Nash Bargaining Solution under Externalities*

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August 31, 2011

Abstract

We define a Nash bargaining solution (NBS) for partition function games (PFGs). Based on a PFG, we define an extensive game (EG), which is a propose-respond sequential game where the first rejecter of a proposal exits from the game with a positive probability. We show that the NBS is supported as the expected payoff profile by any stationary subgame perfect equilibrium (SSPE) of the EG such that in any subgame, the coalition of all active players immediately forms. We provide a necessary and sufficient condition for such an SSPE to exist.

Keywords: Nash bargaining solution, Partition function game, Noncooperative foundation, Rejecter-exit partial breakdown

JEL classification codes: C72, C73, C78

*This version is extreee$\cdots$eeemely preliminary.

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1 Introduction

When we regard coalition formation as a bargaining problem (BP), one natural disagreement situation is that if any player disagreed, each player would stand alone. In this situation, each player's thread payoff is her payoff when every player stands alone. Another plausible disagreement situation is that if a player disagreed, the other players would remain to cooperate and she would be isolated. In this situation, each player's thread payoff is her payoff when she is isolated. When there is no externality in coalition formation, i.e., coalition formation is represented by a characteristic function game (CFG) (N, v), both situations generate the same thread point $(v(\{i\}))_{i \in N}$.

However, if there may be externalities in coalition formation, i.e., coalition formation is represented by a partition function game (PFG) (N, V), the two situations above may generate different thread points. In the situation that every player stands alone in disagreement, player i's disagreement results in coalition structure $\{\{j\} \mid j \in N\}$, her thread payoff is $V(\{i\}, \{\{j\} \mid j \in N\})$, and thus, the thread point is $(V(\{i\}, \{\{j\} \mid j \in N\}))_{i \in N}$. In the situation that each disagreer is isolated, player i's disagreement results in coalition structure $\{\{i\}, N \setminus \{i\}\}$, her thread payoff is $V(\{i\}, \{\{i\}, N \setminus \{i\}\})$, and thus, the thread point is $(V(\{i\}, \{\{i\}, N \setminus \{i\}\}))_{i \in N}$. Therefore, in the former and latter situations, we define the Nash bargaining solutions (NBSs) for the PFG as the NBSs for the BP such that players split the worth of the grand coalition under the thread points $(V(\{i\}, \{\{j\} \mid j \in N\}))_{i \in N}$ and $(V(\{i\}, N \setminus \{i\}))_{i \in N}$, respectively. We refer to the former and latter as the *finest NBS (fNBS) for PFGs* and *coarsest NBS (cNBS) for PFGs*, respectively.

The followings are noteworthy. First, the entries of thread point $V(\{i\}, \{\{i\}, N \setminus \{i\}\})$ are not consistent because for any distinct $i, j \in N$, coalition structures $\{\{i\}, N \setminus \{i\}\}$ and $\{\{j\}, N \setminus \{j\}\}$ do not coexist. However, this inconsistency is reasonable because the players' disagreements and the coalition structures by the disagreements are hypothetical and they do not actually disagree. Secondly, if positive externalities are strong, the cNBS for PFG does not exist because $\sum_{i \in N} V(\{i\}, \{\{i\}, N \setminus \{i\}\}) >$

$V(N, \{N\})$. Thirdly, if there is no externality, the cNBS and fNBS for PFGs coincide with the NBS for CFGs that are naturally reduced from the PFGs.

According to [Gom05], the fNBS for PFGs is supported as the limit of a sequence of stationary subgame perfect equilibrium (SSPE) payoff profiles in extensive games (EGs).[1] However, the cNBS has not been given any noncooperative foundation.

Our paper will give the cNBS for PFGs a noncooperative foundation. Based on a PFG, we define an EG, which is a propose-respond sequential bargaining game where the first rejecter exits from the game with a positive probability (*rejecter-exit partial breakdown*). We show that the expected payoff profile by any full-coalition SSPE (SSPE such that in any subgame, the coalition of all active players immediate forms) is equal to the cNBS. We also provide a necessary and sufficient condition for a full-coalition SSPE to exist.

The fNBS and cNBS are also defined by the following two-step approach: first, define a CFG based on the PFG, secondly, let the NBS for the CFG be the NBS for the PFG. For the fNBS (cNBS, resp.), in the first step, CFG (N, v) based on PFG (N, V) is defined as for any $S \in 2^N \setminus \{\emptyset\}$, $v(S) = V(S, \pi)$, where π is the finest (coarsest, resp.) partition of N such that $S \in \pi$, i.e., $\pi = \{S\} \cup \{\{i\} \mid i \in N \setminus S\}$ ($\pi = \{S\} \cup \{N \setminus S \mid i \in N \setminus S\}$,[2] resp.). We refer to this way as the *finest way* (*coarsest way*, resp.). The approach to define CFGs from PFGs is used to define the Shapley value and core for PFGs. [dCS08] and [McQ09] axiomatize the Shapley values for PFGs defined by the finest and coarsest ways, and they are called the *externality-free Shapley value* and the *extended, generalized Shapley value*, respectively. They point out that the externality-free Shapley value and the extended, generalized Shapley value are supported as equilibrium payoff profiles in EGs in [HMC96] and [Gul89], respectively. [Haf07] defines cores for PFGs by the finest and coarsest ways, and he called the *core with singleton expectations* and the *core with merging expectations*, respectively.

In the standard bargaining problem, the disagreement point does not depend on

[1] [Oka10] investigates EGs based on strategic games. He shows that the strategic-game counterpart to the fNBS for PFGs is supported as the limit of a sequence of stationary SSPE payoff profiles in EGs.

[2] If $S \neq N$, $\{S\} \cup \{N \setminus S \mid i \in N \setminus S\} = \{S, N \setminus S\}$, and otherwise, $\{S\} \cup \{N \setminus S \mid i \in N \setminus S\} = \{N\}$.

who disagrees (*anonymous disagreement*). On the other hand, in the present paper, the disagreement situation may depend on who disagrees (*nonanonymous disagreement*). Several papers consider BPs with nonanonymous disagreements. [KT10] investigate BPs with nonanonymous disagreements in a cooperative approach. In [KT10], each player's disagreement determines an allocation in disagreement. On the other hand, in the cNBS of the present paper, a player i's disagreement determines her payoff and the worth of coalition of the other players but does not determine the allocation among the other players, which does not matter in defining the cNBS. [CB00] considers a noncooperative bargaining game with nonanonymous disagreements. However, in the model, the number of players is two, and thus, coalition formation is not considered.

A feature of our EGs is the rejecter-exit partial breakdown, in which, if players fail to agree, the first rejecter exits from the game with a certain probability. After player i exits from the game by the partial breakdown in the first round, the other players form coalition $N \setminus \{i\}$ in the full-coalition SSPE, coalition structure $\{\{i\}, N \setminus \{i\}\}$ is realized, and then, player i obtains a payoff of $V(\{i\}, \{\{i\}, N \setminus \{i\}\})$. This is behind the fact that the expected payoff profile by any full-coalition SSPE is equal to the cNBS. [Miy08], [Cal08] and [HMC96] consider partial breakdowns. In [Miy08], a responder is randomly selected and exits from the game. In [Cal08], a player is randomly selected and exits from the game. In [HMC96], the proposer exits from the game. In their models, there is no externality in coalition formation. On the other hand, papers studying coalitional bargaining with externalities have not considered the partial breakdown (e.g., [Blo96] and [RV99]).

