Tullock Contests with Asymmetric Information[*,†]

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Abstract

We show that under standard assumptions every member of a broad class of generalized Tullock contests with asymmetric information has a pure strategy Bayesian Nash equilibrium. Next we study common-value Tullock contests. We show that in equilibrium the expected payoff of a player is greater or equal to that of any other player with less information, i.e., an information advantage is rewarded. Moreover, if there are only two players and one of them has an information advantage, then in the unique equilibrium both players exert the same expected effort, although the less informed player wins the prize more frequently. These latter properties do not extend to contests with more than two players. Interestingly, players may exert more effort in a Tullock contest than in an all-pay auction.

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1 Introduction

In a *Tullock contest* – see Tullock (1980) – each player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races) which are strategically equivalent to a Tullock contest. Tullock contests also arise by design, e.g., in sport competition, internal labor markets. A number of studies have provided an axiomatic justification to such contests, see, e.g., Skaperdas (1996) and Clark and Riis (1998).

There is an extensive literature studying Tullock contests, and their variations, with complete information. Perez-Castrillo and Verdier (1992), Baye Kovenock and de Vries (1994), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) and Chowdhury and Sheremeta (2009) study existence and uniqueness of equilibrium. Skaperdas and Gan (1995), Glazer and Konrad (1999), Konrad (2002), Cohen and Sela (2005) and Franke *et al.* (2013) look into the effects of changes in the payoff structure on the behavior of players, and Schweinzer and Segev (2012) and Fu and Lu (2013) study optimal prize structures. See Konrad (2009) for a general survey. Tullock contests with asymmetric information, the topic of the present paper, received more limited attention – see Fey (2008) and Wasser (2013) for a recent analysis of rent-seeking games under asymmetric information.[1]

In our setting, each player's value for the prize as well as his cost of effort depend on the state of nature. The set of states of nature is finite.[2] Players have a common prior belief, but upon realization of a state of nature, and prior to taking action, each player observes some event that contains the realized state of nature. The information of each player at the moment of taking action is described by a partition of the set of states of nature. A contest is therefore formally described by a set of players, a probability space describing players' uncertainty and their prior belief, a collection of partitions of the state space describing the players' information, a collection of state-dependent functions describing the players' values and costs, and

[1] Hurley and Shogren (1998), Warneryd (2003), Malueg and Yates (2004), Schoonbeen and Winkel (2006) are among other key contributions.

[2] Our main results, Theorems 1 and 2, also hold for an infinite state space – see Remark 2 and Footnote 8.

a success function specifying the probability distribution that is used to allocate the prize for each profile of efforts. This representation is equivalent to Harsanyi's model of Bayesian games that uses players' types – see Jackson (1993) and Vohra (1999). (In a similar framework, Einy *et al* (2001, 2002) ([11] and [12]), Forges and Orzach (2011), and Malueg and Orzach (2009, 2012) study common-value first- and second-price auctions.)

We show that if players' cost functions are strictly increasing, convex and continuous at zero, then a Tullock contest has a pure strategy Bayesian Nash equilibrium. Moreover, the class of contests for which we establish existence of equilibrium extends far beyond Tullock contests. Every contest in a broad class that we term *generalized Tullock contests* has a pure strategy Bayesian Nash equilibrium. This class is characterized by the following three properties of the success function: (i) when the total effort is positive, each player's probability of winning the prize is continuous with respect to the efforts of all players; (ii) each player's probability of winning is non-decreasing and concave in his own effort (and hence exhibits decreasing returns); (iii) if only one player exerts positive effort, then his probability of winning is 1 – i.e., his effort is perfectly discriminated. The proportional success function defining the Tullock contest obviously satisfies (i)-(iii), but so do more general versions of it, such as the case where each player's probability to win is defined by the ratio between the *score* given to that player's effort and the total score of all players, provided each player's score function – translating his efforts into scores – is strictly increasing and concave. Furthermore, our existence result applies regardless of whether players have private or common values, or whether their costs of effort is the same or different, and makes no assumptions about the players' private information. (The literature has focused on particular cases of interest. For example, Ryvkin (2010) establishes existence of a symmetric Bayesian Nash equilibrium in Tullock contests in an independent private values setting, and Warneryd (2012) assumes that each player is either completely informed, or has no information, about the continuously distributed common value.)

We establish existence of a pure strategy equilibrium by using the result of Reny (1999). The main step of the proof consists of showing that a generalized Tullock contest is a *better-reply-secure* game, which is a premise for Reny's theorem. The discontinuity of the expected payoff functions when all efforts are equal to zero in some state of nature is what prevents us from using the standard Nash's equilibrium existence theorem.[3]

Next we study Tullock contests in which players have a common value for the prize and a common state independent linear cost function, to which we refer simply as *common-value Tullock contests*. We show that in any equilibrium of a common-value Tullock contests, the payoff of a player is greater or equal to that of any other player with less information. Thus, a common-value Tullock contests rewards any information advantage. This result, which is a direct implication of a theorem of Einy, Moreno and Shitovitz (2002) showing that in any Bayesian Cournot equilibrium of an oligopolistic industry a firm's information advantage is rewarded, is established by observing the formal equivalence between a common-value Tullock contest and a certain oligopoly with asymmetric information.

We then proceed to study other properties of the equilibria of common-value Tullock contests. We show that a two-player contest in which one player has an information advantage over his opponent (i.e., the partition of one player is finer than that of his opponent) has a unique (pure strategy) Bayesian Nash equilibrium, which we explicitly describe[4]. In equilibrium both players exert the same expected effort, although the player with less information wins the prize more frequently. Furthermore, assuming that the distribution of the players' values for the prize is not too disperse, we show that when one player is better informed than the other, the total effort exerted by the players is smaller (and thus the share of the total surplus they capture is larger) than when both players have the same information. These properties have been established by Warneryd (2003) in the framework where players' common value

[3]The use of Nash's existence theorem is still a viable alternative. Einy *et al* (2013) ([10]), the discussion paper upon which the current work is based, provides another (somewhat longer) proof that first considers "truncated" contests in which players choose efforts from a compact interval with a positive lower bound, on which the expected payoff functions are continuous, thereby allowing the use of the Nash's theorem to deduce existence of equilibrium. The crux of this proof is to show that a limit point of the sequence of equilibria of truncated contests with a lower bound on players' efforts approaching zero is an equilibrium in the original contest.

[4]Explicit description of equilibrium in a common-value Tullock contest also appears in Section 5.2 of Warneryd (2003), but only for the binary case where there are two possible values of winning – low and high.

is a *continuous* random variable, and one player observes the value precisely while the other player does not observe anything. Warneryd's results do not automatically carry over into the setting with a discrete state space considered here, however. We establish these properties in our setting as corollaries of the explicit characterization of equilibrium strategies.

It turns out that the equilibrium properties mentioned above need not hold when there are more than two players. Specifically, we construct a three-player contest in which two of the players have symmetric information, which is superior to that of the third player, and the expected efforts exerted by the players differ. We also provide an example of a contest in which a player that has an information advantage over all other players (who are symmetrically informed) wins the prize with a greater ex-ante probability than that of any other player.

