Stable Coalition Structures in Symmetric Majority Games:

A Coincidence between Myopia and Farsightedness

Takaaki Abe

Waseda University, Japan

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Outline

Definitions

- **Terminology**
- Motivating questions
- Power index
- Stability concepts

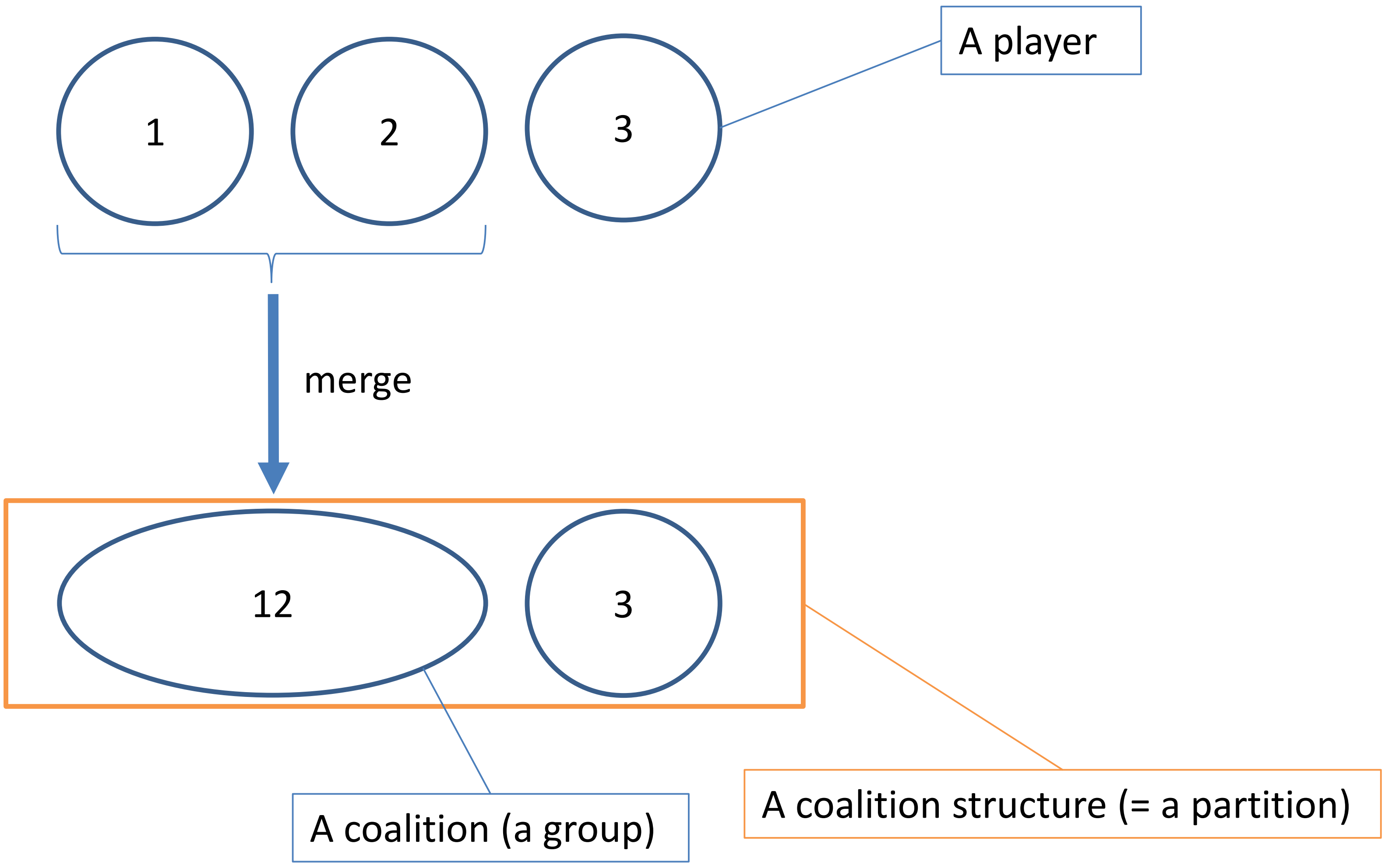
Results

- Myopic stability
- Farsighted stability
- Individual stability

Conclusion

- Summery
- Future work

Terminology



Motivating Question

In a majority game…

Symmetric majority games

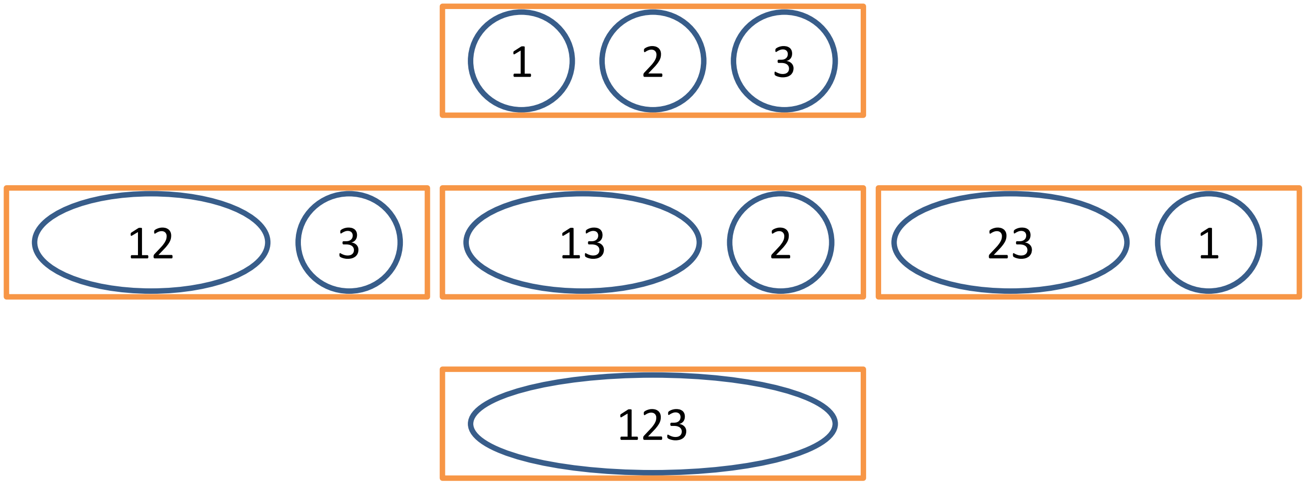
Let N be the player set.
For any coalition $S \subseteq N$,

$$v(S) = \begin{cases} 1 & if\ |S| \geq k \\ 0 & otherwise, \end{cases}$$

where k is the number of exact majority (the minimal number for a coalition to win).