The remainder of the paper is organized as follows: Section 2 defines NBSs for PFGs, Section 3 presents an EG based on PFG, Section 4 shows that the cNBS is supported by the expected payoff profile by any efficient SSPE, Section 5 provides a necessary and sufficient condition that there exists an SSPE in which all players cooperate immediately, and Section 6 concludes the paper.

2 Nash bargaining solution

For any function $f \in Y^X$, for any $x \in X$, let f_x be the image of x under f, i.e., $f_x := f(x)$. For any nonempty set N, let Π^N be the set of partitions of N. For any partition π, for any $i \in \bigcup \pi$, let $[i]_\pi$ be the equivalence class of i by π. For any nonempty set N and any $S \in 2^N \setminus \{\emptyset\}$, let $\underline{\pi}_S^N$ ($\overline{\pi}_S^N$, resp.) be the finest (coarsest, resp.) partition of N such that $S \in \pi$, i.e., $\underline{\pi}_S^N := \{S\} \cup \{\{i\} \mid i \in N \setminus S\}$ ($\overline{\pi}_S^N := \{S\} \cup \{N \setminus S \mid i \in N \setminus S\} = \{S, N \setminus S\} \setminus \{\emptyset\}$, resp.).

A *bargaining problem (BP)* is a triple (N, B, d) such that N is a nonempty finite set, $B \subset \mathbb{R}^N$ and $d \in \mathbb{R}^N$. For any BP (N, B, d), a *Nash bargaining solution (NBS)* of (N, B, d) is a solution of $\max_{x \in B} \prod_{i \in N} (x_i - d_i)$ s.t. $x \geq d$. A *characteristic function game (CFG)* is a pair (N, v) such that N is a nonempty finite set and v is a function from $2^N \setminus \{\emptyset\}$ to $\mathbb{R}$. For any CFG (N, v), a *Nash bargaining solution (NBS)* of (N, v) is an NBS of BP $\left(N, \left\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq v_N\right\}, \left(v_{\{i\}}\right)_{i \in N}\right)$. A *partition function game (PFG)* is a pair (N, V) such that N is a nonempty finite set and V is a function from $\left\{(S, \pi) \in 2^N \times \Pi^N \mid S \in \pi\right\}$ to $\mathbb{R}$.

Definition 1. For any PFG (N, v), a *fine Nash bargaining solution (fNBS)* (*coarse Nash bargaining solution (cNBS)*, resp.) of (N, V) is an NBS of CFG (N, v) such that for any $S \in 2^N \setminus \{\emptyset\}$, $v_S = V_{\left(S, \underline{\pi}_S^N\right)}$ ($v_S = V_{\left(S, \overline{\pi}_S^N\right)}$, resp.).

Remark 1. There exists an fNBS (cNBS, resp.) if and only if $\sum_{i \in N} V_{\left(\{i\}, \underline{\pi}_{\{i\}}^N\right)} \leq V_{(N, \{N\})}$ ($\sum_{i \in N} V_{\left(\{i\}, \overline{\pi}_{\{i\}}^N\right)} \leq V_{(N, \{N\})}$, resp.). If the grand coalition is efficient, i.e., for any $\pi \in \Pi^N$, $V(N, \{N\}) \geq \sum_{S \in \pi} V(S, \pi)$, then an fNBS exists, but a cNBS does not necessarily exist.

Remark 2. If there exists an fNBS (cNBS, resp.), it is unique and is a solution of

$$\max_{x \in \left\{y \in \mathbb{R}^N \mid \sum_{i \in N} y_i \leq V_{(N, \{N\})}\right\}} \prod_{i \in N} \left(x_i - V_{(\{i\}, \pi_i)}\right),$$

which is given by $\left(\frac{V_{(N,\{N\})} - \sum_{j \in N} V_{\left(\{j\}, \pi_j\right)}}{|N|} + V_{(\{i\}, \pi_i)}\right)_{i \in N}$, where for any $i \in N$, $\pi_i = \underline{\pi}_{\{i\}}^N$ ($\pi_i = \overline{\pi}_{\{i\}}^N$, resp.).

Behind the fNBS, there is the situation that if any player disagreed, each player would stand alone. Behind the cNBS, there is the situation that if a player disagreed, the other players would remain to cooperate and she would be isolated.

Example 1 (Cournot competition)**.** Let $P : \mathbb{R}_+ \to \mathbb{R}_+$ be the inverse demand function. For simplicity, suppose that for any $Q \in \mathbb{R}_+$, $P(Q) = \mathbf{1}_{Q \leq 1}(1 - Q)$. Let N be the set of firms. For any $S \in 2^N \setminus \{\emptyset\}$, let $c_S \in \mathbb{R}_+$ be the marginal and average cost for the merged firm of firms in S. Suppose that for any $S, T \in 2^N \setminus \{\emptyset\}$, if $S \subseteq T$, then $c_S \geq c_T$. If $S \subset T$ and $c_S > c_T$, it means the synergy effect of reducing cost by merger. For any $\pi \in \Pi^N$, let $G(\pi)$ be the strategic game defined as follows: the set of players is π; for any $S \in \pi$, the set of player S's strategies is $\mathbb{R}_+$; for any $S \in \pi$, player S's payoff function is $\mathbb{R}_+^\pi \ni q \mapsto \left(P\left(\sum_{S \in \pi} q_S\right) - c_S\right) q_S \in \mathbb{R}$. Thus, $G(\pi)$ is a Cournot game played by merged firms. For inner solutions to be ensured, suppose that for any $\pi \in \Pi^N$ and any $S \in \pi$, $\frac{1+\sum_{T \in \pi} c_T}{|\pi|+1} \geq c_S$. For any $\pi \in \Pi^N$, there uniquely exists a Nash equilibrium in $G(\pi)$, and player S's equilibrium payoff is $\left(\frac{1+\sum_{T \in \pi} c_T}{|\pi|+1} - c_S\right)^2$. Let V be a map from $\left\{(S, \pi) \in 2^N \times \Pi^N \mid S \in \pi\right\}$ to $\mathbb{R}$ such that for any $\pi \in \Pi^N$ and any $S \in \pi$, $V_{(S,\pi)}$ is player S's payoff by the Nash equilibrium in $G(\pi)$. Since $\sum_{i \in N} V_{(\{i\},\{\{j\}|j \in N\})} \leq V_{(N,\{N\})}$, there uniquely exists an fNBS of (N, V). The fNBS is $x \in \mathbb{R}^N$ such that for any $i \in N$,

$$x_i = \frac{\frac{(1-c_N)^2}{4} - \sum_{j \in N}\left(\frac{1+\sum_{k \in N} c_{\{k\}}}{|N|+1} - c_{\{j\}}\right)^2}{|N|} + \left(\frac{1 + \sum_{k \in N} c_{\{k\}}}{|N| + 1} - c_{\{i\}}\right)^2.$$

