = \{1, 2, ..., n\}$ be the finite set of persons with at least two. Let $X$ be the finite set of social states with at least three. Let $\succcurlyeq_i$ be the preference of person $i$. We assume that $\succcurlyeq_i$ is complete and transitive on $X$[1]. The strict and indifferent preferences associated with $\succcurlyeq_i$ are denoted by $\succ_i$ and $\sim_i$ respectively. Let $P(X)$ be the set of all preferences. A profile $\succcurlyeq$ is the list of individual preferences $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$, so the set of profiles is $P(X)^n$. A social choice rule $F$, simply a rule, is a mapping that associates with each profile $\succcurlyeq \in P(X)^n$ a social preference $\succcurlyeq_F$, a complete binary relation on $X$. The strict and indifferent social preference associated with $\succcurlyeq_F$ are denoted by $\succ_F$ and $\sim_F$ respectively.

A social state $x \in X$ is characterized by welfare and non-welfare attributes. Given a profile, the welfare attribute of $x$ is the profile itself and all the concepts

*This research was financially supported by the Kansai University Fund for Domestic Research Support, 2017.

†Department of Economics, 3-3-35 Yamatecho Suita 564-8680 Japan

‡The School of Political Science and Economics, 1-6-1 Nishi-Waseda Shinjuku Tokyo 169-8050 Japan

[1] We say that $\succcurlyeq_i$ is complete on $X$ if and only if for all $x, y \in X$, $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$ , and $\succcurlyeq_i$ is transitive on $X$ if and only if for all $x, y, z \in X$, $x \succcurlyeq_i y \succcurlyeq_i z$ implies $x \succcurlyeq_i z$.

derived from the profile such as utilities of $x$, the Borda numbers of $x$ and so on, which depend on profiles. On the other hand non-welfare attributes are intrinsic to $x$ independently from profiles. Let non-welfare attributes be given. We assume that all the social states are classified into subgroups in which each member is thought of as identical from the viewpoint of the non-welfare attributes. Thus $X$ has a partition $\{X_\lambda\}_{\lambda\in\Lambda}$, i.e., $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ and $X_\lambda \cap X_{\lambda'} = \emptyset$ for all $\lambda \neq \lambda'$. If $x, y \in X_\lambda$, we cannot distinguish between $x$ and $y$ from the viewpoint of the non-welfare attribute. We call $X_\lambda$ an attribute set. We assume that there exist at least two attribute sets.

**Remark 1** A weaker definition of attribute sets is that $X$ has a covering, i.e., $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ holds, but not necessarily $X_\lambda \cap X_{\lambda'} = \emptyset$ for all $\lambda \neq \lambda'$. This definition is however essentially the same as our definition by letting all the intersections be new attribute sets. The example below helps us to understand this point. Let $R$ and $B$ be the sets of red-colored objects and blue-colored objects respectively. If there exist some objects that look red and blue, i.e., the intersection $R \cap B$ is nonempty, we let $P = R \cap B$ be a new attribute set called the set of purple-colored objects.

**Example 1** Lady Chatterley's Lover (Sen 1969)
$X = \{r_{AB}, r_A, r_B, r_0\}$.
The non-welfare attribute: Read or not, a kind of morality
The attribute sets: either $\{r_{AB}\}, \{r_A, r_B\}, \{r_0\}$ or $\{r_{AB}, r_A, r_B\}, \{r_0\}$.

**Example 2** Marriage (Gibbard 1974)
$X = \{w_E, w_J, w_o\}$.
The non-welfare attribute: Marriage or not, one of customs
The attribute sets:$\{w_E, w_J\}, \{w_o\}$.

**Example 3** Mac or Windows
$X = \{(m, m), (m, w), (w, m), (w, w)\}$.
The non-welfare attribute: Corporate or not.
The attribute sets:$\{(m, m), (w, w)\}, \{(m, w), (w, m)\}$.

