Viable economic states in a dynamic model of taxation

J. B. Krawczyk[*] K.L. Judd[†]

November 4, 2012

Abstract

Viability theory is the study of dynamical systems that asks what is the set of initial conditions that generate evolutions which obey the laws of motion of a system and some state constraints, for the length of the evolution. We apply viability theory to Judd's (JPE, 1987) dynamic tax model to identify which are the economic states today that are compatible with only slightly constrained tax-rate adjustments in the future, when the dynamic budget constraint and consumers' TVC in infinity are satisfied. The set of such states we call the economic viability kernel. We observe, unsurprisingly, that a very high consumption economy lies outside such kernels, at least for annual tax-adjustment levels limited by 20%; higher consumption levels can only be sustained when capital is abundant. Furthermore, we notice that the sizes of the kernel slices, for a given taxation level, do not diminish as the tax rate-rate rises, hence high taxation economies are not necessarily more prone to explode, or implode, than their low taxation counterparts. In fact, higher tax rates are necessary to keep many consumption choices viable, when capital approaches the constraint-set boundary. In broad terms, knowledge of the viability kernel can tell the planner what economic objectives are achievable and assists the choice of suitable controls to effectuate them.

.

Keywords: taxation policy, macroeconomic modeling, dynamic systems, viability theory; VIKAASA
JEL: C61, E61, E62

1 Introduction

This paper uses viability theory (Aubin (1997)), to examine basic problems in dynamic public finance. For specificity, we use the model studied in Judd (1987)[1].

Viability theory is the study of dynamical systems that asks what is the set of possible paths that obey the laws of motion of a system and remain in some

[**]Victoria University of Wellington, New Zealand
[†]Stanford University, CA
[1]This paper draws from Krawczyk and Judd (2012).

state-constraint set. In one example in our paper, we compute the set of possible consumption levels today given a fixed level of government expenditure in the future and making only loose restrictions on tax policy. Another way of putting this is that we perform a kind of robustness analysis to answer the question *what are the possible consumption levels today if all we know is that tax policy will satisfy the dynamic budget constraint and that consumers' transversality conditions in infinity will be satisfied?* The usual perfect foresight analysis specifies one future path for taxes. The viability theory approach asks how much does the perfect foresight result depends on having perfect foresight. For example, suppose that we have some debt today and know the future path of tax rates and government expenditure. Then, there would (likely) be only one consumption and capital combination, which would be *viable* i.e., this economic evolution could originate from. In that case, viability reduces to equilibrium. On the other hand, a viability analysis is about establishing *all* pairs of consumption and capital (c, k) such that there is some future tax-rate path that obeys the restrictions we put on the change in tax rate and is consistent with equilibrium with initial condition (c, k). We assert the collection of all such initial conditions, we will say *viability kernel*, generalizes the notion of equilibrium, which is one theme of viability theory.

We find that if the only tax is a proportional income tax, then uncertainty about future tax policy does not affect consumption much. However, in other tax systems, such as one that taxes labor and capital differently, uncertainty about future tax policy may lead to much greater uncertainty about current consumption.

This paper focuses on some specific questions in a simple dynamic model of expenditure and taxation. However, there is a much more ambitious agenda behind this paper, which is to present viability theory as an important tool for economic problems' solution .[2] Its main machinery consists of the formulation and solution of differential inclusions. That is, in viability theory, the system's dynamics is represented as a *set* of the directions of motion of the system that depends at any moment on the state. The concept of solution is a path of sets instead of a path of points, where the "tube" formed by those sets is the union of all possible paths that stay in the tube but also satisfy the usual terminal constraints and some additional state restriction. Viability theory is part of set-valued analysis.

Solving viability problems is computationally intensive. However, thanks to some specialized software, solving simple models, of 2 – 4 state variables and 1 – 2 controls, is possible. The software we use is VIKAASA (see Krawczyk and Pharo (2011) and Krawczyk and Pharo (2012)).

Here is how the paper is organized. We expound viability theory in Section 2. Following Judd (1987), we introduce a simple model of expenditure and taxation

[2]So far, viability theory has been applied to a handful of economic and financial problems. For applications to *environmental economics* see Martinet and Doyen (2007), De Lara, Doyen, Guilbaud, and Rochet (2006) and Martinet, Thébaud, and Doyen (2007); *finance* – Pujal and Saint-Pierre (2006); *managerial economics* – Krawczyk, Sissons, and Vincent (2012); *macroeconomics* – Krawczyk and Kim (2009), Bonneuil and Saint-Pierre (2008), Bonneuil and Boucekkine (2008), Krawczyk and Kim (2004), Krawczyk and Sethi (2007), Clément-Pitiot and Saint-Pierre (2006), Clément-Pitiot and Doyen (1999); *microeconomics* – Krawczyk and Serea (2011). However, many of the above publications are working papers of limited circulation.

in Section 3. In Section 4, we make an assumption that the only tax charged in this model will be a proportional income tax and calibrate the model according to this assumption. Further, in Section 5, we compute viability kernels and comment on their topology. We also show (in Section 6) a few possible time profiles of the *debt-to-GDP* ratio and observe that a high value of the ratio does not have to imply non-viability. The paper ends with concluding remarks.

2 A brief on viability theory and viable solutions

2.1 An introduction to viability theory

Viability theory is a relatively new part of continuous mathematics, see e.g., Aubin (1991, 1992, 1997, 2001). Viability problems concern systems that evolve over time, where the concern is to identify *viable evolutions* – trajectories that do not violate some set of viability constraints over a given (possibly infinite) time-frame. A *viability domain* that is the set of initial states from which viable trajectories originate and, in particular, the *viability kernel* that is the *largest* domain, hence become useful tools for analyzing such problems.

The basic feature of the viability kernel is that it provides us with the information necessary to determine whether or not a given state-space position has a viable trajectory proceeding from it, i.e., whether starting at that position, the system can be maintained within its constraints, or not. In what follows, we give a more technical explanation of viability theory, including a formal definition of the *viability kernel*.

The core ingredients of a viability problem are (compare Krawczyk and Pharo (2011)):

1. A continuum of time[3] values, $\Theta \equiv [0, T] \subseteq \mathbb{R}_+$, where T can be finite or infinite.

2. A vector of n real-valued state variables, $x(t) \equiv [x_1(t), x_2(t), \ldots, x_n(t)]' \in \mathbb{R}^n$, $t \in \Theta$ that together represent the dynamic system in which we are interested.

3. A *constraint set*, $K \subset \mathbb{R}^n$, which is a closed set representing some normative constraints to be imposed on these state variables. Violation of these constraints means that the system has become non-viable. Thus in seeking viable trajectories, we want to ensure that $\forall t (t \in \Theta)\ x(t) \in K$.

4. A vector of real-valued controls, $u(t) \equiv [u_1(t), u_2(t), \ldots, u_m(t)]' \in \mathbb{R}^m$, $t \in \Theta$.

5. Some normative constraints on the controls, so that

$$\forall t\ \forall x\ t \in \Theta \wedge x \in K,\ u(t) \in U(x(t)) \subset \mathbb{R}^m.$$

   So, $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is a set-valued function, which gives the set of control vectors available in each state. Thus the control vector at time t can be constrained according to the state $x(t)$ of the system.

[3] A similar formulation could be made for a viability problem in discrete time.

6. A set of real-valued first-order differential inclusions,

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \in \left\{ \psi(x,u) = \begin{bmatrix} \psi_1(x,u) \\ \psi_2(x,u) \\ \vdots \\ \psi_n(x,u) \end{bmatrix} \right\}_{u \in U} . \tag{1}$$

Each function $\psi_i : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, $i = 1, 2 \dots n$ is a poin-to-set mapping, which specifies the range (cone) of velocities of the corresponding variable x_i, for any pair (x, u), where $x \in \mathbb{R}^n$ is a position in the state space, and $u \in U \subset \mathbb{R}^m$ is a control choice. Some, but not all, inclusions can be equalities.