There exists a cNBS if and only if $\frac{(1-c_N)^2}{4} \geq \sum_{i \in N} \frac{\left(1-2c_{\{i\}}+c_{N\setminus\{i\}}\right)^2}{9}$, i.e., the profit under monopoly is greater than or equal to the sum of the profits when all the firms stand alone. If there exists a cNBS, it is $x \in \mathbb{R}^N$ such that for any $i \in N$,

$$x_i = \frac{\frac{(1-c_N)^2}{4} - \sum_{j \in N} \frac{\left(1-2c_{\{j\}}+c_{N\setminus\{j\}}\right)^2}{9}}{|N|} + \frac{\left(1 - 2c_{\{i\}} + c_{N\setminus\{i\}}\right)^2}{9}.$$

Suppose that $|N| \geq 3$. Suppose that for some $c \in \left[0, \frac{2}{|N|+1}\right]$, for any $i \in N$, $c_{\{i\}} = c$ and for any $S \in 2^N \setminus \{\emptyset\}$ with $|S| \geq 2$, $c_S = 0$. Then, there exists a cNBS if and only if $c \geq \frac{2|N|-3\sqrt{|N|}}{4|N|}$, which means that the cost synergy by merger is significant.

Note that $\frac{2|N|-3\sqrt{|N|}}{4|N|} > 0$ and if $|N| \leqq 7.5$, $\frac{2|N|-3\sqrt{|N|}}{4|N|} \leqq \frac{2}{|N|+1}$. Then, if there are eight or more firms, regardless of the magnitude of the cost synergy, the cNBS does not exist.

Example 2 (Bertrand competition)**.** Let $\epsilon \in \mathbb{R}_{++}$ be the price unit. Let $P = \{\epsilon i \mid i \in \mathbb{Z}_+\}$ be the set of prices. Let $Q : P \to \mathbb{R}_+$ be the demand function. For simplicity, suppose that for any $p \in P$, $Q(p) = \mathbf{1}_{p \leq 1}(1-p)$. Let N be the set of firms. For any $S \in 2^N \setminus \{\emptyset\}$, let $c_S \in \mathbb{R}_+$ be the marginal and average cost for the merged firm of firms in S. Suppose that for any $S, T \in 2^N \setminus \{\emptyset\}$, if $S \subseteq T$, then $c_S \geq c_T$. For any $\pi \in \Pi^N$ and any $S \in \pi$, let $c^\pi_{-S} := \min\{c_T \mid T \in \pi \setminus \{S\}\} \cup \left\{\frac{1+c_N}{2}\right\}$. If $S \neq N$, c^π_{-S} is the cost of the most efficient competitor for the merged firm of S, and if $S = N$, c^π_{-S} is the monopoly price. For simplicity, suppose that for any $S \in 2^N \setminus \{\emptyset\}$, $c_S \in P$. For any $\pi \in \Pi^N$, let $G(\pi)$ be the strategic game defined as follows: the set of players is π; for any $S \in \pi$, the set of player S's strategies is P; for any $S \in \pi$, player S's payoff function is $P^\pi \ni p \mapsto \mathbf{1}_{p_S = \min_{T \in \pi} p_T}(p_S - c_S)\frac{Q(p_S)}{|\arg\min_{T \in \pi} p_T|} \in \mathbb{R}$. Thus, $G(\pi)$ is a Bertrand game played by merged firms. Suppose that for any $S \in 2^N \setminus \{\emptyset\}$, $c_S \leq \frac{1}{2}$. By this supposition, in any Nash equilibrium in $G(\pi)$ with $\pi \neq \{N\}$, no player enjoys the monopoly profit. For any $\pi \in \Pi^N$, there exists a Nash equilibrium without weakly dominated strategies in $G(\pi)$, and in any such equilibrium, player S's payoff is $\mathbf{1}_{c_S \leq c^\pi_{-S}}\left(c^\pi_{-S} - c_S\right)\left(1 - c^\pi_{-S}\right)$. Let V be a map from $\left\{(S,\pi) \in 2^N \times \Pi^N \mid S \in \pi\right\}$ to $\mathbb{R}$ such that for any $\pi \in \Pi^N$ and any $S \in \pi$, $V_{(S,\pi)}$ is player S's payoff by a Nash equilibrium without weakly dominated strategies in $G(\pi)$. Let $i \in \arg\min_{j \in N} c_{\{j\}}$. Since $\sum_{i \in N} V_{(\{i\},\{\{j\} | j \in N\})} \leq V_{(N,\{N\})}$, there uniquely exists an fNBS of (N, V). The fNBS is $x \in \mathbb{R}^N$ such that $x_i = \frac{(1-c_N)^2 + 4(|N|-1)\left(c^{\{\{j\}|j \in N\}}_{-\{i\}} - c_{\{i\}}\right)\left(1 - c^{\{\{j\}|j \in N\}}_{-\{i\}}\right)}{4|N|}$ and for any $j \in N \setminus \{i\}$, $x_j = \frac{(1-c_N)^2 - 4\left(c^{\{\{j\}|j \in N\}}_{-\{i\}} - c_{\{i\}}\right)\left(1 - c^{\{\{j\}|j \in N\}}_{-\{i\}}\right)}{4|N|}$. Since $\sum_{i \in N} V_{(\{i\},N \setminus \{i\})} \leq \sum_{i \in N} V_{(\{i\},\{\{j\}|j \in N\})} \leq V_{(N,\{N\})}$, there uniquely exists a cNBS of (N, V). Note that for any $j \in N \setminus \{i\}$, $c_{\{j\}} \geq c_{\{i\}} \geq c_{N \setminus \{j\}}$. The cNBS is $x \in \mathbb{R}^N$ such that $x_i = \frac{(1-c_N)^2 + 4(|N|-1)\mathbf{1}_{c_{N\setminus\{i\}} > c_{\{i\}}}\left(c_{N \setminus \{i\}} - c_{\{i\}}\right)\left(1 - c_{N \setminus \{i\}}\right)}{4|N|}$ and for any $j \in N \setminus \{i\}$, $x_j = \frac{(1-c_N)^2 - 4\mathbf{1}_{c_{N\setminus\{i\}} > c_{\{i\}}}\left(c_{N \setminus \{i\}} - c_{\{i\}}\right)\left(1 - c_{N \setminus \{i\}}\right)}{4|N|}$. Then, if $c_{N \setminus \{i\}} < c^{\{\{j\}|j \in N\}}_{-\{i\}}$ and $c_{\{i\}} < c^{\{\{j\}|j \in N\}}_{-\{i\}}$, player i's share in the fNBS is greater than her share in the cNBS; otherwise, the fNBS is equal to the cNBS. Due to the cost synergy, the cost of firm i's (most efficient) competitor in her thread is not greater in the cNBS than in the fNBS, and the other players' payoff in their threads are zero in the both fNBS and cNBS. Thus, player i's share is not less in the fNBS than in the cNBS.