**Example 4** Building a commercial complex (C) or protecting natural environment (E)
$X = \{C, E\} \times \prod_{i=1}^{n} X_i$.
The non-welfare attribute: Environment.
The attribute sets: $\{C\} \times \prod_{i=1}^{n} X_i, \{E\} \times \prod_{i=1}^{n} X_i$.

# 3 Axioms

A rule $F$ satisfies Conditional Full Rationality (CFR) if for any $\succcurlyeq \in P(X)^n$, any $X_\lambda, X_{\lambda'}, \lambda \neq \lambda'$ and any $\{x, y, z\} \subset X_\lambda \cup X_{\lambda'}$, $x \succcurlyeq_F y \succcurlyeq_F z$ implies $x \succcurlyeq_F z$. Note that transitivity of $\succcurlyeq_F$ does not always hold on $\{x, y, z\}$ if each of the three belongs to a different attribute set. It is easy to see that a rule $F$ satisfies CFR if and only if for any $X_\lambda, X_{\lambda'}, \lambda \neq \lambda'$ and any $\{x, y, z\} \subset X_\lambda \cup X_{\lambda'}$,(i) $x \sim_F y \sim_F z$ implies $x \sim_F z$ and (ii) either $x \succ_F y \succcurlyeq_F z$ or $x \succcurlyeq_F y \succ_F z$ implies $x \succ_F z$.

There are four cases for CFR.

(0) either $x, y, z \in X_\lambda$ or $x, y, z \in X_{\lambda'}$;

(1) $x \in X_\lambda, y \in X_\lambda, z \in X_{\lambda'}$;

(2) $x \in X_\lambda, y \in X_{\lambda'}, z \in X_{\lambda'}$;

(3) $x \in X_\lambda, y \in X_{\lambda'}, z \in X_\lambda$.

Case (0) is essentially equivalent to Full Rationality (FR) imposed on Arrovian rules. A rule $F$ satisfies FR if for any $\succcurlyeq \in P(X)^n$ and any $x, y, z \in X$, $x \succcurlyeq_F y \succcurlyeq_F z$ implies $x \succcurlyeq_F z$. An everyday example illustrates Case (1).

**Example 5** Let a non-welfare attribute be religion. Let $x, y, z$ be social states as follows.

$x$ :We are Christian with a piece of bread per day;

$y$ :We are Christian with no bread per day; and

$z$ :We are not religious with bread as much as we like per day.

Then it looks natural that $x \succcurlyeq_F y \succcurlyeq_F z$ implies $x \succcurlyeq_F z$. If we Christians like having one piece of bread better than no bread ($x \succcurlyeq_F y$) and if we like being a Christian with no bread better than being a rich with no religious faith ($y \succcurlyeq_F z$), then we like being a Christian with one piece of bread better than a rich with no religious faith ($x \succcurlyeq_F z$).

The formal meaning of Case (1) is as follows. Note that $\{x, z\}$ and $\{y, z\}$ have no difference in non-welfare attributes; $x \in X_\lambda$ and $z \in X_{\lambda'}$ whereas $y \in X_\lambda$ and $z \in X_{\lambda'}$. Thus if we have $x \prec_F z$ whereas $y \succcurlyeq_F z$, this implies that the welfare attributes of $y$ are more highly praised in social preference than that of $x$. But this is a contradiction because $x \succcurlyeq_F y$ was made only by welfare attributes. In this case we can ignore non-welfare attributes since $x$ and $y$ have no difference in non-welfare attributes. Therefore $x \succcurlyeq_F z$ should be made. Case (2) can be justified as well.

A slight modification of the everyday example in Case (1) illustrates Case (3).

**Example 6** Let $x, y, z$ be social states as follows.

$x$ :We are Christian with a piece of bread per day;

$y$ :We are not religious with bread as much as we like per day; and

$z$ :We are Christian with no bread per day.

If we like being a Christian with a piece of bread better than being a rich with no religious faith ($x \succcurlyeq_F y$) but we might as well discard the faith as starve to death ($y \succcurlyeq_F z$), then we Christian like having food better than no food ($x \succcurlyeq_F z$).