Finally, we study the relative effectiveness of Tullock contests and all-pay auctions in inducing the players to exert effort when one player has an information advantage. Einy *et al* (2013) ([13]) characterize the unique equilibrium of a two-player common-value all-pay auction, which is in mixed strategies, and provide an explicit formula that allows to compute the players' total effort. Using the results in Einy *et al* (2013) ([13]) and our results we show that the sign of the difference in the total effort exerted by players in a Tullock contest and an all-pay auction is undetermined, and may be either positive or negative depending on the distribution of the players' value for the prize. (Fang (2002), Epstein, Mealem and Nitzan (2011), and Dubey and Sahi (2012) compare the outcomes of Tullock contests and all-pay auction under complete information.[5])

The rest of the paper is organized as follows: in Section 2 we describe the general setting. In Section 3 we establish that every generalized Tullock contest has a pure strategy Bayesian Nash equilibrium. In Section 4 we study common-value Tullock contests. Section 5 studies the relative effectiveness of common-values Tullock contests and all-pay auction in inducing players to exert effort.

[5] Dubey and Sahi (2012) also consider the incomplete information case, but in a binary framework (with just two possible efforts) and a common certain value for the prize.

2 Contests with Asymmetric Information

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by choosing a level of *effort* in $\mathbb{R}_+$. Players' uncertainty about the state of nature is described by a probability space (Ω, p), where Ω is a finite set and p is a probability distribution over Ω describing the players' common prior belief about the realized state of nature. W.l.o.g. we assume that $p(\omega) > 0$ for every $\omega \in \Omega$. The private information about the state of nature of player $i \in N$ is described by a partition Π_i of Ω. The value for the prize of each player i is given by a random variable $V_i : \Omega \to \mathbb{R}_{++}$, i.e., if $\omega \in \Omega$ is realized then player i's ("private") value for the prize is $V_i(\omega)$. The cost of effort of each player $i \in N$ is described by a function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, such that for any $\omega \in \Omega$ the function $c_i(\omega, \cdot)$ is strictly increasing, continuous, convex, and vanishes at 0.

A contest starts by a move of nature that selects a state ω from Ω according to the distribution p. Every player $i \in N$ observes the element $\pi_i(\omega)$ of Π_i which contains ω – the set of states of nature between which i cannot distinguish given ω. Then players simultaneously choose their effort levels, which results in a profile of efforts $(x_1, ..., x_n) \in \mathbb{R}^n_+$. The prize is awarded to the players in a probabilistic fashion, according to a state-dependent *success function* $\rho : \Omega \times \mathbb{R}^n_+ \to \Delta^n$ that attributes to each $\omega \in \Omega$ and profile of effort levels $x \in \mathbb{R}^n_+$ a probability distribution $\rho(\omega, x)$ in the n-simplex Δ^n, according to which the prize recipient is chosen if ω is realized. Hence, the payoff of player $i \in N$, $u_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$, is given for every $\omega \in \Omega$ and $x \in \mathbb{R}^n_+$ by

$$u_i(\omega, x) = \rho_i(\omega, x) V_i(\omega) - c_i(\omega, x_i). \tag{1}$$

Thus, **a** contest is described by a collection $(N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$.

In a contest, a pure strategy of player $i \in N$ is a Π_i-measurable function $X_i : \Omega \to \mathbb{R}_+$ (i.e., X_i is constant on every element of Π_i), that represents i's choice of effort in each state of nature following the observation of his private information. We denote by S_i the set of strategies of player i, and by $S = \times_{i=1}^n S_i$ the set of strategy profiles. For any strategy $X_i \in S_i$ and $\pi_i \in \Pi_i$, $X_i(\pi_i)$ stands for the constant value that $X_i(\cdot)$ takes on π_i. Also, given a strategy profile $X = (X_1, ..., X_n) \in S$, we denote by X_{-i} the profile obtained from X by suppressing the strategy of player $i \in N$. Throughout the paper we restrict attention to pure strategies.

Let $X = (X_1, ..., X_n)$ be a strategy profile. We denote by $U_i(X)$ the expected payoff of player i, i.e.,

$$U_i(X) \equiv E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot))].$$

For $\pi_i \in \Pi_i$, we denote by $U_i(X \mid \pi_i)$ the expected payoff of player i conditional on π_i, i.e.,

$$U_i(X \mid \pi_i) \equiv E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot)) \mid \pi_i].$$

An N-tuple of strategies $X^* = (X_1^*, ..., X_N^*)$ is a (Bayesian Nash) equilibrium if

$$U_i(X^*) \geq U_i(X_{-i}^*, X_i) \tag{2}$$

for every player $i \in N$, and every strategy $X_i \in S_i$; or equivalently,

$$U_i(X^* \mid \pi_i) \geq U_i(X_{-i}^*, x_i \mid \pi_i) \tag{3}$$

for every $i \in N$, every $\pi_i \in \Pi_i$, and every effort $x_i \in \mathbb{R}_+$ of player i (viewed here as a strategy in S_i with the constant value x_i on the set π_i).

3 Existence of Equilibrium in Generalized Tullock Contests

In this section we show that every contest in a class to which refer as *generalized Tullock contests* has a pure strategy Bayesian Nash result. This class is characterized by contest success functions satisfying some simple properties. For $x \in \mathbb{R}_+^n$ we denote by $x_{-i} \in \mathbb{R}_+^{n-1}$ the profile of efforts obtained from x by suppressing the effort of player i, and by $\mathbf{0} \in \mathbb{R}_+^n$ the zero vector (i.e., the profile of zero efforts in our context). A *generalized Tullock contest* is a contest in which the success function ρ has the following properties at each $\omega \in \Omega$:

(i) $\rho(\omega, \cdot)$ is continuous on $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$;

(ii) for each $i \in N$ and $x_{-i} \in \mathbb{R}_+^{n-1}$, $\rho_i(\omega, x_{-i}, x_i)$ is non-decreasing and concave in the effort x_i of player i; and

(iii) for each $i \in N$ and $x_i > 0, \rho_i(\omega, \mathbf{0}_{-i}, x_i) = 1$, i.e., if all players but i make zero effort at ω, any positive effort by i guarantees that he gets the prize with probability

1 at that state of nature.[6]

A *Tullock contest* is a particular case of a generalized Tullock contest, in which the state-independent success function ρ^T is given for each $x \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ and $i \in N$ by

$$\rho_i^T(x) = \frac{x_i}{\overline{x}}, \tag{4}$$

where $\bar{x} \equiv \sum_{k=1}^{N} x_k$ is the total effort exerted by the players. It is easy to see that ρ^T satisfies conditions (i), (ii) and (iii). More generally, conditions (i)–(iii) are satisfied by any success function ρ that is given for any $\omega \in \Omega$, $x \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ and $i \in N$ by

$$\rho_i(\omega, x) = \frac{g_i(\omega, x_i)}{\sum_{j=1}^{n} g_j(\omega, x_j)}, \tag{5}$$

where for every $\omega \in \Omega$ and $j \in N$ the state-dependent *score function* $g_j(\omega, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, translating player j's efforts into *scores,* is strictly increasing, continuous, concave, and vanishes at 0. Thus, contests with success functions given by (5) are also generalized Tullock contests.[7]

