Risk and Ambiguity in Asset Returns
– Cross-Sectional Differences –

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Outline

Ambiguity in asset markets

- Explore implications of ambiguity and ambiguity aversion on portfolio choices and asset returns (prices).
- Motivated to explain some phenomena that cannot be explained by expected utility functions.
- Unlike those working on the equity premium puzzle, we do not aggregate stock returns in a single index such as S&P500.
- We concentrate on the composition of stocks in optimal portfolios. Cf. Chen and Epstein (2002) and Epstein and Miao (2003).

Our "stocks": FF6 portfolios

1. Sort out the stocks traded on NYSE, AMEX, and NASDAQ in terms of the market equity (market value, market capitalization) and the ratio of the book equity (book value) to the market equity.
2. Partition them into six groups, according to whether the ME belongs to the top or bottom 50%, and whether the BE/ME belongs to the top or bottom 30%, or neither.
3. Form the ME-weighted portfolio for each of the six groups:

	Bottom 50% of ME	Top 50% of ME
Bottom 30% of BE/ME	SL	BL
Middle 40% of BE/ME	SN	BN
Top 30% of BE/ME	SH	BH

Return on the FF6 portfolios

The means, variances, and covariances of the monthly returns in % of the FF6 portfolios, and the mean of the risk-free rates, from 1926 to 2014.

	Mean (%)	SL	SN	SH	BL	BN	BH
risk-free	0.28						
SL	0.98	57.36	50.77	55.76	34.61	35.62	43.74
SN	1.28	50.77	49.64	55.73	31.89	35.74	45.07
SH	1.48	55.76	55.73	67.64	34.64	41.20	53.89
BL	0.91	34.61	31.89	34.64	28.62	27.43	31.63
BN	0.97	35.62	35.74	41.20	27.43	32.89	38.32
BH	1.19	43.74	45.07	53.89	31.63	38.32	50.95

The Small and High portfolios have higher means and variances.

Mean-variance-efficient portfolio and market portfolio

The proportions of the total investment allocated to the FF6 portfolios.

	MVE portfolio	MKT portfolio
SL	−3.3641	0.0246
SN	3.4532	0.0295
SH	1.1213	0.0208
BL	1.8397	0.5074
BN	−1.0806	0.3120
BH	−0.9694	0.1057
Total	1.0000	1.0000

The MVE portfolio involves large long and short positions.

Introducing ambiguity to rationalize the market portfolio

- In the CARA-normal setting, the investor would hold a MVE portfolio.
- For what kind of utility functions is the MKT portfolio optimal?
- We use the ambiguity-averse utility functions of Klibanoff, Marinnacci, and Mukerji (2005).
- In particular, we extend the CARA-normal setting to the case where the expected asset returns are ambiguous but the covariance matrix is not, and the second-order belief of expected asset returns is also a multivariate normal distribution.

Old results of ours

- Identified "basis portfolios," which may constitute mutual funds.
- Proved that for every portfolio, there is an ambiguity-averse investor for whom the portfolio is optimal if and only if the expected rate of return of the portfolio exceeds the risk-free rate.
- For each such portfolio, identified a class of minimally ambiguity-averse investors for whom it is optimal.
- Proposed two notions of, and found, the least ambiguity-averse investor among them.

New results of ours

- Discuss why it is important to ask whether the observed choice is optimal for a reasonably ambiguity-averse investor.
- Use a criterion to decide whether the investor for whom the observed choice is optimal is reasonably ambiguity-averse, and argue that it is better than criteria that have been proposed in the literature.
- Investigate whether the representative investor is reasonably ambiguity-averse according to this criterion using the FF6 portfolios.

Outline

Ambiguity and ambiguity aversion

- Represent the returns of N assets by a random vector X.
- Denote the risk-free rate by R.
- Conditional on a random vector M, X has mean vector M: $X|M \sim \mathcal{N}(M, \Sigma_{X|M})$.
- Suppose that $M \sim \mathcal{N}(\mu_M, \Sigma_M)$. It is the second-order belief.
- An ambiguity-averse utility function $U_{\gamma,\theta}$ is defined by

$$U_{\gamma,\theta}\left(a^\top X + bR\right) = E\left[u_\gamma\left(u_\theta^{-1}\left(E\left[u_\theta\left(a^\top X + bR\right)|M\right]\right)\right)\right],$$

  where u_γ and u_θ have CARA γ and θ. If $\gamma > \theta$, then $U_{\gamma,\theta}$ is ambiguity-averse.