The formal meaning of Case (3) is explained as well as in Case (1). Note that $\{x, y\}$ and $\{y, z\}$ have no difference in non-welfare attributes; $x \in X_\lambda$ and $y \in X_{\lambda'}$ whereas $y \in X_{\lambda'}$ and $z \in X_\lambda$. Thus $x \succcurlyeq_F y \succcurlyeq_F z$ implies that the welfare attributes of $x$ are not less praised in social preference than that of $z$. Since there exists no difference in non-welfare attributes between $x$ and $z$, the social preference on $\{x, z\}$ should be made only by the welfare attributes so that we conclude $x \succcurlyeq_F z$.

**Example 7** Let $X = \{x, y, z\}$. The attribute sets are $\{x, y\}$ and $\{z\}$. The table below shows that (1)-(3) are independent each other.

| | (1) | (2) | (3) |
|---|---|---|---|
| $x \sim_F z \sim_F y \succ_F x$ | yes | yes | no |
| $z \sim_F x \sim_F y \succ_F z$ | yes | no | yes |
| $y \sim_F x \sim_F z \succ_F y$ | no | yes | yes |

As we noted before, transitivity of $\succcurlyeq_F$ does not always hold if three social states belong to different attribute sets. The example below illustrates this point.

**Example 8** There exist three non-welfare attributes, religion, health and sex. Let $x, y, z$ be social states as follows:

$x$ :We are Christian and smokers, and same-sex marriage is not legalized;

$y$ :We are Non-Christian and nonsmokers, and same-sex marriage is not legalized; and

$z$ :We are are Non-Christian and smokers, and same-sex marriage is legalized.

The attribute sets are $\{x\}, \{y\}, \{z\}$. In this example $x \succcurlyeq_F y \succcurlyeq_F z$ does not imply $x \succcurlyeq_F z$. Note that the social decision for any two social states are made by two non-welfare attributes; $x \succcurlyeq_F y$ is made by religion and health whereas $y \succcurlyeq_F z$ is made by health and sex. Similarly $x \succcurlyeq_F z$ has to be made by sex and religion. But no information needed for this decision is derived from $x \succcurlyeq_F y$ and $y \succcurlyeq_F z$.

A rule $F$ satisfies Independence (I) if for any $\succcurlyeq, \succcurlyeq' \in P(X)^n$ and any $x, y \in X$, if $\succcurlyeq_i \cap \{x, y\}^2 = \succcurlyeq'_i \cap \{x, y\}^2$ for all $i \in N$, then $\succcurlyeq_F \cap \{x, y\}^2 = \succcurlyeq'_F \cap \{x, y\}^2$. A rule $F$ satisfies Pareto (P) if for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$, if $x \succ_i y$ for all $i \in N$, then $x \succ_F y$. A rule $F$ satisfies Indifference Pareto (IP) if for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$, if $x \sim_i y$ for all $i \in N$ then $x \sim_F y$. A rule $F$ satisfies $\lambda$-Indifference Pareto ($\lambda$IP) if for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$, if $x \sim_i y$ for all $i \in N$ and $x, y \in X_\lambda$ for some $\lambda$ then $x \sim_F y$. A rule $F$ is the Pareto extension rule if and only if for all $\succcurlyeq \in P(X)^n$ and all $x, y \in X$, $x \succcurlyeq_F y \iff \neg(y \succ_i x \; \forall i \in N)$. A person $i$ is decisive for $(x, y)$ if for any $\succcurlyeq \in P(X)^n$, $x \succ_i y$ implies $x \succ_F y$. A person $i$ is dictator on $Y \subset X$ if he is decisive for any pair in $Y \times Y$. A person $i$ is dictator if he is dictator on