3 Extensive games

In the following sections, fix a PFG (N, V). For any $(\delta, p) \in [0,1]^2 \setminus \{(1,1)\}$, define an extensive game $G(\delta, p)$ as follows. A *prestate* is π such that for some $S \in 2^N \setminus \{\emptyset\}$, $\pi \in \Pi^S$, or $\pi = \emptyset$. π represents a coalition structure of inactive players. For any prestate π, let $A^\pi = N \setminus \bigcup \pi$. A^π represents the set of active players. A *state* is (π, R) such that π is a prestate and R is a complete system of representatives of π if $\pi \neq \emptyset$ and $\emptyset$ otherwise. R represents the set of owners of coalitions in π: for any $i \in R$, inactive player i possesses coalition $[i]_\pi$. Each active player i owns coalition $\{i\}$. In a round with state (π, R) with $A^\pi \neq \emptyset$, bargaining proceeds as follows. Player $i \in A^\pi$ is selected with probability $\frac{1}{|A^\pi|}$. Player i offers a proposal (S, x) such that $i \in S \in 2^{A^\pi}$, $x \in \mathbb{R}^S$ and $\sum_{j \in S} x_j = 0$ (the proposal means that player i offers monetary term x_j for player j's resource). Each player $j \in S \setminus \{i\}$ announces her acceptance or rejection of the proposal according to some predetermined order until a responder rejects it or all responders accept it. If all responders accept it, the state is updated to $(\pi \cup \{S\}, R \cup \{i\})$. Otherwise, the state remains (π, R) with probability p and is updated to $(\pi \cup \{\{j\}\}, R \cup \{j\})$ with probability $1-p$, where j is the rejecter of the proposal. In a round with state (π, R) with $A^\pi = \emptyset$, no bargaining occurs, and the state remains (π, R). In each case, the game proceeds to a new round with the updated state. The game starts from a round with state $(\emptyset, \emptyset)$. In the game, there are four types of players: active players, players who became inactive by rejecting proposals, players who became inactive by their proposals being accepted and players who became inactive by accepting proposals. In state (π, R), the set of the first type of players is A^π, and the set of the first three types of players is $A^{\pi} \cup R$, which is the set of players who possess coalitions. For any complete history h, player i's payoff is defined as follows. For any $t \in \mathbb{N}$, let $\left(\pi^t, R^t\right)$ be the state in the end of the tth round in h and $\left(\hat{\pi}^t, \hat{R}^t\right)$ be a pair such that $\hat{\pi}^t = \pi^t \cup \left\{\{i\} \mid i \in A^{\pi^t}\right\}$ and $\hat{R}^t = R^t \cup A^{\pi^t}$. For any $t \in \mathbb{N}$ and any $i \in N$, let x_i^t be the transfer to player i in the tth round in h. Then, player i's payoff in h is $\sum_{t \in \mathbb{N}} \delta^{t-1} \left((1-\delta)\, \mathbf{1}_{i \in \hat{R}^t} V_{\left([i]_{\hat{\pi}^t}, \hat{\pi}^t\right)} + x_i^t\right)$.

Definition 2. A strategy profile s is a *stationary subgame perfect equilibrium (SSPE)* if s is a subgame perfect equilibrium, and in s, in any round with the same prestate, players take the same actions.

Definition 3. A strategy profile is a *full-coalition* strategy profile if in the strategy profile, in every subgame starting with prestate $\pi \neq \emptyset$, coalition A^{π} is immediately formed.

We say that a PFG (M, U) is a *subgame* of (N, V) if $M \subseteq N$ and for some $f : \Pi^M \to \Pi^N$ such that $\pi \subseteq f(\pi)$ for any $\pi \in \Pi^M$ and $f(\pi) \setminus \pi = f(\pi') \setminus \pi'$ for any $\pi, \pi' \in \Pi^M$, for any $(S, \pi) \in 2^M \times \Pi^M$ with $S \in \pi$, $U_{(S,\pi)} = V_{(S, f(\pi))}$. If in any subgame (M, U) of (N, V), the grand coalition is efficient, i.e., $U(M, \{M\}) \geq \sum_{S \in \pi} V(S, \pi)$ for any $\pi \in \Pi^M$, then, for any $(\delta, p) \in [0, 1) \times [0, 1]$, full-coalition strategy profiles coincide subgame-efficient strategy profiles, i.e., strategy profiles that are Pareto efficient in any subgame of $G(\delta, p)$. The subgame-efficiency is defined by [Oka96]. [Haf07] shows that if (N, V) is convex, i.e., for any subgame (M, U) of (N, V) and any $S, T \in 2^M \setminus \{\emptyset\}$ with $S \cup T = M$, $U_{(S \cup T, \{S \cup T\})} + U_{(S \cap T, \{S \cap T, S \setminus T, T \setminus S\})} \geq U_{(S, \{S, T \setminus S\})} + U_{(T, \{T, S \setminus T\})}$, then, in any subgame of (N, V), the grand coalition is efficient.

4 Support for Nash bargaining solution

We show that any full-coalition SSPE brings the same expected payoff profile in any subgame starting with the same prestate, and explicitly characterize the expected payoff profile. By the characterization, the expected payoff profile by any full-coalition SSPE of $G(1,p)$ ($G(\delta,1)$, resp.) is proved to be the cNBS (fNBS, resp.).

For any prestate π with $A^\pi \neq \emptyset$, let $\underline{v}^\pi$ ($\overline{v}^\pi$, resp.) be a function from $2^{A^\pi} \setminus \{\emptyset\}$ to $\mathbb{R}_+$ such that for any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, $\underline{v}^\pi_S = V_{\left(S, \pi \cup \underline{\pi}^{A^\pi}_S\right)}$ ($\overline{v}^\pi_S = V_{\left(S, \pi \cup \overline{\pi}^{A^\pi}_S\right)}$, resp.).

Theorem 1. *Let $(\delta,p) \in [0,1]^2 \setminus \{(1,1)\}$. Let s be a full-coalition SSPE of $G(\delta,p)$. Then, for any prestate π, for any $i \in A^\pi$, player i's expected payoff by s in the subgame of $G(\delta,p)$ starting with prestate π is*

$$\begin{aligned}&\frac{\overline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi} \frac{(1-\delta)\underline{v}^\pi_{\{j\}} + (1-p)\delta \overline{v}^\pi_{\{j\}}}{1-p\delta}}{|A^\pi|} + \frac{(1-\delta)\underline{v}^\pi_{\{i\}} + (1-p)\delta\overline{v}^\pi_{\{i\}}}{1-p\delta} \\ &= \frac{(1-\delta)\left(\frac{\underline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi} \underline{v}^\pi_{\{j\}}}{|A^\pi|} + \underline{v}^\pi_{\{i\}}\right) + (1-p)\delta\left(\frac{\overline{v}^\pi_{A^\pi} - \sum_{j\in A^\pi} \overline{v}^\pi_{\{j\}}}{|A^\pi|} + \overline{v}^\pi_{\{i\}}\right)}{1-p\delta}.\end{aligned}$$

Corollary 1. *If there exists a cNBS, for any $p \in [0,1)$, the expected payoff profile by any full-coalition SSPE of $G(1,p)$ is equal to the cNBS. If there exists an fNBS, for any $\delta \in [0,1)$, the expected payoff profile by any full-coalition SSPE of $G(\delta,1)$ is equal to the fNBS.*

Suppose that $\delta = 1$. In the first round, by rejection, responder i is excluded from the society with probability $1-p$, and in the next round, the coalition of the other players forms in any full-coalition SSPE and she obtains a payoff of $V_{(\{i\}, N \setminus \{i\})}$. Thus, the thread for player i in the first round is $V_{(\{i\}, N\setminus\{i\})}$. In the first round, in any full-coalition SSPE, all players cooperate and share $V_{(N,\{N\})}$. Therefore, the expected payoff profile by any full-coalition SSPE is the cNBS.