Optimal portfolio

- The utility function $U_{\gamma,\theta}$ can be rewritten as

$$u_\gamma^{-1}(U_{\gamma,\theta}(a^\top X + bR)) = \mu_M^\top a + Rb - \frac{\theta}{2} a^\top \Sigma_{X|M} a - \frac{\gamma}{2} a^\top \Sigma_M a.$$

Cf. Maccheroni, Marinacci, and Ruffino (2013)

- The first-order condition for an optimal portfolio is

$$\mu_M - R\mathbf{1} = (\theta \Sigma_{X|M} + \gamma \Sigma_M) a = \theta(\Sigma_X + \eta \Sigma_M) a$$

$$\text{thus, } a = \frac{1}{\theta}(\Sigma_X + \eta \Sigma_M)^{-1}(\mu_M - R\mathbf{1}), \tag{1}$$

where $\eta = \gamma/\theta - 1$ and $\Sigma_X = \Sigma_{X|M} + \Sigma_M$.

Role of ambiguity in asset composition

- The optimal portfolio a is a scalar multiple of the MVE portfolio $(\mathbf{1}^\top \Sigma_X^{-1}(\mu_M - R\mathbf{1}))^{-1}\Sigma_X^{-1}(\mu_M - R\mathbf{1})$ when $\eta\Sigma_M = 0$.
- It is so even when $\Sigma_M = \lambda\Sigma_X$ for some $\lambda \in [0,1]$. Indeed, then,

$$a = \frac{1}{\theta(1+\lambda\eta)}\Sigma_X^{-1}(\mu_M - R\mathbf{1}).$$

- It is so as long as $\Sigma_M a = \lambda\Sigma_X a$ for some $\lambda \in [0,1]$.
- The expected excess return is always strictly positive:

$$a^\top(\mu_M - R\mathbf{1}) = \frac{1}{\theta}(\mu_M - R\mathbf{1})^\top(\Sigma_X + \eta\Sigma_M)^{-1}(\mu_M - R\mathbf{1}) > 0.$$

The converse also holds

We take Σ_X as objective and observable, and Σ_M as subjective and unobservable; and so is the decomposition $\Sigma_X = \Sigma_{X|M} + \Sigma_M$.

Theorem 1. For every portfolio $a \in \boldsymbol{R}^N$, if $a^\top(\mu_M - R\mathbf{1}) > 0$, then there is a (Σ_M, η, θ) for which (1) holds.

- We have characterized the set of all such (Σ_M, η, θ)'s by finding:
  1. the supremum $\bar{\theta}$ of the coefficients of risk aversion equal to $a^\top(\mu_M - R\mathbf{1})/(a^\top \Sigma_X a)$; and
  2. for each $\theta \in (0, \bar{\theta})$, a unique $(\Sigma_M^\theta, \eta^\theta)$ that are smaller than any other (Σ_M, η) such that (Σ_M, η, θ) belongs to the set.
- With the data of FF6 portfolios,

$$\min_{\theta \in (0,\bar{\theta})} \eta^\theta = 9.31.$$

Cf. $\varphi(y) = (u_\gamma \circ u_\theta^{-1})(y) = -(-y)^{\eta+1}$.

Outline

Should we be fully content with these results?

- The condition of a strictly positive expected excess return seems weak.
- This implies that the predictive power of ambiguity-averse utility functions also seems weak.
- Thus, we should ask whether a given portfolio can be optimal for a "reasonably" ambiguity-averse investor.
- How can we determine whether an investor is reasonably ambiguity-averse?

Will introspection or experiments guide us?

- Expected utility is determined solely by the distribution of consumption levels, and the preference over these distributions, or lotteries, is assumed to "travel" with the subject across settings. Cf. Mehra and Prescott (1985), Kocherlakota (1996), Lucas (2003).
- However, ambiguity or ambiguity aversion may not travel with the subject from laboratories to asset markets.
- Ambiguity aversion has been found more compatible with experimental results than expected utility. Cf. Ellsberg (1961), Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010), Ahn, Choi, Gale, and Kariv (2014), Attanasi, Gollier, Montesano, and Pace (2014).
- However, different parameter values of ambiguity-averse utility functions of the same type have rarely been compared.

