組合せ構造上の協力ゲームとシャープレイ値

藤本 勝成　(Katsushige FUJIMOTO)
福島大学 共生システム理工学類　(School of Symbiotic Systems Science)
fujimoto@sss.fukushima-u.ac.jp

1　はじめに

協力ゲーム (TU ゲーム) における提携のあり方に対して，主要な 2 つの批判がある．

(I) (無意味な提携)　プレイヤー間のあらゆる提携が実現可能であることを仮定している．しかしながら，このようなケースは一般的とは言えない．提携を実現するためには，提携内のメンバーが互いに，コミュニケーションチャンネルを持つ必要がある．また，たとえコミュニケーションチャンネルを持っていたとしても，空間的制約，立場やイデオロギーなど様々な要因から実現不能な提携も存在する．例えば，プレイヤーが政党である場合，右翼政党と左翼政党の提携は通常考えられない．

(II) (白黒はっきりするとは限らない)　プレイヤーは，提携に参加するか否かの二者択一を強いられる．しかしながら，もう少し緩やかな参加形態も考えられる．実際，投票行動では，賛成，反対に加え第 3 の選択肢「棄権」が存在する．

これらの批判に対応すべく，通常の協力ゲームを拡張・一般化する形で様々なゲームとそのゲームにおける解概念が提案されてきている．たとえば，(I) の批判に対して，Myerson [18] は，プレイヤー間相互のコミュニケーション構造を無向グラフで表現し，提携をこのグラフネットワークによって実現可能なもののみに制限したコミュニケーションゲーム (communication game, network-restricted game) とその解 (いわゆるマイヤーソン値) を提案した．(II) の批判に対しては，Hsiao と Raghavan [12] は, 提携に対する多段階の参加レベルを考慮した多選択肢ゲーム (multichoice game) を Machover と Felsenthal [7] は賛成・反対・棄権の 3 択投票ゲーム (ternary voting game) を提案した．その後，Faigle と Kern [6] は，(I) の批判に対して，プレイヤー間の階層・順序関係を考慮した新たなゲーム構造と解概念を提案した．このゲームの構造は多選択肢ゲームの拡張概念にもなっている．一方，Bilbao [2] らは，Myerson のコミュニケーション構造をより一般化した数学的構造上 (union stable set, convex geometry, matroid,...) でのゲームとその解概念を提案した．

現在では，Myerson [18] に端を発するコミュニケーションゲームの文脈における様々な解概念の研究 (e.g., ポジション値 [3]，Hamiache 値 [11] など) と Fagle と Kern [6] による階層・順序構造を考慮したゲームに端を発する様々な組合せ構造上のゲームの研究 (e.g., games on union stable systems [1], on convex geometries [2], 3 択投票ゲーム [7], 双容量 [9] など) が 2 つの代表的な研究の方向となっている．

以下，本稿を通して，N を空でない有限な n 要素集合とし，$N = \{1, \ldots, n\}$ と要素をナンバリングして表す．また，$\emptyset \in \mathcal{F}$ なる $\mathcal{F} \subseteq 2^N$ を N 上の集合系 (set system) と呼ぶ．また，特に混乱が生じる恐れがない場合，表記の複雑さを避けるため，集合を表す記号 $\{\ \}$, $\{\ ,\ \}$ を省略し，$\{i\}$ や $\{i, j\}$ の代わりに，i や ij の記法を用いることがある．

2　組合せ構造上へのゲームの拡張

2.1　コミュニケーションネットワークに基づく種々のゲーム

定義 1 (提携型ゲーム) プレイヤー全体の集合を N とし，その部分集合 $S \subseteq N$ を**提携**と呼ぶ．また，$v(\emptyset) = 0$ なる関数 $v : 2^N \to \mathbb{R}$ を N 上の**特性関数**と呼び，これらからなる (N, v) を**提携型ゲーム**と呼ぶ．また，単に v を N 上のゲームと呼ぶこともある．また，特別なゲームとして

$$u_T(S) = \begin{cases} 1 & \text{if} \quad S \supseteq T, \\ 0 & \text{otherwise} \end{cases}$$

を $T \subseteq N$ における **unanimity game** と呼ぶ.

定義 2 (連絡網・連絡状況) (N, v) を提携型ゲームとするとき，N を節点の集合とする無向グラフ (N, L) を N の**連絡網** (communication network) と呼ぶ．ただし，$L \subseteq \{ij \subseteq N \mid i \neq j\}$ とする．また，(N, L) の $T \subseteq N$ への制限を $(T, L(T))$ と記し，

$$L(T) := \{ij \in L \mid ij \subseteq T\}$$

と定義する．また，(N, v^N, L) を**連絡状況** (communication situation) と呼び，N 上の連絡状況全体を CS^N と記す.