Suppose that $p = 1$. In the first round, by rejection, responder i obtains an instant payoff of $(1-\delta) V_{(\{i\},\{\{j\}|j\in N\})}$. Thus, the thread for player i is $V_{(\{i\},\{\{j\}|j\in N\})}$. In the first round, in any full-coalition SSPE, all players cooperate and share $V_{(N,\{N\})}$. Therefore, the expected payoff profile by any full-coalition SSPE is the fNBS.

5 Conditions for full coalition formation

We provide a necessary and sufficient condition for a full-coalition SSPE to exist.

Theorem 2. *Let* $(\delta, p) \in [0,1]^2 \setminus \{(1,1)\}$*. Then, there exists a full-coalition SSPE of* $G(\delta, p)$ *if and only if for any prestate* π *with* $A^\pi \neq \emptyset$ *and any* $S \in 2^{A^\pi} \setminus \{\emptyset\}$*,*

$$\frac{\overline{v}^\pi_{A^\pi} - \sum_{k \in A^\pi} \frac{(1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}}}{1-\delta p}}{|A^\pi|} \geq \frac{(1-\delta)\underline{v}^\pi_S + \delta \overline{v}^\pi_S - \sum_{k \in S} \frac{(1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}}}{1-\delta p}}{\delta p |S| + (1-\delta p)|A^\pi|} \tag{1}$$

and for any prestate π *with* $|A^\pi| \geq 2$ *and any distinct* $i, j \in A^\pi$*,*

$$\begin{aligned} &\overline{v}^\pi_{A^\pi} - \sum_{k \in A^\pi} \frac{(1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}}}{1-\delta p} + \frac{\delta(1-p)}{1-\delta p}\overline{v}^\pi_{\{i\}} \\ &\geq \frac{\delta(1-p)}{1-\delta p} \frac{\overline{v}^{\pi \cup \{\{j\}\}}_{A^\pi \setminus \{j\}} - \sum_{k \in A^\pi \setminus \{j\}} \frac{(1-\delta)\underline{v}^{\pi \cup \{\{j\}\}}_{\{k\}} + \delta(1-p)\overline{v}^{\pi \cup \{\{j\}\}}_{\{k\}}}{1-\delta p}}{|A|^\pi - 1} \\ &\quad + \frac{\delta(1-p)}{1-\delta p} \frac{(1-\delta)\underline{v}^{\pi \cup \{\{j\}\}}_{\{i\}} + \delta(1-p)\overline{v}^{\pi \cup \{\{j\}\}}_{\{i\}}}{1-\delta p}. \end{aligned} \tag{2}$$

Corollary 2. *For some* $\bar{p} \in [0,1)$*, for any* $p \in [\bar{p}, 1)$*, there exists a full-coalition SSPE of* $G(1,p)$ *if and only if for any prestate* π *with* $A^\pi \neq \emptyset$ *and any* $S \in 2^{A^\pi} \setminus \{\emptyset\}$*,*

$$\frac{\overline{v}^\pi_{A^\pi} - \sum_{k \in A^\pi} \overline{v}^\pi_{\{k\}}}{|A^\pi|} \geq \frac{\overline{v}^\pi_S - \sum_{k \in S} \overline{v}^\pi_{\{k\}}}{|S|} \tag{3}$$

and for any prestate π *with* $|A^\pi| \geq 2$ *and any distinct* $i, j \in A^\pi$*,*

$$\overline{v}^\pi_{A^\pi} - \sum_{k \in A^\pi} \overline{v}^\pi_{\{k\}} + \overline{v}^\pi_{\{i\}} \geq \frac{\overline{v}^{\pi \cup \{\{j\}\}}_{A^\pi \setminus \{j\}} - \sum_{k \in A^\pi \setminus \{j\}} \overline{v}^{\pi \cup \{\{j\}\}}_{\{k\}}}{|A^\pi| - 1} + \overline{v}^{\pi \cup \{\{j\}\}}_{\{i\}}. \tag{4}$$

For some $\bar{\delta} \in [0,1)$*, for any* $\delta \in \left[\bar{\delta}, 1\right)$*, there exists a full-coalition SSPE of* $G(1,\delta)$ *if and only if for any prestate* π *with* $A^\pi \neq \emptyset$ *and any* $S \in 2^{A^\pi} \setminus \{\emptyset\}$*,*

$$\frac{\underline{v}^\pi_{A^\pi} - \sum_{k \in A^\pi} \underline{v}^\pi_{\{k\}}}{|A^\pi|} \geq \frac{\underline{v}^\pi_S - \sum_{k \in S} \underline{v}^\pi_{\{k\}}}{|S|}. \tag{5}$$

For any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, (3) holds if and only if the NBS for $(A^\pi, \overline{v}^\pi)$ is in the core of $(A^\pi, \overline{v}^\pi)$.[3] (3) is a condition for any player to offer a proposal with the full

[3] Suppose that $\pi = \emptyset$. Then, for any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, (3) holds if and only if the cNBS for (N, V) is in the core with merging expectations of (N, V), which is defined by [Haf07].

coalition in the round with prestate π. (4) is a condition for any player to offer a proposal to be accepted in the round with prestate π.

