

# Precautionary Saving and Ambiguity\*

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## Abstract

This paper considers a two period consumption-saving model in which future income is uncertain. If the future income is also ambiguous, in the sense of having multiple priors, then ambiguity attitudes also affect the saving decision. Unlike one-period decision problems, ambiguity attitude does more than just distort the probabilities of the various priors. It also distorts the relative importance of second-period consumption, which in turn affects precautionary demand for saving. These effects can either reinforce or counteract the well known effects based upon risk attitudes in expected utility models.

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## 1 Introduction

The role of "prudence" in expected-utility models of consumption and saving has been known ever since the seminal works of Leland (1968), Sandmo (1970) and Dreze and Modigliani (1972). Kimball (1990) formalized the concept of prudence. Essentially, a convex marginal utility ("prudence") increases the benefit of saving when future income is risky. This extra saving is referred to as "precautionary saving."<sup>1</sup>

In this paper, we explore what might happen in a two-date consumption-saving model when there are several competing prior distributions for future income. In other words, the distribution of future income is *ambiguous*. Under expected utility, individuals are ambiguity neutral and simply aggregate the probabilities. But if individuals are ambiguity averse, how will it affect saving decisions?

Using the smooth ambiguity aversion models of Klibanoff, Marinacci and Mukerji (2005, 2009) – hereafter KMM (2005) and KMM (2009) respectively, we show how ambiguity can affect saving decisions. KMM (2009) embeds ambiguity into a model of dynamic decision making. However, each stage of the process is a two-date static optimization. Thus,

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<sup>1</sup>See Kimball (1992) for an overview of this topic.

our focus on a two-date model of consumption and saving has a structure similar to this dynamic framework.

Essentially, we show that ambiguity aversion distorts one's preferences for consumption timing ("impatience") as well as the relative weights given to the competing prior distributions. The importance of consumption timing is similar to the discussion in Strzalecki (2013). This effect can also be found in the two-date self protection model of Berger (2011b) and in the social-discount-rate model of Gierlinger and Gollier (2011).

The distortion of probability weights for competing priors is analyzed within a one-period framework by Gollier (2011). As will be shown, the distortion is more complicated to interpret in our consumption-saving framework, since our distorted weights need not sum to one; i.e. the distortion embodies more than just a shift in probabilities.

For an expected-utility maximizer with utility  $u$  and with zero prudence ( $u''' = 0$ ), there is zero precautionary demand for saving. Such is not always the case in the presence of ambiguity aversion. Indeed, even with zero risk prudence ( $u''' = 0$ ), we show that precautionary saving can be either positive or negative under ambiguity aversion. Although Kimball (1990) showed that convex marginal utility,  $u''' > 0$ , guarantees a precautionary demand for saving under expected utility, the same does not hold under ambiguity aversion. Again, negative precautionary saving is also a possibility. This possibility exists even if the second order utility of Klibanoff et al. (2005),  $\phi$ , has a marginal utility that is convex ( $\phi''' > 0$ ). In other words, both  $u''' > 0$  and  $\phi''' > 0$  are not sufficient to guarantee a

precautionary demand for saving.<sup>2</sup>

In each of the above settings, what matters about ambiguity aversion is how its local intensity changes in response to increases in expected utility, where the local measure of absolute ambiguity aversion,  $-\phi''/\phi'$ , is as defined in KMM (2005). In many competing models of ambiguity aversion, it follows from Grant and Polak (2013) that  $-\phi''/\phi'$  is constant. For example, Strzalecki (2011) shows how this follows in models with so-called "multiplier preferences," as introduced by Hansen and Sargent (2001).

However, the model of KMM (2005) imposes no such restriction. When  $-\phi''/\phi'$  is an increasing function, we show below how future uncertainty becomes less important. This by itself would tend to decrease saving, so that it is possible for an ambiguity-averse individual with both  $u''' > 0$  and  $\phi''' > 0$  to actually have a negative demand for precautionary saving.

The above complications make it difficult to compare saving behaviors of different individuals. Given the same underlying (first-order) utility function, a more ambiguity-averse individual facing the same consumption-saving decision will not always save more. However, when the measure of absolute ambiguity aversion is constant, we show that a more ambiguity averse individual will indeed increase her level of precautionary saving.

The following section introduces our basic model of consumption and saving. We next examine an application of ambiguity without using the dynamic framework of KMM (2009)

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<sup>2</sup>This claim holds only in our model with certainty equivalence. Using KMM (2005) without certainty equivalence, as in our section 3, these two conditions are sufficient for precautionary saving. A proof follows along lines similar to an example of precautionary saving in Baillon (2013).

for ambiguity aversion; after all, our model is not a dynamic one. We show how this model has some undesirable properties, which suggests trying an alternate model. We then allow for certainty equivalence as proposed in KMM (2009) and establish conditions under which precautionary saving will exist, as well as conditions under which precautionary saving will be more than (or less than) would be the case absent any ambiguity aversion. Finally, we examine conditions under which comparative ambiguity aversion gives us definitive qualitative results.

## 2 Precautionary Saving and Expected Utility

There are two dates indexed  $t = 0$  and  $t = 1$ . An individual earns a sure income  $w$  at date  $t = 0$ , but faces a random income  $\tilde{w}_1$  at date  $t = 1$ . At date  $t = 0$ , the individual decides how much of her wealth  $w$  to consume. The rest is saved at a risk-free rate of interest. To keep the exposition simple, we assume for now that  $\tilde{w}_1 = w + \tilde{\epsilon}$ , where  $\tilde{\epsilon}$  is a zero-mean random income shock. We also assume that the risk-free rate of interest is zero and that the individual is not impatient, in the sense that future utility need not be discounted. All of these assumptions can be relaxed, but add complicated nuances to the basic model.

