Is Majority Consistency Possible?

By

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Abstract

The most well-known approaches to decision rules are inspired by "majority-based" and "ranking-based” utilitarianism. The long lasting discussion on the appropriate collective decision mechanism is based on the merits of the rules consistent with these two approaches. Focusing on conformity with qualified majority, we propose Single-Approval Multiple-Rejection (SAMR) as a plausible flexible scoring rule narrowing the gap between the two approaches. Given k alternatives, such a mechanism permits approval of a single alternative and rejection of at most (k-2) alternatives allowing any relative significance of the approved vs. the rejected alternatives. SAMR is the unique type of rule that spans the whole spectrum of the qualified majority-based utilitarian rules, independent of k. Our first characterization result exposes the relationship between its consistency with any predetermined α-majority based rule, $\alpha > 1/2$, and the best/worst (approval/rejection) relative weight p. Our second result establishes that the plurality rule is the unique scoring rule consistent with any α-majority, $\alpha < 1/2$. These results imply the impossibility of universal scoring-rule consistency with any ideal or real α- majority.

Keywords: single-approval multiple-rejection (SAMR), plurality rule, consistency with α -majority-based rules.

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1. Introduction

A decision rule inspired by "ranking-based" utilitarianism selects an alternative that maximizes the sum of the decision makers' utilities where every agent's utility is the same weakly monotone increasing function of the alternatives' ranking. A decision rule inspired by "majority-based" utilitarianism has to satisfy the following requirement: select the α-majority consensus, the alternative ranked first by an α-majority, when such a majority exists. This is a plausible minimal requirement for decision rules inspired by majority-based utilitarianism which implies that the selected alternative maximizes the sum of utilities of individuals who constitute a majority of the decision makers who share the same first-best alternative. Majority-based utilitarianism is restricted in two senses. First, it applies only to situations where such a majority exists. Second, in those situations the sum of individual utilities is taken only over the majority-group members (equal weights are assigned to the utilities of the majority group and zero weights are assigned to the utilities of the remaining decision makers). Of course, the existence of a more than 1/2-majority winner is unlikely, but this is the precise reason why consistency with such majority is such a plausible, reasonable, weak and very relevant condition. The particlar case of consistency with 1-majority, namely the unanimity condition, most clearly illustrates our point. The same is true in the other extreme case of the property of Condorcet consistency, a plausible widely discussed property in the social choice literature, which is stronger than consistency with simple majority . As is well known, it is highly unlikely to have a Condorcet winner. This very drawback has created the need to insist on Condorcet-consistent rules. Had the existence of a Condorcet winner been a likely event, the need for insisting on Condorcet consistency would have been considerably diminished. A similar argument applies, perhaps even more forcefully, with respect to the weaker condition of α-majority consistency. Since the existence of an α -majority winner is not a common, (yet in some cases required) case, the insistence on α-majority-consistent rules is most plausibe from a majoriarian point of view.

The most well-known decision rules advocated by the supporters of "majority-based" and "ranking-based" utilitarianism are, respectively, the widely used simple and special majority rules and scoring rules. The most common scoring rules are the rigid ones: plurality rule and the Borda method of counts and the flexible one: approval voting.

The "majority-based" rules respect a plausible majoritarian principle, but in order to be well defined decision rules, they must yield an outcome given all possible preference profiles and, in particular, in those situations where a majority does not exist. In contrast to these rules, by definition, the scoring rules are well defined; they result in a collective choice under all possible preference profiles. The on-going two-century debate on the appropriate decision system (Risse (2005), Saari (2003), (2006)) is based on the significance assigned to the majoritarian principle, despite its well-known drawbacks, relative to the merits of scoring rules, despite their well-known disadvantages.[1]

This paper proposes Single-Approval Multiple-Rejection (SAMR) as a resolution to the disagreement between majority-based and ranking-based utilitarianism. Given k alternatives, SAMR is a flexible scoring rule that permits approval of a single alternative and rejection (disapproval) of at most (k-2) alternatives allowing the relative score of the best vs. the worst (approved vs. the rejected) alternative to be equal to p. Since the rule depends on the parameter p, it should actually be denoted SAMR(p) – SAMR where p is the approval/rejection relative weight (score). To simplify the notation, we nevertheless use the notation SAMR with few exceptions in section 4. SAMR is a simple operative rule that enables the decision makers to reveal their (single) best, intermediate and worst alternatives. We prove that it is the unique type of scoring rules that spans the whole spectrum of majoritarian rules being consistent with *any* predetermined α - majority rule (α >1/2), independent of k. That is, it picks the α-majority consensus when one exists. As explained above, this is a plausible minimal requirement for decision rules inspired by majority-based utilitarianism.[2] Our first result exposes the relationship between