定義 3 (連絡可能性・提携の実現可能性) 連絡網 (N, L) において，プレイヤー $j \in S$ と $k \in S$ が S において**連絡可能**であるとは,

$$\begin{aligned} & j = k \qquad \text{または} \\ & \qquad \exists \{i_1, \cdots, i_m\} \subseteq S \quad s.t. \quad j = i_1,\ k = i_m,\ \{i_t, i_{t+1}\} \in L \quad \forall t \in \{1, \cdots, m-1\} \end{aligned}$$

を満すことをいい，$j \sim_S k$ と書く．この L より導かれる同値関係 $\sim_S$ は，S の分割 $S/L := S/\sim_S$ を導く．また，任意の $j,\ k \in S$ が S において連絡可能 (つまり，$(S, L(S))$ が連結グラフ) であるとき提携 S は**実現可能**であるという.

例 4 $N = \{1, 2, 3, 4, 5, 6, 7\}$, $L_1 = \{12, 15, 26, 37, 47, 56\}$ からなる連絡網 (N, L_1) を考える (図 1).

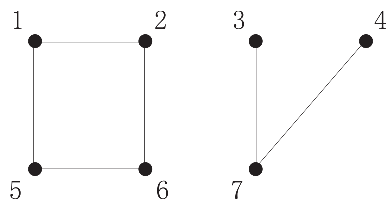

図 1: 連絡網 (N_1, L_1)

また，$v(S)$ を提携 S によって実現可能な最大利得とする．この場合，メンバー $\{1, 2, 6\}$ は互いに連絡可能であり，提携 $\{1, 2, 6\}$ が実現可能となる．よって，$v(\{1, 2, 6\})$ を獲得することができる．一方で，メンバー $\{1, 2, 3, 4\}$ では，1 と 2 は連絡可能であるが，3 と 4 はいずれも，他のどのメンバーとも連絡可能ではない．よって，このメンバーで実現可能な提携は $\{1, 2\}$, $\{3\}$, $\{4\}$ であり，メンバーとして獲得できる最大利得は $v(\{1, 2\}) + v(\{3\}) + v(\{4\})$ となる．つまり，一般に，メンバー S によって獲得可能な最大利得は

$$\sum_{T \in S/L} v(T)$$

ということになる.

定義 5 (連絡網制限ゲーム [18]) 連絡状況 (N, v, L) における**連絡網制限ゲーム** (network restricted game) とは，以下の特性関数

$$v^L(S) := \sum_{T \in S/L} v(T) \quad \text{for each } S \subseteq N$$

を持つゲーム (N, v^L) として定義される．

あるコミュニケーション構造において実行可能な 2 つの提携 S と T があり，これらの提携に共通なプレイヤーが存在した場合 (i.e., $S \cap T \neq \emptyset$)，この共通したプレイヤーを仲介者として，これらの提携の合併 $S \cup T$ のプレイヤーらは互いにコミュニケーションをとることができる．つまり，$S \cup T$ もまた実現可能となる．これを基に，Bilbao[2] らは，**合併安定系 (union stable system)** を定義した．

定義 6 (合併安定系) 集合系 $\mathcal{U} \subseteq 2^N$ が以下の条件を満たすとき，$\mathcal{U}$ は，**合併安定系** (union stable system) と呼ばれる：

$$S, T \in \mathcal{U},\ S \cap T \neq \emptyset \quad \Rightarrow \quad S \cup T \in \mathcal{U}.$$

定義 7 ($\mathcal{U}$-成分) $\mathcal{U} \subseteq 2^N$ を合併安定系とし，$S \subseteq N$ とする．$T \subseteq S$ が以下の条件を満たすとき，T を S の $\mathcal{U}$-成分であるという．

$$\text{(i) } T \in \mathcal{U} \quad \text{and} \quad \text{(ii) } \neg \exists U \in \mathcal{U} \text{ s.t. } T \subsetneq U \subseteq S.$$

また，S の $\mathcal{U}$-成分全体の集合を $\mathcal{C}_{\mathcal{U}}(S)$ で記す．

定義 8 ($\mathcal{U}$-制限ゲーム) (N, v) を通常の提携型ゲーム，$\mathcal{U}$ を合併安定系とする．このとき，**$\mathcal{U}$-制限ゲーム** ($\mathcal{U}$-restricted game) とは，以下の特性関数

$$v^{\mathcal{U}}(S) := \sum_{T \in \mathcal{C}_{\mathcal{U}}(S)} v(T) \quad \text{for each } S \subseteq N$$