In an expected utility (EU) setting, the individual wants to choose a savings level  $s$  to maximize lifetime expected utility

$$U(s) \equiv u(w - s) + Eu(w + \tilde{\epsilon} + s), \tag{1}$$

where  $u$  is an increasing and strictly concave utility function. Concavity connotes both a preference for intertemporal consumption smoothing as well as intra-period risk aversion. If  $\tilde{\epsilon}$  is degenerate (with a zero variance), then optimal saving is zero. If  $\tilde{\epsilon}$  is not degenerate, then optimal saving is positive if the individual is prudent, i.e. if  $u'''(w) > 0$ , see for example Kimball (1990). If this inequality is reversed, the individual is said to be imprudent and she borrows at the risk-free rate (a negative saving). If utility is quadratic with  $u''' = 0$ , then the optimal saving remains at zero. Any extra saving for the case where  $\tilde{\epsilon}$  is non-degenerate is so-called "precautionary saving."

### 3 Ambiguity Aversion

We now introduce ambiguity and suppose that there are  $n \geq 2$  possible distributions for  $\tilde{\epsilon}$ , each leading to a conditional random variable  $\tilde{\epsilon}_\theta$  for  $\theta = 1, \dots, n$ , with distribution functions  $F_\theta(\epsilon)$ . We assume for now that  $E\tilde{\epsilon}_\theta = 0$  for each  $\theta$  and that each  $F_\theta$  has a support contained in the interval  $(a, b)$ . We will relax the assumption that  $E\tilde{\epsilon}_\theta = 0$  for each  $\theta$  later in the paper. Over the space of potential distributions of  $\tilde{\theta}$ , the individual chooses a subjective set of probabilities  $q_\theta$  for the likelihood of  $\tilde{\epsilon}_\theta$  being the true random income. Under expected utility, we simply set  $\tilde{\epsilon}$  equal to the lottery that yields  $\tilde{\epsilon}_\theta$  with probability  $q_\theta$  for  $\theta = 1, \dots, n$ . As is well known, ambiguity has no effect on saving decisions in an EU setting.

Following KMM (2005), we can define an increasing second-order utility  $\phi$  over the

(first-order) utility of wealth. The function  $\phi$  is assumed to be thrice differentiable. For each wealth  $w$  and each  $\tilde{\epsilon}_\theta$ ,  $\phi(Eu(w + \tilde{\epsilon}_\theta))$  denotes the second order utility derived from the expected utility  $Eu(w + \tilde{\epsilon}_\theta)$ . Under ambiguity, expected second-order utility is thus  $\sum_\theta q_\theta \phi(Eu(w + \tilde{\epsilon}_\theta))$ . If  $\phi$  is linear, ambiguity has no effect on underlying intra-period preferences, since  $\phi$  only induces an affine transformation of utility  $u$ . If  $\phi$  is strictly concave, then

$$\sum_\theta q_\theta \phi(Eu(w + \tilde{\epsilon}_\theta)) < \phi(\sum_\theta q_\theta Eu(w + \tilde{\epsilon}_\theta)) = \phi(Eu(w + \tilde{\epsilon})), \quad (2)$$

indicating an aversion to the ambiguity.

More recently, the literature has also examined the certainty equivalent for second order utility:  $\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w + \tilde{\epsilon}_\theta))]$ . It is important to note that this particular "certainty equivalence" is with respect to first-order utility  $u$  and *not* with respect to consumption.<sup>3</sup> With no ambiguity, we simply revert to first-order utility (EU). In a one-period optimization problem with ambiguity, such as the static portfolio choice model examined by Gollier (2011), certainty equivalence is not particularly relevant; since maximizing  $\phi^{-1}[\sum_\theta q_\theta \phi(Eu(\tilde{w}_\theta(\alpha)))]$ , with an endogenous wealth  $\tilde{w}_\theta(\alpha)$ , is equivalent to maximizing  $\sum_\theta q_\theta \phi(Eu(\tilde{w}_\theta(\alpha)))$ . However, the same cannot be said for a multiperiod model, as was also noted by Berger (2011b), Gierlinger and Gollier (2011) and Strzalecki (2013).

Consider the following example without certainty equivalence. Let  $u''' = 0$ , so that

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<sup>3</sup>A few papers have also expressed this certainty equivalent in terms of consumption. See, for example, Itzhakian and Benninga (2011).

optimal saving is zero in our two-period model of precautionary saving under EU. Since  $\phi$  does transform utility, we apply it in both periods to find  $s$  to maximize

$$\phi(u(w - s)) + \sum_{\theta} q_{\theta} \phi(Eu(w + \tilde{\epsilon}_{\theta} + s)). \quad (3)$$

This yields a first-order condition<sup>4</sup>

$$-\phi'(u(w - s))u'(w - s) + \sum_{\theta} q_{\theta} \phi'(Eu(w + \tilde{\epsilon}_{\theta} + s))Eu'(w + \tilde{\epsilon}_{\theta} + s) = 0. \quad (4)$$

Since we assume  $u''' = 0$ , we have  $Eu'(w + \tilde{\epsilon}_{\theta} + s) = u'(w + s) \forall \theta$ . It thus follows in a straightforward manner that if  $\phi$  and  $u$  are both strictly concave,  $s^* > 0$ . Ambiguity aversion alone (even with  $u''' = 0$ ) would seem to imply a precautionary motive for saving.

But consider the case where  $q_1 = 1$ , so there is no ambiguity. We still have  $\phi'(u(w)) < \phi'(Eu(w + \tilde{\epsilon}_1))$ . Thus, with  $u''' = 0$ , it follows from (4) that the optimal level of saving is still positive,  $s^* > 0$ . Utility gives no grounds for a precautionary saving motive, since  $u''' = 0$ . Moreover, there is no ambiguity. Yet, we still get precautionary saving, which may seem like a puzzling result.<sup>5</sup>

Although we can explore more results in this setting, using certainty equivalence seems to provide a "cleaner" set of results for the saving model, as we show in the next section.