[1] The drawbacks of the majoritarian principle are, mainly, its potential non-determinacy due to instability/cyclicity, its exclusive focus on pair-wise comparisons that results in the loss of significant information about the true will of the decision makers and its annoying possible disregard of the preferences of the minority, Baharad and Nitzan (2002), (2007), Campbell and Kelly (2000), Fishburn (1973), (1974), Mueller (2003), Nurmi (1999), (2002), Saari and Sieberg (2001). The disadvantages of scoring rules are, mainly, vulnerability to strategic behavior, arbitrariness of the applied scores, the potential violation of Condorcet consistency and the existence of various collective decision anomalies (paradoxes), Alcantud and Laruelle (2014), Baharad and Nitzan (2003), Brams and Fishburn (2002), Chebotarev and Shamis (1998), Ching (1996), Garcia-Lapresta et al. (2010), Ju (2005), Lehrer and Nitzan (1985), Merrill (1984), Nurmi (1999), (2002), Richelson (1978), Saari (1990), Saari and Sieberg (2001), Young (1974).

[2] The only requirement for majority rule is to assign a positive weight to the majority members' vote and a zero weight to the other's vote.

consistency with an α-majority rule, $½<\alpha \leq 1$, and the approval/rejection relative weight p.

Our study mitigates the debate between the supporters of majoritarian and positional rules by identifying a flexible scoring rule that respects the fundamental very plausible majoritarian principle, viz., consistency with α-majority**,** (α >1/2). Furthermore, this simple operative rule enables the decision makers to reveal not only their most preferred alternative, but also their worst and intermediate ones, while avoiding the likely psychological difficulty associated with the requirement to rank *all* the alternatives, Dummet (1984), Garcia-Lapresta et al. (2010). The resulting stronger incentives to take part in the decision making process, in particular, in elections may clearly increase turnout, which is yet another possible advantage of using SAMR.

One particular sub-type of SAMR, Single-Approval Single-Rejection (SASR) can be defended by arguments similar to those justifying the most common Plurality Rule. A special case of SASR is supported by psychological evidence regarding the prevailing relative weight of best-worst alternatives. It should also be noted that our result mitigates Gardenfors' (1973) result, according to which no scoring rule is Condorcet consistent: whenever there is a majority of more than 0.5, SAMR is Condorcet consistent.

If consistency with a qualified majority is replaced with consistency with a "down to earth" α-majority, α < ½, then the widely used plurality rule turns out to be the unique rule consistent with one such majority – the minimal possible majority. Our two characterization results therefore imply the impossibility of universal consistency with any ideal or real majority, $0 \leq \alpha \leq 1$.

In the next section we present the formal framework. The relationship of SAMR to alternative related decision rules that attracted much attention in the literature is discussed in Section 3. The main result (the consistency of SAMR with any predetermined α–majority, (α>1/2)) is presented in Section 4 together with the discussion on the particular sub-type SASR and the plausibility of one of its members. We then discuss in Section 5 one possible extension of SAMR, its relationship to alternative attempts to narrow the gap between the majority-based and ranking-based utilitarian approaches and its relationship to the amelioration of majority tyranny. Section 6 presents the characterization of the plurality rule as the unique scoring rule consistent with some

α-majority, α < ½ and the impossibility of universal consistency with any ideal or real majority. A brief summary of the results including the advantages of SAMR is presented in the concluding Section 7.

2. The framework

Let N={1,…,n}, $n \geq 3$ (n is odd), denote a finite set of decision makers and A a finite set consisting of k distinct alternatives, $k \geq 3$. A decision rule is a mapping from the set of orderings (complete, transitive and asymmetric relations) over A to the set of non-empty subsets of A.