Note that we have formulated viability problems above in terms of *differential inclusions* whereby the evolution of some or all of the system's variables is *set-valued*. That is, for a given $x(t)$ we have an array of possible controls $U(x(t))$ to choose from and hence have a *set* of velocities $\psi(x(t), u(x(t)), u(x(t)) \in U(x(t))$, associated with state $x(t)$. Symbol ψ denotes then a point-to-set map, or correspondence, from states x to velocities $\psi(x, U)$. We will abbreviate the notation and write $\psi(x)$ instead of $\psi(x, U)$. For small $\sigma > 0$, $x(t) + \sigma\psi(x(t))$ defines points in the state space, which can be reached in time $t + \sigma$. We are then concerned with confiming existence of those members of the set U, for which the trajectories are viable i.e., $x(t) + \sigma\psi(x(t)) \in K$ for all $t \in \Theta$.

Given such a problem, we can attempt to find one or more *viability domains*, $D \subseteq K$, where each viability domain is a set of initial conditions $x(0)$, for which there exist viable trajectories. That is, for every element $x(0) \in D$, there exists a function (or feedback rule) $g : \mathbb{R}^n \mapsto \mathbb{R}^m$, such that for each element k of constraint set $K \subset \mathbb{R}^n$, $g(k) \in U(k)$ and $\forall t \in T, x(t) \in K$ where $x(t)$ is a solution to (1) with $u(t) = g(x(t))$. In other words, for every initial state in D, there must exist sufficient control from U to prevent violation of the viability set K, over $t \in \Theta$. The problem's *viability kernel*, $V \subseteq K$ is then the *largest* possible viability domain (or the union of all viability domains), giving all initial conditions in K, for which a set of controls in U exists to prevent the system from exiting K over $t \in \Theta$.

More formally, we use the notion of *viability* introduced in Quincampoix and Veliov (1998) (for existence and characterisation of feedback controls assuring viability see Veliov (1993)). We will characterize a viability domain using the Viability Theorem from Cardaliaguet, Quincampoix, and Saint-Pierre (1999a) (Theorem 2.3):

Proposition 1. *Assume D is a closed set in $\mathbb{R}^N$. Suppose that $\psi : \mathbb{R}^N \times U \to \mathbb{R}^N$ is a continuous function, Lipschitz in the first variable; furthermore, for every x we define a set valued map $\psi(x, U) = \{\psi(x, u); u \in U\}$, which is supposed to be Lipschitz continuous with convex, compact, nonempty values.*

Then the two following assertions are equivalent [4]*:*

1.

$$\forall x \in D, \quad \forall p \in \mathcal{NP}_D(x), \quad \min_u \langle \psi(x,u), p \rangle \leq 0 \tag{2}$$

[4] Here $\mathcal{NP}_D(x)$ denotes the set of *proximal normals* to D at x i.e., the set of $p \in \mathbb{R}^N$ such that the distance of $x + p$ to D is equal to $\|p\|$.

(respectively, $\max_u \langle \psi(x,u), p\rangle \le 0$*)* ;

2. *there exists* $u \in U_{[t,T]}$ *such that*

   (respectively, for all $u \in U_{[t,T]}$*)*

$$\textit{the solution of} \quad \begin{cases} c\dot{x}(s) = \psi(x(s), u(s)) \textit{ for almost every } s \\ x(t) = x \end{cases} \tag{3}$$

   remains in D.

Notice that the inequality $\min_u \langle \psi(x,u), p\rangle \le 0$ in (2) means that there *exists* a control for which the system's velocity $\dot{x}$ "points inside" the set D. Respectively, $\max_u \langle \psi(x,u), p\rangle \le 0$ means that the system's velocity $\dot{x}$ "points inside" the set D for *all* controls from U.

When 1. (or 2.) holds we say that D is a *viability domain* (or, respectively, D is an *invariance domain*) for the dynamics ψ.

This introduces the classical notion of viability (respectively, invariance) domain Aubin (2001), as opposed to viability domains in problems with *targets*, see Quincampoix and Saint-Pierre (1995).

Definition 2.1. *Let K is a closed set in* $\mathbb{R}^N$*. We call* ***viability kernel*** *in K, for the dynamics* ψ*, denoted:*

$$\mathcal{V}_\psi(K)$$

the largest closed subset of K, which is a viability domain for ψ*.*

It was proved (see e.g., Aubin (1992) or Quincampoix and Veliov (1998)) that $\mathcal{V}_\psi(K)$ is *the* set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \psi(x(s)) \tag{4}$$

starting from x, which is defined on $[0, \infty)$ and $x(s) \in K$ for all $s \ge 0$.

2.2 A method for the determination of viability kernels

We will approximate $\mathcal{V}_\psi(K)$ by looking for solutions to (4). If ψ is the collective vector of right hand sides like in (1) then the problem that we want to *solve* is

$$\textit{establish viability kernel} \quad \mathcal{V}_\psi(K) \quad \textit{for the dynamics } \psi\,. \tag{5}$$

In Gaitsgory and Quincampoix (2009) we can find the base for how to establish $\mathcal{V}_\psi(K)$ using the solutions to (4). In broad terms, they say that if *an* optimal control problem can be solved for $x \in K$ and $x(t) \in K\ \forall t$, then x is viable. Our method consists of solving a truncated optimal stabilization problem, rather than some general optimization problem, for each $x^h \in K^h \subset K$ where K^h is a suitably discretized K.

VIKAASA, see Krawczyk and Pharo (2011) and Krawczyk and Pharo (2012) (also Krawczyk, Pharo, and Simpson (2011)), is a computational tool based on the above method, which can be used to create approximate viability kernels (actually, domains[5]) for the class of viability problems introduced in Section 2.1.

In short, VIKAASA divides the problem into a discrete set of points, and then assesses whether, when starting from each point, the dynamic evolution of the system can be slowed to a (nearly) steady state without leaving the constraint set in finite time. Those points that can be brought close enough to such a state are included in the kernel by the algorithm, whilst those that are not are excluded.

In Section 5 we present some results from running the algorithm on the taxation problem.[6]

3 The tax model

Our goal in this paper is to use viability theory for an analysis of a tax model based on Judd (1987).

In that model, capital, labor, consumption, debt, marginal utility of consumption and tax rates are all variables of time. However, to unburden the notation we will drop the time argument on each of them.

The fundamental law of motion for capital k is determined by net output i.e., $y - \delta k$, where y is output and $\delta > 0$ is the rate of depreciation, diminished by consumption $c > 0$; government expenditure is $g \geq 0$. If so and assuming a Cobb-Douglas type production function for output, we get, in continuous time t,

$$\frac{dk}{dt} = A \cdot k^{\alpha} \ell^{1-\alpha} - \delta k - c - g \,. \tag{6}$$

As usual, $\ell > 0$ is labor, $A > 0$ — total factor productivity and α, $0 < \alpha < 1$ — output elasticity of capital. In this model, expenditure g is assumed constant but several values of g will be checked in the computations.

Let the utility of consumption of a representative agent be

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{7}$$

and the disutility of labor

$$v(\ell) = V \frac{\ell^{1+\eta}}{1+\eta} \tag{8}$$

where V, γ, η are positive. If $\lambda > 0$ is the private marginal value of capital at time t, then it follows from maximization of the utility function $u(c) - Vv(\ell)$, on an infinite horizon with some discount rate $\rho > 0$, that[7]

$$\frac{d\lambda}{dt} = \lambda(\rho - \bar{r}) \,. \tag{9}$$

[5] Our method will miss some viable points if they are viable only because the evolutions starting at them are large orbits. In our experiments, we have not encountered points like that.

[6] VIKAASA has produced results in Krawczyk et al (2011) that coincide with with those from Krawczyk and Serea (2009), where a method based directly on Gaitsgory and Quincampoix (2009) was applied to the same monetary economics problem. In turn, the outputs in Krawczyk and Serea (2009) coincide with those published in Krawczyk and Kim (2009).

[7] Just write the Hamiltonian and the adjoint state equation. Also notice that λ is the agent's marginal utility of consumption; see (15).