Motivating Question

Players deviate from one coalition structure to another to get more **“power”**.



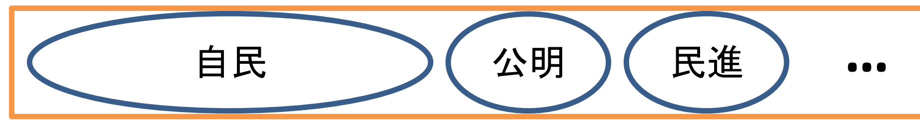
Which coalition structure is “stable” ?

Motivating Question

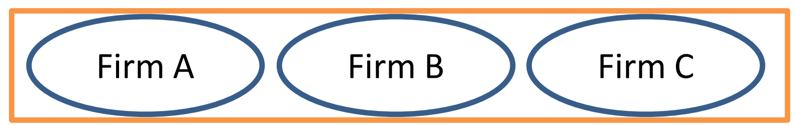
<u>Which coalition structure is **“stable”** ?</u>

This question might be applied to

- Structure of political parties



- Partition of stakeholders



- Social isolation in a classroom/office

Motivating Question

Question: Which coalition structure is "stable" ?

Preceding works:

Hart and Kurz (1984)

- They apply the α-, β-, γ-, and δ- stability concepts.
- Their analysis is only offered for $n \leq 10$.

Bloch (1996) also offers his analysis for $n \leq 10$.

No general answer!

The present paper answers this question.

Power index / **Stability**

Introduction

- <u>Power index</u>
- Stability

Power Index

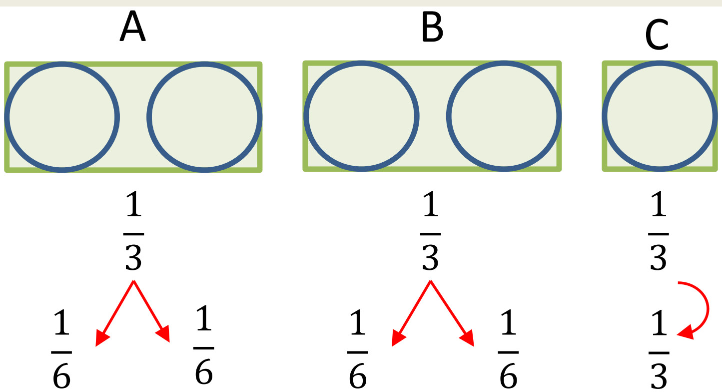
- We extend **the Shapley-Shubik power index**.

> What's that?
> The Shapley-Shubik power index is an index to measure the powers of players in a voting situation.

- For a given partition $\mathcal{P}$, we assign "power" ϕ_i to each player $i \in N$.

ϕ_1 ϕ_2 ϕ_3

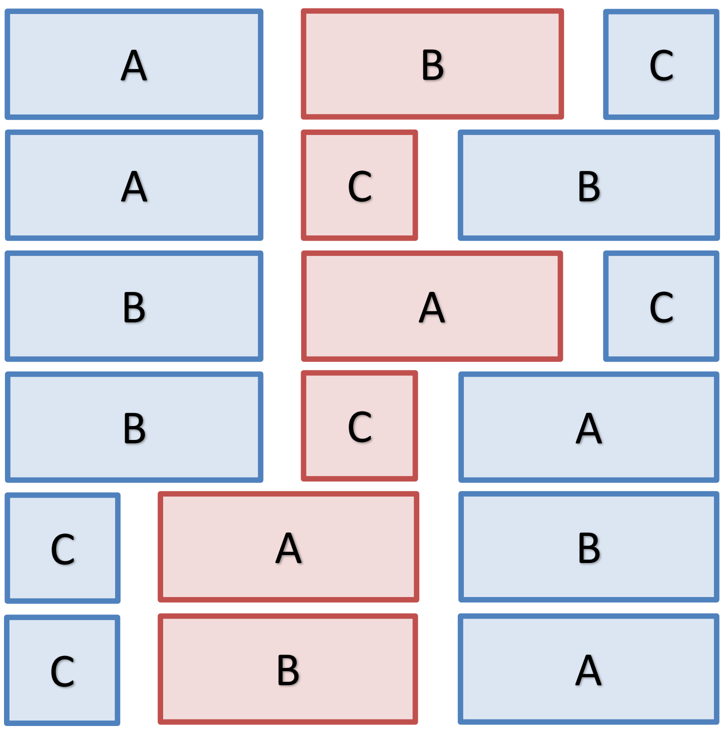
Power Index



Each coalition's power

Each player's power

= the Owen power index



A pivot coalition

Let k be the number of exact majority.
Let ρ be an order of coalitions in $\mathcal{P}$.