In this study we focus on two types of decision rules (and, more specifically, on the relationships between them). The first type inspired by ranking-based utilitarianism is usually referred to as (standard or flexible) scoring rules. In standard scoring rules, the scores do not vary across decision makers. In flexible scoring rules, the scores assigned by different decision makers to the alternatives can vary; although the permissible scores are given, the decision makers are allowed flexibility in their assignment.[3] We refer to a scoring rule as α –majority consistent, if it selects an α–majority winner when one exists. The notion of α–majority consistency implies two requirements. First, respect for α–majority reflected by the necessity to select an α–majority consensus. Second, insistence on α–majority reflected by no obligation to choose any alternative that is less than an α –majority consensus. Such an alternative can be chosen, but not necessarily because the scores take into account not only the voters' most preferred alternative, but their ranking of the other alternatives. Given these two aspects of α–majority consistency, a scoring-rule consistency is determined by the minimal α–majority associated with it. Hence, if a scoring rule is α–majority consistent, then it is β–majority consistent for $\beta > \alpha$. In particular, the plurality rule which is consistent with simple majority (α=(n+1)/2n>1/2) is also consistent with any larger α-majority. The opposite is obviously not true, that is, if a scoring rule is α–majority consistent, then it is not β–majority consistent for $\beta < \alpha$.

[3] Standard scoring rules are sometimes referred to as positional rules, Gardenfors (1973) or as point-voting schemes, Mueller (2003). They are uniquely characterized by anonymity, unanimity, reinforcement and continuity, see Young (1974) and Brams and Fishburn (2002). For a comprehensive survey of alternative axiomatic characterizations of standard scoring rules see Chebotarev and Shamis (1998). Flexible scoring rules are discussed, for example, in Brams and Fishburn (1978).

In particular, a rule that is 0.7-majority consistent *need not* choose an alternative that has a 0.6 majority support, but it *can* choose that alternative.

A flexible scoring rule is characterized by $S=\{S_1, S_2,\ldots, S_k\}$ - a weakly decreasing sequence of real numbers, $S_1 \geq S_2 \geq \ldots \geq S_k$ and $S_1 > S_k$. In addition, each of the n decision makers is allowed to assign a score of $S_i \in S$ to the alternative ranked in the *i's* position, such that the assigned scores are weakly decreasing with i, the score S_1 must be assigned to the best alternative which is ranked first, the score S_k must be assigned to the least preferred (worst) alternative and the scores assigned to the alternatives ranked i, i=2, …, k-1, can vary across decision makers as long as they are picked from $\{S_2,\ldots, S_{k-1}\}$ and are weakly decreasing with the position of the alternatives. With no loss of generality, we normalize the scores such that $S_1 = p > 0$, and $S_k = -1$. Hence, $S=\{S_1, S_2,\ldots, S_k\}=\{p, S_2,\ldots, ,-1\}$.The selected alternative(s) is that (are those) receiving the maximal total score.

This study draws attention to the Single-Approval Multiple-Rejection (SAMR) flexible scoring rule, a rule that hitherto has not attracted almost any attention. Under this flexible rule every decision maker assigns a score of S_1 to the single best (approved) alternative, S_2 to the second-best alternative and he can assign the score S_k to up to k-2 alternatives he wants to reject. Since the number of rejected alternatives must be smaller than k-1, a distinction is made between the rejected (worst) alternatives and at least one (the second-best) intermediate alternative. More formally, **a Single-Approval Multiple-Rejection (SAMR)** is a flexible scoring rule such that, $S^{\text{SAMR}}=\{S_1, S_2,\ldots, S_k\}$ satisfies $S_1 > S_2 > S_k$.[4] With no loss of generality, we normalize the scores assuming that $S_1 = p > 0$, $S_2 = 0$ and $S_k = -1$. Hence, $S^{\text{SAMR}}=\{S_1, S_2,\ldots, S_k\}=\{p,0,S_3,\ldots,S_{k-1},-1\}$. The informational requirements of this rule are not necessarily modest. Nevertheless, it can be easily implemented by the decision maker in case only a single rejection is allowed (everybody is indifferent among the k-2 non best or worst alternatives). A revelation of the complete individual's ranking of the alternatives, as, for example, in the rigid Borda scoring rule, is not required; the decision maker must only point out the single alternative he approves and at least one alternative must be rejected, but if one wishes he can reject up to k-2

[4] Recall that, by assumption, $S_j \geq S_{j+1}$ for $2 \leq j \leq n-1$.

alternatives. The unique properties of this scoring rule and its special appeal are discussed in the sequel.