$X$[2]. Neutrality holds on $Y \subset X$ if for any $\succcurlyeq \in P(X)^n$ and any $x, y, z, w \in Y$, $\{i \in N : x \succcurlyeq_i y\} = \{i \in N : z \succcurlyeq_i w\}$ and $\{i \in N : x \preccurlyeq_i y\} = \{i \in N : z \preccurlyeq_i w\}$ imply $x \succcurlyeq_F y \Longleftrightarrow z \succcurlyeq_F w$. If Neutrality holds on $X$, we say simply a rule $F$ satisfies Neutrality (N). A rule $F$ satisfies $\not\lambda$-Neutrality ($\lambda$N) if for any $\succcurlyeq \in P(X)^n$ and any $x, y, z, w \in X$, $\{i \in N : x \succcurlyeq_i y\} = \{i \in N : z \succcurlyeq_i w\}$ and $\{i \in N : x \preccurlyeq_i y\} = \{i \in N : z \preccurlyeq_i w\}$ and $x, y \in X_\lambda$, $z, w \in X_{\lambda'}$ for some $\lambda$, $\lambda'$ ($\lambda = \lambda'$ is possible) imply $x \succcurlyeq_F y \Longleftrightarrow z \succcurlyeq_F w$.

For any $x \in X$, let $X(x)$ be the attribute set containing $x$. We say that a rule $F$ uses non-welfare attributes if either (i) there exist some $\succcurlyeq \in P(X)^n$ and some $x, y \in X$ such that $X(x) \neq X(y)$, $x \sim_i y$ for all $i$ and $x \nsim_F y$ or (ii) there exist some $\succcurlyeq \in P(X)^n$ and some $x, y, z, w \in X$ such that

(ii-a) $\{i \in N : x \succcurlyeq_i y\} = \{i \in N : z \succcurlyeq_i w\}$ and $\{i \in N : x \preccurlyeq_i y\} = \{i \in N : z \preccurlyeq_i w\}$;

(ii-b) $z \notin X(x) \cup X(y)$ or $w \notin X(x) \cup X(y)$ ; and

(ii-c) $x \succcurlyeq_F y \Longleftrightarrow z \succcurlyeq_F w$ does not hold.

Note that $x = z$ & $y = w$ never happens at (ii) because of (ii-b). Note also that if a rule uses non-welfare attributes it violates N. We say that a rule $F$ satisfies the Use of Non-Welfare Attributes (UNWA) if it uses non-welfare attributes. Note that if a rule satisfies UNWA then it violates N, but not vice versa. The Borda rule violates N but does not satisfy UNWA. In contrast, it looks natural to impose $\lambda$N on rules satisfying UNWA.

# 4 Results

**Theorem 1** *(1) Suppose that there exists some attribute set with at least two elements. Then if a rule $F$ satisfies CFR, I and P, there exists a person $i$ who is decisive for any pair $(x, y)$ except for all the pairs such that $\{x\} = X_\lambda$ and $\{y\} = X_{\lambda'}$.*

*(2) Suppose that there exists at most one attribute set that is singleton. Then if a rule $F$ satisfies CFR, I and P, there exists dictator.*

**Proof.** (1). Let $X_\lambda$ be the set with $x$ and $y$. Take $X_{\lambda'} (\lambda' \neq \lambda)$ and $z \in X_{\lambda'}$ arbitrarily. First we show that $i$ is dictator on $X_\lambda \cup X_{\lambda'}$. Thanks to CFR, $\succcurlyeq_F$ is complete and transitive on $\{x, y, z\}$ and hence Arrow's Theorem is applied. Thus there exists a dictator $i$ on $\{x, y, z\}$. This further implies that $i$ is dictator on $X_\lambda$ and decisive for any pairs in $(X_\lambda \times X_{\lambda'}) \cup (X_{\lambda'} \times X_\lambda)$. The only remaining thing to prove is that $i$ is dictator on $X_{\lambda'}$ if $X_{\lambda'}$ contains at least two elements. Let $z, w \in X_{\lambda'}$ and $z \succ_i w$. We can let $z \succ_i x \succ_i w$ and $x \in X_\lambda$. Since $i$ is decisive on $\{x, z\}$ and $\{x, w\}$, we have $z \succ_F x \succ_F w$. By CFR, we have $z \succ_F w$, the desired result. By noting that this holds for any $X_{\lambda'}$, this completes the proof of (1).