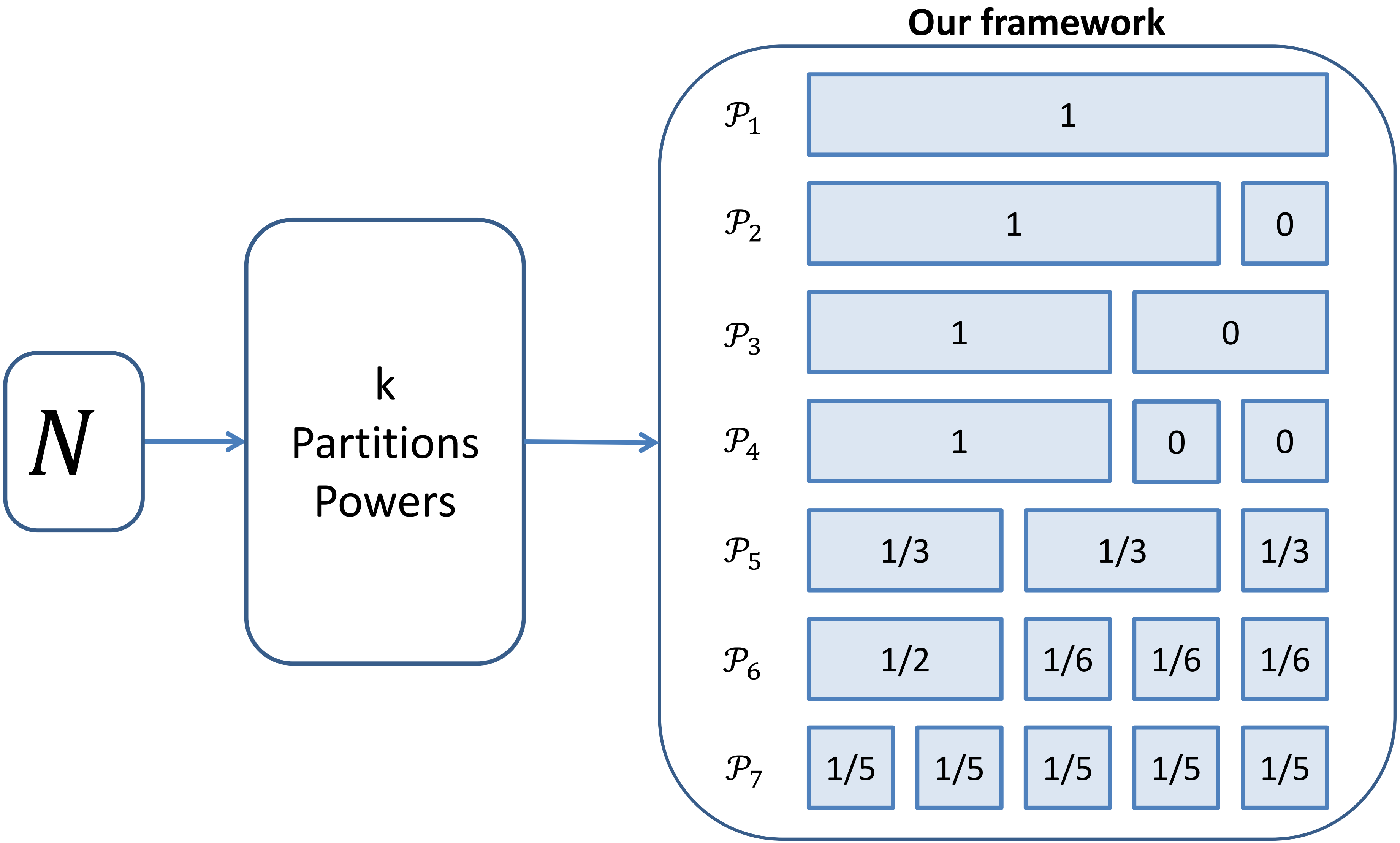
A coalition $S \in \mathcal{P}$ is ***a pivot coalition*** in ρ if the coalition S contains the kth player in ρ.

Power Index

$$\phi(\mathcal{P}) = (\phi_1, \dots, \phi_5)$$

$\mathcal{P}_1$	1	$\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$
$\mathcal{P}_2$	1 | 0	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0$
$\mathcal{P}_3$	1 | 0	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0$
$\mathcal{P}_4$	1 | 0 | 0	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0$
$\mathcal{P}_5$	1/3 | 1/3 | 1/3	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}$
$\mathcal{P}_6$	1/2 | 1/6 | 1/6 | 1/6	$\frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$
$\mathcal{P}_7$	1/5 | 1/5 | 1/5 | 1/5 | 1/5	$\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$

Power Index


Our framework
N
k
Partitions
Powers
$\mathcal{P}_1$
1
$\mathcal{P}_2$
1
0
$\mathcal{P}_3$
1
0
$\mathcal{P}_4$
1
0
0
$\mathcal{P}_5$
1/3
1/3
1/3
$\mathcal{P}_6$
1/2
1/6
1/6
1/6
$\mathcal{P}_7$
1/5
1/5
1/5
1/5
1/5

Power Index

1. For any partition $\mathcal{P}$, let $\phi_i(\mathcal{P})$ denote player i's power in $\mathcal{P}$.

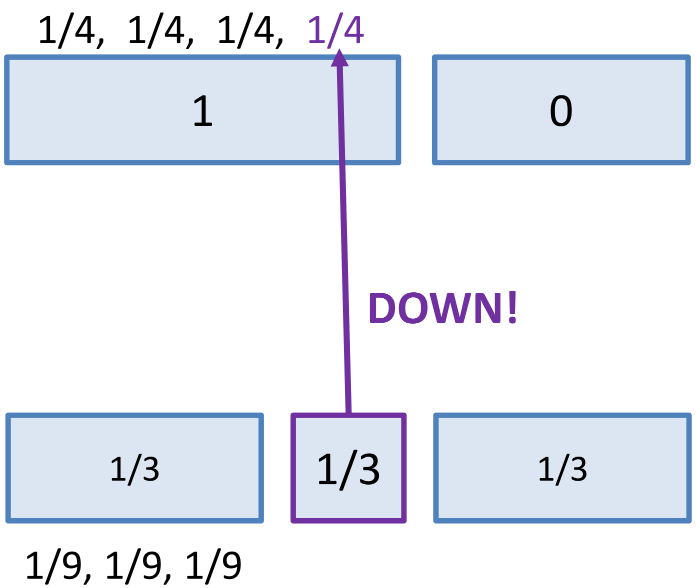
2. Players deviate from one partition to another to get more power.

3. Let $\succsim_i$ be player i's preferences based on $\phi_i(\mathcal{P})$:

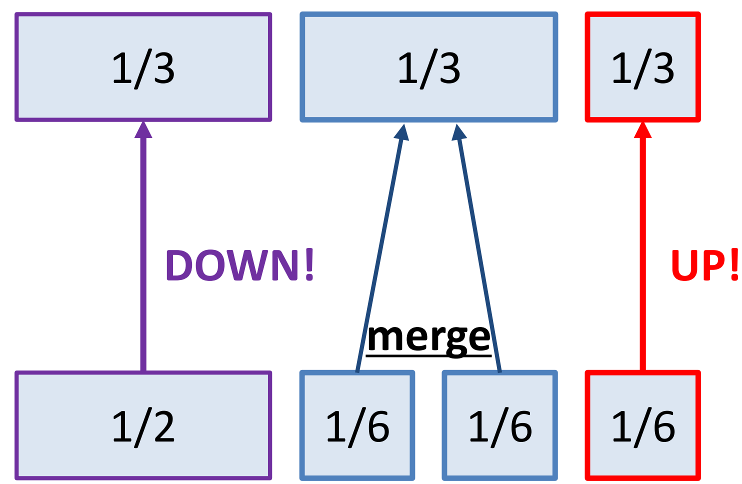
$$\mathcal{P} \succsim_i \mathcal{P}' \Leftrightarrow \phi_i(\mathcal{P}) \geq \phi_i(\mathcal{P}').$$

What's difficult?