A special case of SAMR is **the Single-Approval Single-Rejection (SASR) where** $S^{\text{SASR}}=\{S_1, S_2,\dots, S_k\}=\{p,0, S_3,\dots,S_{k-1},-1\}$ and $S_{k-1} > -1$.

Alternative scoring rules are the standard rigid ones, such as the Plurality Rule (PR), the Inverse Plurality Rule (IPR) and the Borda Rule and the flexible ones such as the Approval Voting (AV) and the Approval-Disapproval Voting (ADV).

Under the **plurality rule (PR)**, $S^{\text{p}}=\{S_1, S_2,\dots, S_k\}=\{1,0,\dots,0,0\}$. By definition, no flexibility is allowed and the alternative which is ranked first by the largest number of decision makers is elected. The plurality rule is the most commonly used scoring rule.[5] This rule plays a major role when insisting on α<1/2 –majority consistency, as we show in the sequel.

Under the **inverse plurality rule (IPR)**, $S^{\text{ip}}=\{S_1, S_2,\dots, S_k\}=\{1, 1,\dots,1, 0\}$.[6] Again, in this case, by definition, no flexibility is allowed and the alternative which is ranked last by the smallest number of decision makers is elected.

Under the **Borda rule**, $S^{\text{B}}=\{S_1, S_2,\dots, S_{k\text{-}1}, S_k\}= \{k\text{-}1, k\text{-}2,\dots,1,0\}$ the rigid nature of the rule is reflected by the additional requirement that the score S_{i} must be assigned to the alternative which is ranked in the i'th place.[7]

Under **Approval Voting (AV)** [8], $S^{AV}=\{S_1, S_2,\dots, S_k\}=\{1,1,\dots,0,0\}$. This scoring rule is flexible and the decision makers can approve any number of alternatives (at most k-1).

Under **Approval-Disapproval Voting (ADV)** [9], $S^{ADV}=\{S_1, S_2,\dots, S_k\}=\{1,1,\dots,0, -1,-1\}$. That is, the decision maker is allowed a flexible trichotomous preference expression. He can approve or disapprove any (at most k-1) number of alternatives.

The other type of decision rules discussed in this study is the α**-majority rules**. Under these rules, an alternative is selected provided that it is the first choice of a fraction

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[5] For an axiomatic characterization of the plurality rule, see Richelson (1978) and Ching (1996).

[6] For an axiomatic characterization of the inverse plurality rule, see Baharad and Nitzan (2005b).

[7] For axiomatic characterizations of the Borda rule, see Nitzan and Rubinstein (1981), Saari (1990), Young (1974). For a discussion on the Condorcet consistency of this rule, see Baharad and Nitzan (2003).

[8] For an axiomatic characterization of AV, see Brams and Fishburn (1978). Footnote 8 in Baharad and Nitzan (2002) mentions that AV is not a standard rigid scoring rule. The same it true with respect to SAMR and ADV (defined below).

[9] For an axiomatic characterization of Approval Disapproval Voting, see Alcantud and Laruelle (2014), Garcia-Lapresta et al. (2010) and Ju (2005).

of at least α ($0.5 < \alpha \leq 1$) out of the n decision makers. As already noted, in order to be well defined, such rules must specify an outcome under any preference profile. That is, they need to specify the chosen alternative/s independently of the existence of α-majority. A typical such decision rule selects an alternative if it is the best for an α –majority and otherwise the rule always selects one particular alternative (e.g., the status quo). The issue of narrowing the gap between scoring rules inspired by ranking-based utilitarianism and α-majority rules inspired by majority-based utilitarianism is the focus of our study.

3. SAMR and alternative related decision rules

SAMR is strongly related to some of the above decision rules that attracted considerable attention in the social choice and collective decision making literature: The PR (Plurality Rule), IPR (Inverse Plurality Rule), Borda's Rule, AV (Approval Voting), ADV (Approval- Disapproval Voting), α-majority rule, and Condorcet Rule.[10] The study of these decision rules focused mostly on their justification, in particular, by attempting to provide their axiomatic characterization.