The sketch of proof of necessity of Theorem 2 is as follows. Suppose that for sufficiently large p, there exists a full-coalition SSPE s of $G(1,p)$. Take a sufficiently large p. Then, the following argument approximately holds. By her rejection, player j's continuation payoff by her rejection is her expected payoff u_i^π by s in the subgame with prestate π, In full-coalition SSPE s, since player i offers a proposal with the full coalition to be accepted, she obtains a payoff of $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} u_j^\pi$. If she deviates to offering a proposal with coalition S to be accepted, in full-coalition SSPE s, $A^\pi\setminus S$ forms in the next round. Thus, by the deviation, she can obtain $\overline{v}_S^\pi - \sum_{j\in S\setminus\{i\}} u_j^\pi$. Thus, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi \geq \overline{v}_S^\pi - \sum_{j\in S\setminus\{i\}} x_j^\pi$, i.e., $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi} u_j^\pi \geq \overline{v}_S^\pi - \sum_{j\in S} u_j^\pi$. Note that since s is a full-coalition SSPE, $\sum_{j\in A^\pi} u_j^\pi = \overline{v}_{A^\pi}^\pi$. Then, $\overline{v}_S^\pi \geq \sum_{j\in S} u_j^\pi$. Note that S is arbitrary and by Theorem 1, the expected payoff profile is the NBS for $(A^\pi, \overline{v}^\pi)$. Thus, the NBS for $(A^\pi, \overline{v}^\pi)$ is in the core of $(A^\pi, \overline{v}^\pi)$. If player i deviates to offering a proposal to be accepted by player j, the prestate is updated to prestate $\pi \cup \{\{j\}\}$ with probability $1-p$ and it remains π with probability p. Thus, by the deviation, she can obtain $(1-p)\, u_i^{\pi\cup\{\{j\}\}} + p u_i^\pi$. Thus, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} \left((1-p)\, v_{\{j\}}^\pi + p u_j^\pi\right) \geq (1-p)\, u_i^{\pi\cup\{\{j\}\}} + p u_i^\pi$, i.e., $\overline{v}_{A^\pi}^\pi - p\sum_{j\in A^\pi} u_j^\pi - (1-p)\sum_{j\in A^\pi\setminus\{i\}} v_{\{j\}}^\pi \geq (1-p)\, u_i^{\pi\cup\{\{j\}\}}$. Note that since s is a full-coalition SSPE, $\sum_{j\in A^\pi} u_j^\pi = \overline{v}_{A^\pi}^\pi$. Then, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} v_{\{j\}}^\pi \geq u_i^{\pi\cup\{\{j\}\}}$. Note that by Theorem 1, $u_i^{(A^\pi\setminus\{j\},\pi)}$ is player i's share in the NBS for $\left(A^\pi \setminus \{j\}, \overline{v}^{\pi\cup\{\{j\}\}}\right)$. Then, (4) holds.

For any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, (5) holds if and only if the NBS of $(A^\pi, \underline{v}^\pi)$ is in the core of $(A^\pi, \underline{v}^\pi)$.[4]

Example 3 (Cournot competition)**.** Consider Example 1 again. Suppose that $N = \{1,2,3\}$. Suppose that for some $c \in \left[0, \frac{1}{2}\right]$, for any $i \in N$, $c_{\{i\}} = c$ and for any $S \in 2^N \setminus \{\emptyset\}$ with $|S| \geq 2$, $c_S = 0$. Then, for some $\bar{p} \in [0,1)$, for any $p \in [\bar{p}, 1)$, there exists a full-coalition SSPE of $G(1,p)$ if and only if $\frac{2-\sqrt{3}}{4} \leq c \leq \frac{-2+\sqrt{6}}{2}$. Note that $0 < \frac{2-\sqrt{3}}{4} < \frac{-2+\sqrt{6}}{2} < \frac{1}{2}$. The reason for this condition to be required is as

[4] Suppose that $\pi = \emptyset$. Then, for any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, (5) holds if and only if the fNBS for (N, V) is in the core with singleton expectations of (N, V), which is defined by [Haf07].

follows: (i) if c is sufficiently large, the advantage of a two-firm coalition over the other isolated firm, this coalition is profitable, and thus, in prestate π with $A^\pi = N$, the grand coalition fails to be formed; (ii) if c is sufficiently small, by the merger paradox, a two-firm coalition is not profitable, and thus, in prestate π with $|A^\pi| = 2$, the full coalition fails to be formed; (iii) if c is sufficiently small, by the positive externalities, an isolated firm's profit is large, in prestate π with $A^\pi = N$, each responder's continuation payoff is large, and thus, the grand coalition fails to be formed. For some $\bar{\delta} \in [0,1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, 1)$ if and only if $17 - 12\sqrt{2} \leq c \leq \frac{-2+\sqrt{6}}{2}$. Note that $0 < 17 - 12\sqrt{2} < \frac{-2+\sqrt{6}}{2}$. The reason for this condition to be required is the same as (i) and (ii) above. Under $p = 1$, each responder's continuation payoff in prestate π with $A^\pi = N$ is her profit when every player stands alone, the positive externalities do not affect the continuation payoff, and thus, (iii) does not matter. Hence, the condition for a full-coalition SSPE to exist is stronger under $\delta = 1$ than $p = 1$ ($17 - 12\sqrt{2} < \frac{2-\sqrt{3}}{4}$).

Example 4 (Bertrand competition)**.** Consider Example 2 again. Suppose that for any $S, T \in 2^N \setminus \{\emptyset\}$, $c_S = c_T$. Then, for any $S \neq N$, $V(S, \pi) = 0$. Thus, for any $(\delta, p) \in [0,1]^2 \setminus \{(1,1)\}$, there exists a full-coalition SSPE of $G(\delta, p)$. Suppose that $N = \{1, 2, 3\}$. Suppose that for some $c \in \mathbb{R}_+$, $c_{\{1\}} = c_{\{2\}} = c$ and for any $S \neq \{1\}, \{2\}$, $c_S = 0$. Then, for some $\bar{p} \in [0,1)$, for any $p \in [\bar{p}, 1)$, there exists a full-coalition SSPE of $G(1, p)$ if and only if $c \leq \frac{3-\sqrt{3}}{6}$. If c is large, the advantage of a two-firm coalition that firm 1 belongs to over the other isolated firm is large, this coalition is profitable, and thus, the grand coalition fails to be formed. On the other hand, for some $\bar{\delta} \in [0,1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, 1)$. Under $p = 1$, the profit of the two-firm coalition that firm 1 belongs to is large, but since player 1's thread payoff is large, the player 1's share in the fNBS is large. Thus, the two-firm coalition cannot block the fNBS. Therefore, regardless of c, the full-coalition SSPE exists.

6 Conclusion

We defined two kinds of NBS for PFGs (fNBS and cNBS). Based on any PFG, we defined an EG, which is a propose-respond sequential game where the first rejecter exits from the game with a positive probability. We showed that the NBS is supported as the expected payoff profile by any SSPE of the EG such that in any subgame, the coalition of all active players is immediately formed. We also provided a necessary and sufficient condition for such an SSPE to exist.

Appendix

A Lemmas

Let $(\delta, p) \in [0,1]^2 \setminus \{(1,1)\}$. Let s be a full-coalition SSPE of $G(\delta, p)$.

Lemma 1. *Let π be a prestate with $A^\pi \neq \emptyset$. For any $i \in A^\pi$, let u_i be the payoff of player i by s at her proposing node in any round with prestate π. Let (S,x) be a proposal in a round with prestate π. Then, in the round with prestate π, if for any $i \in S$, $x_i > (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$, then, (S,x) is accepted; if for some $i \in S$, $x_i < (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$, then, (S,x) is rejected.*

Proof. Suppose that for any $i \in S$, $x_i > (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$. Given the other actions in s, the last responder i obtains x_i by accepting (S,x) and $(1-\delta)V_{\left(\{i\},\underline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)} + \delta(1-p)V_{\left(\{i\},\overline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)} + \delta p u_i = (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$ by rejecting it. Thus, she accepts it in s. Let i be a responder. Suppose that any follower j of i accepts (S,x) in s. Then, given the other actions in s, responder i obtains x_i by accepting (S,x) and $(1-\delta)V_{\left(\{i\},\underline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)} + \delta(1-p)V_{\left(\{i\},\overline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)} + \delta p u_i = (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$ by rejecting it. Thus, she accepts it in s. Therefore, by the mathematical induction, (S,x) is accepted.