Here, $\bar{r} = (1 - \tau_K)\dfrac{\partial y}{\partial k}$ is the after tax ($0 < \tau_K < 1$) marginal product of capital, so

$$\frac{d\lambda}{dt} = \lambda\left(\rho - (1 - \tau_K)\left(\alpha A\left(\frac{\ell}{k}\right)^{1-\alpha} - \delta\right)\right). \tag{10}$$

To characterize the economy at hand, we will also use debt B, which grows in g and diminishes with tax T as follows:

$$\frac{dB}{dt} = \bar{r}B - T + g \tag{11}$$

where, as above, $\bar{r}$ is the net-of-tax interest rate. In this economy, tax rates on capital and labor are τ_K and τ_L ($0 < \tau_L < 1$, $0 < \tau_K < 1$), respectively; if so, the expression for total tax T in (11) at time t becomes

$$\begin{aligned} T &= \tau_K \alpha A k^\alpha \ell^{1-\alpha} + \tau_L (1-\alpha) A k^\alpha \ell^{1-\alpha} \\ &= \Big(\alpha(\tau_K - \tau_L) + \tau_L\Big) A k^\alpha \ell^{1-\alpha}. \end{aligned} \tag{12}$$

Combing (12) and (11) results in the following debt dynamics

$$\frac{dB}{dt} = \bar{r}B - \Big(\alpha(\tau_K - \tau_L) + \tau_L\Big) A k^\alpha \ell^{1-\alpha} + g, \tag{13}$$

where $\bar{r} = (1 - \tau_K)(\alpha A k^{-(1-\alpha)} \ell^{1-\alpha} - \delta)$ will be included in this expression later. In simple terms, we see that debt can diminish if output is large or if the tax rates are high (and when output is not too small).

While the private value of capital, λ, can adequately characterize the consumer's behavior, it lacks an easy economic interpretation. We will replace the equation for $\dfrac{d\lambda}{dt}$, (9), by a differential equation for consumption, easily interpretable.

The marginal utility of consumption (see (7)) is

$$\frac{du}{dc} = \frac{1}{c^\gamma}; \tag{14}$$

on the other hand, λ is the marginal utility of consumption, so

$$\frac{du}{dc} = \lambda \tag{15}$$

hence,

$$c = \frac{1}{\lambda^{1/\gamma}}, \tag{16}$$

which, after differentiation in the time domain, yields

$$\frac{dc}{dt} = \frac{1}{\gamma} \cdot \frac{1}{\lambda^{1+1/\gamma}} \cdot \frac{d\lambda}{dt} = \frac{1}{\gamma} c^{1+\gamma} \frac{d\lambda}{dt}. \tag{17}$$

Using (10), after some simplifications, we get

$$\frac{dc}{dt} = -c \cdot \frac{\rho + \left(\delta - \alpha A k^{\alpha-1} \ell^{1-\alpha}\right)(1 - \tau_K)}{\gamma} \tag{18}$$

We can see that consumption has one trivial steady state and will grow if ρ (discount rate) and/or δ (depreciation) are "small".

We will now write the three equations of motion (6), (18), (13) together, for a better look at the economy we want to analyze:

$$\frac{dk}{dt} = Ak^{\alpha}\ell^{1-\alpha} - \delta k - c - g \tag{19}$$

$$\frac{dc}{dt} = -c \cdot \frac{\rho + \left(\delta - \alpha\, A\, k^{\alpha-1}\, \ell^{1-\alpha}\right)(1-\tau_K)}{\gamma} \tag{20}$$

$$\frac{dB}{dt} = \bar{r}B - \left(\alpha(\tau_{_K} - \tau_{_L}) + \tau_{_L}\right)Ak^{\alpha}\ell^{1-\alpha} + g\,. \tag{21}$$

The system of differential equations (19) - (21) is the basic representation of the economy at hand, for which we want to establish the viability kernel i.e., the loci of economic states, from which moderate tax adjustments can guarantee a balanced evolution of the economy.

We recognize that this system is nonlinear with multiple steady states. We can see that, expectantly, the consumption growth or decline can be moderated by adjusting the capital tax rate while debt will (mainly) depend on the labor tax rate. If the rates were identical ($\tau_{_L} = \tau_{_K}$), then increasing them/it will slow down the consumption rate and diminish debt. With high taxation rate, consumption and debt will naturally diminish and and capital will grow (because labor increases, see below). We also notice that debt will grow very fast for large B and non-excessive capital taxation.

We now want to express labour ℓ through capital and consumption and thus "close" the dynamic system (19) - (21).

Let w denote (time-dependent) wages; they equal to the marginal product of labour:

$$w = \frac{dy}{d\ell} = \frac{(1-\alpha)k^{\alpha} A}{\ell^{\alpha}} \tag{22}$$

In equilibrium, the marginal utility of consumption weighted by the after-tax wages must be equal to the marginal disutility from labor:

$$\frac{(1-\tau_{_L})w}{c^{\gamma}} = \ell^{\eta} V\,. \tag{23}$$

Substituting wages and solving for labor yields,

$$\ell = \left(\frac{(1-\tau_{_L})(1-\alpha)Ak^{\alpha}}{c^{\gamma}V}\right)^{1/(\alpha+\eta)}\,, \tag{24}$$

from which we see that labor is determinable by capital and consumption.

We could now use (24) to substitute labor in (19) - (21) but, the resulting formulae are more complicated than the original equations, even if they contain one variable less. We will not show them here. We will use them though in the computations, after we have calibrated the equations. Here, we can observe that if $\gamma > \alpha$ than labor decreases in consumption faster than it grows in capital. Allowing for this tells us that the sign of (20) will be negative for large discount and depreciation rates hence high consumption levels will quickly diminish. Large consumption will also contribute to a decline of capital and a rise of debt. However, this multiple downturn may be avoided by an "early" (preemptive) drop of taxes on capital. We will see from which states such an preventive drop can be efficient, after we have computed the viability kernel for this economy, in Section 5 .

To fully describe the tax model dynamics, the equations (19) - (21) (with (24)) need be completed by two differential inclusions for the two tax rates τ_L and τ_K:

$$\frac{d\tau_L}{dt} = u_L \in [-d_L, d_L] = U_L \tag{25}$$

and

$$\frac{d\tau_K}{dt} = u_K \in [-d_K, d_K] = U_K \tag{26}$$

where d_L, d_K are positive numbers. The inclusions represent bounds on the speed at which tax rates can change. This corresponds to the government policy of "smooth" tax rates adjustments determined by d_L and d_K.

In the current version of the model we will assume that the only tax is a proportional income tax so, the tax rate on labour and capital are equal i.e., $\tau_L = \tau_K = \tau$. So, the above inclusions (25), (26) collapse to

$$\frac{d\tau}{dt} = u \in [-d, d] = U, \, d \geq 0\,. \tag{27}$$

4 Model calibration

We propose that neglecting depreciation will not greatly affect the economic dynamics so, $\delta = 0$. Government expenditure g is assumed to be constant. We will construct a couple of different calibrations for the model, each with a different level of government expenditure. First, we set g at 10% of no-tax steady-state output.

We will assume $\rho = 0.04$, $\alpha = 0.3$, $\eta = 1$ and $\gamma = 0.5$ that, in broad terms, characterize a reasonably industrialized economy composed of rational agents interested in the near future (notably, $exp(-0.04 \cdot 10) = 0.67$ and $exp(-0.04 \cdot 50) = 0.13$), drawing a fair satisfaction from consumption and feeling, quite strongly, the burden of labor.

We will use a stylized steady state $\underline{k} = \underline{\ell} = 1$ with no taxes and no government expenditure to calibrate A and V. Setting the right hand sides of (6) and (9) to zero yields

$$A = \underline{c}, \quad \text{and} \quad A = \frac{\rho}{\alpha} \quad \text{hence} \quad A = \underline{c} = 0.1333 \tag{28}$$

where $\underline{c}$ is the no-tax consumption steady state. Then, we get from (24) that

$$V = (1-\alpha)\left(\frac{\rho}{\alpha}\right)^{1-\gamma} \quad \text{hence} \quad V = 0.2556\,. \tag{29}$$

Finally, in our initial calibration, $g = 0.1A = 0.0133$.

As said in Section 2, we also need to set boundaries that the economy should *not* cross. We propose that

I. **capital** should be between 10% and 200% of no-tax steady state capital stock i.e., $k \in [0.1, 2]$;

II. **consumption** should range between 1/5 of and 5 times the no-tax steady state consumption $\underline{c}$ i.e., $c \in [0.0267, 0.6667]$;

III. **debt** may be allowed to grow to 150-200% of the maximum no-tax steady-state capital stock and also drop below zero so, in this study, $B \in [-1, 3.5]$;

IV. **tax rate** $\tau \in [0. 0.8]$;

V. **tax-rate adjustment speed** i.e., the amount by which the regulator can change the current tax-rate level within a year will be between -20 and 20 percentage points so, $u \in [-0.2, 0.2]$, where u is the adjustment speed.