Theorem 1. *Every generalized Tullock contest has a (pure strategy) Bayesian Nash equilibrium.*

Proof. Let $C = (N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$ be a generalized Tullock contest. Since the cost function of each player is strictly increasing and convex in the player's effort, it follows from (1) that there exists a sufficiently large $Q > 0$ so that $u_i(\cdot, x) < 0$ for every $i \in N$ and every $x \in \mathbb{R}^n_+$, provided $x_i \geq Q$. Consider a bounded variant of the contest, denoted by $\overline{C}$, in which the effort set of each player i is restricted to be the bounded interval $[0, Q]$. In $\overline{C}$, the set of strategies of player i, $\overline{S}_i$, is identifiable with the *compact* set $[0, Q]^{\Pi_i}$ via the the bijection $X_i \longleftrightarrow (X_i(\pi_i))_{\pi_i \in \Pi_i}$, and player i's expected payoff function U_i is *concave* in i's own strategy (as the state-dependent payoff function $u_i(\cdot, x)$ is concave in the variable x_i, which follows from condition (ii) and the convexity of costs).

The expected payoff function U_i is not continuous on $\overline{S} = \times_{i=1}^{n} \overline{S}_i$, but it is continuous on $\overline{S}_+$, where $\overline{S}_+ \subset \overline{S}$ consists of strategy-profiles X such that $X(\cdot) \neq \mathbf{0}$ on

[6] Notice that (iii) implies (ii) for $x_{-i} = \mathbf{0}_{-i}$ (due to the assumption that $\rho_i(\omega, \mathbf{0}) \leq 1$), and hence it would have sufficed to state property (ii) only for $x_{-i} \in \mathbb{R}^{n-1}_+ \diagdown \{\mathbf{0}_{-i}\}$.

[7] Existence of equilibrium in contests with success functions belonging to this class was established for the complete information case by Szidarovszky and Okuguchi (1997).

Ω. (The function U_i is continuous on $\overline{S}_+$ since the success function ρ is continuous on $\mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ by condition (i), and the state-dependent cost function is continuous on $\mathbb{R}_+$.) Moreover, each U_i is *lower semi-continuous* in the variable $X_i \in \overline{S}_i$; i.e., for a fixed $X_{-i} \in \overline{S}_{-i} \equiv \times_{j \neq i} \overline{S}_j$ and every sequence $\left(X_i^k\right)_{k=1}^{\infty} \subset \overline{S}_i$ that converges (pointwise) to X_i, $\liminf_{k \to \infty} U_i\left(X_{-i}, X_i^k\right) \geq U_i\left(X_{-i}, X_i\right)$, since the i^{th} component of the success function, ρ_i, is lower semi-continuous in $x_i \in \mathbb{R}_+$ as follows from conditions (i) and (iii).

Given the compactness of $\overline{S}_i$ and the concavity of U_i in the variable $X_i \in \overline{S}_i$, for each $i \in N$, existence of equilibrium in $\overline{C}$ is guaranteed by Theorem 3.1 of Reny (1999), provided $\overline{C}$ is in addition *better-reply-secure*: if (a) $\left(X^k\right)_{k=1}^{\infty} \subset \overline{S}$ is a sequence such that the (pointwise) limit $X \equiv \lim_{k \to \infty} X^k$ exists and X is not a Bayesian Nash equilibrium in $\overline{C}$; and (b) $w_i \equiv \lim_{k \to \infty} U_i(X^k)$ exists for every $i \in N$, then there must be some player i that can *secure* a payoff greater than w_i at X, i.e., there exist $Y_i \in \overline{S}_i$, $z_i > w_i$, and an open neighborhood $W \subset \overline{S}_{-i}$ of X_{-i} such that $U_i(X'_{-i}, Y_i) \geq z_i$ for every $X'_{-i} \in W$.

We will show that $\overline{C}$ is indeed better-reply-secure. Let $\left(X^k\right)_{k=1}^{\infty}$, X, and $(w_i)_{i \in N}$ be as above. If $X \in \overline{S}_+$, then the functions $(U_i)_{i \in N}$ are continuous at X and hence $w_i = U_i(X)$ for every $i \in N$. Since X is not an equilibrium by assumption, there exist $i \in N$ and $Y_i \in \overline{S}_i$ such that

$$U_i(X_{-i}, Y_i) > w_i + \varepsilon \tag{6}$$

for some $\varepsilon > 0$. It can be assumed w.l.o.g. that Y_i is strictly positive, as U_i is lower semi-continuous in the i^{th} variable. By the continuity of U_i at $(X_{-i}, Y_i) \in \overline{S}_+$, $U_i(X'_{-i}, Y_i) \geq z_i \equiv w_i + \frac{\varepsilon}{2}$ for every X'_{-i} in some open neighborhood W of X_{-i}, and thus i can secure at X a payoff greater than w_i.

Assume now that $X \in \overline{S} \setminus \overline{S}_+$; thus, $X(\omega^*) = \mathbf{0}$ for some $\omega^* \in \Omega$. Consider an accumulation point $(\widetilde{p}(\omega))_{\omega \in \Omega}$ of the sequence $\{\left(\rho\left(\omega, X^k(\omega)\right)\right)_{\omega \in \Omega}\}_{k=1}^{\infty}$, and assume w.l.o.g. (passing to a subsequence if necessary) that $\lim_{k \to \infty} \left(\rho\left(\omega, X^k(\omega)\right)\right)_{\omega \in \Omega} = (\widetilde{p}(\omega))_{\omega \in \Omega}$. Define, for every $\omega \in \Omega$ and $i \in N$,

$$\widetilde{w}_i(\omega) \equiv \widetilde{p}_i(\omega) V_i(\omega) - c_i(\omega, X_i(\omega)).$$

By the continuity of the cost function, $w_i = E(\widetilde{w}_i(\cdot))$.