(2). (1) completes the proof. ■

[2] We can say that $i$ is dictator if he is decisive for all pairs in $X$.

**Theorem 2** *(1) If a rule satisfies CFR, I, and UNWA, then it violates either FR or IP, and if either (i) there exist only two attribute sets or (ii) there exists at most one attribute set that is singleton, the rule satisfies FR and violates IP. If (ii) holds, the rule is a dictatorial rule with a decision hierarchy. See Appendix for the definition.*

*(2) For any rule satisfying CFR and I, it satisfies $\lambda$N if and only if it satisfies $\lambda$IP.*

**Proof.** The first part of the statement in (1) and (2) follows from a well known fact that any rule satisfying FR, I and IP satisfies N (Sen1970). Noting that FR is reduced to CFR for two attribute sets case, we establish (i) of (1). See Appendix for (ii). If IP is satisfied, Theorem 3 in Appendix shows that the rule is a dictatorial rule with a decision hierarchy which obviously violates UNWA (and satisfies FR). ■

(ii) of Theorem 2 says that if there exists at most one attribute set that is singleton, there exists no rule satisfying CFR, I, P, IP and UNWA. For rules satisfying all the axioms of Theorems 1 and 2, there are eight cases that are logically possible. Table 1 lists the cases.

Table 1

| | (1) of Th. 1 | (2) of Th. 1 |
|---|---|---|
| only two attribute sets<br>FR is satisfied and IP is violated | Case 1 | Case 2 |
| three or more attribute sets<br>FR is violated and IP is satisfied | Case 3 | Case 4 |
| three or more attribute sets<br>FR is satisfied and IP is violated | Case 5 | Case 6 |
| three or more attribute sets<br>neither FR nor IP is satisfied | Case 7 | Case 8 |

Theorem 2 says that Case 4 is impossible. Note also that there exists dictator in Case 5, which therefore is impossible. See Appendix for more detailed argument on the remaining cases.

We show independence of the axioms. Each attribute set is indexed by $X_\tau$ ($\tau = 1, ..., t$). For any $x \in X$, let $\tau(x) \in \{1, ..., t\}$ be such that $x \in X_{\tau(x)}$.

**Example 9** (The simple majority rule weighted by non-welfare value) Let $N(x, y, \succcurlyeq) = \#\{i \in N : x \succ_i y\}$. Let a rule $F$ be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$x \succ_F y \Longleftrightarrow N(x, y, \succcurlyeq) > N(y, x, \succcurlyeq)$ or $[N(x, y, \succcurlyeq) = N(y, x, \succcurlyeq)$ and $\tau(x) > \tau(y)]$

$x \sim_F y \Longleftrightarrow N(x, y, \succcurlyeq) = N(y, x, \succcurlyeq)$ and $\tau(x) = \tau(y)$.

This rule has no dictator and satisfies all the axioms except for CFR.

**Example 10** (The Borda rule weighted by non-welfare value) Let $\beta(x, \succcurlyeq) = \sum_{i=1}^{n} \#\{y \in X : x \succcurlyeq_i y\}$. Let $k > 0$ be such that $n + k > kt$. Let a rule $F$ be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$$x \succcurlyeq_F y \Longleftrightarrow \beta(x, \succcurlyeq) + k\tau(x) \geq \beta(y, \succcurlyeq) + k\tau(y).$$

This rule has no dictator and satisfies all the axioms except for I. Note that P is assured by the condition $n + k > kt$.

**Example 11** (The non-welfare value first rule) Let a rule $F$ be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$x \succ_F y \Longleftrightarrow [\tau(x) > \tau(y)]$ or $[\tau(x) = \tau(y) \& x \succ_1 y]$

$x \sim_F y \Longleftrightarrow \tau(x) = \tau(y) \& x \sim_1 y$

This rule has no dictator and satisfies all the axioms except for P.