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<sup>4</sup>The second order condition is trivial to verify.

<sup>5</sup>KMM (2005) and most other models do not fully separate "ambiguity beliefs" from "ambiguity preferences." Removing multiple priors need not revert to expected utility. So our results above also may not seem so puzzling. A more general investigation of results in this KMM (2005) setting, without certainty equivalence, including a brief look at precautionary saving, can be found in Baillon (2013).

## 4 Introducing Certainty Equivalence

Since we will want to examine properties of the second order utility  $\phi$ , we will follow Baillon (2013) and refer to properties related to utility  $u$  as "risk" properties, such as "risk prudence," and properties related to  $\phi$  as "ambiguity" properties, such as "ambiguity prudence." A model of ambiguity aversion with certainty equivalence was established by KMM (2009). Using this version of ambiguity preferences, we can rewrite the objective function (3) as

$$V(s) \equiv u(w - s) + \phi^{-1}[\sum_{\theta} q_{\theta} \phi(Eu(w + \tilde{\epsilon}_{\theta} + s))]. \quad (5)$$

Reconsidering the case where  $u''' = 0$  and  $q_1 = 1$ , so that there is no ambiguity and zero risk prudence, it follows trivially that optimal saving remains at zero. In other words, with zero risk prudence and no ambiguity, there is no demand for precautionary saving.

In models without certainty equivalence, see for example Baillon (2013), the property of ambiguity prudence  $\phi''' > 0$  is sufficient to generate precautionary saving, whenever we also have risk prudence,  $u''' > 0$ .<sup>6</sup> To see that this result need not hold with certainty equivalence, we first consider the case where we have zero risk prudence,  $u''' = 0$ . Again in this case, we have that  $Eu'(w + \tilde{\epsilon}_{\theta}) = u'(w) \forall \theta$  and  $\forall w$ . We can thus evaluate  $V'(s)$  at

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<sup>6</sup>Baillon (2013) sets up a very general model of higher order ambiguity attitudes, without certainty equivalence. His focus is a static one-period framework, although he does consider an application to precautionary saving. In this framework, ambiguity aversion alone is enough to distort preference towards consumption smoothing – even absent any ambiguity.

$s = 0$ :

$$V'(0) \equiv -u'(w) + \frac{\sum_{\theta} q_{\theta} \phi'(Eu(w + \tilde{\epsilon}_{\theta}))}{\phi'[\phi^{-1}[\sum_t q_t \phi(Eu(w + \tilde{\epsilon}_t))]]} u'(w). \quad (6)$$

To evaluate (6), we implicitly define the *ambiguity premium*  $\pi_A$  as in Berger (2011a):  $\sum_{\theta} q_{\theta} \phi(z(\theta)) \equiv \phi[\sum_{\theta} q_{\theta} z(\theta) - \pi_A]$ , where  $z(\theta)$  denotes (first order) expected utility when the true random income risk is  $\tilde{\epsilon}_{\theta}$ . The ambiguity premium is positive whenever the individual is ambiguity averse,  $\phi'' < 0$ .<sup>7</sup> Here, we also wish to consider a precautionary premium, as was defined by Kimball (1990) for utility  $u$ . In particular, we implicitly define the *ambiguity precautionary premium*  $\psi_A$  via  $\sum_{\theta} q_{\theta} \phi'(z(\theta)) \equiv \phi'[\sum_{\theta} q_{\theta} z(\theta) - \psi_A]$ . The ambiguity precautionary premium  $\psi_A$  is easily seen to be positive if  $\phi'$  is strictly convex. In such a case, we adopt the terminology of Baillon (2013) and say that preferences are *ambiguity prudent*. Similarly,  $\psi_A$  is easily seen to be zero if  $\phi'$  is linear.

An example of the ambiguity premium is illustrated in Figure 1. In the figure, we assume that there are only two competing prior distributions,  $\theta = 1$  and  $\theta = 2$ . The analogy with Pratt's (1964) risk premium should be apparent. Rather than converting wealth to utility, here we convert first order utility to second order utility.

The expression  $\phi^{-1}[\sum_{\theta} q_{\theta} \phi(z(\theta))]$  is the certainty equivalent of  $z(\tilde{\theta})$  under  $\phi$ . Thus, we obtain  $\phi^{-1}[\sum_{\theta} q_{\theta} \phi(z(\theta))] = \sum_{\theta} q_{\theta} z(\theta) - \pi_A$ . In our application, we set  $z(\theta) = Eu(w + \tilde{\epsilon}_{\theta})$ . Assuming that  $\phi$  is strictly concave, it follows easily from Pratt (1964) that  $\pi_A > 0$ ; see

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<sup>7</sup>The ambiguity premium  $\pi_A$  as defined here is defined in units of first-order utility. It is not defined in terms of units of consumption. The latter is also possible, as first defined by Itzhakian and Benninga (2011).

also Berger (2011a). It is easy to see that the denominator of the fraction in (6) is equal to  $\phi'(\sum_{\theta} q_{\theta} z(\theta) - \pi_A)$ , while the numerator is equal to  $\phi'(\sum_{\theta} q_{\theta} z(\theta) - \psi_A)$ . Since  $\phi'$  is strictly decreasing, this fraction is greater than 1 iff  $\psi_A > \pi_A$ .