Since $\widetilde{p}(\omega^*)$ is a probability vector, there exists $i \in N$ for whom

$$\widetilde{p}_i(\omega^*) < 1. \tag{7}$$

For any $0 < \varepsilon < Q$, consider a strategy $Y_i^\varepsilon \in \overline{S}_i$ given by $Y_i^\varepsilon(\cdot) \equiv \max\{X_i(\cdot), \varepsilon\}$. (In particular, $Y_i^\varepsilon(\pi_i(\omega^*)) = \varepsilon$.) Then for any $\omega \in \Omega$ with $X(\omega) \neq \mathbf{0}$,

$$\lim_{\varepsilon \to 0+} u_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) = \lim_{\varepsilon \to 0+} [\rho_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) V_i(\omega) - c_i(\omega, Y_i^\varepsilon(\omega))] = \widetilde{w}_i(\omega), \tag{8}$$

since ρ_i is continuous at $X(\omega) \neq \mathbf{0}$ and hence $\lim_{\varepsilon \to 0+} \rho_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) = \lim_{k \to \infty} \rho_i\left(\omega, X^k(\omega)\right) = \widetilde{p}_i(\omega)$. And for any $\omega \in \Omega$ with $X(\omega) = \mathbf{0}$,

$$\lim_{\varepsilon \to 0+} u_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) = \lim_{\varepsilon \to 0+} [\rho_i(\omega, 0_{-i}, \varepsilon) V_i(\omega) - c_i(\omega, \varepsilon)] = V_i(\omega) \geq \widetilde{w}_i(\omega) \tag{9}$$

by property (iii) of ρ, with a *strict inequality* for $\omega = \omega^*$ as follows from (7). It is then implied by (8) and (9) that[8]

$$\lim_{\varepsilon \to 0+} U_i(X_{-i}, Y_i^\varepsilon) > E(\widetilde{w}_i(\cdot)) = w_i. \tag{10}$$

Now fix some $\varepsilon > 0$ for which $U_i(X_{-i}, Y_i^\varepsilon) > w_i + \varepsilon$ (it exists by (10)), and denote $Y_i \equiv Y_i^\varepsilon$. By definition, (X_{-i}, Y_i) satisfies (6), and repeating the arguments following (6) shows that i can secure a payoff greater than w_i. Thus $\overline{C}$ is better-reply-secure, and hence it possesses some Bayesian Nash equilibrium X^*. In particular, X^* satisfies (3) for every $x \in [0, Q]$.

Finally, note that every $x_i > Q$ leads to a negative expected payoff to player i conditional on $\pi_i \in \Pi_i$, which can be improved upon by lowering the effort to zero. Thus, in contemplating a unilateral deviation from $X_i^*(\pi_i)$ conditional on π_i, player i is never worse off by limiting himself to efforts $0 \leq x_i \leq Q$. But this means that X^* satisfies (3) for every $x \in \mathbb{R}_+$. Since this is the case for every $i \in N$ and every $\pi_i \in \Pi_i$, X^* is a Bayesian Nash equilibrium of the original contest C. ■

Theorem 1 makes no assumptions about players' private information, and applies regardless of whether players have private or common values, or whether their costs of effort are the same or different. Theorem 1 also implies existence of a Bayesian

[8] If we were to apply this proof to a contest with a countable state space Ω, as in Remark 2 after this proof, (10) would follow by the bounded convergence theorem.

Nash equilibrium in a generalized Tullock contest in the Harsanyi types model, where each player's uncertain type represents his private information, and players have a common prior distribution over all possible realizations of types. (In the discrete case, these two models of incomplete information games are equivalent – see Jackson (1993) and Vohra (1999).)

Under additional conditions, a generalized Tullock contest has a unique equilibrium. Remark 1 states this result, which follows from Ewerhart and Quartieri (2013)'s Theorem 1.

Remark 1. A generalized Tullock contest in which: (i) the success function ρ is given by (5) for state-dependend score functions $(g_i(\omega,\cdot))_{i\in N,\omega\in\Omega}$ that are twice differentiable; (ii) players' state-dependent cost functions $(c_i(\omega,\cdot))_{i\in N,\omega\in\Omega}$ are twice differentiable; and (iii) $\rho_i(\omega,0) < 1$ for every $i \in N$ and $\omega \in \Omega$, has a unique (pure strategy) Bayesian Nash equilibrium.

Remark 2. Our assumption that the state space Ω is finite is made primarily for simplicity of exposition and notations. Theorem 1 on equilibrium existence holds for a generalized Tullock contest with an infinite Ω, provided: (i) information partitions $(\Pi_i)_{i\in N}$ are at most countable; (ii) the value functions $(V_i)_{i\in N}$ are integrable; (iii) the state-dependent cost function $c_i(\omega,\cdot)$ of each player i is (uniformly in ω) bounded from above and from below by strictly increasing convex functions; and (iv) the state-dependent success function ρ is measurable with respect to $\vee_{i\in N}\Pi_i$ ($\equiv$the coarsest partition that refines each Π_i) in its first variable. Indeed, the positive probability elements of $\vee_{i\in N}\Pi_i$ can be viewed as the state space Ω' instead of Ω, and the values, costs, and probabilities of success can be redefined by taking the conditional expectation of these functions on Ω at each of the new states in Ω'. The induced generalized Tullock contest with the (at most countable) state space Ω' is strategically equivalent to the original contest, and the proof of Theorem 1 applies to the induced contest (due to countability of the state space) with some minor changes in wording (see, e.g., footnote 9).

4 Common-Value Tullock Contests

Henceforth we study Tullock contests (with $\rho = \rho^T$ defined in (4)), in which players have a common value for the prize and a common state-independent linear cost function, i.e., for all $i \in N$, $V_i = V$, and $c_i(\cdot, x) \equiv x$ on Ω. We refer to these contests as *common-value Tullock contests,* and they are described by a collection $(N, (\Omega, p), (\Pi_i)_{i \in N}, V)$.

We say that player $i \in N$ *has an information advantage* over player $j \in N$ if partition Π_i is finer than partition Π_j. Thus, if i has an information advantage over j, then $\pi_i(\omega) \subset \pi_j(\omega)$ for every $\omega \in \Omega$, i.e., player i knows the realized state of nature with at least the same precision as player j.

We begin by establishing in Theorem 2 a general property of common-value Tullock contests: these contests reward information advantage. This is a direct implication of the theorem of Einy, Moreno and Shitovitz (2002), that shows that information advantage is rewarded in any Bayesian Cournot equilibrium of a symmetric oligopolistic industry in which the firms' cost function is linear.

Theorem 2. *Let $X^* = (X_1^*, ..., X_n^*)$ be any equilibrium of an n-player common-value Tullock contest.*[9] *If player i has an information advantage over player j, then $U_i(X^*) \geq U_j(X^*)$.*

Proof. An n-player common-value Tullock contest $(N, (\Omega, p), (\Pi_i)_{i \in N}, V)$ is formally identical to an oligopolist industry $(N, (\Omega, p), P, c, (\Pi_i)_{i \in N})$, where the market demand P and the cost function c are defined for $(\omega, x) \in \Omega \times \mathbb{R}_{++}$ as

$$P(\omega, x) = \frac{V(\omega)}{x},$$

and

$$c(\omega, x) = x,$$

respectively. With this convention, the state-dependent profit of firm $i \in N$ in the industry coincides with the payoff of player $i \in N$ in the contest, i.e., for $\omega \in \Omega$ and

[9]Theorem 2 also holds for common-value Tullock contests with infinite state space Ω (under conditions set forth in Remark 2), with the same proof.

$X \in S$,

$$\begin{aligned} u_i(\omega, X) &= \frac{V(\omega)}{\sum_{s=1}^n X_s(\omega)} X_i(\omega) - X_i(\omega) \\ &= P(\omega, \sum_{s=1}^n X_s(\omega)) X_i(\omega) - c(\omega, X_i(\omega)). \end{aligned}$$

Theorem 2 then follows from the theorem of Einy, Moreno and Shitovitz (2002).[10] ■

Theorem 2 shows that when effort is monetary, the benefits of the flexibility of having superior information, which allows the player with an information advantage to exert a larger effort when the value of the prize is high, outweigh the benefits of committing to exert a relatively high effort that inferior information allows. When the cost of effort is not linear, however, the benefits of commitment may be greater that those of flexibility – see Einy, Moreno and Shitovitz (2002) for an example in an oligopolistic setting.