Suppose that for some $i \in S$, $x_i < (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$. Suppose that (S,x) is accepted. Player $i \in A^\pi$'s payoff by s at her node at which she responds (S,x) in a round with prestate π is x_i. Her payoff by the deviation to rejection is $(1-\delta)V_{\left(\{i\},\underline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)} + \delta(1-p)V_{\left(\{i\},\overline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)} + \delta p u_i = (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i$. Thus, the payoff by s is greater than that of the deviation. This is a contradiction. Thus, (S,x) is rejected. Q.E.D.

Lemma 2. *Let π be a prestate with $A^\pi \neq \emptyset$. For any $i \in A$, let u_i be player i's payoff by s at her proposing node in any round with prestate π and $\left(S^i, x^i\right)$ be player i's proposal in any round with prestate π. Then, for any $i \in A^\pi$, $S^i = A^\pi$ and for any $j \in S^i \setminus \{i\}$, $x^i_j = (1-\delta)\underline{v}^\pi_{\{j\}} + (1-p)\delta\overline{v}^\pi_{\{j\}} + p\delta u_j$.*

Proof. Since s is full-coalition, $S^i = A^\pi$. Since s is full-coalition, by Lemma 1, for any $j \in S^i$, $x^i_j \geq (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$. Suppose that for some $j \in S^i$, $x^i_j > (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$. Let $\epsilon := \frac{x^i_j - (1-\delta)\underline{v}^\pi_{\{j\}} - \delta(1-p)\overline{v}^\pi_{\{j\}} - \delta p u_j}{2}$. Let y be a member in $\mathbb{R}^{S^i}$ such that $y_j = x^i_j - \epsilon$ and for any $k \in S^i \setminus \{j\}$, $y_k = x_k + \frac{\epsilon}{2(|S^i|-1)}$. Then, for any $k \in S^i$, $y_k > (1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}} + \delta p u_k$. Thus, by Lemma 1, $\left(y, S^i\right)$ is accepted in s. Hence, by the deviation to proposing $\left(y, S^i\right)$, player i's payoff at her proposing node in any round with prestate π increases from x^i_i to y_i, which is a contradiction. Therefore, for any $j \in S^i \setminus \{i\}$, $x^i_j = (1-\delta)\underline{v}^\pi_{\{j\}} + (1-p)\delta\overline{v}^\pi_{\{j\}} + p\delta u_j$. Q.E.D.

B Proof of Theorem 1

Let $(\delta, p) \in [0,1]^2 \setminus \{(1,1)\}$. Let s be a full-coalition SSPE of $G(\delta, p)$. Let π be a prestate. For any $i \in A^\pi$, let u_i be player i's expected payoff by s in a round with prestate π. Then, by Lemma 2, for any $i \in A^\pi$,

$$\begin{aligned}
u_i = &\frac{V_{(A^\pi, \pi\cup\{A^\pi\})} - \sum_{j\in A^\pi\setminus\{i\}}\left((1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j\right)}{|A^\pi|} \\
&+ (|A^\pi|-1)\frac{(1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i}{|A^\pi|} \\
= &\frac{\overline{v}^\pi_{A^\pi} - \sum_{j\in A^\pi}\left((1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j\right)}{|A^\pi|} + (1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}} + \delta p u_i.
\end{aligned}$$

and thus,

$$u_i = \frac{\frac{\overline{v}^\pi_{A^\pi} - \delta p\sum_{j\in A^\pi} u_j}{1-\delta p} - \sum_{j\in A^\pi}\frac{(1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}}}{1-\delta p}}{|A^\pi|} + \frac{(1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}}}{1-\delta p}.$$

Note that since s is a full-coalition SSPE, $\sum_{i\in A^\pi} u_i = V_{(A^\pi, \pi\cup\{A^\pi\})} = \overline{v}^\pi_{A^\pi}$. Then, we have the conclusion. Q.E.D.

C Proof of Theorem 2

Let $(\delta, p) \in [0,1]^2 \setminus \{(1,1)\}$. For any prestate π with $A^\pi \neq \emptyset$ and any $i \in A^\pi$, let $u_i^\pi := \frac{\overline{v}_{A^\pi}^\pi - \sum_{j \in A^\pi} \frac{(1-\delta)\underline{v}_{\{j\}}^\pi + \delta(1-p)\overline{v}_{\{j\}}^\pi}{1-\delta p}}{|A^\pi|} + \frac{(1-\delta)\underline{v}_{\{i\}}^\pi + \delta(1-p)\overline{v}_{\{i\}}^\pi}{1-\delta p}$ and $x_i^\pi := (1-\delta)\,\underline{v}_{\{i\}}^\pi + \delta\,(1-p)\,\overline{v}_{\{i\}}^\pi + \delta p u_i^\pi$. For any prestate π with $A^\pi \neq \emptyset$, any $i \in A^\pi$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$ with $S \ni i$, let

$$
\begin{aligned}
a_{iS}^\pi &:= \left(\overline{v}_{A^\pi}^\pi - \sum_{k \in A^\pi \setminus \{i\}} x_k^\pi\right) - \left((1-\delta)\,\underline{v}_S^\pi + \delta \overline{v}_S^\pi - \sum_{k \in S \setminus \{i\}} x_k^\pi\right) \\
&= \frac{\delta p\,|S| + (1-\delta p)\,|A^\pi|}{|A^\pi|}\left(\overline{v}_{\{i\}}^\pi - \sum_{k \in A^\pi} \frac{(1-\delta)\,\underline{v}_{\{k\}}^\pi + \delta\,(1-p)\,\overline{v}_{\{k\}}^\pi}{1-\delta p}\right) \\
&\quad - \left((1-\delta)\,\underline{v}_S^\pi + \delta \overline{v}_S^\pi - \sum_{k \in S} \frac{(1-\delta)\,\underline{v}_{\{k\}}^\pi + \delta\,(1-p)\,\overline{v}_{\{k\}}^\pi}{1-\delta p}\right).
\end{aligned}
$$