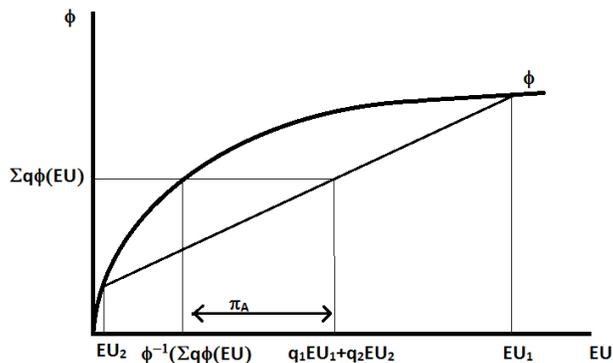


Figure 1. The Ambiguity Premium

First note that if  $\phi'' < 0$  and  $\phi''' = 0$ , then  $\psi_A = 0 < \pi_A$ . It thus follows from (6) that we must have a negative optimal level of saving,  $s^* < 0$ . By continuity of preferences, it follows that for a small enough increase in  $\phi'''$ , we will still obtain  $s^* < 0$ . In other words,  $u''' = 0$  and  $\phi''' > 0$  are *not sufficient* to guarantee precautionary saving.

Let  $\lambda(z) \equiv -\phi''(z)/\phi'(z)$  denote the coefficient of absolute ambiguity aversion, as defined by KMM (2005). Again, analogous to the risk premium and the risk precautionary

premium, it is easy to show<sup>8</sup> that  $\psi_A > \pi_A$  for all possible random  $z(\tilde{\theta})$  iff the coefficient of absolute ambiguity aversion  $\lambda(z)$  is decreasing in  $z$  (*decreasing absolute ambiguity aversion* or *DAAA*). When  $\lambda(z)$  is constant (*constant absolute ambiguity aversion* or *CAAA*), we obtain  $\psi_A = \pi_A$  and when  $\lambda(z)$  is increasing in  $z$ , we obtain  $\psi_A < \pi_A$ . Our earlier assumption that  $\phi''' = 0$  is an example of this last case, with so-called *increasing absolute ambiguity aversion* (*IAAA*). We thus obtain the following result:

**Proposition 1:** *If  $u''' = 0$ , then*

- (i) *precautionary saving is positive under DAAA*
- (ii) *precautionary saving is zero under CAAA*
- (iii) *precautionary saving is negative under IIIA*

We wish to reiterate that  $\phi''' > 0$  is not sufficient to guarantee precautionary saving. When  $u''' = 0$ , any precautionary effect stems strictly from the fraction in (6) being greater than one. Such a circumstance is equivalent to increasing the weight of utility received at date  $t = 1$ . In a certain sense, DAAA exacerbates the importance of any income risk at a later date. This is quite similar to the dynamic model of Strzalecki (2013) who labels this effect "a preference for the earlier resolution of uncertainty."<sup>9</sup>

It also follows from Proposition 1 (i) and the smoothness of preferences that even with slight risk imprudence (i.e.  $u'''$  is very slightly negative) we can obtain a positive

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<sup>8</sup>See, for example, Gollier (2001), who shows the analogous result for EU.

<sup>9</sup>See Strzalecki (2013) Theorem 4 and set his rate of time preference  $\beta = 1$ . His "preference for an earlier resolution of uncertainty" can be restated as an increased dislike for uncertainty that is resolved in later periods.

precautionary demand for saving under DAAA. Although an EU maximizer would have a negative saving, the increased disutility of uncertainty later in life under DAAA might cause overall effect to be an increase in saving.

In a similar manner, we can have  $u'''$  slightly positive with IAAA and have  $s^*$  negative. The risk-prudent EU maximizer would choose a positive level of saving, but the IAAA preferences mitigate the importance of the future income uncertainty. From Proposition 1 (iii) and the continuity of preferences, it follows that  $u''' > 0$  together with  $\phi''' > 0$  need not guarantee a positive level of saving.

## 5 Precautionary Saving under Ambiguity Aversion

We now turn to cases in which the individual is risk prudent, with  $u''' > 0$ . This complicates the model in that we no longer have  $Eu'(w + \tilde{\epsilon}_\theta)$  equal to the same constant for each  $\theta$ . We also relax the assumption that  $E\tilde{\epsilon}_\theta = 0$  for all  $\theta$ . Instead, we define  $\mu_\theta \equiv E\tilde{\epsilon}_\theta$  and we assume only that  $\sum_\theta q_\theta \mu_\theta = 0$ . This last assumption insures that the expected-utility maximizer with  $u''' = 0$  would still desire zero saving.

Under risk prudence, the convexity of  $u'$  yields  $Eu'(w + \tilde{\epsilon}_\theta) > u'(w + \mu_\theta)$  for all  $\theta$ . The first order condition for (5) is now

$$V'(s) \equiv -u'(w - s) + \frac{\sum_\theta q_\theta \phi'(Eu(w + \tilde{\epsilon}_\theta + s))}{\phi'[\phi^{-1}[\sum_t q_t \phi(Eu(w + \tilde{\epsilon}_t + s))]]} Eu'(w + \tilde{\epsilon}_\theta + s) = 0. \quad (7)$$

In the previous section, we already showed that  $\phi''' > 0$  together with  $u''' > 0$  is not

sufficient to guarantee a precautionary saving. In this section, we examine conditions under which ambiguity increases the level of precautionary saving. Since we assume risk prudence,  $u''' > 0$ , and since we assume that  $\sum_{\theta} q_{\theta} \mu_{\theta} = 0$ , it follows that the EU maximizer will choose a positive level of saving. Let  $s_0 > 0$  denote the solution to (1). Thus, for the EU maximizer,  $s_0$  satisfies the first-order condition

$$-u'(w - s_0) + \sum_{\theta} q_{\theta} Eu'(w + \tilde{\epsilon}_{\theta} + s_0) = 0. \quad (8)$$

From (7), it follows trivially that if the individual is ambiguity neutral,  $\phi'' = 0$ , she would choose the same optimal level of saving  $s_0$ .