Internal effect	External effect
Not "monotonic"!	Not "negative externalities"!



(seven players)



(five players)

Stability Concepts

- Power index
- <u>Stability</u>

Stability Concepts

Myopic Stability

Projective core
Optimistic core
Pessimistic core

Farsighted Stability

Farsighted vNM stable set

From now on

Individual Stability

Nash stability
The individual stability

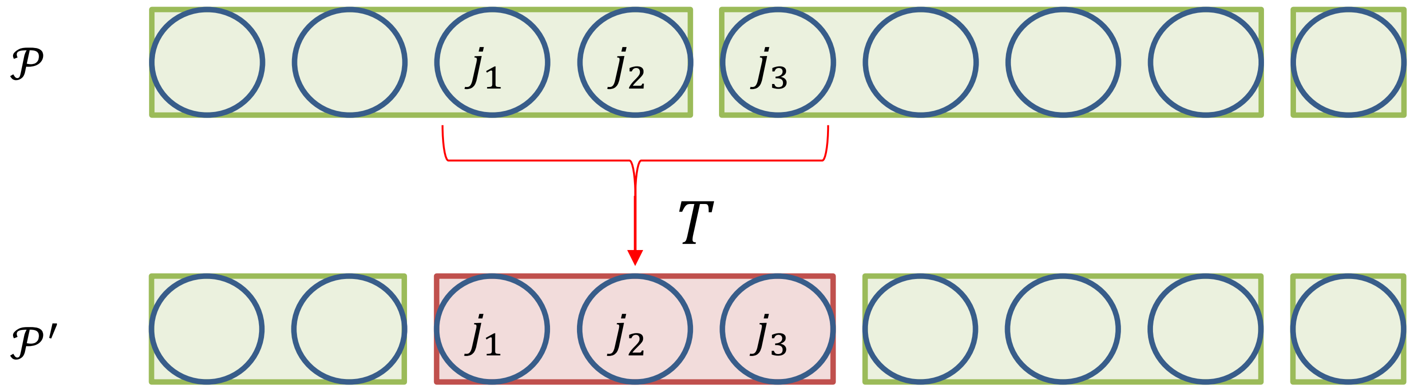
Later!

Myopic Stability Concepts

The projective core

A partition $\mathcal{P}$ is in the *projective core* if there is no T such that for all $j \in T$,

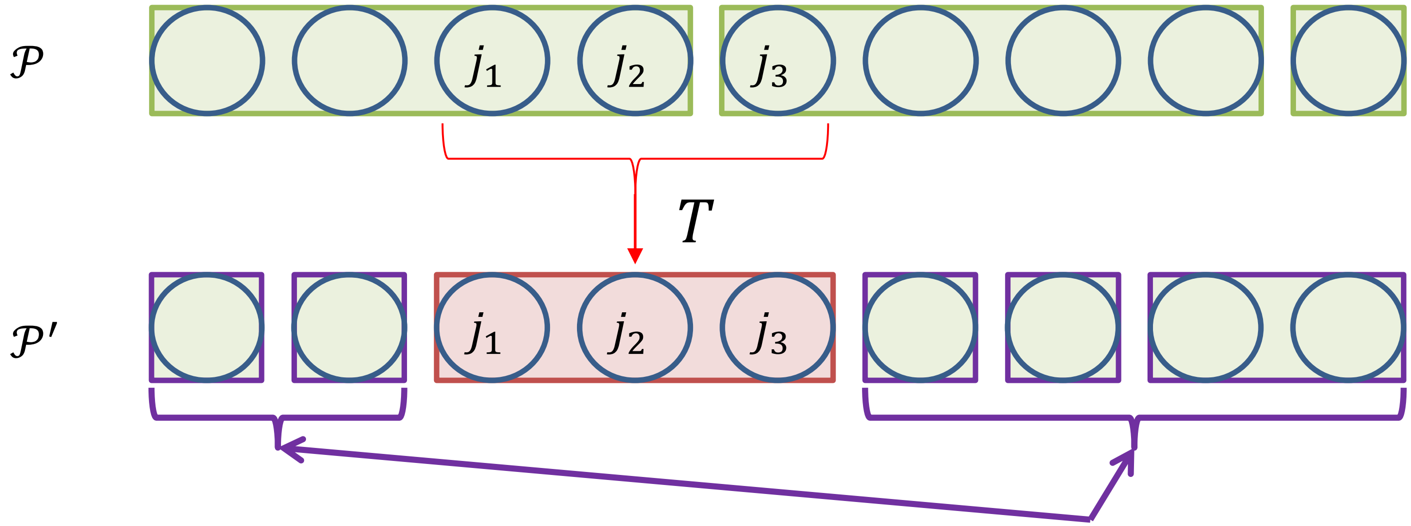
$\mathcal{P}' \succ_j \mathcal{P}$ where $\mathcal{P}' = T \cup \left(\mathcal{P}|_{N \setminus T}\right)$.



Myopic Stability Concepts

The pessimistic core

$$C^{\mathrm{pes}} = \{\mathcal{P}' \in \Pi(N) |\ \nexists S \subseteq N \text{ s.t. } [\forall \mathcal{P} \text{ with } S \in \mathcal{P} \text{ and } \forall i \in S, \mathcal{P} \succ_i \mathcal{P}']\}$$