を持つゲーム $(N, v^{\mathcal{U}})$ として定義される．

ここで，連絡網 (N, L) における

$$\mathcal{F} := \{S \subseteq N \mid S : \text{実現可能}\}$$

は，合併安定系であり，すべての $S \subseteq N$ に対して，$\mathcal{C}_{\mathcal{F}}(S)$ は S/L に一致する．よって，$v^{\mathcal{F}}$ と v^L も一致する．

例 4 では，与えられた連絡状況において，それぞれのプレイヤーの集合が制限されたコミュニケーション環境の下で，どれだけの利得を得ることが可能かについて見てきた．次は，コミュニケーション構造が変化した際，全体で得ることができる最大利得がどのように変化するのかについて考える．

例 9 図 1 の連絡状況の下で，メンバー全体として $\{1, 2, 5, 6\}$ と $\{3, 4, 7\}$ の提携が実現できるので

$$v^{L_1}(N) = v(\{1, 2, 5, 6\}) + v(\{3, 4, 7\})$$

の利得が獲得可能である．一方で，何らかの理由で，プレイヤー 3 と 7 の間の連絡がとれなくなったとすると，連絡網は $L_2 = \{12, 15, 26, 47, 56\}$ となり，全体 N での獲得利得は

$$v^{L_2}(N) = v(\{1, 2, 5, 6\}) + v(\{3\}) + v(\{4, 7\})$$

となる．一般に，コミュニケーション構造が (N, L) によって与えられるとき，メンバー全体で獲得可能な最大利得は，

$$\sum_{T \in N/L} v(T)$$

となる．つまり，コミュニケーション構造を変数として (つまり，各プレイヤー間のリンク $l \in L$ をプレイヤーとして)，プレーヤー全体により獲得可能な利得によってゲームを表現する立場も考えられる．

定義 10 (リンクゲーム [3]) 連絡状況 (N, v, L) 下における**リンクゲーム** (link game) とは，以下の特性関数

$$\gamma^v(M) := v^M(N) = \sum_{T \in N/M} v(T) \quad \text{for each } M \subseteq L$$

を持つゲーム (L, γ^v) として定義される．ただし，$\gamma^v(\emptyset) = 0$ とするため，(N, v) はゼロ正規化されている必要がある．

以上の紹介したいずれのゲームも (N, L) **が完全グラフであるとき，ベースとなっている提携型のゲーム** (N, v) **に一致する．**

2.2 実現可能提携上に制限された種々のゲーム

前節では，コミュニケーションネットワークに基づく**実現可能な提携**を考え，これらの実現可能な提携を基に，新たな提携型のゲームを構築した．一方で，特性関数の定義域を実行可能な提携全体とするゲームへの一般化もなされてきている．本節ではこれらの紹介を行う．

定義 11 (凸幾何集合系 [5]) 集合系 $\mathcal{L} \subseteq 2^N$ が以下の条件を満たすとき，$\mathcal{L}$ は**凸幾何集合系** (convex geometry) とであるという：

$$\textbf{(C1)} : S, T \in \mathcal{L} \quad \Rightarrow \quad S \cap T \in \mathcal{L},$$

$$\textbf{(C2)} : S \in \mathcal{L}, S \neq N \quad \Rightarrow \quad \exists j \in N \setminus S \quad \text{s.t.} \quad S \cup j \in \mathcal{L}.$$

ここでは，全体提携 N が実現可能である状況を想定している（逆に，直接，間接を問わず，コミュニケーションを取ることができる最大範囲を N と考えても良い)．実現可能提携 S,T において，コミュニケーションの仲介を行うグループ $S \cap T$ はそれ自身実現可能であると考えることが自然であり (C1)，N のメンバー全員が連絡を取り合うことができるのであれば，$S \subsetneq N$ のメンバーの誰かは，S の外部のメンバーの誰かと連絡が取れなければならないと考えるのが自然である (C2)．

また，(N, L) が連結ブロックブラフであるとき，

$$\mathcal{F} := \{S \subseteq N \mid S : \text{実現可能}\}$$

は凸幾何集合系となる．

定義 12 (マトロイド [23]) 集合系 $\mathcal{M} \subseteq 2^N$ が以下の条件を満たすとき，$\mathcal{L}$ は**マトロイド** (matroid) であるという：

$$\textbf{(M1)} : S \in \mathcal{M}, T \subseteq S \quad \Rightarrow \quad T \in \mathcal{M},$$

$$\textbf{(M2)} : S, T \in \mathcal{M}, |S| = |T| + 1 \quad \Rightarrow \quad \exists j \in S \setminus T \quad \text{s.t.} \quad T \cup j \in \mathcal{M}.$$

ここでは，ある提携 S は，共通の興味 (interest) をもって構成されている．よってその部分提携 $T \subseteq S$ においても，同一の興味をもって構成されるはずであると考えている (M1).