**Example 12** (The hierarchical dictatorial rule weighted by non-welfare value) Let each alternatives be indexed, $1, 2, ..., q$, where $\#X = q$. Let $\mu(x) \in \{1, 2, ..., q\}$ be the number of $x$. Let a rule $F$ be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$$x \succ_F y \Longleftrightarrow \begin{cases} \exists k \in \{1, ..., n\} \text{ s.t. } x \sim_i y \ \forall i \leq k-1 \ \& x \succ_k y \\ \text{or} \\ x \sim_i y \ \forall i \ \& \ \mu(x) > \mu(y). \end{cases}$$

This is a dictatorial rule where Person 1 is dictator, and satisfies all the axioms except for $\lambda$IP.

**Example 13** (The complete dictatorial rule) Let a rule $F$ be such that there exists some $i \in N$, called complete dictator, such that $x \succcurlyeq_F y \Longleftrightarrow x \succcurlyeq_i y$ for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$. This rule satisfies all the axioms except for UNWA.

# 5 Conclusion

# 6 Appendix

1. We obtain a refinement of the Arrow's impossibility theorem when we impose IP on rules.

Let $D$ be a nonempty subset of $P(X)^n$. We say that person $i$ is dictator, complete dictator, converse dictator, and complete converse dictator for $D$ if the followings holds respectively: For any $\succcurlyeq \in D$ and any $x, y \in X$, $x \succ_i y \Longrightarrow x \succ_F y$ (dictator); $x \succcurlyeq_i y \Longleftrightarrow x \succcurlyeq_F y$ (complete dictator); $x \succ_i y \Longrightarrow x \prec_F y$ (converse dictator); and $x \succcurlyeq_i y \Longleftrightarrow x \preccurlyeq_F y$ (complete converse dictator).

A rule $F$ is a dictatorial rule with a decision hierarchy if there exist persons $i_1, i_2, ..., i_{k-1}, i_k$ $(1 \leq k \leq n)$ such that $i_1$ is dictator, $i_2$ is dictator or converse dictator for $D_{i_1} = \{\succcurlyeq \in P(X)^n : x \sim_{i_1} y \text{ for all } x, y \in X\}$, $i_3$ is dictator or converse dictator for $D_{i_1 i_2} = \{\succcurlyeq \in P(X)^n : x \sim_{i_1} y \text{ and } x \sim_{i_2} y \text{ for all } x, y \in X\}$,..., $i_{k-1}$ is dictator or converse dictator for $D_{i_1 i_2 \cdots i_{k-2}} = \{\succcurlyeq \in P(X)^n : x \sim_{i_1} y, x \sim_{i_2} y, ..., x \sim_{i_{k-2}} y \text{ for all } x, y \in X\}$ and $i_k$ is complete dictator or complete converse dictator for $D_{i_1 i_2 \cdots i_{k-1}} = \{\succcurlyeq \in P(X)^n : x \sim_{i_1} y, x \sim_{i_2} y, ...,$ $x \sim_{i_{k-1}} y$ for all $x, y \in X\}$.

There is a vast variety of decision hierarchy. The shortest decision hierarchy consists of only one person $i_1$ who is complete dictator whereas all the persons take part in the longest decision hierarchy. Thanks to I, social preferences induced from a dictatorial rule with a decision hierarchy are lexicographic order; for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$x \succ_F y \Longleftrightarrow x \succ_{i_1} y$ or $[\exists k' \leq k$ s.t. $x \sim_{i_1} y, x \sim_{i_2} y, ..., x \sim_{i_{k'-1}} y$ & $(x \succ_{i_{k'}} y$ or $x \prec_{i_{k'}} y)]$.

$x \sim_F y \Longleftrightarrow x \sim_{i_1} y, x \sim_{i_2} y, ..., x \sim_{i_{k-1}} y$ and $x \sim_{i_k} y$.