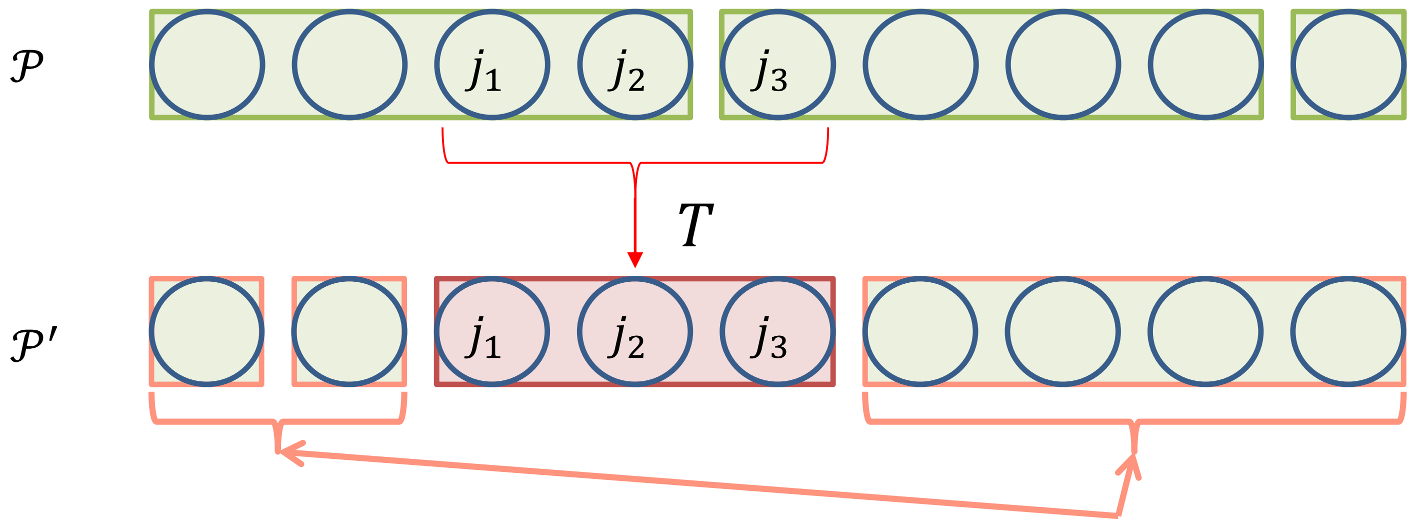


The *worst* reaction (coalition structure) for T

Myopic Stability Concepts

The optimistic core

$$C^{\mathrm{opt}} = \{\mathcal{P}' \in \Pi(N) \mid \nexists S \subseteq N \text{ s.t. } \exists \mathcal{P} \text{ s.t. } [S \in \mathcal{P} \text{ and } \forall i \in S, \mathcal{P} \succ_i \mathcal{P}']\}$$



The *best* reaction (coalition structure) for T

Myopic Stability Concepts

Remark

$$C^{\mathrm{opt}} \subseteq C^{\mathrm{pro}} \subseteq C^{\mathrm{pes}}$$

Farsighted Stability Concepts

$\mathcal{P}'$ indirectly dominates $\mathcal{P}$

$\exists \mathcal{P}^1, \dots, \mathcal{P}^k$ with $\mathcal{P}^1 = \mathcal{P}$ and $\mathcal{P}^k = \mathcal{P}'$ and $\exists S^1, \dots, S^{k-1}$ such that

for every $j = 1, \dots, k-1$,

i. $\mathcal{P}^{j+1} = \{S^j\} \cup \left(\mathcal{P}^j|_{N \setminus S^j}\right)$

ii. $\mathcal{P}' \succ_{S^j} \mathcal{P}^j$

Notation
$\mathcal{P} \succ_S \mathcal{P}'$ if and only if $\forall i \in S, \mathcal{P} \succ_i \mathcal{P}'$

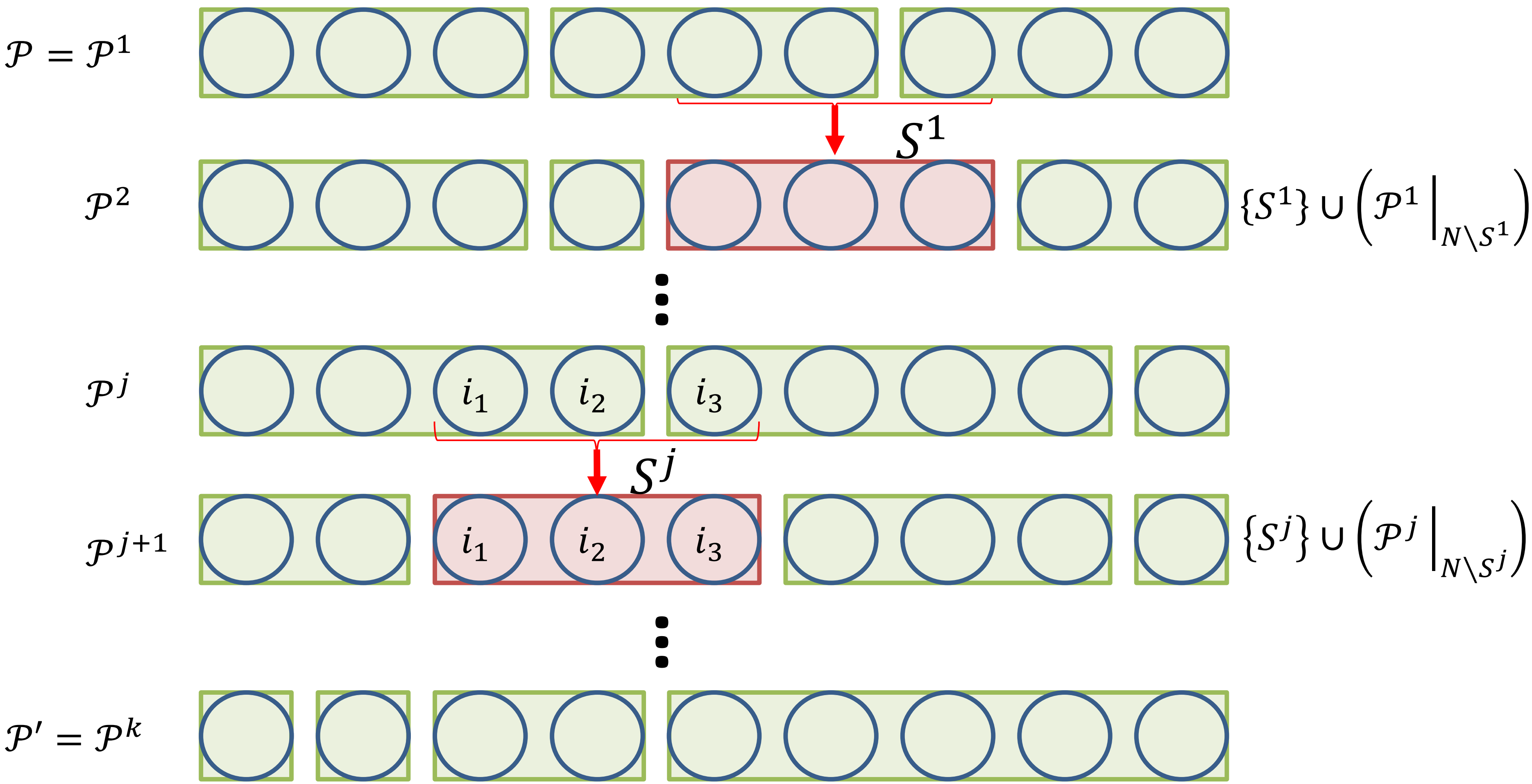
The vNM stable set

The vNM stable set V is the set of partitions satisfying

1. $\forall \mathcal{P}, \mathcal{P}' \in V$, $\mathcal{P}'$ does not indirectly dominates $\mathcal{P}$. (Internal Stability)

2. $\forall \mathcal{P} \in \Pi(N) \setminus V$, $\exists \mathcal{P}' \in V$ such that $\mathcal{P}'$ indirectly dominates $\mathcal{P}$. (External Stability)

$\mathcal{P}'$ indirectly dominates $\mathcal{P}$



Results

<u>Results</u>

Symmetric majority games

Notation

- Let k be the size of the exact majority.
- Let K be a coalition whose size is k.
- Assume $|N| \geq 3$.
- $[T]$ is the partition of coalition T into singletons.