定義 13 (アンチマトロイド [23]) 集合系 $\mathcal{A} \subseteq 2^N$ が以下の条件を満たすとき，$\mathcal{A}$ は**アンチマトロイド** (antimatroid) であるという：

(A1)：$\{C \subseteq N \mid N \setminus C \in \mathcal{A}\}$ が 凸幾何集合系

定義 14 $\mathcal{F}$ を集合系とする．$A \subsetneq B$ なる $A, B \in \mathcal{F}$ に対して，$A \subsetneq C \subsetneq B$ となるような $C \in \mathcal{F}$ が存在しないとき，A は B に**被覆** (cover) されている, あるいは B は A を被覆しているといい, $A \prec B$ または $B \succ A$ と書く.

定義 15 (正則集合系 [13]) 集合系 $\mathcal{R}$ が以下の条件を満たすとき，$\mathcal{R}$ は**正則** (regular) であるという：

(R1): $N \in \mathcal{R}$,

(R2): $S, T \in \mathcal{R}, S \prec T \quad \Rightarrow \quad |T \setminus S| = 1.$

凸幾何集合系およびアンチマトロイドは，いずれも正則集合系である.

定義 16 (鎖・極大鎖) $\mathcal{F}$ を集合系とする．任意の $S, T \in \mathcal{F}$ が $S \subseteq T$ または $S \supseteq T$ を満たすとき, $\mathcal{F}$ を**鎖** (chain) と呼ぶ. $\mathcal{F}$ の部分集合, $\mathscr{C} \subseteq \mathcal{F}$ が鎖であるとき, $\mathscr{C}$ を $\mathcal{F}$ **の鎖**と呼ぶ．$N \in \mathfrak{N}$ なる集合系 (鎖とは限らない)$\mathfrak{N}$ の鎖 $\mathscr{C} = \{C_0, C_1, \ldots, C_m\}$ が $\emptyset = C_0 \prec C_1 \prec \cdots \prec C_m = N$ を満たすとき，$\mathscr{C}$ を $\mathcal{F}$ **の極大鎖** (maximal chain) と呼ぶ. 例えば, $\mathfrak{N} = 2^N$ の極大鎖は $n!$ 個存在し，それらの長さはすべて n である. また，$\mathfrak{N}$ の極大鎖全体の集合を $\mathcal{M}(\mathfrak{N})$ と書く.

また，以下の命題が直ちに導かれる.

命題 17 $\mathfrak{N}$ を N 上の集合系とする．このとき，任意の $S \in \mathfrak{N}$ に対して，ある極大鎖 $\mathscr{C} \subseteq \mathfrak{N}$ が存在して, $S \in \mathscr{C}$ となる.

ここで，上の凸幾何集合系 $\mathcal{L}$，マトロイド $\mathcal{M}$，正則集合系 $\mathcal{R}$ を比較する (図 2). $\mathcal{L}$ においては，任意の実現可能提携 $S \in \mathcal{L}$ に対して，S にプレイヤーを１人ずつ増やしながら全体提携 N が構成できることを保証している．また，$\mathcal{M}$ においては，任意の実現可能提携 $S \in \mathcal{M}$ に対して，どのようなプレイヤー $i \in S$ からも，プレイヤーを１人ずつ増やしながら S が構成できることを保証している．最後に $\mathcal{R}$ においては，任意の実現可能提携 $S \in \mathcal{M}$ に対して，あるプレイヤー $i \in S$ からプレイヤーを１人ずつ増やしながら S が構成でき，さらに，S にプレイヤーを１人ずつ増やしながら全体提携 N が構成できることを保証している.