Next we study other properties of equilibrium of common-value Tullock contests. We begin by considering two-player common value Tullock contests in which one player has an information advantage over his rival. Existence and uniqueness of equilibrium in these contests follows by Remark 1 whenever $\rho_i^T(\mathbf{0}) < 1$ for every $i \in N$. We show that equilibrium exists and is unique – even when $\rho_i^T(\mathbf{0}) = 1$ for some i – by calculating it explicitly (Proposition 1). The formulas that we obtain for equilibrium efforts of the players allow us to establish some basic properties of equilibrium (Proposition 2). We then show by means of examples that these properties do not extend to contests with more than two players.

Let us index the set of states of nature as

$$\Omega = \{\omega_1, ..., \omega_m\}.$$

For $k = 1, ..., m$, write

$$p(\omega_k) = p_k \text{ and } V(\omega_k) = v_k,$$

[10]The demand function $P(\omega, x)$ is not differentiable at $x = 0$ – it is not even defined at 0 – and therefore does not formally satisfy the assumptions of Einy, Moreno and Shitovitz (2002). However, it is easy to see that in any equilibrium X of a common-value Tullock contest the total effort is positive in all states of nature, i.e., $X(\cdot) \neq \mathbf{0}$. Thus the non-differentiability at 0 is irrelevant, and the proof of the theorem in Einy, Moreno and Shitovitz (2002) applies in this case with no change.

and w.l.o.g. assume that

$$0 < v_1 \leq v_2 \leq ... \leq v_m.$$

Assume further that player 2 has an information advantage over player 1. Then we may postulate w.l.o.g. that the only information player 1 has about the state is the common prior belief, i.e., $\Pi_1 = \{\Omega\}$, whereas player 2 has perfect information about the state of nature, i.e., $\Pi_2 = \{\{\omega_1\}, ..., \{\omega_m\}\}$.[11] In such contests a strategy profile is a pair (X, Y), where X can be identified with a number $x \in \mathbb{R}_+$ specifying player 1's unconditional effort, and Y can be identified with a vector $(y_1, ..., y_m) \in \mathbb{R}_+^m$ specifying the effort of player 2 in each of the m states of nature. Abusing notation, we shall write $X = x$ and $Y = (y_1, ..., y_m)$ whenever appropriate.

Proposition 1 below describes the unique equilibrium in a two-player common-value Tullock contest in which player 2 has an information advantage over player 1. In doing so we extend the exercise of Warneryd (2003, section 5.2), who computed equilibrium strategies in the binary framework (with two states of nature, representing a high and a low value for the prize), into the general setting with m states of nature.

The following notation will be needed in characterizing the pure strategy Bayesian Nash equilibrium. For $k \in \{1, ..., m\}$ write

$$A_k = \left(\sum_{s=k}^{m} p_s \sqrt{v_s}\right) \left(1 + \sum_{s=k}^{m} p_s\right)^{-1}. \tag{11}$$

Note that

$$A_1 = \frac{E(\sqrt{V})}{2}.$$

Lemma 1 establishes a key property of the sequence $\{A_k\}_{k=1}^{m}$.

Lemma 1. *If* $\sqrt{v_{\bar{k}}} > A_{\bar{k}}$ *for some* $\bar{k} < m$, *then* $\sqrt{v_k} > A_k$ *and* $A_{\bar{k}} > A_k$ *for all* $k > \bar{k}$.

Proof. Assume that $\sqrt{v_{\bar{k}}} > A_{\bar{k}}$ for some $\bar{k} < m$. From (11),

$$\left(1 + \sum_{s=\hat{k}}^{m} p_s\right) A_{\hat{k}} = \sum_{s=\hat{k}}^{m} p_s \sqrt{v_s}$$

[11] This assumption entails no loss of generality, since, if Π_1 is a non-trivial partition of Ω, a separate anlalysis can be caried out *conditionally* on the realization of each $\pi_1 \in \Pi_1$. In the conditional analysis player 1 has the nill information endowment.

and

$$\left(1+\sum_{s=\hat{k}+1}^{m} p_s\right) A_{\hat{k}+1} = \sum_{s=\hat{k}+1}^{m} p_s\sqrt{v_s},$$

which implies

$$\left(1+\sum_{s=\hat{k}+1}^{m} p_s\right) \left(A_{\hat{k}} - A_{\hat{k}+1}\right) = p_{\hat{k}}\left(\sqrt{v_{\hat{k}}} - A_{\hat{k}}\right).$$

Since by assumption $\sqrt{v_{\hat{k}}} > A_{\hat{k}}$, we have: (i) $A_{\hat{k}} > A_{\hat{k}+1}$, and (ii) $\sqrt{v_{\hat{k}+1}} > A_{\hat{k}+1}$ (as $v_{\hat{k}+1} \geq v_{\hat{k}}$). The statement of the lemma follows by induction. ■

Let $k^* \in \{1, ..., m\}$ be the smallest index such that $\sqrt{v_k} > A_k$. Since

$$\sqrt{v_m} > \frac{p_m}{(1+p_m)}\sqrt{v_m} = A_m,$$

k^* is well defined.

Proposition 1. *A two-player common-value Tullock contest in which player 2 has an information advantage over player 1 has a unique Bayesian Nash equilibrium* $(X^*, Y^*) = (x^*, y_1^*, ..., y_m^*)$, *which is given by* $x^* = A_{k^*}^2$, $y_k^* = 0$ *for* $k < k^*$, *and* $y_k^* = A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right)$ *for* $k \geq k^*$. *Moreover, if he distribution of values is not too disperse, i.e.,* $\sqrt{v_1} > E(\sqrt{V})/2$, *then the equilibrium is interior (i.e.,* $k^* = 1$*).*

Proof. Let (X, Y), where $X = x$ and $Y = (y_1, ..., y_m)$, be a Bayesian Nash equilibrium, whose existence is guaranteed by Theorem 1. We show that $x > 0$. If $x = 0$, then $\rho_2^T(0) = 1$, since otherwise player 2 does not have a best response against $x = 0$. But then $y_1 = y_2 = ... = y_m = 0$, and therefore player 1 can profitably deviate by exerting an arbitrarily small effort $\varepsilon > 0$. Hence $x > 0$. Moreover, $y_k > 0$ for some $k \in \{1, ..., m\}$ since otherwise $x > 0$ is not a best response of player 1.