Symmetric majority games

Proposition 1

$$V = C^{\mathrm{pes}}$$

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7					+	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0						+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0						+
1234|567	1/4	1/4	1/4	1/4	0	0	0		+		+		
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3						
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6						
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6						
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10						
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15						
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15						
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7						

Symmetric majority games

Proposition 2

$$\mathcal{P} \in C^{\text{pes}} \Leftrightarrow K \in \mathcal{P}$$

Why does it result in V?

Internal stability in the seven-player symmetric majority game

Internal stability = Any partition does not indirectly dominate another.

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0

These players get 1/4 >>> No incentive to deviate to another.

Why does it result in V?

Internal stability in the seven-player symmetric majority game

Internal stability = Any partition does not indirectly dominate another.

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0

They get zero and have an incentive to deviate!
However, they cannot increase their powers as long as 1,2,3,4 form their winning coalition.

Why does it result in V?

Internal stability in the seven-player symmetric majority game

Internal stability = Any partition does not indirectly dominate another.

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0

Mixture

Either 1,2,3, or 4 does not agree,
because he already gets 1/4.

External stability in the seven-player symmetric majority game

External stability = For any partition outside V, some partition in V indirectly dominates it.

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
V: 1234|567	1/4	1/4	1/4	1/4	0	0	0
V: 1234|56|7	1/4	1/4	1/4	1/4	0	0	0
V: 1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

External stability in the seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

External stability in the seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

External stability in the seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

The pessimistic core in the seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

Less than 1/4.

No incentive to deviate (even in the pessimistic view).

The pessimistic core in the seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

Less than 1/4.

No incentive to deviate (even in the pessimistic view).

Greater than 1/4!

Someone may want to deviate.

The pessimistic core in the seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12345|67	1/5	1/5	1/5	1/5	1/5	0	0
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0
1234|567	1/4	1/4	1/4	1/4	0	0	0
1234|56|7	1/4	1/4	1/4	1/4	0	0	0
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7

Less than 1/4.

No incentive to deviate (even in the pessimistic view).

Greater than 1/4!

Someone may want to deviate.

But,..

In the pessimistic view, <u>he anticipates zero</u>, because zero is the minimum for one person coalition.

Symmetric majority games

To sum up,

- Proposition 1 shows that they coincide with each other.
- Proposition 2 specifies the set of partitions they coincide with.
- Moreover, Proposition 2
  - characterizes C^{pes} (and V) with exact majority coalitions.
  - guarantees the nonemptiness of C^{pes} (and V).

Symmetric majority games

For your reference,
$C^{\mathrm{opt}} \subseteq C^{\mathrm{pro}} \subseteq C^{\mathrm{pes}}$.

Proposition 3

Partition $\{K\} \cup [N \setminus K]$ is always in the projective core.

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7					+	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0						+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0						+
1234|567	1/4	1/4	1/4	1/4	0	0	0		+		+		
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3						
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6						
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6						
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10						
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15						
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15						
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7						

Symmetric majority games

<u>Remark 1</u>

For $n = 3,4,5,6,8,10,$ $\qquad C^{\mathrm{pro}} = C^{\mathrm{pes}}.$

For $n = 7,9,11,\ldots,$ $\qquad C^{\mathrm{pro}} \subsetneq C^{\mathrm{pes}}.$

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7					+	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0						+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0						+
1234|567	1/4	1/4	1/4	1/4	0	0	0		+		+		
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3						
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6						
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6						
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10						
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15						
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15						
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7						

The seven-player symmetric majority game

Not in C^{pro}!

Because player 4 deviates to form one person coalition.

Partition 123|4|567 forms, and 4 gets 1/3 (>1/4).

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7	
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0	
12345|67	1/5	1/5	1/5	1/5	1/5	0	0	
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0	
1234|567	1/4	1/4	1/4	1/4	0	0	0	
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3	
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6	
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6	
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10	
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0	
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15	
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15	
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	