For the ambiguity averse individual, the variation in  $Eu'(w + \tilde{\epsilon}_{\theta} + s_0)$  as  $\theta$  changes requires us to make some other assumptions, if we wish to obtain definitive effects of ambiguity aversion. To this end, we assume that the prior distributions for  $\tilde{\epsilon}$  can be ranked by  $N^{th}$  order stochastic dominance (NSD), for some integer  $N \geq 1$ . In particular, we assume that  $\tilde{\epsilon}_{\theta+1}$  is riskier than  $\tilde{\epsilon}_{\theta}$  in the sense of NSD for  $\theta = 1, \dots, n-1$ . For example, suppose that  $N = 3$ , but that all of the  $\tilde{\epsilon}_{\theta}$  have a zero mean and that all have the same variance. This would imply that  $\tilde{\epsilon}_{\theta+1}$  has more downside risk than  $\tilde{\epsilon}_{\theta}$ , as described by Menezes, et al. (1980). This would be consistent with a model in which the ambiguity is over the skewness of the distribution, where a higher  $\theta$  indicates a more negative skew.<sup>10</sup>

If  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, \dots, N + 1$  and if the  $\tilde{\epsilon}_{\theta}$  risks can be ranked via  $N^{th}$

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<sup>10</sup>The effects of *NSD* changes in the future income risk are examined within an EU setting by Eeckhoudt and Schlesinger (2008).

order stochastic dominance, we know that  $Eu(w + \tilde{\epsilon}_\theta + s)$  will be decreasing in  $\theta$ , whereas  $Eu'(w + \tilde{\epsilon}_\theta + s)$  will be increasing in  $\theta$ .<sup>11</sup> We will thus assume that the first order utility function satisfies the property:  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 2, \dots, N + 1$ . As an example, when  $N = 2$ , we only require risk aversion and risk prudence, which we are already assuming. If  $N = 3$ , we need to add the assumption of temperance ( $u''' < 0$ ).

To see whether the solution  $s^*$  to (7) is greater than  $s_0$ , we evaluate  $V'(s)$  when  $s = s_0$ .

For our ambiguity averter, we have

$$V'(s_0) = -u'(w - s_0) + \frac{\sum_\theta q_\theta \phi'(Eu(w + \tilde{\epsilon}_\theta + s_0))}{\phi'[\phi^{-1}[\sum_t q_t \phi(Eu(w + \tilde{\epsilon}_t + s_0))]]} \sum_\theta q_\theta Eu'(w + \tilde{\epsilon}_\theta + s_0) \quad (9)$$

$$+ \frac{\text{cov}(\phi'(Eu(w + \tilde{\epsilon}_\theta + s_0)), Eu'(w + \tilde{\epsilon}_\theta + s_0))}{\phi'[\phi^{-1}[\sum_t q_t \phi(Eu(w + \tilde{\epsilon}_t + s_0))]]}.$$

Comparing (8) and (9), we can easily describe the difference via two effects.

Since the term  $Eu(w + \tilde{\epsilon}_\theta + s_0)$  is decreasing in  $\theta$  and the term  $Eu'(w + \tilde{\epsilon}_\theta + s_0)$  is increasing in  $\theta$ , and since  $\phi'$  is a strictly decreasing function, it follows that the covariance term in (9) must be strictly positive. This effect is due to making "worse priors" more heavily weighted under ambiguity aversion. Looking back to equation (7), this positive covariance implies that  $Eu'(w + \tilde{\epsilon}_\theta + s)$  is relatively overweighted at high  $\theta$ , and relatively underweighted at lower  $\theta$ . This effect causes our ambiguity averse consumer to increase her level of precautionary saving. In other words, although she has the subjective probability  $q_\theta$  for  $\tilde{\epsilon}_\theta$  being the "true" income risk, she takes a bit more precautionary saving to protect

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<sup>11</sup>Our notation here uses  $u^n$  to denote the  $n$ th derivative of the function  $u$ .

against the "worse priors."

The second effect is a "timing of uncertainty effect." To understand this effect, define

$\beta(s) \equiv \frac{\sum_{\theta} q_{\theta} \phi'(Eu(w + \tilde{\epsilon}_{\theta} + s))}{\phi'[\phi^{-1}[\sum_t q_t \phi(Eu(w + \tilde{\epsilon}_t + s))]]}$ . From arguments made in the previous section, we can write  $\beta(s) \equiv \frac{\phi'[\sum_{\theta} q_{\theta} Eu(w + \tilde{\epsilon}_{\theta} + s) - \psi_A]}{\phi'[\sum_t q_t Eu(w + \tilde{\epsilon}_t + s) - \pi_A]}$ . One can then rewrite (9) as follows:

$$V'(s_0) = -u'(w - s_0) + \beta(s_0) \sum_{\theta} q_{\theta} Eu'(w + \tilde{\epsilon}_{\theta} + s_0) + \frac{cov(\phi'(Eu(w + \tilde{\epsilon}_{\theta} + s_0)), Eu'(w + \tilde{\epsilon}_{\theta} + s_0))}{\phi'[\phi^{-1}[\sum_t q_t \phi(Eu(w + \tilde{\epsilon}_t + s_0))]]}. \quad (10)$$

As argued in demonstrating Proposition 1,  $\beta(s_0)$  is greater than one iff  $\psi_A > \pi_A$  for the risk  $\sum_{\theta} q_{\theta} Eu(w + \tilde{\epsilon}_{\theta} + s)$ . This effect captures the change in the importance of date  $t = 1$  income risk (*vis-a-vis* the EU maximizer). When  $\beta(s_0) = 1$  under CAAA, then this affect is nil and we have only the covariance effect of overweighting worse priors. Under DAAA,  $\beta(s_0) > 1$  and both effects imply that saving should increase.