SAMR is inspired by all of the above decision rules and, in fact, shares certain fundamental characteristics of these rules. It respects the majoritarian principle, in the sense that it is consistent with α -majority, as clarified below in our main result. Hence it is strongly related to the α-majority rules, $\alpha > ½$. Notice that since the existence of a simple majority is a stronger property than the existence of a Condorcet winner (there is an alternative that beats any other alternative by a simple majority), Condorcet consistency is a stronger property than simple-majority consistency. In turn, if a decision rule is Condorcet consistent, then it is consistent with any simple majority rule. As will be shown in the next section, under certain conditions, the outcome of SAMR is identical to

[10] On majority rules, see Campbell and Kelly (2000), Dummet (1984), Fishburn (1973), May (1952), Mueller (2003), Saari (2001), (2003), Saari and Sieberg (2001). For the PR see Ching, (1996), Richelson(1978). The AV is discussed in Baharad and Nitzan (2005a), Brams and Fishburn (1978), (2002), the IPR in Baharad and Nitzan (2002), (2005b), (2007). For Borda's Rule see Baharad and Nitzan (2003), Chebotarev and Shamis (1998), Farkas and Nitzan (1979), Lehrer and Nitzan (1985), Nitzan and Rubinstein (1981), Saari (1990), (2001), (2006), Young (1974). Special majority rules are studied in Buchanan and Tullock (1962), Fishburn (1973), Mueller (2003) and Condorcet Rule is studied in Mueller (2003), Nurmi (1999), Risse (2005), Saari (2001), Saari and Sieberg (2001).

that of any Condorcet- consistent rule. Being a flexible scoring rule, SAMR certainly resembles the alternative standard rigid scoring rules PR, IPR and Borda's Rule. It allows the assignment of special weight to the best alternative, like PR, and to the worst alternative, like IPR and it enables distinction between the best, the worst and the other (intermediate) alternatives, like Borda's Rule. Allowing the revelation of more information than that revealed by PR and IPR, it shares a basic characteristic of the flexible scoring rules AV and ADV. Although it shares the flexibility that characterizes AV and ADV, it does not enable approval of more than a single alternative. Under AV, different decision makers can approve a different number of alternatives. SAMR requires that all decision makers approve a single alternative. AV is in some sense generalized because, in addition to approving an alternative, decision makers are allowed to reject alternatives. Under ADV, different decision makers can approve and reject a different number of alternatives. SAMR does not allow approval of more than a single alternative, yet it enables rejection of many alternatives and rejection of a different number of alternatives by different decision makers. Furthermore, under SAMR, the relative significance of the best vs. the worst alternative can vary, although once set, it must be applied by all decision makers. In other words, flexibility is taken one step forward: not only flexibility for approving and rejecting, but also flexibility in the expressed uniform ratio between the approved and rejected alternatives. To sum up, SAMR is indeed inspired by and related to PR, IPR, Borda's Rule, AV, ADV, *m*-majority rules and the Condorcet Rule.

4. Narrowing the gap between ranking-based and majority-based utilitarianism

4.1 Main result

To the best of our knowledge, the first and only anticipation of our result for the simple special case of the plurality rule is Saari (2003). He correctly suggests that "Supporters of the plurality vote, for instance, could argue that it is the unique positional method which always elects a majority alternative when one exists". In our terms, indeed, SAMR assuming that p is a sufficiently large real number, henceforth denoted SAMR (p_∞), is consistent with simple majority. Applying a similar reasoning, advocates of the inverse plurality vote could argue that it is the unique scoring rule which is consistent only with a

100%-majority alternative when one exists. That is, the inverse plurality rule is the only scoring rule that provides full protection to any single individual: even a majority of all other decision makers cannot always impose the selection of the single individual's worst alternative. That is, SAMR assuming that $p \to 0$, henceforth denoted SAMR ($p \to 0$), is consistent with 1-majority. In the symmetric case where p is assumed to be equal to 1, we get that SAMR(p=1) is consistent with the ⅔ -majority. In general, the choice of p determines the α – majority SAMR(p) is consistent with. In fact, by our first result, adherents of SAMR(p) – SAMR where p is the approval/rejection relative weight (score), could claim that it is the unique scoring rule that is consistent with any predetermined simple or qualified α - majority rule.

Theorem 1: A flexible scoring rule is consistent with any predetermined α-majority, $½ < \alpha \leq 1$, via the choice of p, if and only if it is a SAMR.