The calibrated system's movements can be learned from Figure 1, which presents vector fields in the *capital-consumption* state space, for no debt, for two different tax levels. The no-tax, no government expenditure steady state is shown as the big dot in the left panel. We observe in each panel that there is a large central area of consumption choices, for which the economy will be stable. We also notice that consumption above 0.2 appears unsustainable in the long-run because it causes capital to quickly diminish or vanish. With this observation, we will reduce the top consumption level to 0.225.

Finally, the constraint set K, for which we will seek the viability kernel, is

$$K = [0.1, 2] \times [0.0267, 0.225] \times [-1, 3.5] \times [0, 0.8]\,. \tag{30}$$

The viability problem is then to determine the kernel $\mathcal{V} \in K \subset \mathbf{R}^4$ for the dynamics $\psi(\cdot, U)$ defined through the vector differential inclusion[8] (19) - (21), (27) (with (24)). We will use VIKAASA to compute $\mathcal{V}$.

[8] Because of (27), system (19) - (21) is now a differential inclusion in $\mathbf{R}^4$.

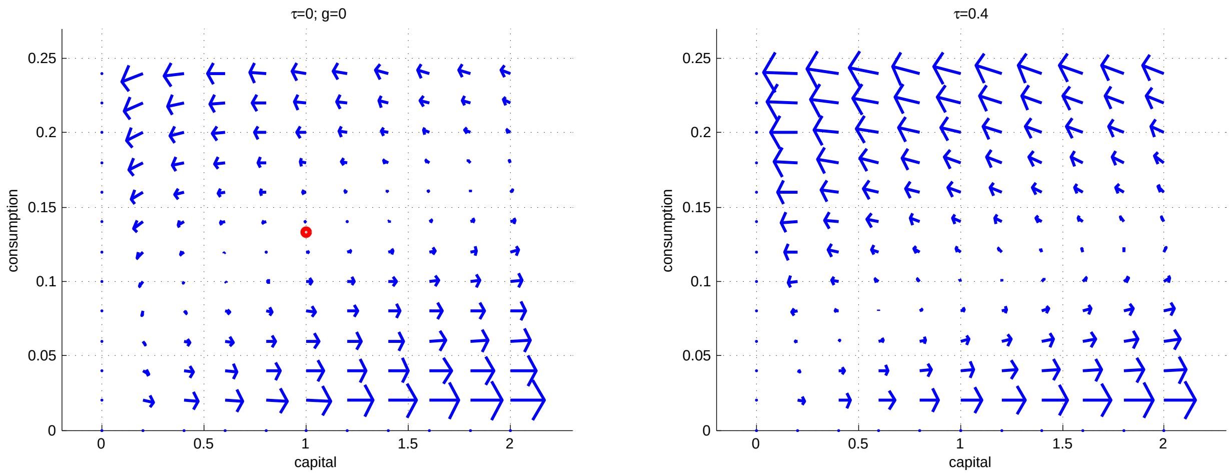

Figure 1: k, c-vector fields for $\tau = 0$ $g = 0$, left panel and $\tau = 0.4$, right panel.

5 The viability kernel

We will show several viability kernel *slices* for the following two situations:

- $\overline{B} = 3.5$ and $g = 0.0133$, as introduced in Section 4;
- government expenditure doubles to $g = 0.0266$.

5.1 How to interpret 3D *slices* of the 4D kernel?

Given that $\mathcal{V} \subset K \subset \mathbf{R}^4$ where we cannot display sets, the analysis will be conducted using 3D (sometimes 2D) slices of $\mathcal{V}$.

Explanation Box 1.

To analyze the tax policy, we will use 3D slices of the 4D space (k, c, B, u) where evolutions of the economy "live". The first such a slice is shown in Figure 2. The three dimensions, for which the slice is cut, are labelled along the respective axes (here: capital, consumption and tax rate); the fourth dimension is kept constant (here: debt=1.25). The rectangular box in each figure delimits a 3D projection of $K \subset \mathbf{R}^4$ where K is the constraint set, within which the economy is supposed to remain. A 3D body ("boulder") is a snapshot of the viability kernel taken for a particular value of the fourth dimension, written down in the caption or as the figure's title. If there is a line (trajectory) shown in the figure, then each point of this line corresponds to a different value of the fourth dimension; we can say that the 3D line is parametrized in the fourth dimension.

Explanation Box 2.

By the kernel definition:
• for each economic state represented as a point in the boulder, there exists a smooth tax-rate policy ($u \in [-0.2, 0.2]$), which maintains the economy in the constraint set K;
• the points outside the boulder are the economic states that cannot be controlled to remain in K by this policy.

A smooth tax-rate policy that maintains the economy in K, keeps it also in $\mathcal{V}$. (This is because we deal with infinite-horizon viability problems.) Henceforth, we can apprise where the economy will be in the future even if our knowledge about the economy today is only of debt and capital, given the restrictions we put on the change in tax rate.

5.2 Maximum allowable debt $B = 3.5$

Figure 2 shows two kernel slices for a medium debt level, $B = 1.25$. We first observe that some low consumption levels (see the far right bottom corner along *capital*) and a lot of high consumption levels ($c \geq 0.14$) are not viable. This is so because the former would lead to overcapitalization of the economy; the latter would decapitalize the economy.

This is visible from the right panel. Three exemplary evolutions show what can happen to the economy depending on the "initial" state. If the state is $[1.6833, 0.0598, 1.2500, 0.4000] \in \mathcal{V}$ then there are smooth[9] tax-rate strategies, for which the evolution remains contained in $\mathcal{V} \in K \subset \mathbf{R}^4$, see the solid line. Actually, the evolution stabilizes when $B = 0$, i.e., within a *different* kernel slice, not shown here.

[9] I.e.,$u \in [-0.2, 0.2]$.

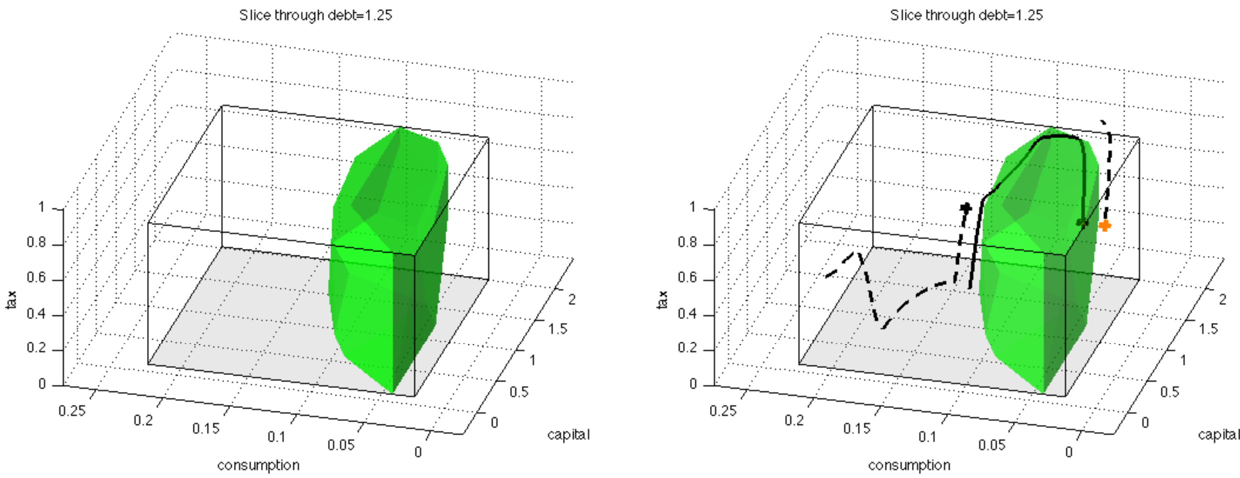

Figure 2: Kernel slices for $B = 1.25$.