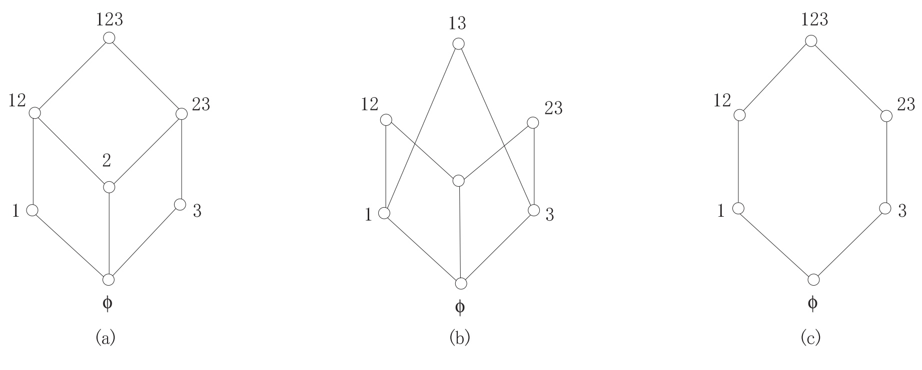

図 2: (a):凸幾何, 正則，(b):マトロイド，(c):正則

定義 18 (集合系上のゲーム) $\mathcal{F} \subseteq 2^N$ を集合系とする．このとき，$S \in \mathcal{F}$ を**実現可能な提携**と呼び，$v(\emptyset) = 0$ なる関数 $v : \mathcal{F} \to \mathcal{R}$ を $\mathcal{F}$ 上の**特性関数**と呼ぶ．これらからなる $(N, v, \mathcal{F})$ を**集合系 $\mathcal{F}$ 上のゲーム**と呼ぶ．ただし，特に混乱の生じる恐れがないときは，$(N, v, \mathcal{F})$ と v を同一視し，v を $\mathcal{F}$ 上のゲームと呼ぶ．また，$\mathcal{F}$ 上のすべてのゲームの集合を $\mathcal{G}(\mathcal{F})$ で表す．つまり，$v \in \mathcal{G}(2^N)$ は v が N 上の通常の提携型ゲームであることを表す．

2.3 多選択肢を許容するにおけるゲームの構造

定義 19 (３選択肢投票ゲーム [7]) N をプレイヤー全体の集合とし，$3^N := \{(F, A) \mid F, A \subseteq N, F \cap A = \emptyset\}$ とする．ここで，F は賛成投票者，A は反対投票者, $N \setminus F \setminus A$ は棄権者を表す．ここで，以下の条件を満たす関数 $v : 3^N \to \{-1, 1\}$ と N からなる (N, v) を３選択肢投票ゲーム (ternary voting game) と呼ぶ．

(T1) : $v(N, \emptyset) = 1$,

(T2) : $v(\emptyset, N) = -1$,

(T3) : $F_1 \subseteq F_2 \subseteq N,\ N \supseteq A_1 \supseteq A_2 \quad \Rightarrow \quad v(F_1, A_1) \leq v(F_2, A_2)$.

また，(T1),(T2) および $b(\emptyset, \emptyset) = 0$ を満たす $b : 3^N \to [-1, 1]$ は **bi-cooperative game** [**?**] と呼ばれ，(T3) を満たす bi-cooperative game は**双容量** (bi-capacity) と呼ばれる [9].

定義 20 (多選択肢ゲーム [12]) プレイヤー全体の集合を N とする．$L_i = \{0, 1, \dots, l_i\}$ を，それぞれのプレイヤー $i \in N$ の実現可能な (多選択肢ゲームへの) **参加レベル**の集合とし，$L = L_1 \times \cdots, \times L_n$ とする．また，$\boldsymbol{s} = (s_1, \dots, s_n) \in L$ を (多選択肢ゲームへの) **参加プロファイル**または**多選択肢提携**と呼ぶ．このとき，$\boldsymbol{l} = \{l_1, \dots, l_n\}$ は，最大参加レベルプロファイルと呼ばれ，通常の協力ゲームにおける全体提携に対応する．逆に，$\mathbf{0} = (0, \dots, 0)$ は空提携に対応する．ここで，$v(\mathbf{0}) = 0$ を満たす関数 $v : L \to \mathbb{R}$ を L 上の特性関数と呼び，(N, L, v) を**多選択肢ゲーム** (multichoice game) と呼ぶ．特に混乱の生じる恐れがない場合，単に v を (N, L) 上の多選択肢ゲームと呼ぶ．

通常のゲーム (N, v) は，全てのプレイヤー $i \in N$ が参加レベルの集合として $L_i = \{0, 1\}$ を持つ多選択肢ゲームとみなすことができる．多選択肢ゲーム (N, v, L) の形式を用いて次のような状況を表現することができる．

例 21 今，共同プロジェクトがあり，合計 n 社の企業体 ($N = \{1, \dots, n\}$) がこのプロジェクトに参加可能である．それぞれの企業体 $i \in N$ は，このプロジェクトに l_i 人までのスタッフの投入が可能である ($L_i = \{0, 1, \dots, l_i\}$)．また，このプロジェクトにおける利得は，各社の投入人数 ($(a_1, \dots, a_n) \in L_1 \times \cdots \times L_n$) によって決まり，利得関数 v によって ($v(a_1, \dots, a_n)$) と表される．

これら２つのゲームはいずれも分配束上のゲームとして一般化される．

定義 22 ($\vee$ 既約元) $(L, \leq, \vee, \wedge, \top, \bot)$ を束とする．ただし，$(L, \leq)$ は空でない有限半順序集合，$\vee$, $\wedge$ は上限 (結び)，下限 (交わり)，$\top$, $\bot$ は最大元，最小元をそれぞれ表すものとする．$x \in L$ が次の条件を満たすとき，x を L の **$\vee$-既約元** (join- irreducible element) と呼ぶ:

1. $x \neq \bot$,
2. 任意の $a, b \in L$ に対して, $x = a \vee b$ ならば $x = a$ または $x = b$ である．

また，L の $\vee$-既約元全体の集合を $\mathcal{J}(L)$ で表す．

定義 23 (分配束上のゲーム) $(L, \leq, \vee, \wedge, \top, \bot)$ を分配束とする．このとき，$\mathcal{J}(L)$ をプレイヤー全体の集合とし，L の要素 $a \in L$ を**提携**と呼ぶ．また，$v(\bot) = 0$ なる関数 $v : L \to \mathbb{R}$ を L 上の**特性関数**と呼び，$(L, \leq, \vee, \wedge, \top, \bot, v)$ を**分配束上のゲーム**と呼ぶ．一般化協力ゲームの時と同様に，単に v を分配束 L 上のゲームと呼ぶこともある．

命題 24 L が分配束であるとき，任意の $a \in L$ は，

$$a = \bigvee_{b \in \eta(a)} b$$

と表される．ただし，

$$\eta(a) := \{x \in \mathcal{J}(L) \mid x \leq a\} \tag{1}$$

とする．また，

$$a = \bigvee_{b \in \eta^*(a)} b \tag{2}$$

と表される最小な $\eta^*(a) \subseteq \eta(a)$ が一意に存在する．このとき (2) を $a \in L$ の最小分解と呼ぶ．

また，η は $(L, \leq, \vee, \wedge, \top, \bot)$ から $(\eta(L), \subseteq, \cup, \cap, \mathcal{J}(L), \emptyset)$ への束同型写像となる [13]．そして，この $(\mathcal{J}(L), \eta(L))$ を**束 L より導かれる集合系**と呼ぶ．

定義 25 $(L, \leq, \vee, \wedge, \top, \bot, v)$ を束上のゲームとする．このとき，束 L より導かれる集合系 $(\mathcal{J}(L), \eta(L))$ 上のゲーム $v^\eta \in \mathcal{G}(\mathcal{J}(L), \eta(L))$ を以下のように定義する．

$$v^\eta(S) := \eta \circ v(S) := v(\eta^{-1}(S)), \quad S \in \eta(L). \tag{3}$$

ただし，η は (1) 式によって与えられる束同型写像とする．

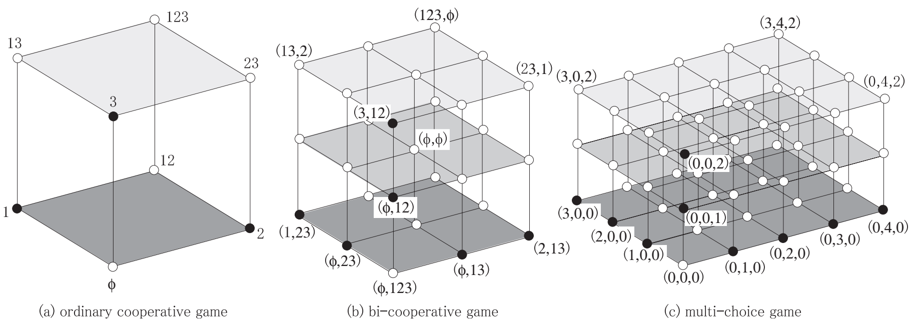

図 3: Examples of games on lattices: elements indicated by black circles are join-irreducible.

2.4 提携形成におけるプレイヤーの順序に制約があるゲーム

ここでは，プレイヤーに提携形成に関するある順序制約がある場合のゲームを考える．ここでは，プレイヤーの集合 N 上にある順序 $\preceq$ が与えられている状況を考える．つまり，プレイヤーの集合を半順序集合 (poset) $(N, \preceq)$ として考える．

定義 26 N をプレイヤーの集合，$(N,\preceq)$ を半順序集合とする．このとき，$(N,\preceq)$ における提携とは，以下の条件を満たすような N の部分集合 $S\subseteq N$ である．

$$i\in S,\ j\preceq i \quad\Rightarrow\quad j\in S.$$

これは，あるプレイヤー $i\in N$ が提携に加わる際には，必ず $j\preceq i$ なるプレイヤーは i と行動を共にしなければならないことを意味している．ここで，$(N,\preceq)$ における提携全体の集合を $\mathfrak{C}(N)$ とするとき，$v(\emptyset)=0$ なる関数 $v:\mathfrak{C}(N)\to\mathbb{R}$ を特性関数とする $(N,\preceq)$ 上のゲーム $(N,\preceq,v)$ が定義できる．