Proof:

1. $\leftarrow$ (sufficiency) SAMR is consistent with any α-majority rule via the choice of p:

Let S_i denote the score of the alternative ranked at the i'th place. By definition of SAMR, $S_1 > S_2 > S_k$. We show that this rule is consistent with any α-majority rule. Suppose that the winner gets the support of αn decision makers. Under SAMR (p), the minimal score obtained by the winner is: $\alpha n S_1 + (1-\alpha) n S_k$.

That is, the winner gets the support of αn decision makers which endows him with a score of $\alpha n S_1$, while getting the minimal score of S_k by $(1-\alpha)n$ decision makers.

The maximal score of the second ranked alternative is obtained when it gets the support of all the $(1-\alpha)n$ decision makers and is ranked second by the αn majority, which endows him with the total score $\alpha n S_2$. The score of the winner is higher than that of a loser, that is,

$$\alpha n S_1 + (1-\alpha) n S_k > (1-\alpha) n S_1 + \alpha n S_2$$

or equivalently,

$$\alpha > \frac{S_1 - S_k}{2S_1 - S_2 - S_k}$$

Since, by assumption, $S_1 = p > 0$, $S_2 = 0$ and $S_k = -1$, we get that

$$\alpha > \frac{p+1}{2p+1}$$

Since the range of the function $m(p) = \frac{p+1}{2p+1}$ for $p>0$ is the open-closed interval (½,1], indeed SAMR is consistent with any predetermined α-majority rule.

2. $\rightarrow$ (necessity) If a flexible scoring rule is consistent with any α -majority rule via the choice of *p*, it is SAMR:

Suppose, to the contrary, that a scoring rule is consistent with any α -majority rule and it is not SAMR. Hence, it violates one of the inequalities satisfied by SAMR, namely, $S_1 > S_2 > S_k$.

Possibility 1: $S_1 = S_2$. In such a case, independent of the scores $S_3, ..., S_k$, we get a contradiction because the scoring rule is consistent *only* with 100%-majority. Suppose there are *n* decision makers and 3 alternatives: *a*,*b*,*c*. The preferences of *n*-1 decision makers are given by *a*>*b*>*c*, and those of the remaining decision maker are given by *b*>*c*>*a*. In such a case, *b* is selected, although *a* is preferred by the maximal possible majority of *n*-1 decision makers.

Possibility 2*:* $S_1 > S_2 = S_k$: In such a case we get a contradiction because this rule is identical with the plurality rule, and is consistent *only* with the simple majority.

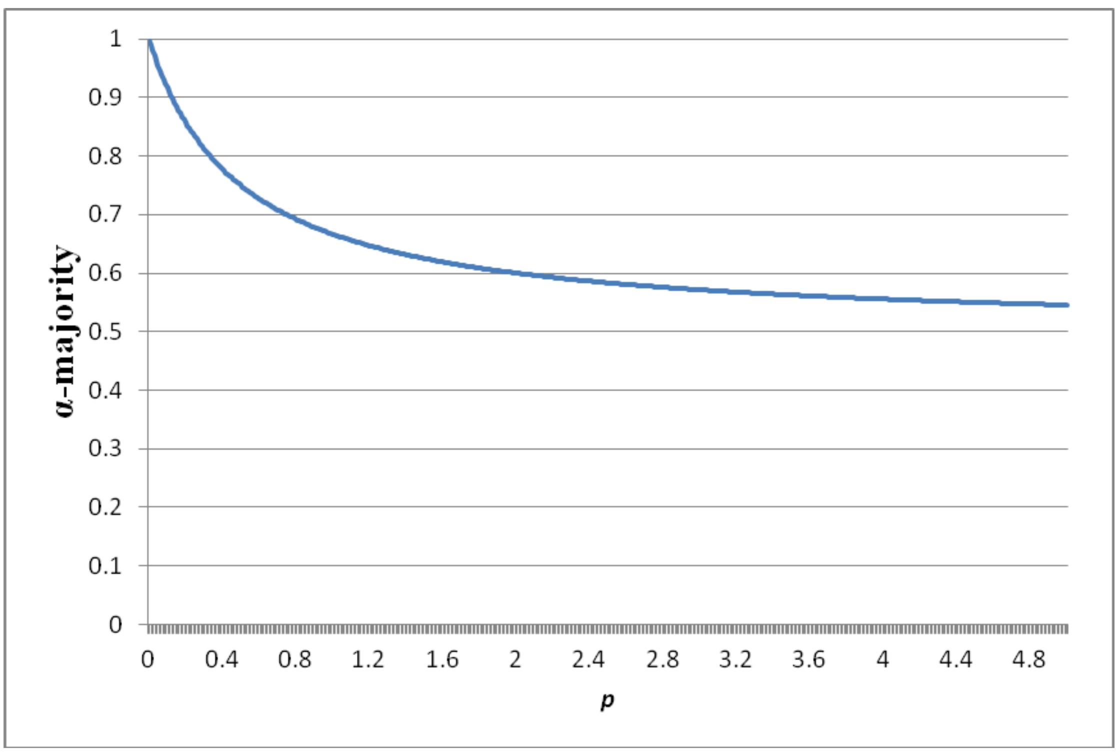
Q.E.D.