Only in the case where we have IAAA do these two effects act in conflicting directions. When  $\beta(s_0) < 1$ , second period uncertainty becomes less important, implying that the consumer might wish to decrease her level of saving. However, the ambiguity averse consumer still overweights the worse priors (the covariance effect), implying that she might wish to increase her level of saving. As is common in economics, this case leads to two opposite effects so that the total effect depends on which effect dominates.<sup>12</sup>

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<sup>12</sup>Gierlinger and Gollier (2011) look at the related problem (somewhat of a dual) of finding a socially optimal discount rate in a two-period Lucas economy. Although their breakdown is somewhat different, they essentially also find these two types of effects.

These results are summarized as follows:

**Proposition 2:** *Suppose  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, \dots, N + 1$  and the  $\tilde{\epsilon}_\theta$  risks can be ranked via  $N^{\text{th}}$  order stochastic dominance. For the ambiguity averse consumer, precautionary saving*

(i) *will increase in the presence of ambiguity under DAAA or CAAA*

(ii) *may either increase or decrease in the presence of ambiguity under IAAA.*

Under CAAA,  $\beta(s) = 1 \forall s$ . Hence, the no ambiguity level of saving  $s_0$  remains optimal, as in Proposition 1. When  $u''' > 0$ , ambiguity amplifies  $Eu'$  for the riskier distributions of  $\tilde{\epsilon}_\theta$ , our "covariance effect," which leads to increased saving.

Under DAAA,  $\beta(s) > 1 \forall s$ . This has the opposite effect as would "impatience" in consumption. Consumption at date  $t = 1$  becomes relatively more important than at date  $t = 0$ . This effect reinforces the effect re-weighting the  $Eu'$  terms, leading to increased saving.

Under IAAA,  $\beta(s) < 1 \forall s$ . Thus, when  $u''' > 0$ , one effect of ambiguity is to increase impatience, and hence reduce saving; but a second effect re-weights the  $Eu'$  terms in such a way as to induce more saving. Thus, we cannot determine *a priori* whether saving will be higher or lower under ambiguity for this case. When  $u'''$  is only very slightly positive, we might still obtain negative saving, which is less than the positive saving of an EU maximizer with  $u''' > 0$ . But we can also have increased saving or no change in saving

under ambiguity with IAAA. Consider the following example.

**Example:** Let  $u(w) \equiv -e^{-aw}$ . Thus  $u$  exhibits CARA and is prudent. Let  $\phi(z) \equiv -(-z)^2$ . Since  $z$  is negative here, it follows that  $\phi' > 0$ ,  $\phi'' < 0$  and  $\phi''' = 0$ . It also follows that  $\lambda(z) \equiv -\phi''(z)/\phi(z) = (-z)^{-1}$ , which exhibits IAAA. Also, it is easy to solve for  $\phi^{-1}(\phi) = -(-\phi)^{1/2}$ , where  $\phi$  is of course negative here. Some calculation shows that

$$\phi^{-1}[\sum_{\theta} q_{\theta} \phi(Eu(w + \tilde{\epsilon}_{\theta} + s_0))] = -[\sum_{\theta} q_{\theta} (E \exp(-a(w + \tilde{\epsilon}_{\theta} + s_0)))^2]^{\frac{1}{2}} \quad (11)$$

Differentiating with respect to  $s$  and comparing  $V'(s_0)$  in (9) and (8), it follows that optimal saving will increase if

$$\sum_{\theta} q_{\theta} E(\exp(-a(w + \tilde{\epsilon}_{\theta} + s_0)))^2 - [\sum_{\theta} q_{\theta} E(\exp(-a(w + \tilde{\epsilon}_{\theta} + s_0)))]^2 > 0. \quad (12)$$

But this inequality always holds, since the left-hand side of inequality (12) is simply the variance of  $x(\tilde{\theta})$ , where  $x(\theta) \equiv E \exp(-a(w + \tilde{\epsilon}_{\theta} + s_0))$ . Thus, we have  $s^* > s_0$ , even though we have IAAA with  $u''' > 0$ .

We can extend the results in Proposition 2 to cases where the  $\tilde{\epsilon}_{\theta}$  are ranked via  $N^{th}$  degree increases in risk, as defined by Ekern (1980). In such a case, we still assume that  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, 2$ , but we only need to (perhaps) additionally require that  $(u^n) = (-1)^{n+1}$  for  $n = N, N + 1$ . For example, if  $N = 5$ , we require both  $u^5 > 0$  (which monotonically ranks the  $Eu$  terms in  $\theta$ ) and  $u^6 < 0$  (which monotonically ranks the  $Eu'$

terms in  $\theta$ ), so that the covariance term in (9) remains strictly positive. See, for example, Eeckhoudt and Schlesinger (2008). Thus, we obtain the following extension of Proposition 2:

**Corollary 1:** *Let  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, 2$  and for  $n = N, N + 1$ , and assume that the  $\tilde{\epsilon}_\theta$  risks can be ranked via  $N^{\text{th}}$  degree increases in risk, as defined by Ekern (1980).*

*Then precautionary saving*

*(i) will increase in the presence of ambiguity under DAAA or CAAA*

*(ii) may either increase or decrease in the presence of ambiguity under IAAA.*

## 6 Comparative Ambiguity Aversion

From Proposition 1, which compares ambiguity neutrality to ambiguity aversion, it follows that neither higher ambiguity aversion nor higher ambiguity prudence will necessarily lead to more saving. This follows since we cannot compare them when  $u''' = 0$ . One person may have CAAA and the other DAAA or IAAA for example. Using continuity arguments, we also cannot compare individuals if  $u'''$  is slightly positive.

Unlike in Gollier (2011), our  $\beta(s)$  as defined in the previous section need not equal 1. Therefore, our first order condition (7) is not a simple distortion of probability weights, where more weight is placed on "worse priors." Only for the special case of CAAA do we obtain  $\beta(s) = 1$ . We examine here whether or not a more ambiguity averse individual

will have more precautionary saving under an assumption of CAAA. Of course, if  $u''' = 0$ , we obtain zero precautionary saving. We assume here that the individual is risk prudent with  $u''' > 0$ .