Since $x > 0$ maximizes player 1's payoff given Y,

$$\frac{\partial}{\partial x}\left(\sum_{s=1}^{m} p_s\left(v_s\frac{x}{x+y_s} - x\right)\right) = \sum_{s=1}^{m} p_s v_s \frac{y_s}{(x+y_s)^2} - 1 = 0. \tag{12}$$

And since y_s maximizes player 2's payoff in state ω_s given x,

$$\frac{\partial}{\partial y_s}\left(v_s\frac{y_s}{x+y_s} - y_s\right) = v_s\frac{x}{(x+y_s)^2} - 1 \leq 0, \tag{13}$$

(with equality if $y_s > 0$) for each $s = 1, ..., m$.

Notice next that if $y_k > 0$ for some $k < m$, then $y_{k'} > 0$ for all $k' > k$. Since $x > 0$, if $y_k > 0$ then $y_k = \sqrt{x}\left(\sqrt{v_k} - \sqrt{x}\right)$ by (13), and since $v_{k'} \geq v_k$ for all $k' > k$, $\sqrt{x}\left(\sqrt{v_{k'}} - \sqrt{x}\right) > 0$, i.e.,

$$v_{k'}\frac{x}{x^2} - 1 > 0,$$

for all $k' > k$. Then $y_{k'} = 0$ would violate inequality (13) for $s = k'$. Hence $y_{k'} > 0$.

Let k° be the smallest index such that $y_k > 0$. Thus, (12) implies

$$\sum_{s=1}^{m} p_s v_s \frac{y_s}{(x+y_s)^2} = \sum_{s=k^\circ}^{m} p_s v_s \frac{y_s}{(x+y_s)^2} = 1,$$

and (13) implies $y_{k'} = \sqrt{x}\left(\sqrt{v_{k'}} - \sqrt{x}\right) > 0$ for all $k' \geq k^\circ$. Hence $x = A^2_{k^\circ}$, $y_k = A_{k^\circ}\left(\sqrt{v_k} - A_{k^\circ}\right)$ for all $k \geq k^\circ$, and $y_k = 0$ for all $k < k^\circ$.

We now show that $k^\circ = k^*$, which establishes that the profile $(x^*, y^*_1, ..., y^*_m)$ identified in Proposition 1 is the unique equilibrium. Assume first that $k^\circ < k^*$. Then $\sqrt{v_{k^\circ}} \leq A_{k^\circ}$ since k^* is the smallest index such that $\sqrt{v_k} > A_k$, and hence $y_{k^\circ} = \sqrt{x}\left(\sqrt{v_{k^\circ}} - \sqrt{x}\right) = A_{k^\circ}\left(\sqrt{v_{k^\circ}} - A_{k^\circ}\right) \leq 0$, a contradiction as $y_{k^\circ} > 0$ by the definition of k°. Assume next that $k^\circ > k^*$. In this case, $y_{k^*} = 0$. Since $\sqrt{v_{k^*}} > A_{k^*}$, by Lemma 1

$$A^2_{k^*} > A^2_{k^\circ} = x, \tag{14}$$

and therefore

$$v_{k^*}\frac{x}{x^2} - 1 = \frac{A^2_{k^\circ}}{A^4_{k^\circ}}\left(v_{k^*} - A^2_{k^\circ}\right) > 0.$$

This stands in contradiction to (13), as $y_{k^*} = 0$ by the definition of $k^\circ (> k^*)$. We conclude that indeed $k^\circ = k^*$. ■

Proposition 1 in particular implies uniqueness and symmetry of equilibrium in the complete information case, i.e., when $m = 1$. In this case $k^* = 1$, and therefore $y^*_1 = A_1(\sqrt{v_1} - A_1) = v_1/4 = A^2_1 = x^*$ (this result is well known in the literature). This observation implies that a two-player common value Tullock contest with incomplete but symmetric information has a unique equilibrium, which is symmetric and involves players exerting effort equal to $E(V)/4$ in expectation.

With asymmetric information, i.e., when $m > 1$, our sufficient condition for an interior equilibrium, the inequality $\sqrt{v_1} > E(\sqrt{V})/2$, holds when, e.g., $v_m < 4v_1$. However, when the distribution of values is very disperse, we may well get a corner equilibrium with $k^* > 1$: player 2 will be inactive in all "low-value" states of nature (those in which $v_k < v_{k^*}$), but will exert positive effort in all "high-value" states of nature (those in which $v_k \geq v_{k^*}$).[12]

Having explicit formulae for equilibrium strategies of the players allows a qualitative comparison of players' chances to win and their conditional expected payoffs across the states of nature where player 2 is active, i.e., when $k \geq k^*$ and hence $y_k^* > 0$. The equilibrium probability that player 1 wins the prize when the state is ω_k is

$$\rho_{1k}^* := \rho_1^T(x^*, y_k^*) = \frac{A_{k^*}^2}{A_{k^*}^2 + A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right)} = \frac{A_{k^*}}{\sqrt{v_k}}$$

when $k \geq k^*$, whereas the probability that player 2 wins the prize is $\rho_{2k}^* = 1 - \rho_{1k}^*$. Thus, the larger is the realized value of the prize, the smaller (larger) is the probability that player 1 (player 2) wins the prize, i.e., $\rho_{1k'}^* \leq \rho_{1k}^*$ and $\rho_{2k'}^* \geq \rho_{2k}^*$ for $k' > k \geq k^*$, with a strict inequality if $v_{k'} > v_k$. Of course, the larger is the realized value of the prize, the larger is the effort of player 2, i.e.,

$$y_{k'}^* = A_{k^*}\left(\sqrt{v_{k'}} - A_{k^*}\right) \geq A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right) = y_k^*.$$

for $k' > k \geq k^*$ (with a strict inequality if $v_{k'} > v_k$). Additionally, for $k' > k \geq k^*$,

$$\rho_{1k'}^* v_{k'} = A_{k^*}\sqrt{v_{k'}} \geq A_{k^*}\sqrt{v_k} = \rho_{1k}^* v_k$$

(with a strict inequality if $v_{k'} > v_k$), i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 1; also,

$$\rho_{2k'}^* v_{k'} \geq \rho_{2k}^* v_{k'} \geq \rho_{2k}^* v_k$$

(with a strict inequality if $v_{k'} > v_k$), i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 2.

Proposition 2 below establishes other basic properties of the equilibrium of a two-player common value Tullock contest in which a player has an information advantage: (1) both players exert the same expected effort, (2) the player with an information advantage wins the prize less frequently than the less informed player, and (3) players' total effort is below the effort they exert when they have symmetric information. We will later present examples showing that parts 1 and 2 of Proposition 2 do not extend to contests with more than two players. (Recall that, on the contrary, Theorem 2

[12] See Example 1 that illustrates this result when $m = 2$.

applies to contests with more than two players: when a player has an information advantage over *some* other player, then in any equilibrium the expected payoff of the former is greater or equal to that of the latter.)