Issues specific to KMM utility functions

- KMM contend that a given economic situation determines ambiguity, and ambiguity aversion refers to the decision maker's sensitivity to it.
- However, Epstein (2010) assets that such separation is impossible.
- Collard, Mukerji, Sheppard, and Tallon (2015) asked with which value of risk aversion an ambiguity-neutral investor would have the same total uncertainty premium as the ambiguity-averse investor.
- However, the notion is not useful, because, in our case, it hinges on which CARA coefficients are deemed as "reasonable".
- Thimme and Völkert (2015) and Gallant, Jahan-Parvar, and Liu (2015) estimated ambiguity aversion coefficients.
- However, ambiguity structure is fixed and assumed to be represented by the risk-free rates, price-dividend ratios, expected consumption and dividend growth rates, etc.

Our criterion of reasonable parameter values

- Let a be the MKT portfolio and choose a rationalizing (Σ_M, η, θ).
- Then, we decompose the expected excess returns into two parts

$$\mu_M - R\mathbf{1} = \text{(Risk Part)} + \text{(Ambiguity Part)}$$

  Cf. Chen and Epstein (2002), Ui (2011), and Thimme and Völkert (2015).
- We (wish to) find a "minimal" ambiguity part by varying (Σ_M, η, θ).

Why should we use this criterion?

- It depends only on the data of asset markets.
- It is valid even when ambiguity and ambiguity aversion cannot be separated.
- It is consistent with an equilibrium comparative statics for a model with an ambiguity-averse representative investor.
- It admits a beta representation along the lines of the arbitrage pricing theory of Ross and the multi-factor model of Fama.

Outline

Basis portfolios

How does the optimal portfolio vary as η increases while Σ_M is fixed?

Theorem 2. There are a positive integer K, K distinct elements $\lambda_1, \lambda_2, \ldots, \lambda_K$ of $[0, 1]$, and K portfolios $v_1, v_2, \ldots, v_K$ such that:

1. $\Sigma_M v_k = \lambda_k \Sigma_X v_k$ for every k;
2. $v_k^\top \Sigma_X v_\ell = 0$ whenever $k \neq \ell$ (the returns are independent); and
3. for every (θ, η), the optimal portfolio for the investor with the coefficients θ and η of risk and ambiguity aversion coincides with

$$\frac{1}{\theta} \sum_{k=1}^{K} \frac{1}{1 + \lambda_k \eta} v_k. \tag{2}$$

Risk-ambiguity decomposition of expected excess returns

If $\eta = 0$, then the investor has a CARA expected utility function and his optimal portfolio coincides with

$$\frac{1}{\theta}\Sigma_X^{-1}(\mu_M - R\mathbf{1}).$$

Thus

$$\frac{1}{\theta}\sum_{k=1}^{K} v_k = \frac{1}{\theta}\Sigma_X^{-1}(\mu_M - R\mathbf{1}), \text{ that is, } \mu_M - R\mathbf{1} = \sum_{k=1}^{K}\Sigma_X v_k.$$

We decompose the expected excess returns into

$$\sum_{k=1}^{K}\frac{1-\lambda_k}{1+\lambda_k\eta}\Sigma_X v_k + \sum_{k=1}^{K}\frac{\lambda_k+\lambda_k\eta}{1+\lambda_k\eta}\Sigma_X v_k. \tag{3}$$

"Equilibrium" interpretation of the decomposition

- The first term of (3) is the expected excess return that would induce the investor to hold (2) if the ambiguity were completely removed and the covariance matrix of asset returns were $\Sigma_X - \Sigma_M$.
- The second term of (3) is the expected excess return that would induce the investor to hold (2) if the pure risk were completely removed and the covariance matrix of asset returns were Σ_M.

This decomposition depends on (Σ_M, η, θ). Among all the (Σ_M, η, θ)'s with which the market portfolio a is optimal, we wish to know the one that "minimizes" the second term.

Notion of the minimal ambiguity part

Definition. The ambiguity part is minimal if its norm with respect to Σ_X^{-1},

$$\left(\left(\sum_{k=1}^{K} \frac{\lambda_k + \lambda_k \eta}{1 + \lambda_k \eta} \Sigma_X v_k\right) \Sigma_X^{-1} \left(\sum_{k=1}^{K} \frac{\lambda_k + \lambda_k \eta}{1 + \lambda_k \eta} \Sigma_X v_k\right)\right)^{1/2}$$

$$= \left(\sum_{k=1}^{K} \left(\frac{\lambda_k + \lambda_k \eta}{1 + \lambda_k \eta}\right)^2 v_k^\top \Sigma_X v_k\right)^{1/2},$$

is minimized over all (Σ_M, η, θ) with which the market portfolio is optimal.

The use of the norm with respect to Σ_X^{-1} seems justifiable because it

- coincides with the standard deviation of the underlying portfolio; and
- weights the N coordinates in inverse proportion to the variances of their returns, in line with GMM of Hansen.