Consider two risk-prudent individuals with the same underlying utility function  $u$  and the same subjective assessment of the ambiguity.<sup>13</sup> Both individuals are ambiguity averse with second order utilities  $\phi_1(z)$  and  $\phi_2(z)$  respectively. We assume that individual 2 is more ambiguity averse than individual 1. Thus, we assume constants  $\lambda_2 > \lambda_1$  such that  $\frac{-\phi_2''(z)}{\phi_2'(z)} = \lambda_2$  and  $\frac{-\phi_1''(z)}{\phi_1'(z)} = \lambda_1 \forall z$ . Let  $s_1$  denote the optimal saving for individual 1. This level of saving is positive,  $s_1 > 0$ , by Proposition 2. We now wish to determine whether the optimal saving for individual 2,  $s_2$ , is higher than  $s_1$ .

In the special case of CAAA, the ambiguity premium and the ambiguity precautionary premium are equal to each other. It follows that, for  $i = 1, 2$ ,

$$\phi_i'[\phi_i^{-1}[\sum_{\theta} q_{\theta} \phi_i(Eu(w + \tilde{\epsilon}_{\theta} + s))]] = \sum_{\theta} q_{\theta} \phi_i'(Eu(w + \tilde{\epsilon}_{\theta} + s)). \quad (13)$$

This allows us to rewrite  $V_i'(s_1)$  from (7) as follows:

$$V_i'(s_1) = -u'(w - s_1) + \sum_{\theta} \hat{q}_{i\theta} Eu'(w + \tilde{\epsilon}_{\theta} + s_1), \quad (14)$$

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<sup>13</sup> A nice attribute of KMM (2005) is its ability to separate ambiguity from ambiguity aversion. Obviously being able to empirically determine how two individuals might agree on the level of ambiguity is a daunting task. An alternative is to view the current setting as a single individual who retains her view about the ambiguity, but becomes more ambiguity averse.

where

$$\widehat{q}_{i\theta} = q_\theta \frac{\phi'_i(Eu(w + \tilde{\epsilon}_\theta + s_1))}{\sum_t q_t \phi'_i(Eu(w + \tilde{\epsilon}_t + s_1))}. \quad (15)$$

Note that  $\widehat{q}_{i\theta}$  is now a type of ambiguity-neutral probability, as introduced by Gollier (2011).

In other words, individual  $i$  acts like an EU maximizer with transformed probabilities for each  $\tilde{\epsilon}_\theta$ . Of course (14) is zero for individual 1, since  $s_1$  is optimal.

Since  $\phi_2(z)$  is more ambiguity averse than  $\phi_1(z)$ , there exists a strictly increasing and concave function  $h$  such that  $\phi_2(z) = h(\phi_1(z)) \forall z$ , as shown by KMM (2005). Hence, we can write  $\phi'_2(z) = h'(\phi_1(z))\phi'_1(z)$ . For every  $\theta$ , we thus have

$$\frac{\widehat{q}_{2\theta}}{\widehat{q}_{1\theta}} = h'(\phi_1(Eu(w + \tilde{\epsilon}_\theta + s_1))) \frac{\sum_t q_t \phi'_1(Eu(w + \tilde{\epsilon}_t + s_1))}{\sum_t q_t \phi'_2(Eu(w + \tilde{\epsilon}_t + s_1))}. \quad (16)$$

Suppose  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, \dots, N$  and the  $\tilde{\epsilon}_\theta$  risks can be ranked via  $N^{\text{th}}$  order stochastic dominance. Since  $Eu(w + \tilde{\epsilon}_\theta + s_1)$  is decreasing in  $\theta$ , and since the composite function  $h' \circ \phi_1$  is decreasing, it follows that the ratio  $\widehat{q}_{2\theta}/\widehat{q}_{1\theta}$  is increasing in  $\theta$ . Thus, the ambiguity-neutral probabilities  $\widehat{q}_{2\theta}$  dominate the ambiguity-neutral probabilities  $\widehat{q}_{1\theta}$  via the monotone likelihood ratio property.

Let  $\tilde{\epsilon}(i) \equiv \sum_\theta \widehat{q}_{i\theta} \tilde{\epsilon}_\theta$  for  $i = 1$  and  $i = 2$ . It follows from Gollier (2011, Proposition 1) that  $\tilde{\epsilon}(1)$  is dominated by  $\tilde{\epsilon}(2)$  via  $N^{\text{th}}$  order stochastic dominance. As a result, since  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, 2, \dots, N$ , it follows that  $V'_2(s_1) > V'_1(s_1) = 0$ .<sup>14</sup> This

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<sup>14</sup>Gollier's (2011) Proposition 1 only considers  $N = 1$  and  $N = 2$ , but it is straightforward to extend his proof to any arbitrary  $N$ . Essentially,  $\widehat{q}_{2\theta}$  puts relatively more weight on the worse (higher  $\theta$ ) priors than does  $\widehat{q}_{1\theta}$ . It then follows from arguments similar to those in Eeckhoudt and Schlesinger (2008) for first

establishes the following result:

**Proposition 3:** *Let  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, 2, \dots, N$ , and assume that the  $\tilde{\epsilon}_\theta$  risks can be ranked via  $N^{\text{th}}$  order stochastic dominance. Further assume that ambiguity preferences satisfy CAAA. Then an individual who is more ambiguity averse in the sense of KMM (2005) will choose a higher level of precautionary saving.*

We can also easily extend Proposition 3 to cases where the  $\tilde{\epsilon}_\theta$  can be ranked by an Ekern (1980)  $N^{\text{th}}$  degree increases in risk. This leads to the following result.