例えば，図 4 に示すような半順序集合 (a) $(N_a,\preceq_a)$, (b) $(N_b,\preceq_b)$, (c) $(N_c,\preceq_c)$ を考えた場合，それぞれの提携の集合は以下のようになる．

$$\mathfrak{C}(N_a)=\{\emptyset,1,3,13,34,123,134,1234\}$$
$$\mathfrak{C}(N_b)=\{\emptyset,1,2,3,12,13,23,123\}$$
$$\mathfrak{C}(N_c)=\{\emptyset,1,12,123\}$$

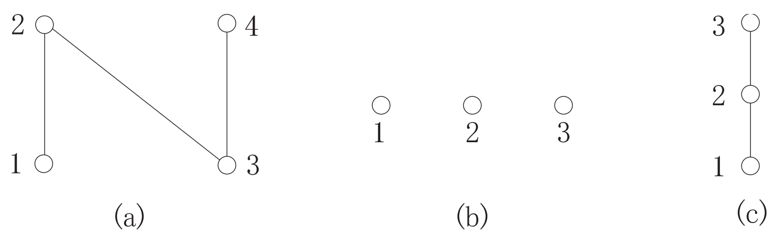

図 4: 順序構造をもつプレイヤーの集合

3 組合せ構造上のゲームの解 –シャープレイ値の拡張–

Shapley [19] は以下のような協力ゲームにおける 1 つの解概念 (協力ゲームにおける，各プレイヤーの平均的価値)，いわゆるシャプレイ値を与えた．

定義 27 (シャプレイ値) 協力ゲーム $v\in\mathcal{G}(N,2^N)$ に関するシャプレイ値 $\Phi(v)=(\Phi_1(v),\ldots,\Phi_n(v))\in\mathbb{R}^n$ は次のように定義される．

$$\Phi_i(v):=\sum_{\substack{S\subseteq N\\ S\ni i}}\frac{(s-1)!\,(n-s)!}{n!}(v(S)-v(S\setminus i)),$$

ただし, $n=|N|, s=|S|$ とする．

3.1 コミュニケーションネットワークに基づく種々のゲームの解

定義 28 (マイヤーソン値 [2, 18]) 連絡状況 (N,v,L) におけるマイヤーソン値 $\Psi(N,v,L)\in\mathbb{R}^n$ は，連絡網制限ゲーム v^L を通して，

$$\Psi(N,v,L)=\Phi(v^L)$$

として定義される．また，安定集合系 $\mathcal{U}$ 上のゲーム $(N,v,\mathcal{U})$ におけるマイヤーソン値 $\Psi(N,v,\mathcal{U})\in\mathbb{R}^n$ は,

$$\Psi(N,v,\mathcal{U})=\Phi(v^{\mathcal{U}})$$

として定義される．

他のコミュニケーションネットワークに基づく種々のゲームに対するシャープレイ値の拡張は [8] にまとめてあるので参照してください.（今回の資料に同梱しています．）

3.2 実現可能提携上に制限された種々のゲーム

ここでは，凸幾何集合系，マトロイド，正則集合系上のゲームに関するシャープレイ値の拡張概念を紹介する．これらの解は，いずれも，それぞれの集合系における，ある種の**線形性**，**効率性**，**対称性**，**ダミープレーヤーの独立性**，および Faigle と Kern [6] による**プレイヤーの階層に関する公理**によって特徴付けられている．個々の公理系に関しては正則集合系の場合を除いて本稿では省略する．

定義 29 (凸幾何集合系上のゲームに関するシャープレイ値 [2]) $\mathcal{L}$ を凸幾何集合系とする．このとき，$(N, v, \mathcal{L})$ に関するシャープレイ値 $\Phi(N, v, \mathcal{L})$ は以下のように定義される：

$$\Phi_i(N, v, \mathcal{L}) := \sum_{\substack{S \in \mathcal{L} \\ ex(S) \ni i}} \frac{c(S \setminus i)\ c([S, N])}{c(N)} (v(S) - v(S \setminus i)).$$

ただし，$ex(S) := \{i \in S \mid S \setminus i \in \mathcal{L}\}$，$c([S, T])$ は S から $T \supseteq S$ までの極大鎖の数 (ただし，$c([S, S]) := 1$, $c(S) := c([\emptyset, S])$ とする．

定義 30 (マトロイド上のゲームに関するシャープレイ値 [2]) $\mathcal{M}$ をマトロイドとする．このとき，$(N, v, \mathcal{M})$ に関するシャープレイ値 $\Phi(N, v, \mathcal{M})$ は以下のように定義される：

$$\Phi_i(N, v, \mathcal{M}) := \sum_{\substack{S \in \mathcal{M} \\ S \ni i}} \frac{b_S}{b_N} \frac{(r_N - r_S)!\ (r_S - 1)!}{r!} (v(S) - v(S \setminus i)).$$