In the extreme situations, $p \to 0$ and p_∞, the consistency of SAMR can be clearly understood, respectively, with the ½- majority and 1-majority rules. The intuition behind the convexity of the function α (*p*)- the majority with which SAMR(*p*) is consistent with (see the following figure)- can be interpreted in terms of the compromise-aversion of SAMR. That is, the α -majority consistent with a convex combination of two best/worst relative weights p_1 and p_2, $mp_1 + (1-m)p_2$, is always smaller than the convex combination $m\alpha_1 + (1-m)\alpha_2$ of the two α -majorities α_1 and α_2 corresponding to p_1 and p_2.

4.2 Policy implication

Our main result clarifies the special appeal of SAMR. This family of decision-making systems is very special, yet still very rich. The degree of freedom in selecting an appealing particular member of this family can be considerably reduced applying two practical considerations. First, the empirical plausibility of *p*, the approval/rejection relative weight. Second, the simplicity of the decision rule. On the basis of these considerations we propose the selection of SASR (p=½) which is consistent with the ¾-majority rule.

Viewing the decision-making process (e.g., elections) as a gamble with an uncertain outcome, the winning of the least preferred alternative as a loss and the winning of the most preferred alternative as a gain, the selection of p=½ is plausible psychologically because it can be inferred from the findings of Tversky and Kahneman (1992): "a prospect will only be acceptable if the gain is at least twice as large as the loss".



In terms of practicality, this decision rule is most appealing because of its extreme simplicity; it requires that every decision maker reveals just his most and least preferred alternatives. In light of the universal prevalence of the plurality rule in democratic environments, it seems plausible to expect that SASR (p=½) can become as appealing and common decision-making system.

5. Discussion

5.1 SAMR and majority tyranny

α-majority tyranny is a strong condition. It requires that the majority can guarantee the selection of its agreed upon most favorable alternative, under any given preference profile. In the strategic case, this condition requires considerable coordination among the majority members which has to take into account the exact rank of every alternative in the preferences of every member of the majority, see Baharad and Nitzan (2002). Consistency with α-majority is a much weaker condition. Its implementation does not involve such demanding informational requirements concerning the position of every alternative within all decision makers' preferences. On the one hand, its existence implies robustness to any coalitional manipulation of the minority, and, on the other hand, it does not require any manipulation from the majority members. While Baharad and Nitzan (2002) suggest a mechanism to construct a scoring rule that is robust to a given α-majority tyranny, the current study takes a different and almost opposite view. Its main concern is the identification of the family of scoring rules that are consistent with any α-majority. The two studies are thus complementing each other. While the emphasis in the former study is on securing minimal protection of the minority, the latter current study focuses on the preservation of the majoritarian-based utilitarian principle (by ensuring consistency with α-majority).

5.2 SAMR with flexible *p*

SAMR can be extended by allowing not only control of the crucial parameter p by the decision-making system designer, but by allowing the decision makers to apply different p's. That is, every decision maker may select his individual relative weight between his approved and rejected alternatives. Such extra flexibility may have substantial effect on the outcome of the elections. The relationship between SAMR with a fixed uniform predetermind p and the extended SAMR with a flexible individually controlled p is in the spirit of the relationship between the plurality rule and approval voting. While in this latter context, the decision maker is allowed flexibility in terms of the number of alternatives he can approve, in the proposed extension of SAMR, the decision maker is no longer subjected to an exogenously determined p, but he can report in addition to his preferences regarding the alternatives his preferred p. The study of this interesting generalization and, in particular, the axiomatization of the extended SAMR, is beyond the scope of the current study and is left for future research.