Symmetric majority games

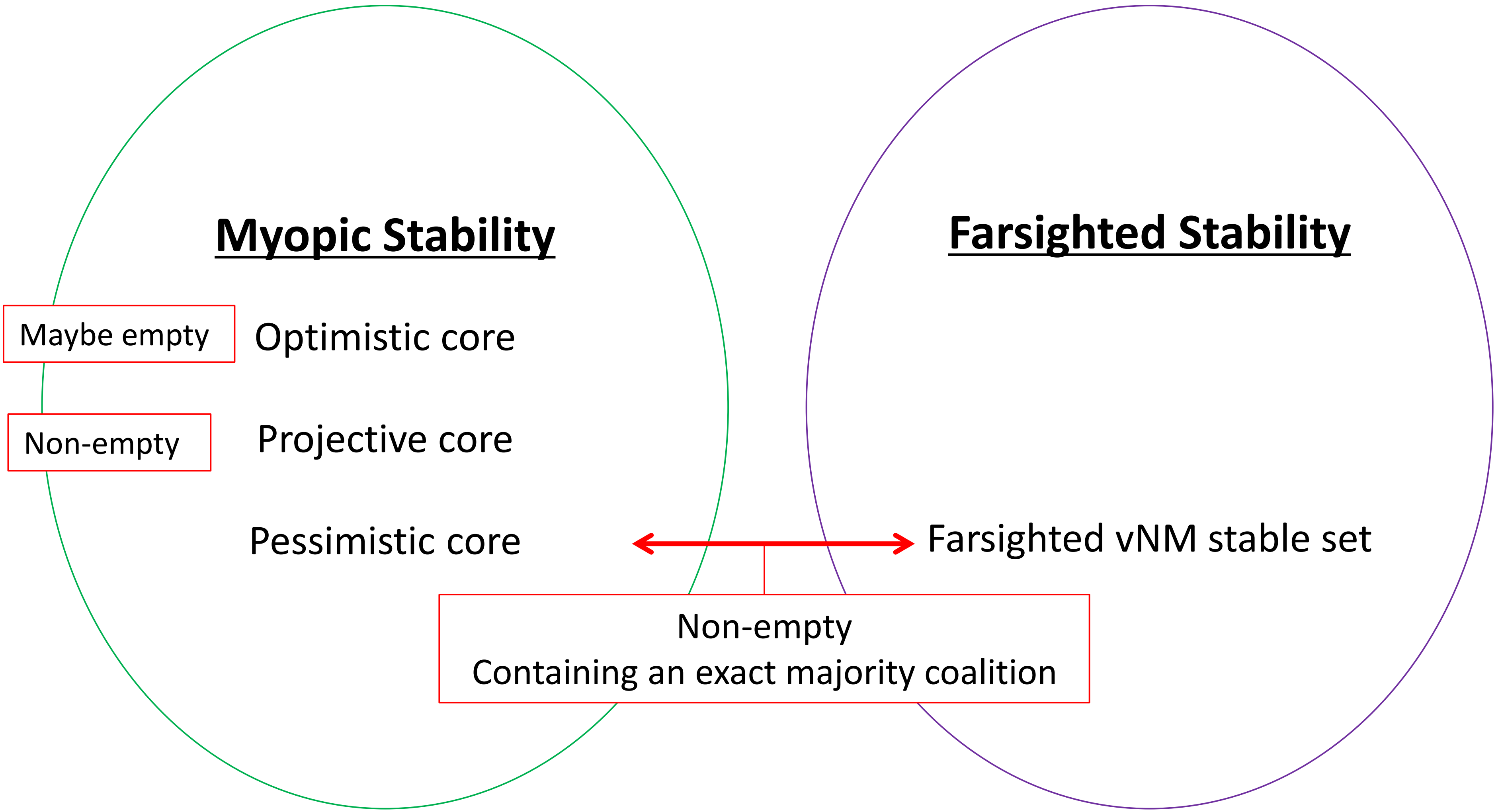
Proposition 4

For any symmetric majority game with $n \geq 5$, the optimistic core is empty. For $n = 3,4$, $C^{\mathrm{opt}} = C^{\mathrm{pro}} = C^{\mathrm{pes}}$.

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7					+	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0						+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0						+
1234|567	1/4	1/4	1/4	1/4	0	0	0		+		+		
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3						
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6						
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6						
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10						
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15						
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15						
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7						

Summary


Myopic Stability
Farsighted Stability
Maybe empty
Optimistic core
Non-empty
Projective core
Pessimistic core
Farsighted vNM stable set
Non-empty
Containing an exact majority coalition

Stability Concepts

Myopic Stability

Optimistic core
Projective core
Pessimistic core

Farsighted Stability

Farsighted vNM stable set

Individual Stability

Nash stability
The individual stability

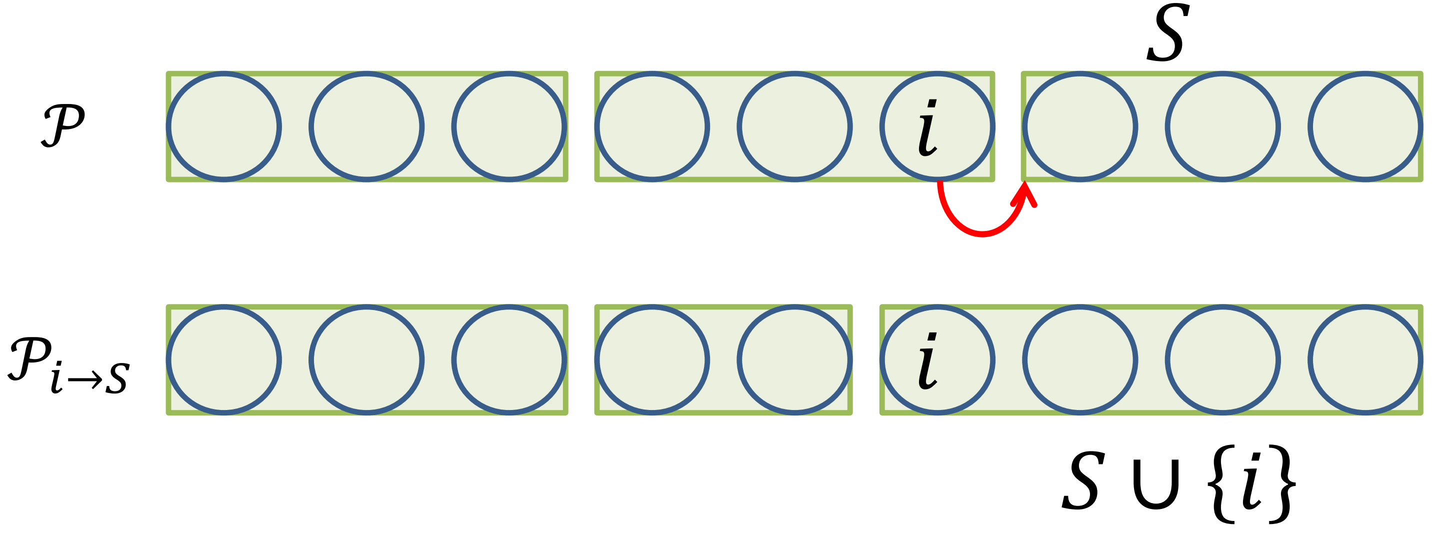
From now on!

Individual Stability Concepts

<u>Nash stability</u>

A partition $\mathcal{P}$ is *Nash stable* if there is no i and $S \in \mathcal{P} \cup \{\emptyset\}$ such that

$$\mathcal{P}_{i \to S} \succ_i \mathcal{P}.$$



Individual Stability Concepts

Individual stability

A partition $\mathcal{P}$ is *individually stable* if there is no i and $S \in \mathcal{P} \cup \{\emptyset\}$ such that

$$\mathcal{P}_{i \to S} \succ_i \mathcal{P}$$

and for every $j \in S$, $\mathcal{P}_{i \to S} \succsim_j \mathcal{P}$.

Remark

Nash ⊆ IS

Symmetric majority games

<u>Proposition 5</u>

Partition $\{N\}$ is always Nash stable.

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7					+	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0						+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0						+
1234|567	1/4	1/4	1/4	1/4	0	0	0		+		+		
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3						
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6						
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6						
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10						
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15						
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15						
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7						

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	$Nash$
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0	

If a player (say 7) deviates from N, he gets zero.