For any prestate π with $|A^\pi| \geq 2$, any $i \in A^\pi$ and any $j \in A^\pi \setminus \{i\}$, let

$$
\begin{aligned}
b_{ij}^\pi &:= \left(\overline{v}_{A^\pi}^\pi - \sum_{k \in A^\pi \setminus \{i\}} x_k^\pi\right) - \left((1-\delta)\,\underline{v}_{\{i\}}^\pi + \delta\,(1-p)\,u_i^{\pi \cup \{\{j\}\}} + \delta p u_i^\pi\right) \\
&= (1-\delta p)\left(\overline{v}_{A^\pi}^\pi - \sum_{k \in A^\pi} \frac{(1-\delta)\,\underline{v}_{\{k\}}^\pi + \delta\,(1-p)\,\overline{v}_{\{k\}}^\pi}{1-\delta p}\right) \\
&\quad - \delta\,(1-p)\,\frac{\overline{v}_{A^\pi \setminus \{j\}}^{\pi \cup \{\{j\}\}} - \sum_{k \in A^\pi \setminus \{j\}} \frac{(1-\delta)\underline{v}_{\{k\}}^{\pi \cup \{\{j\}\}} + \delta(1-p)\overline{v}_{\{k\}}^{\pi \cup \{\{j\}\}}}{1-\delta p}}{|A^\pi| - 1} \\
&\quad - \delta\,(1-p)\left(\frac{(1-\delta)\,\underline{v}_{\{i\}}^{\pi \cup \{\{j\}\}} + \delta\,(1-p)\,\overline{v}_{\{i\}}^{\pi \cup \{\{j\}\}}}{1-\delta p} - v_{\{i\}}^\pi\right).
\end{aligned}
$$

Necessity Suppose that there exists a full-coalition SSPE s of $G(\delta, p)$. Let π be a prestate with $A^\pi \neq \emptyset$. Let $S \in 2^{A^\pi} \setminus \{\emptyset\}$. Let $i \in S$. Since s is a full-coalition SSPE, by Lemma 2 and Theorem 1, player i's payoff by s conditional on being a proposer in the round with prestate π is $V_{(A^\pi, \pi \cup \{A^\pi\})} - \sum_{j \in A^\pi \setminus \{i\}} x_j^\pi = \overline{v}_{A^\pi}^\pi - \sum_{j \in A^\pi \setminus \{i\}} x_j^\pi$. For any $\epsilon \in \mathbb{R}_{++}$, let y^ϵ be a member in $\mathbb{R}^S$ such that for any $j \in S \setminus \{i\}$, $y_j^\epsilon = x_j^\pi + \epsilon$. Then, by Lemma 1 and Theorem 1, for any $\epsilon \in \mathbb{R}_{++}$, player i's proposal (S, y^ϵ) is accepted in s. Thus, for any $\epsilon \in \mathbb{R}_{++}$, player i's payoff by the deviation to proposal (S, y^ϵ) conditional on being a proposer in the round with prestate

π is $(1-\delta) V_{\left(S,\pi\cup\underline{\pi}_S^{A^\pi}\right)} + \delta V_{\left(S,\pi\cup\overline{\pi}_S^{A^\pi}\right)} - \sum_{j\in S\setminus\{i\}} \left(x_j^\pi + \epsilon\right) = (1-\delta)\underline{v}_S^\pi + \delta\overline{v}_S^\pi - \sum_{j\in S\setminus\{i\}} x_j^\pi + (|S|-1)\,\epsilon =: a^\epsilon$. Then, for any $\epsilon \in \mathbb{R}_{++}$, since s is a subgame perfect equilibrium, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi \geq a^\epsilon$. Hence, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi \geq \lim_{\epsilon\to 0} a^\epsilon$. Thus, $a_{iS}^\pi \geq 0$, which is equivalent to (1). Player i's payoff by the deviation to proposal rejected by player j in s conditional on being a proposer in the round with prestate π is $(1-\delta) V_{\left(\{i\},\pi\cup\underline{\pi}_{\{i\}}^{A^\pi}\right)} + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + p u_i^\pi = (1-\delta)\underline{v}_{\{i\}}^\pi + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + p u_i^\pi$. Then, since s is a subgame perfect equilibrium, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi \geq (1-\delta)\underline{v}_{\{i\}}^\pi + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + \delta p u_i^\pi$. Thus, $b_{ij}^\pi \geq 0$, which is equivalent to (2).

Sufficiency Suppose that for any prestate π with $A^\pi \neq \emptyset$, any $i \in A^\pi$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$ with $S \ni i$, (1) holds and for any prestate π with $|A^\pi| \geq 2$, any $i \in A^\pi$ and any $j \in A^\pi \setminus \{i\}$, (2) holds. Then, for any prestate π with $A^\pi \neq \emptyset$, any $i \in A^\pi$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$ with $S \ni i$, $a_{iS}^\pi \geq 0$, and for any prestate π with $|A^\pi| \geq 2$, any $i \in A^\pi$ and any $j \in A^\pi \setminus \{i\}$, $b_{ij}^\pi \geq 0$. Construct a strategy profile s of $G(\delta, p)$ as in any round with any prestate π, players' actions described as follows. Any proposer i proposes (A^π, x) such that for any $j \in A^\pi \setminus \{i\}$, $x_j = x_j^\pi$. Responses to any proposal (S, x) are recursively defined. The last responder j accepts it if and only if $x_j \geq x_j^\pi$. Let $k \in S \setminus \{j\}$. If all followers of k accepts it, responder k accepts a proposal (S, x) if and only if $x_k \geq x_k^\pi$; otherwise, she accepts it if and only if $(1-\delta)\underline{v}_{\{k\}}^\pi + \delta(1-p)\, u_k^{\pi\cup\{\{l\}\}} + \delta p u_k^\pi \geq x_k^\pi$, where l is the first follower who rejects it. Then, any player's proposal in s is accepted in s. Consider any player i's proposing node with any prestate π. Since her proposal in s is accepted in s, her payoff by s at the node is $V_{(A^\pi,\pi\cup\{A^\pi\})} - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi = \overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi$. Her payoff by the deviation to a proposal (S, x) accepted in s is less than or equal to $(1-\delta) V_{\left(S,\pi\cup\underline{\pi}_S^{A^\pi}\right)} + \delta V_{\left(S,\pi\cup\overline{\pi}_S^{A^\pi}\right)} - \sum_{j\in S\setminus\{i\}} x_j^\pi = (1-\delta)\underline{v}_S^\pi + \delta\overline{v}_S^\pi - \sum_{j\in S\setminus\{i\}} x_j^\pi$. Since $\left(\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi\right) - \left((1-\delta)\underline{v}_S^\pi + \delta\overline{v}_S^\pi - \sum_{j\in S\setminus\{i\}} x_j^\pi\right) = a_{iS}^\pi \geq 0$, she cannot improve her payoff by this deviation. Player i's payoff by the deviation to a proposal rejected by responder j in s is $(1-\delta) V_{\left(\{i\},\pi\cup\underline{\pi}_{\{i\}}^{A^\pi}\right)} + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + \delta p u_i^\pi = (1-\delta)\underline{v}_{\{i\}}^\pi + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + \delta p u_i^\pi$. Since $\left(\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi\right) -$

$\left((1-\delta)\,\underline{v}^{\pi}_{\{i\}} + \delta\,(1-p)\,u_i^{\pi\cup\{\{j\}\}} + pu_i^{\pi}\right) = b^{\pi}_{ij} \geq 0$, she cannot improve her payoff by this deviation. Thus, her proposal at the node is optimal. Players' responses in s is obviously optimal. Hence, by the one-stage deviation principle, s is a subgame perfect equilibrium. Obviously, s is a stationary and full-coalition strategy profile. Therefore, s is a full-coalition SSPE. Q.E.D.

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