However, if the evolution starts at $[1.6833, \mathbf{0.0433}, 1.2500, 0.4000] \notin \mathcal{V}$, then even the fastest tax-rate *growth* cannot prevent overcapitalization and the economy violates the capital upper bound $k = 2$. If the evolution starts at $[1.6833, \mathbf{0.1454}, 1.2500, 0.4000] \notin \mathcal{V}$ then even the fastest tax-rate *decrease* cannot prevent the dramatic capital reduction to below its lower bound $k = 0.2$. [10]

Furthermore, the slice projections onto the planes: *tax-consumption* and *tax-capital*, not shown but easy to spot, are almost rectangular. This implies that, for this moderate debt level (i.e., $B = 1.25$), the income tax-rate "initial" conditions are non-essential for the consumption choices.

Figure 3 shows two kernel slices: for an economy with savings, $B = -0.55$ left panel and a high debt economy, right panel $B = 2.6$. Overall, we notice that while the left slice is slanted toward higher consumption, with respect to the position of the slice in Figure 2, the right panel slice (high debt) is slanted toward lower consumption.

Moreover, the kernel slice for an economy without debt (left panel) appears largest among the so far analyzed slices. This implies that when the debt level is low there are more viable consumption choices for a given level of capital and tax, than when debt is high (or higher). We also notice that viable consumption decisions are different for each level of debt. When debt is low (left panel), there are fewer

[10] See Section 6, page 19, for the *debt-to-GDP* ratio computed for each of these evolutions.

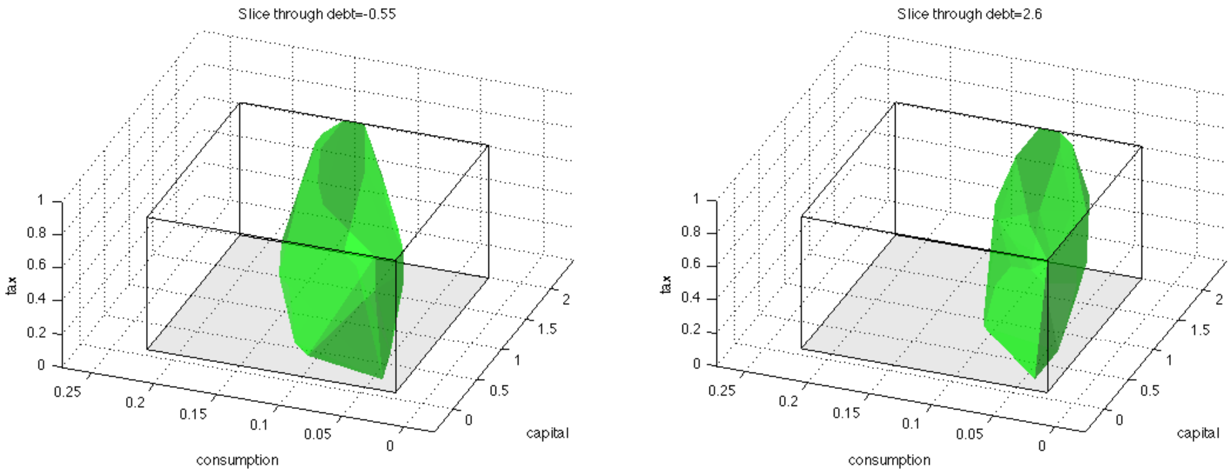

Figure 3: Kernel slices for $B = -0.55$, left panel and $B = 2.6$, right panel.

consumption decision that would de-capitalize the economy, than when debt is high. Also, there are more consumption levels that could lead to overcapitalization in a low tax economy.

The slice projections onto the planes of *tax-consumption* and *tax-capital* are not rectangular. This implies that, for these debt levels (i.e., $B = -0.55$ and $B = 2.6$), the income tax-rate "initial" conditions need be taken into account when the consumption choices are made. This is exemplified in Figure 4 where the slices' cuts for $k = 1.525$ are shown. The left (darker) shape is for the high debt economy, the right one is for the economy with savings. We can see that (for this capital level) viable consumption choices, when debt is big i.e., $B = 2.6$ the left shape, cannot be as "lavish" as when $B = -0.55$ the right shape. Evidently, with higher debt, consumption must be lower.

Some might ask why it is not "viable" to have even lower consumption than c=0.0598, which is the left boundary of the left slice (high debt). In broad terms, the reason is that lower consumption *now*, combined with the restrictions that must be satisfied along the *future* path, which include the rate at which future taxes can

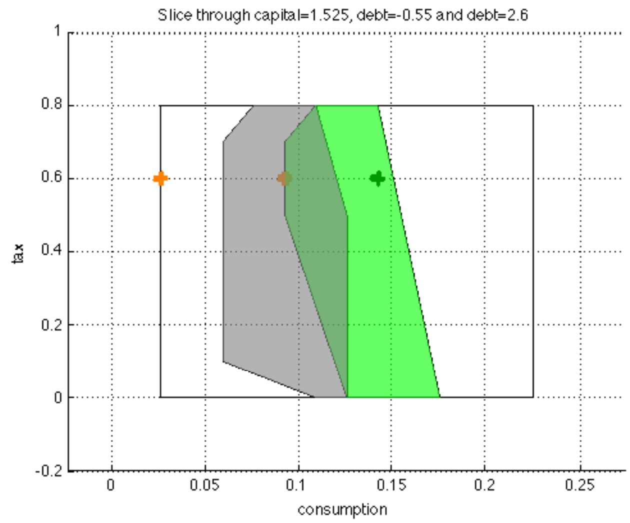

Figure 4: Kernel slices for $c = 1.525$ for $B = -0.55$ and $B = 2.6$.

change, would put the capital accumulation process on an explosive path, which would violate the capital upper bound and TVC-infinity[11]. We illustrate this and show which constraint can be binding and on what slice in an appendix, see A, page 23.

Here, we will examine the impact of tax-rate levels on viable consumption choices. Figure 5 shows two kernel slices for low ($\tau = 0$) and high ($\tau = 0.8$) tax rates. The presented slices appear similar to those shown in Figure 3, which were parametrized by debt. (Notice, we have chosen a different "elevation" for these slices.) As before, we see that higher consumption decisions can be made for larger capital values. Furthermore, for a given capital level, the consumption decisions' *ranges* are wider and the consumption *values* are higher for the low tax rate; they both decrease as the tax-rate increases. In addition, we observe that higher consumption levels are viable when debt is low, for both taxation levels.

In the left panel we start an evolution from $[0.2583, 0.0928, -0.1, \mathbf{0}] \in \mathcal{V}$. The evolution in the right panel begins at $[0.2583, 0.0928, -0.1, \mathbf{0.8}] \notin \mathcal{V}$. We can see that the latter, which illustrates what can happen in a highly taxed economy, crashes

[11]Unless *crisis control* was undertaken, see Cardaliaguet, Quincampoix, and Saint-Pierre (1999b).

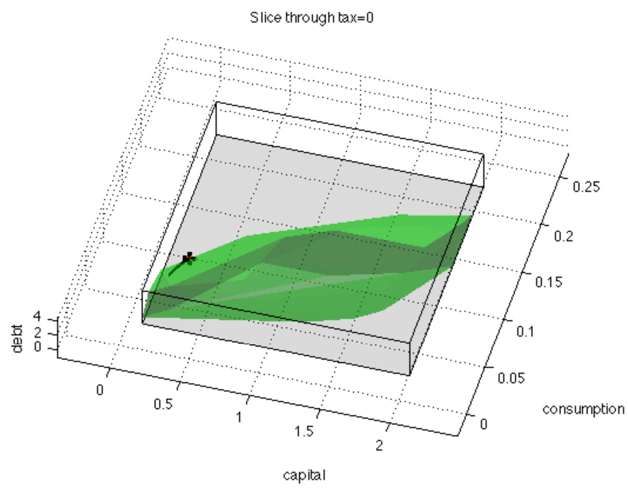

Figure 5: Kernel slices for $\tau = 0$ and $\tau = 0.8$.

through the capital lower boundary. This is because the tax could not drop sufficiently fast to prevent de-capitalization. The former stabilizes at low capital and consumption values.