Our approach

- Instead of minimizing the ambiguity part over all (Σ_M, η, θ)'s with which the market portfolio a is optimal, we minimize it only over all $(\Sigma_M^\theta, \eta^\theta, \theta)$'s, defined after Theorem 1.
- For $(\Sigma_M^\theta, \eta^\theta)$, Theorem 2 holds with $K = 2$, $\lambda_1 = 0$, and $\lambda_2 = 1$. Moreover, (2) can be rewritten as

$$v_1^\theta + \frac{1}{1+\eta^\theta} v_2^\theta,$$

  and, thus, the risk-ambiguity decomposition of asset returns is

$$\mu_M - R\mathbf{1} = \Sigma_X v_1^\theta + \Sigma_X v_2^\theta$$

- Thus, our minimization problem is

$$\inf_{\theta \in (0, \bar{\theta})} \left((v_2^\theta)^\top \Sigma_X v_2^\theta \right)^{1/2}.$$

Solution of our minimization problem

Theorem 3. $\left((v_2^\theta)^\top \Sigma_X v_2^\theta\right)^{1/2}$ is a strictly decreasing function of θ. Moreover, $\Sigma_X v_1^{\bar{\theta}} = \bar{\theta}\Sigma_X a$.

- The minimization problem is "solved" at $\theta = \bar{\theta}$. Moreover, since $a^\top \Sigma_X v_1^{\bar{\theta}} = a^\top(\mu_M - R\mathbf{1})$, the expected excess return of the market portfolio a can be explained completely by the risk part.
- It can be shown that

$$\left((v_2^\theta)^\top \Sigma_X v_2^\theta\right)^{1/2} = \underbrace{\text{Sharpe ratio}}_{\frac{\text{mean}}{\text{standard deviation}}} \text{ of } \left(\frac{1}{\theta}\Sigma_X^{-1}(\mu_M - R\mathbf{1}) - a\right)$$

Numerical result based on FF6 portfolios

When the ambiguity part is minimized, the risk-ambiguity decomposition of returns are as follows:

	μ_M	$\mu_M - R\mathbf{1}$	risk part	ambiguity part
SL	0.98	0.69	0.84	−0.15
SN	1.28	0.98	0.81	0.17
SH	1.48	1.19	0.92	0.27
BL	0.91	0.63	0.65	−0.02
BN	0.97	0.69	0.70	−0.01
BH	1.19	0.92	0.83	0.09
MKT	0.98	0.70	0.70	0.00

The returns of the Small and High portfolios are more ambiguous.
Cf. Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010).

Another numerical result

When the coefficient of ambiguity aversion is minimized ($\eta^\theta = 9.31$), the risk-ambiguity decomposition of returns are as follows:

	μ_M	$\mu_M - R\mathbf{1}$	risk part	ambiguity part
SL	0.98	0.69	0.44	0.25
SN	1.28	0.98	0.39	0.60
SH	1.48	1.19	0.43	0.77
BL	0.91	0.63	0.33	0.30
BN	0.97	0.69	0.35	0.34
BH	1.19	0.92	0.41	0.51
MKT	0.98	0.70	0.35	0.35

The High portfolios are more ambiguous, but the Small ones are not.

Issues on our approach

- As $\theta \to \bar{\theta}$, $\eta^\theta \to \infty$.
  Thus, minimizing the ambiguity part of asset returns and minimizing the coefficient of ambiguity aversion are very different.
- Yet, our approach is a hybrid of the two because we concentrate on the $(\Sigma_M^\theta, \eta^\theta, \theta)$'s.
- Collard, Mukerji, Sheppard, and Tallon (2015) found an ambiguity-neutral investor who has the same certainty equivalents as the calibrated investor to assess whether the latter is reasonably ambiguity-averse by using the former's risk aversion.
- In our model, the ambiguity-neutral investor's CARA is equal to $\bar{\theta}$ for all rationalizing (Σ, η, θ)'s, but whether $\bar{\theta}$ is reasonable is unknown.

Outline

Conclusion

- Extended the CARA-Normal setup to accommodate ambiguity.
- Established a necessary and sufficient condition for a given portfolio to be optimal for some ambiguity-averse investor.
- Discussed some criteria with respect to which the investor is "reasonably" ambiguity-averse.
- Assessed to what extent the representative investor is ambiguity-averse based on the U.S. equity market data.

- Should spell out pros and cons of various criteria in view of "portability" and applications.
- Should separate the issue of ambiguity distribution across different asset classes from that of reasonable ambiguity aversion.