<u>Note</u>
Partition $\{N\}$ is **not the only partition** to be Nash stable:
{{1,2,3}, {4,5,6}, {7,8,9}} is Nash stable.

Symmetric majority games

Proposition sym-7 (IS)

Partition $\{K\} \cup [N \setminus K]$ is always individually stable.

Moreover, any partition which contains a winning coalition whose size is strictly greater than k is individually stable.

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7					+	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0						+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0						+
1234|567	1/4	1/4	1/4	1/4	0	0	0		+		+		
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+	+		+		+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3						
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6						
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6						
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10						
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0						+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15						
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15						
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7						

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	IS
1234567	1/7	1/7	1/7	1/7	1/7	1/7	1/7	+
123456|7	1/6	1/6	1/6	1/6	1/6	1/6	0	+
12345|67	1/5	1/5	1/5	1/5	1/5	0	0	+
12345|6|7	1/5	1/5	1/5	1/5	1/5	0	0	+
1234|567	1/4	1/4	1/4	1/4	0	0	0	
1234|56|7	1/4	1/4	1/4	1/4	0	0	0	+
1234|5|6|7	1/4	1/4	1/4	1/4	0	0	0	+
123|456|7	1/9	1/9	1/9	1/9	1/9	1/9	1/3	
123|45|67	1/9	1/9	1/9	1/6	1/6	1/6	1/6	
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6	
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10	
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0	+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15	
12|3|4|5|6|7	1/6	1/6	2/15	2/15	2/15	2/15	2/15	
1|2|3|4|5|6|7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	

“reasonable” to be IS

The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	IS
123|45|67	1/9	1/9	1/9	1/9	1/6	1/6	1/6	
123|45|6|7	1/6	1/6	1/6	1/12	1/12	1/6	1/6	
123|4|5|6|7	1/5	1/5	1/5	1/10	1/10	1/10	1/10	
12|34|56|7	1/6	1/6	1/6	1/6	1/6	1/6	0	+
12|34|5|6|7	3/20	3/20	3/20	3/20	2/15	2/15	2/15	

Case 1

A player in a two person coalition (say 6) deviates:

to be alone: 2/15 (< 1/6)

to another two person coalition (say {12}): 1/6

to the one person coalition (say {7}): 1/6 in the same partition.

>> No incentive to deviate!

Case 2

A single player (say 7) deviates to one of two person coalitions (say 12).

>> He gets 1/9 (> 0)! However, 1 and 2 **reject** it because 1/9 < 1/6.

Conclusion

Messages

1. Coalition structures containing a super majority coalition are not stable for coalitional deviations.

2. Even if a coalition structure contains an exact majority coalition, some myopic players may deviate.

3. If we focus on individual deviations, some coalition structures with no majority coalition can be stable.

   (e.g., 123|456|789 satisfies Nash; 12|34|56|7 satisfies IS)

The seven-player symmetric majority game

	$\mathcal{P}$	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS
Super majority coalitions	1234567					+	+
	123456|7						+
	12345|67						+
	12345|6|7						+
Exact majority coalitions	1234|567		+		+		
	1234|56|7	+	+		+		+
	1234|5|6|7	+	+		+		+
No majority coalitions	123|456|7						
	123|45|67						
	123|45|6|7						
	123|4|5|6|7						
	12|34|56|7						+
	12|34|5|6|7						
	12|3|4|5|6|7						
	1|2|3|4|5|6|7						

Stability Concepts

Myopic Stability

Projective core
Optimistic core
Pessimistic core

Farsighted Stability

Farsighted vNM stable set

Individual Stability

Nash stability
The individual stability

Talked

To talk from now on

Comparisons with
Hart and Kurz (1984) and Bloch (1996)

Comparisons with Hart and Kurz (1984) and Bloch (1996)

- The approach of **Hart and Kurz (1984)** and our approach are basically the same.

- There are **two differences**:
  1. They use the α-, β-, γ-, and δ- stability concepts.
  2. Their analysis is offered for $n \leq 10$.

- **Bloch (1996)** modeled the process of coalition formation as a non-cooperative game.

- His "stability" is described as the *equilibrium coalition structure*, which is an outcome of his non-cooperative game.

- The analysis is only offered for $n \leq 10$.

The five-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	C^{pro}	C^{pes}	C^{opt}	V	$Nash$	IS	Hart& Kurz	Bloch
12345	1/5	1/5	1/5	1/5	1/5					+	+		
1234|5	1/4	1/4	1/4	1/4	0						+		
123|45	1/3	1/3	1/3	0	0	+	+		+		+	$\alpha\beta$ $\gamma\delta$	$Eq.$
123|4|5	1/3	1/3	1/3	0	0	+	+		+		+	$\alpha\beta$ $\gamma\delta$	$Eq.$
12|34|5	1/6	1/6	1/6	1/6	1/3								
12|3|4|5	1/4	1/4	1/6	1/6	1/6								
1|2|3|4|5	1/5	1/5	1/5	1/5	1/5								

Future Work

“Better be the head of a <u>dog</u> than the tail of a <u>lion</u>”

or

“Better be the head of a <u>chicken</u> than the tail of a <u>cow</u>”

It is better to be the head of a small group than a trivial member of something big.

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Questions

<u>Possible Questions</u>

Question 1

<u>Generalization of the power index</u>

In this paper, we use (a variant of) the Shapley-Shubik power index.

⇩

Our results are available only for this **specific** index.

⇩

How about deriving some properties from the SS index, and making an abstract index?

⇩

An index satisfying
efficiency,
monotonicity,
consistency.....

We apply the abstract index to the same problems.

Question 2

Higher threshold k

In this paper, we assume k is an exact majority: $k = 50\%$

In some actual situations, $k = 75\%$ and 2/3.

How about trying different ks?

Question 3

Asymmetric games

In this paper, players are symmetric.

In United Nations, some countries have more power than the others.

How about applying our approach to asymmetric situations?

Great! But, note that…

Our easy way of computation is no longer equivalent to the Owen index.

Question 4

Other stability concepts

In this paper, we mainly use the myopic cores and the farsighted vNM set.

How about some intermediate notions?

"The farsighted core" and "the myopic vNM stable set".

- The **myopic vNM stable set** is a superset of the farsighted vNM stable set.
- The **farsighted core** is a subset of the myopic cores (the projective core).

Larger

$C^{pes} = V$

C^{pro}

C^{opt}

Smaller