Figure 6 shows that high debt levels are incompatible with low tax. Here again, we see the slice through $\tau = 0$ (for a different elevation). Notice two evolutions starting at $[1.05, 0.1259, 0.35, 0] \in \mathcal{V}$ (low debt, inside slice) and $[1.05, 0.1259, 2.6, 0] \notin \mathcal{V}$ (high debt, outside slice). We see that the high-debt trajectory rises fast in debt and crashes through its upper boundary. This is because the smooth taxation policy cannot generate enough tax to curb the increasing debt. On the other hand, the initially low-debt economy remains almost stationary.

5.3 A higher government expenditure

He we have computed the kernel when the government expenditure is doubled, so $g = 0.0266$. The other parameters are as in Section 5.2.

In Figure 7, we observe that the kernel slice in the right panel (slightly fatter) appears "turned" clockwise, with respect to that in the left panel, computed in Section 5.2 for the lower g. This means that (even) if the economy is in credit i.e., $B = -0.55$, increasing the government expenditure reduces maximum achievable consumption. This is visible from a larger space in the right panel between the K-set wall raised at the consumption upper boundary $c = 0.225$ and the kernel and,

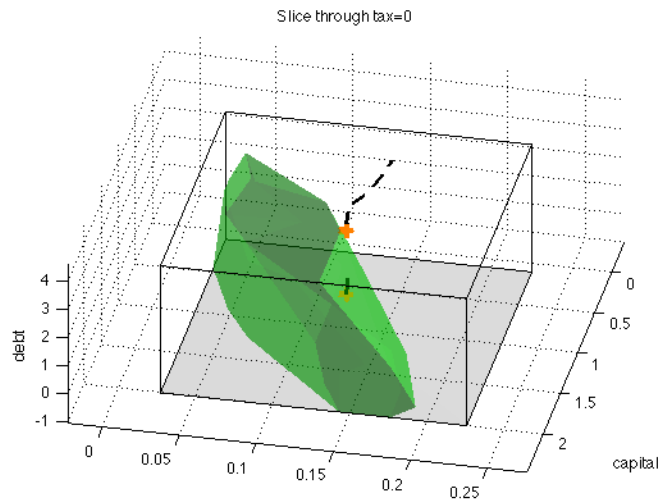

Figure 6: Kernel slice for $\tau = 0$.

also, from a smaller space between the K-set wall raised at the consumption lower boundary $c = 0.0267$ and the kernel.

The same phenomenon is visible in Figure 8, which shows the kernel slices for a high-debt economy, B=2.6 where the left panel is for the lower g. The right-panel empty space between the maximum consumption wall is larger, even if the slice is larger than the one in the left panel. So, again, a higher government expenditure results in that only lower consumption choices are feasible, given the adopted tax policy.

However, there is a feature of the kernel slice in the right panel i.e., of the high government-expenditure economy, absent from the left panel. Here, the higher g economy kernel-slice is clearly not rectangular. This means that, given low capital ($k < 0.6$) or high capital ($k > 1.4$), consumption choices can be viable only if high tax rates are applied.

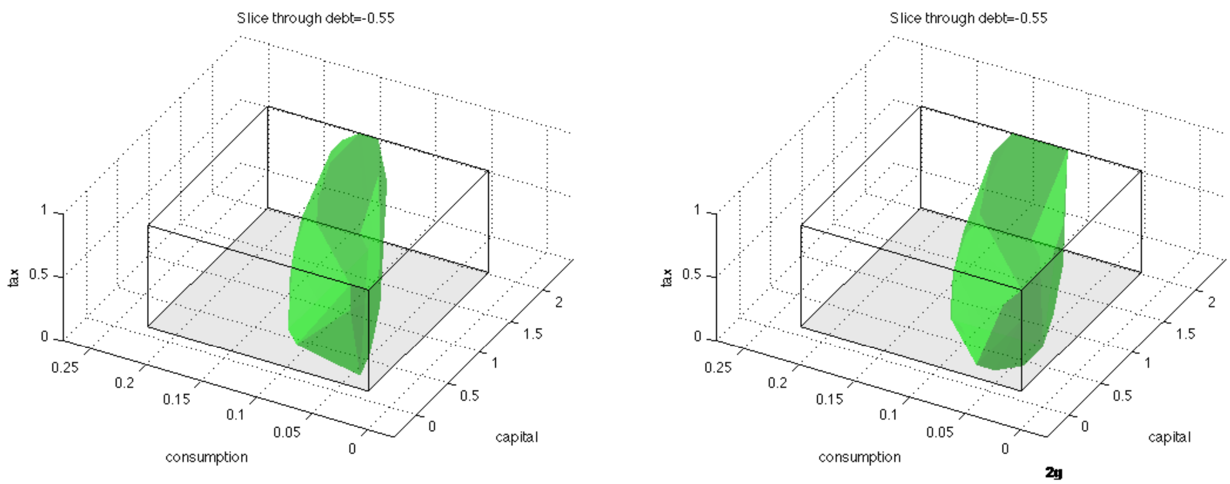

Figure 7: Kernel slices for $B = -0.55$. The left panel is as in Figure 3, the right-panel kernel slice is computed for the doubled g.

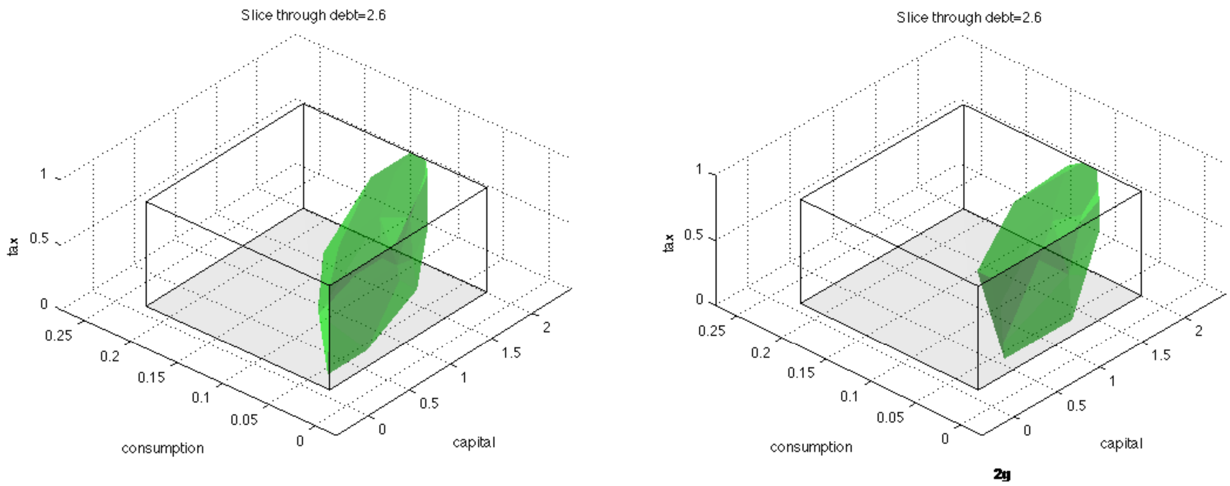

Figure 8: Kernel slices for $B = 2.6$. The left panel is as in Figure 3, the right-panel kernel slice is computed for the doubled g.

6 Debt-to-GDP ratio

An economic evolution could also be characterized by the *debt-to-GDP* ratio (see e.g., Baker, Kotlikoff, and Leibfritz (1999)). We have computed such ratio time profiles for the three evolutions pictured in Figure 2, right panel. Remember that the evolution from $c_0 = 0.0598$ is viable while the two others: from $c_0 = 0.1454$ and $c_0 = 0.0433$ are not. We show the corresponding *debt-to-GDP* ratios in Figure 9.