**Corollary 2:** *Suppose that ambiguity aversion satisfies CAAA and that individual 2 is more ambiguity averse than individual 1. Further suppose that  $\text{sgn}(u^n) = (-1)^{n+1}$  for  $n = 1, 2, 3$  and for  $n = N$ , and that the  $\tilde{\epsilon}_\theta$  risks can be ranked via  $N^{\text{th}}$  degree increases in risk. Then precautionary saving is higher for individual 2 than for individual 1.*

## 7 Concluding Remarks

We considered a simple model of precautionary saving when shocks to future income are ambiguous. A model without certainty equivalence was shown to have properties that might be considered undesirable - mostly the existence of precautionary saving even with  $u''' = 0$  and no ambiguity present.

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order stochastic dominance under EU that  $V_2'(s_1) > V_1'(s_1) = 0$ .

For a model with certainty equivalence, there are two effects that must be compared when measuring the consequences of ambiguity aversion. One effect is a shift in the relative weighting of the various prior distributions of future income. Similar to Gollier's (2011) model of portfolio choice, an ambiguity averse individual shifts more relative weight to "worse" prior distributions. But a second effect re-weights the overall importance of second period consumption and hence the importance of the second-period uncertainty. This effect is analogous to the "preference for the earlier resolution of uncertainty" (under DAAA) found in the dynamic framework of Strzalecki (2013). Our methodology decomposes these two effects.

For an expected utility maximizer, risk prudence ( $u''' > 0$ ) is known to generate a precautionary demand for saving. However, an ambiguity averse individual who is also ambiguity prudent ( $\phi''' > 0$ ) need not have a positive precautionary saving demand. Indeed, perhaps somewhat surprisingly, both  $u''' > 0$  and  $\phi''' > 0$  together are not sufficient for a precautionary demand.

Only in the case of constant absolute ambiguity aversion, do we not get this second effect of re-weighting the importance of second-period consumption. Under CAAA, if the prior distributions of future income can be ranked via second order stochastic dominance, a more ambiguity averse individual will have a higher level of precautionary saving. CAAA seems to be a common feature found in many other models of ambiguity aversion, as examined by Grant and Pollack (2103); although Gierlinger and Gollier (2011) argue that DAAA should be a fairly canonical assumption, similar to decreasing absolute risk aversion.

Although the simple model of saving considered in this paper is not dynamic, it adds much insight into the dynamic frameworks of KMM (2009) and Strzalecki (2013) – which essentially consist of nested sets of static two-period decision models. Indeed, the importance of preference properties such as CAAA, DAAA and IAAA are just becoming understood. Hopefully, the analysis presented here is a good start towards fully understanding such properties.

## References

- [1] Baillon, A., 2013. Prudence (and more) with respect to uncertainty and ambiguity. Erasmus University Working Paper.
- [2] Berger, L., 2011a. Smooth ambiguity aversion in the small and in the large. ECARES Working Paper 2011-020.
- [3] Berger, L., 2011b. Does ambiguity aversion raise the optimal level of effort: a two period model. ECARES Working Paper 2011-021.
- [4] Dreze, J. and Modigliani, F., 1972. Consumption decisions under uncertainty. *Journal of Economic Theory* 5, 308-335.
- [5] Eeckhoudt, L. and Schlesinger, H., 2006. Putting risk in its proper place. *American Economic Review* 96, 280-289.

- [6] Eeckhoudt, L. and Schlesinger, H., 2008. Changes in risk and the demand for saving. *Journal of Monetary Economics* 55, 1329-1336.
- [7] Ekern, S., 1980. Increasing  $N^{th}$  degree risk. *Economics Letters* 6, 329-333.
- [8] Gierlinger, J. and Gollier, C., 2011. Socially efficient discounting under ambiguity aversion. University of Toulouse IDEI working paper no. 561.
- [9] Gollier, C., 2001. *The Economics of Risk and Time*. Cambridge: MIT Press.
- [10] Gollier, C., 2011. Portfolio choice and asset prices: the comparative statics of ambiguity aversion. *Review of Economic Studies* 78, 1329-1344.
- [11] Grant S. and Polak, B., 2013. Mean-dispersion preferences and constant absolute uncertainty aversion. *Journal of Economic Theory* 148, 1361-1398.
- [12] Hansen, L. and Sargent, T., 2001. Robust control and model uncertainty. *American Economic Review* 91, 60-66.
- [13] Itzhakian, Y. and Benninga, S., 2011. The uncertainty premium in an ambiguous economy. *Quarterly Journal of Finance* 1, 323-354.
- [14] Kimball, M., 1990. Precautionary savings in the small and in the large. *Econometrica* 58, 53-73.

- [15] Kimball, M., 1992. Precautionary motives for holding assets, in *The New Palgrave Dictionary of Money and Finance*. P. Newman, M. Milgate, and J. Falwell, eds., MacMillan, London.
- [16] Klibanoff, P.; Marinacci, M. and Mukerji, S., 2005. A smooth model of decision making under ambiguity. *Econometrica* 78, 1849-1892.
- [17] Klibanoff, P.; Marinacci, M. and Mukerji, S., 2009. Recursive smooth ambiguity preferences. *Journal of Economic Theory* 144, 930-976.
- [18] Leland, H., 1968. Saving and uncertainty: The precautionary demand for saving. *Quarterly Journal of Economics* 82, 465-473.
- [19] Menezes, C.; Geiss, C. and Tressler, J. 1980. Increasing downside risk, *American Economic Review* 70, 921-32.
- [20] Pratt, J., 1964. Risk aversion in the small and in the large. *Econometrica* 32, 122-136.
- [21] Rothschild, M. and Stiglitz, J., 1970. Increasing risk: I. A definition." *Journal of Economic Theory* 2, 225-243.
- [22] Sandmo, A., 1970. The effect of uncertainty on saving decisions." *Review of Economic Studies* 37, 353-360.
- [23] Strzalecki, T., 2013. Temporal resolution of uncertainty and recursive models of ambiguity aversion. *Econometrica* 81, 1039-1074.