Warneryd (2003) establishes counterparts to Proposition 2, as well as a version of Theorem 2 for two-player common value Tullock contest in which a player has an information advantage, when the players' common-value V is distributed according to a continuous cumulative distribution. As far as we can see, Warneryd (2003)'s results do not apply to our discrete setting.[13]

Proposition 2. *In the equilibrium of a two-player common-value Tullock contest in which player 2 has an information advantage over player 1:*

1. *Both players exert the same expected effort, i.e.,* $E(Y^*) = x^* = A_{k^*}^2$*, and hence the expected total effort is* $TE = X^* + E(Y^*) = 2A_{k^*}^2$*.*

2. *If* $v_1 < v_2 < ... < v_m$*, then the ex-ante probability that player 2 wins the prize is less than that of player 1.*

3. *If* $v_1 < v_m$ *and* $\sqrt{v_1} > E(\sqrt{V})/2$*, then the players' exert less effort and hence capture a greater share of the surplus than when both players have symmetric information.*

Proof.

Part 1. By Proposition 1,

$$\begin{aligned} E(Y^*) &= \sum_{s=1}^{m} p_s y_s^* \\ &= \sum_{s=k^*}^{m} p_s A_{k^*} \left(\sqrt{v_s} - A_{k^*}\right) \\ &= A_{k^*} \sum_{s=k^*}^{m} p_s \sqrt{v_s} - A_{k^*}^2 \sum_{s=k^*}^{m} p_s \\ &= A_{k^*}^2 \left(1 + \sum_{s=k^*}^{m} p_s\right) - A_{k^*}^2 \sum_{s=k^*}^{m} p_s \\ &= A_{k^*}^2. \end{aligned}$$

[13] Exception being the already mentioned computation of equilibrium in the binary setting in Warneryd (2003, Section 5.2), that our Proposition 1 extends.

Part 2. Given $(y_{k^*}, ..., y_m) \in \mathbb{R}_+^{m-k^*+1}$ define the function

$$\bar{p}_2\left(y_{k^*}, ..., y_m\right) := \sum_{k=k^*}^{m} \frac{p_k y_k}{y_k + \sum_{s=k^*}^{m} p_s y_s}.$$

Denote by $\bar{\rho}_2$ the ex-ante probability that player 2 wins the prize. Since x^* satisfies $x^* = E(Y^*)$ by Part 1, obviously

$$\bar{\rho}_2 = \bar{p}_2\left(y_{k^*}^*, ..., y_m^*\right).$$

We will show that a maximum point $\overline{y}$ of $\bar{p}_2$ on $K = \{(y_{k^*}, ..., y_m) \in \mathbb{R}_+^{m-k^*+1} \mid y_{k*} \leq y_{k*+1} ... \leq y_m\}$ must satisfy $\overline{y}_{k*} = ... = \overline{y}_m$. Hence

$$\max_K \bar{p}_2 = \frac{\sum_{s=k^*}^{m} p_s}{1 + \sum_{s=k^*}^{m} p_s} \leq \frac{1}{2}. \tag{15}$$

Since $y_{k^*}^* < ... < y_m^*$ (the inequalities are strict, which follows from our assumption that $v_1 < v_2 < ... < v_m$ and the expressions for $(y_k^*)_{k=k^*}^m$ given in Proposition 1), (15) will imply

$$\bar{\rho}_2 = \bar{p}_2\left(y_{k^*}^*, ..., y_m^*\right) < \max_K \bar{p}_2 \leq 1/2,$$

and this will yield Part 2 of the proposition.

To this end, differentiate $\bar{p}_2$ with respect to y_k for $k \in \{k^*, ..., m\}$ to obtain

$$\frac{\partial \bar{p}_2}{\partial y_k} = p_k \left(\sum_{t=k^*, t \neq k}^{m} \frac{p_t y_t}{(y_k + \sum_{s=k^*}^{m} p_s y_s)^2} - \sum_{t=k^*, t \neq k}^{m} \frac{p_t y_t}{(y_t + \sum_{s=k^*}^{m} p_s y_s)^2} \right). \tag{16}$$

For every $(y_{k^*}, ..., y_m) \in K$ such that $y_{k^*} < y_{k^*+1} \leq ... \leq y_m$, $\partial \bar{p}_2 / \partial y_{k^*}(y) > 0$, and therefore necessarily $\overline{y}_{k^*} = \overline{y}_{k^*+1}$. Suppose now that it has already been shown that $\overline{y}_{k^*} = \overline{y}_{k^*+1} = ... = \overline{y}_k$, $m-1 \geq k > 1$. We show that $\overline{y}_{k+1} = \overline{y}_k$ as well. Indeed, if $\overline{y}_{k^*} = \overline{y}_{k^*+1} = ... = \overline{y}_k < \overline{y}_{k+1} \leq ... \leq \overline{y}_m$, then by (16) we obtain that $\partial \bar{p}_2 / \partial y_k(\overline{y}) > 0$, a contradiction. Thus $\overline{y}_{k^*} = ... = \overline{y}_m$.

Part 3. When player 2 has an information advantage, $\sqrt{v_1} > E(\sqrt{V})/2$ implies that the equilibrium is interior by Proposition 1, and therefore the expected total effort is $TE = 2A_1^2 = \left(E(\sqrt{V})\right)^2/2$ by Proposition 2.1. As noted previously, when players have symmetric information the expected total effort $\overline{TE}$ is $\overline{TE} = E(V)/2$. Then $v_1 < v_m$ together with Jensen's inequality imply

$$\overline{TE} - TE = \frac{E(V)}{2} - \frac{\left(E(\sqrt{V})\right)^2}{2} > 0. \blacksquare$$

Example 1 illustrates the results of Proposition 2 for a binary state space (Warneryd (2003) demonstrated his central ideas in continuously distributed common value contests by a similar binary example). This example will be useful in our discussion in the next section.

Example 1. Assume $m = 2$. Write $p_1 = 1 - p$, where $p \in (0, 1)$, and normalize the common value so that $v_1 = 1$ and $v_2 = v$ with $v \in (1, \infty)$. Then $E(V) = 1 - p(1 - v)$, $E(\sqrt{V}) = 1 - p(1 - \sqrt{v})$, $A_1 = E(\sqrt{V})/2$, and $A_2 = p\sqrt{v}/(1 + p)$. If $v < (1 + p)^2 / p^2$, then $\sqrt{v_1} = 1 > A_1$ and $k^* = 1$; otherwise $k^* = 2$. In a Tullock contest in which player 2 observes the value but player 1 does not, the unique equilibrium is

$$X^* = A_1^2,\ Y^* = \left(A_1 \left(1 - A_1\right), A_1 \left(\sqrt{v} - A_1\right)\right),$$

and the total effort is $TE = 2A_1^2 = [1 - p(1 - \sqrt{v})]^2/2$ when $v < (1 + p)^2 / p^2$. Otherwise, the unique equilibrium is

$$X^* = A_2^2,\ Y^* = \left(0, A_2 \left(\sqrt{v} - A_2\right)\right),$$

and the total effort is $TE = 2A_2^2 = 2p^2 v/(1+p)^2$. If $v < (1 + p)^2 / p^2$, then the ex-ante probability that player 1 wins the prize $\bar{\rho}_1^*$ is

$$\bar{\rho}_1^* = (1 - p) A_1 + p\frac{A_1}{\sqrt{v}} = \frac{1}{2}\left(p + (1 - p)\sqrt{v}\right)\frac{1 - p + p\sqrt{v}}{\sqrt{v}} \geq \frac{1}{1 + p} > \frac{1}{2}.$$