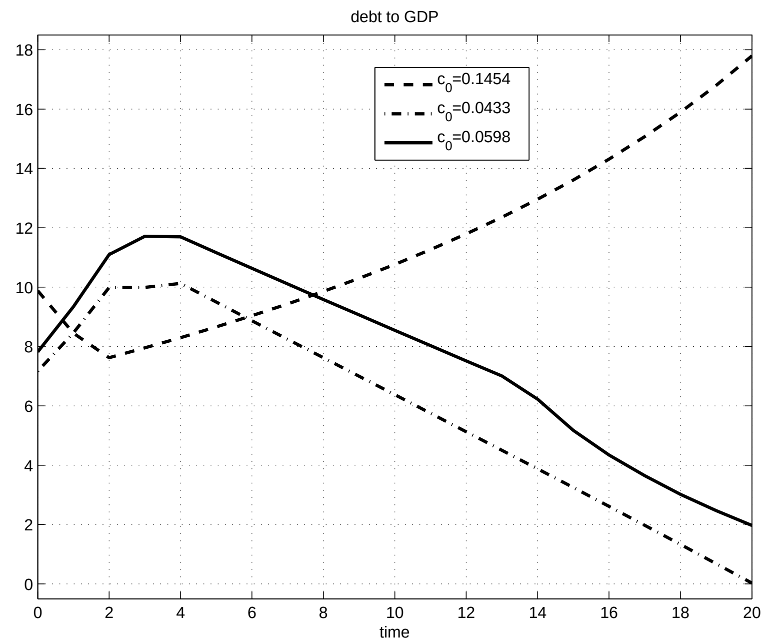

Figure 9: *Debt-to-GDP* ratio time profiles.

An interesting case represents the solid line, which corresponds to the viable trajectory starting at $c_0 = 0.0598$. We see that the *debt-to-GDP* ratio eventually diminishes; however, it transits rather high values of the ratio. The values are numerically high because of the stylized steady-state low value, which is equal to $A = 0.1333$. Under an increasing tax, debt diminishes and, eventually, the steady state is such that capital is large enough to assure a medium-level consumption and a growing GDP (output).

The similarly looking dash-dotted line originating from a lower consumption level $c_0 = 0.0433$ is non-viable because of overcapitalization of the economy, see Figure 2. Here, the *debt-to-GDP* radio (eventually) diminishes because capital grows fast and, as said, the economy becomes over-capitalized. On the other hand, the dash line, which displays the (eventually) growing *debt-to-GDP* ratio, corresponds to an evolution from $c_0 = 0.1454$, which is also non-viable. This is so because even the fastest tax-drop cannot prevent the capital reduction below its lower bound. We observe that *debt-to-GDP* ratio cannot be used as a proxy for viability; on the other hand, a viable evolution can imply a diminishing *debt-to-GDP* ratio.

7 Concluding remarks

We have presented a computational method based on viability theory for a discovery of consumption choices that are compatible with the state variables of the economy at hand. The compatibility means that viable consumption and capital choices will generate a nearly steady-state path for a smooth tax-rate adjustment policy.

Among other findings we report that increasing government expenditure implies that higher tax rates will be needed to preserve viability of many consumptions choices, when capital levels approach the constraint set boundaries.

References

Aubin JP (1991) Viability Theory. Systems & Control: Foundations & Applications, Birkhäuser, Boston

Aubin JP (1992) Viability Theory

Aubin JP (1997) Dynamic Economic Theory – A Viability Approach. Springer Verlag

Aubin JP (2001) Viability kernels and capture basins of sets under differential inclusions. SIAM J Control Optimization, 40:853–881

Baker B, Kotlikoff LJ, Leibfritz W (1999) Generational accounting in new zealand. In: Auerbach AJ, Kotlikoff LJ, Willi L (eds) Generational Accounting around the World, University of Chicago Press, pp 347 – 368

Bonneuil N, Boucekkine R (2008) Sustainability, optimality and viability in the ramsey model, manuscript, 39 pages

Bonneuil N, Saint-Pierre P (2008) Beyond optimality: Managing children, assets, and consumption over the life cycle. Journal of Mathematical Economics 44(3-4):227–241

Cardaliaguet P, Quincampoix M, Saint-Pierre P (1999a) Set valued numerical analysis for optimal control and differential games. Stochastic and differential Games: Theory and Numerical Methods, Ann Internat Soc Dynam Games 4:177–274

Cardaliaguet P, Quincampoix M, Saint-Pierre P (1999b) Set-valued numerical analysis for optimal control and differential games. Annals of the International Society of Dynamic Games 4:177–247

Clément-Pitiot H, Doyen L (1999) Exchange rate dynamics, target zone and viability

Clément-Pitiot H, Saint-Pierre P (2006) Goodwin's models through viability analysis: some lights for contemporary political economics regulations. In: 12th International Conference on Computing in Economics and Finance, Cyprus, conference maker

De Lara M, Doyen L, Guilbaud T, Rochet MJ (2006) Is a management framework based on spawning stock biomass indicators sustainable? A viability approach. ICES Journal of Marine Science

Gaitsgory V, Quincampoix M (2009) Linear programming approach to deterministic infinite horizon optimal control problems with discounting. SIAM J Control Optim forthcoming

Judd KL (1987) The welfare cost of factor taxation in a perfect-foresight model. Journal of Political Economy 95, No. 4:675–709

Krawczyk JB, Judd KL (2012) Viable economic states in a dynamic model of taxation. In: 'Conference Maker' [editorialexpress.com], 18th International Conference on Computing in Economics and Finance, Prague, Czech Republic

Krawczyk JB, Kim K (2004) A viability theory analysis of a macroeconomic dynamic game. In: Eleventh International Symposium on Dynamic Games and Applications, Tucson, AZ, US

Krawczyk JB, Kim K (2009) "Satisficing" solutions to a monetary policy problem: a viability theory approach. Macroeconomic Dynamics 13(1):46–80, URL http://www.vuw.ac.nz/staff/jacek_krawczyk/somepapers/MDY07046.pdf

Krawczyk JB, Pharo A (2011) **Vi**ability **K**ernel **A**pproximation, **A**nalysis and **S**imulation **A**pplication – VIKAASA Manual. SEF Working Paper 13/2011, Victoria University of Wellington, URL http://hdl.handle.net/10063/1878

Krawczyk JB, Pharo A (2012) **Vi**ability **K**ernel **A**pproximation, **A**nalysis and **S**imulation **A**pplication – VIKAASA Code. URL http://code.google.com/p/vikaasa/

Krawczyk JB, Serea OJ (2009) Computation of viability kernels for three and four meta-state monetary-policy macroeconomic problems. In: Workshop on Perturbations, Game Theory, Stochastics, Optimisation and Applications, University of South Australia, URL www.unisa.edu.au/maths/research/JerzyFilar/default.asp

Krawczyk JB, Serea OS (2011) When can it be not optimal to adopt a new technology? a viability theory solution to a two-stage optimal control problem of new technology adoption. Optimal Control Applications and Methods DOI: 10.1002/oca.1030 (published online: 8 Dec 2011)

Krawczyk JB, Sethi R (2007) Satisficing solutions for New Zealand monetary policy. Tech. rep., Reserve Bank of New Zealand, No DP2007/03.

Krawczyk JB, Pharo A, Simpson M (2011) Approximations to viability kernels for sustainable macroeconomic policies. Tech. rep., VUW SEF Working paper: 01/2011, URL http://hdl.handle.net/10063/1531

Krawczyk JB, Sissons C, Vincent D (2012) Optimal versus satisfactory decision making: a case study of sales with a target. Computational Management Science pp 10,287–012, URL http://www.vuw.ac.nz/staff/jacek_krawczyk/somepapers/cms2012.pdf, published online: February 2012, DOI: 10.1007/s10287-012-0141-7

Martinet V, Doyen L (2007) Sustainability of an economy with an exhaustible resource: A viable control approach. Resource and Energy Economics 29(1):17–39

Martinet V, Thébaud O, Doyen L (2007) Defining viable recovery paths toward sustainable fisheries. Ecological Economics in press, [doi:10.1016/j.ecolecon.2007.02.036]

Pujal D, Saint-Pierre P (2006) Capture basin algorithm for evaluating and managing complex financial instruments. In: 12th International Conference on Computing in Economics and Finance, Cyprus, conference maker

Quincampoix M, Saint-Pierre P (1995) An algorithm for viability kernels in hölderian case: Approximation by discrete dynamical systems. Journal of Mathematical Systems, Estimation and Control 5(1):1–13

Quincampoix M, Veliov VM (1998) Viability with a target: Theory and applications. Applications of Mathematical Engineering, Cheshankov B and Todorov M, eds , Heron Press, Sofia, pp 47:47–58

Veliov VM (1993) Sufficient conditions for viability under imperfect measurement. Set-Valued Analysis, 1(3):305–317

A Why some choices can be non-viable

To help understand why some economic states can be non-viable we will consider evolutions from three *capital-consumption-tax* combinations for two different levels of debt $B = -0.55$ and $B = 2.6$. The evolution starting points are represented by the dots shown in Figure 4,

We need to remark that viable evolutions, represented by the solid lines in the following figures, are constructive in that we have found tax-rate adjustments that generate them and lead to a (numerically) steady state. On the other hand, the dash and dash-dotted lines cross a boundary of K in finite time, hence represent nonviable evolutions; they are computed by VIKAASA as "best" in that the sum of their velocities is minimal, but too big to be deemed steady.