**Theorem 3** *If a rule $F$ satisfies FR, I, P and IP, it is a dictatorial rule with a decision hierarchy.*

**Proof.** Let $i_1$ be dictator. Let $x, y \in X$ and $\succcurlyeq \in P(X)^n$ such that $x \sim_{i_1} y$ and $x \succ_i y$ for all $i \neq i_1$ be given. Then we have $x \succ_F y$, $x \prec_F y$ or $x \sim_F y$. Thanks to FR, I and IP, these hold for *all* $x, y \in X$. That is, for any $x, y \in X$ and any $\succcurlyeq \in P(X)^n$, if $x \sim_{i_1} y$ and $x \succ_i y$ for all $i \neq i_1$, then $x \succ_F y$ (Case1), $x \prec_F y$ (Case 2) or $x \sim_F y$ (Case 3).

Case 1: If $n \geq 3$, then letting $D_{i_1}$ be the new domain with the society of $n-1$ persons except for $i_1$, we can apply Arrow's impossibility theorem and show the existence of $i_2$ who is dictator for $D_{i_1}$. If $n = 2$, IP shows that the other person $i_2$ is complete dictator for $D_{i_1}$, which completes the proof.

Case 2: If $n \geq 3$, then letting $D_{i_1}$ be the new domain with the society of $n-1$ persons except for $i_1$, we can apply Arrow's impossibility theorem without Pareto (See Wilson (), Binmore (), and Fountain and Suzumura ()) and show the existence of $i_2$ who is converse dictator for $D_{i_1}$. If $n = 2$, IP shows that the other person $i_2$ is complete converse dictator for $D_{i_1}$, which completes the proof.

Case 3: We show that $i_1$ is complete dictator, which completes the proof. Take three alternatives $a, b, c$ and $\succcurlyeq \in P(X)^n$ such that $a \sim_{i_1} b \sim_{i_1} c$, and $a \succ_i b, a \succ_i c$ for all $i \neq i_1$. Case 3 implies $b \sim_F a \sim_F c$, which by FR further implies $b \sim_F c$. Noting that for any $i \neq i_1$, no preference between $b$ and $c$ is specified, we have the desired result.

If either Case 1 or Case 2 holds, the same proof is repeated by letting $D_{i_1 i_2}$ be the new domain with the society of $n-2$ persons except for $i_1$ and $i_2$. If either Case 1 or Case 2 hold again here, the proof is also repeated again by letting $D_{i_1 i_2 i_3}$ be the new domain with the society of $n-3$ persons except for $i_1$, $i_2$ and $i_3$. The proof completes when it finds person $i_k$ $(k \leq n)$ who is complete dictator. Note also that IP is applied when $k = n$. ■

Note that if a rule is a dictatorial rule with a decision hierarchy, it does not use non-welfare attributes. If we impose strong Pareto (SP), which says $\forall \succcurlyeq \in P(X)^n, \forall\, x, y \in X$, $[x \succcurlyeq_i y\ \forall i\ \&\ x \succ_i y\ \exists i \Longrightarrow x \succ_F y]$, then the decision hierarchy is uniquely characterized, i.e., someone is dictator and each of the rest plays as dictator at each stage in the hierarchy.

2. Cases 1, 2 and 6: Given $\succcurlyeq \in P(X)^n$, we define a lexicographic order $\succcurlyeq_L$ as follows.

For any $x, y \in X$, the asymmetric part of $\succcurlyeq_L$ is defined by

$$x \succ_L y \iff \begin{cases} \exists k \in \{1, ..., n\} \text{ s.t. } x \sim_i y \ \forall i \leq k-1 \ \& x \succ_k y \\ \text{or} \\ x \sim_i y \ \forall i \ \& \ \tau(x) > \tau(y). \end{cases}$$

The symmetric part is defined by $x \sim_L y \iff x \sim_i y \ \forall i \ \& \ \tau(x) = \tau(y)$.

Let a rule $F$ be such that $x \succcurlyeq_F y \iff x \succcurlyeq_L y$ for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$. Person 1 is dictator for $F$. This rule illustrates Cases 1, 2 and 6[3]. It is obvious that $F$ satisfies FR, I, P, $\lambda$IP and UNWA, and violates IP.