Otherwise, this probability is

$$\bar{\rho}_1^* = (1 - p) + p\frac{A_2}{\sqrt{v}} = (1 - p) + \frac{p^2}{1 + p} = \frac{1}{1 + p} > \frac{1}{2}.$$

Hence, consistent with Proposition 2.2, the informed player wins the prize less frequently than the uninformed player. Further, if $v < (1 + p)^2 / p^2$, then

$$\begin{aligned}
2\left[U_2(X^*, Y^*) - U_1(X^*, Y^*)\right] &= (1 - p)\frac{A_1 \left(1 - A_1\right) - A_1^2}{A_1^2 + A_1 \left(1 - A_1\right)} + pv\frac{A_1 \left(\sqrt{v} - A_1\right) - A_1^2}{A_1^2 + A_1 \left(\sqrt{v} - A_1\right)} \\
&= (1 - p)\, p \left(1 - \sqrt{v}\right)^2 \\
&> 0.
\end{aligned}$$

And if $v \geq (1+p)^2/p^2$, then

$$\begin{aligned}
2\left[U_2(X^*,Y^*) - U_1(X^*,Y^*)\right] &= -(1-p) + pv\frac{A_2\left(\sqrt{v}-A_2\right)-A_2^2}{A_2^2+A_2\left(\sqrt{v}-A_2\right)} \\
&= \frac{1-p}{p+1}\left(p(v-1)-1\right) \\
&> \frac{1-p}{p} \\
&> 0.
\end{aligned}$$

That is, consistent with Theorem 2, the payoff of the informed player is greater or equal to that of the uninformed player. Under symmetric information the equilibrium total effort in a Tullock contest is $E(V)/2 > \max\{2A_1^2, 2A_2^2\}$, i.e., the total effort when player 2 has an information advantage is less than when both players have the same information.

The following examples show that the properties established in propositions 2.1 and 2.2 do not extend to common-value Tullock contests with more than two players. In Example 2 player 1 has only the prior information whereas players 2 and 3 have complete information. In equilibrium the expected effort of the uninformed player is *below* that of each of the informed players.

Example 2. Consider a 3-player common-value Tullock contest in which $m = 2$, $p_1 = p_2 = 1/2$, $v_1 = 1$ and $v_2 = 2$. Player 1 has no information, i.e., his information partition is $\Pi_1 = \{\omega_1, \omega_2\}$, and players 2 and 3 have complete information, i.e., their information partitions are $\Pi_2 = \Pi_3 = \{\{\omega_1\}, \{\omega_2\}\}$. In the interior equilibrium of this contest, which is readily calculated by solving the system of equations formed by the players' reaction functions, the effort of player 1 is $X_1^* = 0.30899$ while the efforts of players 2 and 3 are $X_2^* = X_3^* = (0.20342, 0.46933)$. Note that

$$X_1^* = 0.30899 < \frac{1}{2}(0.20342 + 0.46933) = E(X_2^*) = E(X_3^*),$$

i.e., the effort of player 1 is less than the expected effort of players 2 and 3.

In Example 3 there is an informed player and a number of uninformed players. In equilibrium, the ex-ante probability that the informed player wins the prize is *above* that of the uninformed players. Thus, the natural extension of Proposition 2.2 to contests with more than two players does not hold.

Example 3. Consider an eight player common-value Tullock contest in which $m = 2$, $p_1 = p_2 = 1/2$, $v_1 = 1$ and $v_2 = 2$. Players 1 to 7 have no information, i.e., their information partition is $\Pi_i = \{\omega_1, \omega_2\}$ for $i \in \{1, ..., 7\}$, and player 8 is completely informed, i.e., his information partitions is $\Pi_8 = \{\{\omega_1\}, \{\omega_2\}\}$. This contest has a (corner) equilibrium given by

$$X_1^* = ... = X_7^* = 0.15551,\ X_8^* = (0, 0.38694).$$

In equilibrium, the ex-ante probability that player $i \in \{1, 2, ..., 7\}$ wins the prize is

$$\bar{\rho}_i^* = \frac{1}{2}(\frac{1}{7} + \frac{0.155\,51}{7(0.155\,51) + 0.386\,94}) = 0.12413,$$

whereas the ex-ante probability that player 8 win the prize is

$$\bar{\rho}_8^* = 1 - 7(0.12413) = 0.13109.$$

Thus, the informed player wins the prize more frequently than an uninformed player.

5 Common-Value Tullock Contests and All-Pay Auctions

Contests that arise in many economic and political applications are effectively all-pay auctions either by design (e.g., sports or political competition) or by the nature of the problem (e.g., patent races). Here we study whether the players' expected total effort in all-pay auctions and Tullock contests can be ranked.

A *common-value all-pay auction* is a common-value contest in which the success function is given for $x \in \mathbb{R}_+^n$ by $\rho^{APA}(x) = 1/m(x)$ if $x_i = \max\{x_j\}_{j \in N}$, and $\rho^{APA}(x) = 0$ otherwise, where $m(x) = |k \in N : x_k = \max\{x_j\}_{j \in N}|$ (thus, conditions (i) and (ii) on the success function in generalized Tullock contests are not satisfied). Einy, Haimanko, Orzach and Sela (2013) show that in the unique equilibrium of a two-player common-value all-pay auction in which $v_1 < ... < v_m$ and player 2 observes the value while player 1 does not, the players' total expected effort is

$$TE^{APA} = 2\sum_{s=1}^{m} p_s \left(\sum_{k=1}^{s-1} p_k v_k + \frac{1}{2} p_s v_s \right) = 2\sum_{s=1}^{m} p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^{m} p_s^2 v_s.$$

Hence the difference between total efforts in an all-pay auction and a Tullock contest is

$$\Delta := TE^{APA} - TE = 2\sum_{s=1}^{m} p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^{m} p_s^2 v_s - 2A_{k^*}^2.$$

For simplicity, consider the case studied in Example 1, in which $m = 2$. If the equilibrium of the Tullock contest is interior, i.e., $v < (1+p)^2/p^2$, then

$$\begin{aligned} \Delta &= 2(1-p)p + (1-p)^2 + p^2 v - 2A_1^2 \\ &= 2(1-p)p + \frac{1}{2}\left((1-p) - p\sqrt{v}\right)^2 \\ &> 0. \end{aligned}$$

Hence an all-pay auction generates more effort that a Tullock contest. However, if the Tullock contest has a corner equilibrium, then

$$\begin{aligned} \Delta &= 2(1-p)p + (1-p)^2 + p^2 v - 2A_2^2 \\ &= 1 - p^2 - \left(\frac{2}{(1+p)^2} - 1\right) p^2 v. \end{aligned}$$

The sign of Δ may be either positive or negative depending on the distribution of the players' common value. Assume that $p = 1/4$. Then

$$\Delta = \frac{15}{16} - \frac{7}{400}v \gtreqless 0 \Leftrightarrow v \lesseqgtr 375/7.$$

Hence the total effort generated by all-pay auctions and Tullock contests cannot be ranked in general.

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