Consider the following points (from left to right in Figure 4):

1. $[0.0267, 1.525, 2.6, 0.6] \notin \mathcal{V}$
2. $[0.0267, 1.525, -0.55, 0.6] \notin \mathcal{V}$
3. $[0.0928, 1.525, 2.6, 0.6] \in \mathcal{V}$

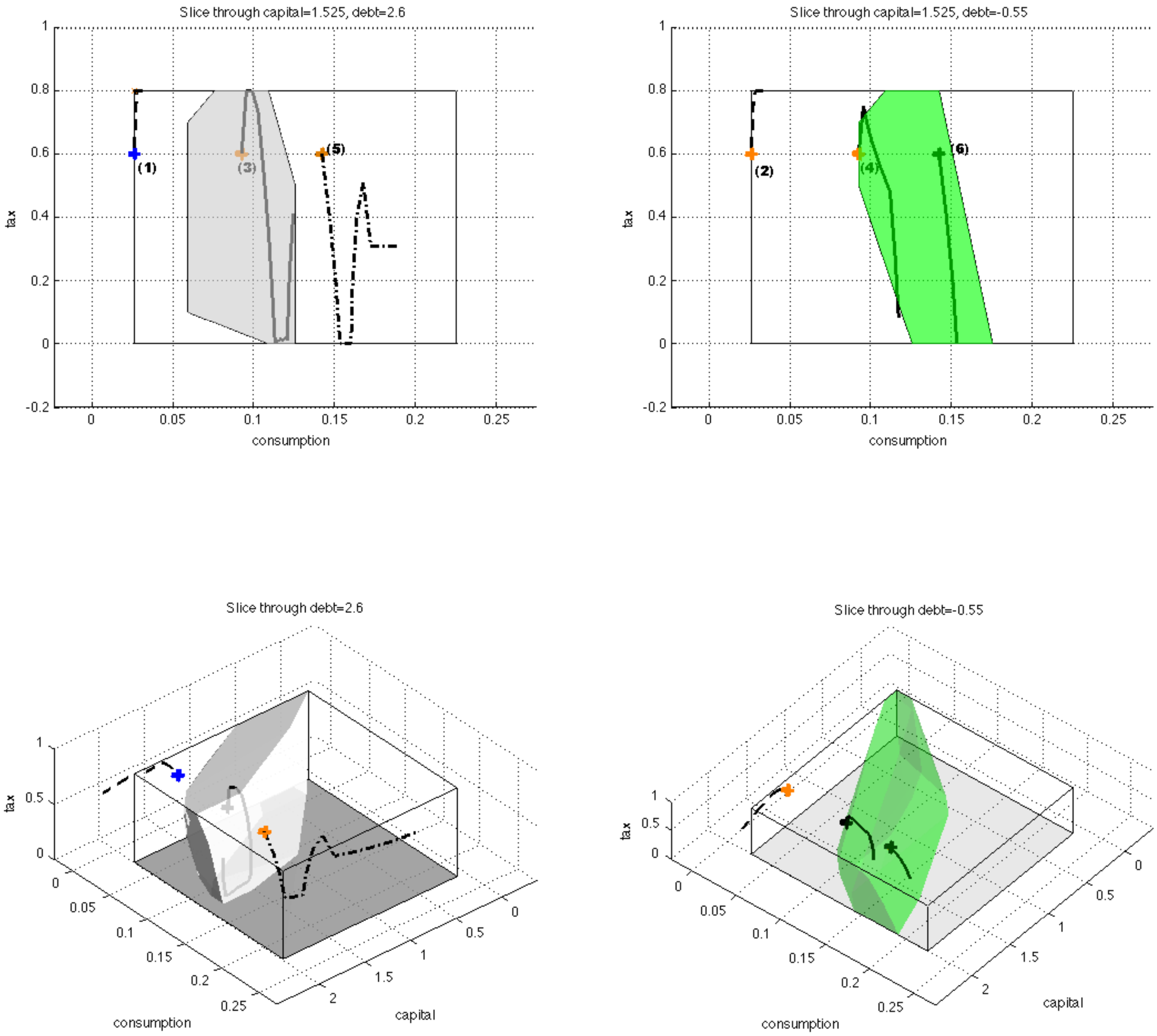

Figure 10: Viable trajectories (solid lines) and non-viable trajectories (dash and dash-dotted lines).

4. $[0.0928, 1.525, -0.55, 0.6] \in \mathcal{V}$
5. $[0.1424, 1.525, 2.6, 0.6] \notin \mathcal{V}$
6. $[0.1424, 1.525, -0.55, 0.6] \in \mathcal{V}$

According to the points' locations, relative to the viability kernel, we expect that the (high-debt) point, numbered "1" is nonviable.

1. We can see in the left panels in Figure 10 that even with the application of the maximum tax rate, the evolution crashes through the capital upper bound, albeit consumption increases.

2. Very similarly to what we have seen in the left panels, we notice in the right panels in Figure 10 that with the application of the maximum tax rate, the evolution also crashes through the capital upper bound (consumption increases too).

The evolution that starts at the (low-debt) point number "2" is also nonviable. However, the evolution that starts at the (high-debt) point numbered "3" is viable.

3. Here, we notice (see the left panels in Figure 10) that with the application of the maximum tax rate, capital decreases and consumption increases faster than from point "1.", especially, after the intermediate tax-rate drop. After the tax-rate hike to 40%, the economy stabilizes.

The evolution that starts at the (low-debt) point number "4" is also viable.

4. Here, we notice (see the right panels in Figure 10) that with a medium size tax-rate hike, capital decreases and consumption increases albeit both processes are slower than under "3". The economy stabilizes with the tax rate below 10%.

The evolution that starts at the (high-debt) point numbered "5" is non-viable.

5. Here, we notice (see the left panels in Figure 10) that with a medium size tax-rate drop, capital decreases and consumption increases however both processes are faster than under "3". After increasing the tax-rate and then decreasing it, capital still diminishes very fast and almost crashes through the lower boundary. However, in this case, debt also grows rapidly and violates the upper limit before capital reaches its border. This is visible from Figure 11.

The evolution that starts at the (low-debt) point number "6" is viable.

6. Here, we notice (see the right panels in Figure 10) that with a big tax-rate drop, capital decreases and consumption increases however both processes are rather slow and the economy stabilizes with zero tax rate.

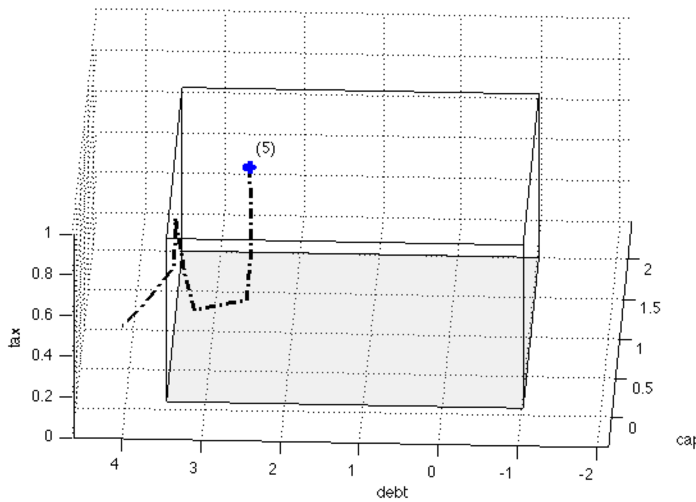

Figure 11: Kernel slice for $c = 0.1424$.