6. Consistency with α < ½ majority

So far, α–majority consistency was applied to the ideal notion of a simple or qualified α–majority, namely, α was assumed to be larger than ½. But in reality α is often smaller than ½. If consistency of a scoring rule is required for any predetermined such realistic α –majority, then the plurality rule emerges as the unique rule satisfying the consistency requirement just with respect to the smallest possible majority ($2/n$ when $n \le k$ and $1/k$ when $n>k$).

Theorem 2: The only scoring rule that is consistent with some predetermined α- majority, α < ½, is the plurality rule; it is consistent with the smallest possible majority, $2/n$, when $n \le k$ and $1/k$ when $n>k$.

Proof:

Let S_i denote the score of the alternative ranked at the i'th place, and let $S_1 \ge S_2 \ge ... \ge S_{k-1} \ge S_k$ and $S_1 > S_k$. W.l.o.g., we normalize the scores such that $S_k = 0$.

Suppose, by negation, that there exists a scoring rule (that is not the plurality rule, hence $S_2 > 0$) that is consistent with α < ½ -majority rule. In such a case, the alternative that gets the support of αn-majority of the voters gets a *minimal* score of $\alpha n S_1$+(1- α)$n S_k$, and the alternative ranked second gets a *maximal* score of $\beta n S_1$ + (1-β)$n S_2$ where $\beta < \alpha$, β=(α-ε), α>0. This is true since this second ranked alternative gets a the support of no more than $(\alpha\text{-}\varepsilon)n$ voters. Since the scoring rule is α-majority consistent, the score of the alternative that gets the support of αn-majority is *always* higher than the one of the second ranked alternative. Thus,

$$\alpha n\, S_1 + (1-\alpha)n\, S_k > (\alpha-\varepsilon)n\, S_1 + (1-(\alpha-\varepsilon))n\, S_2$$

Since, by normalization, $S_k = 0$, we obtain that

$$\alpha n\, S_1 > (\alpha-\varepsilon)n\, S_1 + (1-(\alpha-\varepsilon))n\, S_2$$

which reduces to:

$$\varepsilon\, S_1 > (1-\alpha+\varepsilon)\, S_2$$

This inequality should be satisfied for any ε, which implies that $S_2 = 0$. Since $S_k = 0$ due to normalization, the rule is the plurality rule. Due to the minimality aspect in the definition of majority consistency, the plurality rule is thus associated with the minimal majority, that is $2/n$ when $n \leq k$ and $1/k$ when $n>k$. **Q.E.D**

Theorems 1 and 2 directly imply the impossibility of universal scoring-rule consistency. That is, no scoring rule can span the whole range of α–majority based rules, $0 \leq \alpha \leq 1$. That is,

Theorem 3: No scoring rule can be consistent with any predetermined α-majority, $0 \leq \alpha \leq 1$.

Note that although the plurality rule seems to be consistent with any α–majority that is larger than the minimal possible majority and, in particular, with a qualified α–majority, $\alpha>1/2$, this is not true because of the minimality requirement in the definition of *α-consistency*. That is, if there is an $\alpha - majority$ consensus, α being larger than the minimal possible majority, it will be chosen by the plurality rule, but the same will also hold true with respect to any β–majority, $0 < \beta < \alpha$, contradicting the minimality of α. On the one hand, the plurality rule respects any majority; it selects any α–majority consensus. On the other hand, it completely disregards the weights assigned to the (k-1) alternatives differing from the best one. In contrast, the SAMR can respect any α–

majority, $\alpha > 1/2$, but it does take into account not only the voters' best alternative, but also their (up to *k*-2) worst/rejected ones as well as the best/worst relative weight *p*.

7. Summary

SAMR is characterized by consistency with any predetermined ideal qualified majority which is larger than ½. This is its main advantage. Since this unique property is a plausible minimal requirement for decision rules inspired by majority-based utilitarianism - the alternative selected by such rules maximizes the sum of utilities of individuals who constitute a majority of the decision makers and who share the same first-best alternative- the SAMR is the only flexible scoring rule that respects both majority-based and ranking-based utilitarianism. Another notable advantage of SAMR is the flexibility it allows: flexibility in the relative assessment of the best and worst alternatives, which is determined by the designer as well as flexibility among the decision makers who can reject a different number of alternatives.

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