Case 3: Let a rule $F$ be such that for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$$x \succcurlyeq_F y \iff \begin{cases} x \succcurlyeq_1 y & \text{if } x, y \in X_\lambda \cup X_{\lambda'} \text{ with } \#X_\lambda \geq 2 \text{ or } \#X_{\lambda'} \geq 2 \\ \text{or} & \\ \neg\,(y \succ_i x \ \forall i \in N) & \text{otherwise.} \end{cases}$$

This rule illustrates Case 3. This rule satisfies CFR, I, P, UNWA and IP (and hence $\lambda$IP) , and violates FR. Person 1 is decisive in (1) of Theorem 1. Let $X = \{x, y, z, w\}$ where the attribute sets are $\{x, y\}$, $\{z\}$ and $\{w\}$. According the rule, 1 is complete dictator[4] on $\{x, y, z\}$ and $\{x, y, w\}$ and the Pareto extension rule govern on $\{z, w\}$. Non-welfare attributes are used since $x \succ_1 y$ implies $x \succ_F y$ whereas $z \succ_1 w$ does not always imply $z \succ_F w$. FR is also violated since $z \succ_F x \succ_F w$ does not always imply $z \succ_F w$.

Case 7: All the singleton attribute sets are indexed $X_1, ..., X_m$ $(2 \leq m)$[5]. Let a rule $F$ be such that for any $x, y \in X$ with $\#\,(X(x) \cup X(y)) \geq 3$, $x \succcurlyeq_F y \iff x \succcurlyeq_1 y$, and for any $x, y \in X$ with $\#\,(X(x) \cup X(y)) = 2$, i.e., $X(x)$ and $X(y)$ are singleton, $x \succ_F y \iff (x \succ_i y \ \forall i)$ or $[x \in X_p, y \in X_q, p > q$ and $\neg\,(y \succ_i x \ \forall i)]$.

Note that $x \sim_F y$ never happens if $X(x)$ and $X(y)$ are singleton. This rule illustrates Case 7, satisfying CFR, I, P, $\lambda$IP and UNWA, and violating FR and IP. Person 1 is decisive for any pair $(x, y)$ except for $\{x\} = X(x)$ and $\{y\} = X(y)$, but not dictator.

Case 8: Suppose that there exist at least three attribute sets. Let $A, B, C$ be such that $A = \{x : \tau(x) = 1\}$, $B = \{x : \tau(x) = 2\}$ and $C = \{x : \tau(x) \geq 3\}$. A binary relation $\geq_T$ is defined by its asymmetric parts $>_T$ and symmetric parts $=_T$ as follows:

$$\begin{aligned} x >_T y \quad &\iff \quad [x \in A \& y \in B] \vee [x \in B \& y \in C] \vee [x \in C \& y \in A] \\ x =_T y \quad &\iff \quad [x, y \in A] \vee [x, y \in B] \vee [x, y \in C] \end{aligned}$$

[3] Note that the only possible attribute sets for Case 1 are $X - \{x\}$ and $\{x\}$ for any $x \in X$. Thus Person 1 is dictator for Case 1.

[4] See Example 12 for the definition.

[5] If $m = 1$, the only possible attribute sets are $X - \{x\}$ and $\{x\}$ on which a dictator governs.

Note that $\geq_T$ is complete but not transitive; $>_T$ has cycles such that $x >_T y >_T z >_T x$ where $x \in A$, $y \in B$ and $z \in C$. Let $F$ be such that for any $\succsim \in P(X)^n$ and any $x, y \in X$,

$$\begin{array}{rcl} x \succ_F y & \Longleftrightarrow & x \succ_1 y \text{ or } [x \sim_1 y \text{ and } x >_T y]\,, \\ x \sim_F y & \Longleftrightarrow & [x \sim_1 y \text{ and } x =_T y]\,. \end{array}$$

This rule illustrates Case 8. This is a dictatorial rule satisfying CFR, I, P, $\lambda$IP and UNWA and violating FR and IP. Letting $x \in A$, $y \in B$, $z \in C$, $x \sim_1 y \sim_1 z$, we have $x \succ_F y \succ_F z \succ_F x$. This shows that $F$ violates FR and uses non-welfare attributes. It is easy to check that this rule satisfies CFR.

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