ただし，$r_S := \max\{|T| : T \subseteq S,\ T \in \mathcal{M}\}$，$b_S := |\mathfrak{B}_S(\mathcal{M})|$, $\mathfrak{B}_S(\mathcal{M}) := \{B \in \mathfrak{B}(\mathcal{M}) \mid B \supseteq S\}$, $\mathfrak{B}(\mathcal{M}) := \{B \in \mathcal{M} \mid |B| = r_N\}$ とする．

定義 31 (正則集合系上のゲームに関するシャープレイ値 [13]) $\mathcal{R}$ を正則集合系とする．このとき，$(N, v, \mathcal{R})$ に関するシャープレイ値 $\Phi(N, v, \mathcal{R})$ は以下のように定義される：

$$\Phi_i(N, v, \mathcal{R}) := \frac{1}{c(N)} \sum_{\mathscr{C} \in \mathcal{M}(\mathfrak{R})} (v(\mathscr{C}(i)) - v(\mathscr{C}(i) \setminus i))$$

ただし，$\mathscr{C}(i) = \bigcup_{\substack{C \in \mathscr{C} \\ C \not\ni i}} C \cup \{i\}$.

また，$\mathcal{R}$ が凸幾何集合系である場合，定義 31 は定義 29 に一致する．

3.2.1 正則集合系上のゲームに関するシャープレイ値の公理論的特徴付け

本節では，正則集合系上のゲームに関するシャープレイ値 $\Phi = (\Phi_1, \ldots, \Phi_n)$ として満たすべき諸性質 (つまり，公理系) について議論する．

公理 1 (効率性) 集合系 $\mathcal{R}$ が N の鎖であるとき，任意の $v \in \mathcal{G}(N, \mathcal{R})$ に対して，以下が成り立つ：

$$\sum_{i=1}^{n} \Phi_i(N, v, \mathcal{R}) = v(N)$$

公理 2 (ナルプレイヤーのゼロ評価) 集合系 $\mathcal{R}$ が N の鎖であるとき, 任意の $v \in \mathcal{G}(N, \mathcal{R})$ に対して, プレイヤー $i \in N$ が v に関するナルプレイヤー (i.e., $S \in \mathcal{R}, S \cup \{i\} \in \mathcal{R} \Rightarrow v(S \cup \{i\}) = v(S)$) のとき，以下が成り立つ：

$$\Phi_i(N, \mathcal{R}, v) = 0$$

公理 3 (対称性) σ を N 上の置換とする．集合系 $\mathcal{R}$ が N の鎖であるとき, 任意の $v \in \mathcal{G}(N, \mathcal{R})$ に対して,

$$\Phi_i(N, \mathcal{R}, v) \quad = \quad \Phi_{\sigma(i)}(N, \sigma(\mathcal{R}), \sigma \circ v)$$

ただし, $\sigma(\mathcal{R}) := \{\sigma(S) \mid S \in \mathcal{R}\}, \sigma \circ v(S) := v(\sigma^{-1}(S)), S \in \sigma(\mathcal{R})$ とする．

公理 4 (加法性) 集合系 $\mathcal{R}$ が N の鎖であるとき, 任意の $v_1, v_2 \in \mathcal{G}(N, \mathcal{R})$ に対して，以下が成り立つ：

$$\Phi(N, \mathcal{R}, v_1 + v_2) = \Phi(N, \mathcal{R}, v_1) + \Phi(N, \mathcal{R}, v_2)$$

公理 5 (凸性) 集合系 $\mathcal{R}$ に対して，$\{\mathcal{M}(\mathcal{R}_1), \mathcal{M}(\mathcal{R}_2), \ldots, \mathcal{M}(\mathcal{R}_m)\}$ が $\mathcal{M}(\mathcal{R})$ の分割となるような集合系列 $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m \subseteq \mathcal{R}$ が存在するとき，$\sum_{k=1}^{m} \alpha_k = 1$ なる $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1]$ が存在して, 任意の $v \in \mathcal{G}(N, \mathcal{R})$ に対して，以下が成り立つ：

$$\Phi(N, \mathcal{R}, v) = \sum_{k=1}^{m} \alpha_k \Phi(N, \mathcal{R}_k, v|_{\mathcal{R}_k})$$

ただし, $v|_{\mathcal{R}_k}$ は v の $\mathcal{R}_k$ への制限とする．

公理 5 は，Φ のサブドメインへの分解可能性に関する要請と条件付けを与えている．例えば, $\mathcal{R}$ の任意の極大鎖 $\mathscr{C} = \{\emptyset = C_0, C_1, \ldots, C_m = N\}$ は，それ自身一つの集合系となっている (i.e., $(N, \mathscr{C})$ は集合系). ここで，$\mathcal{M}(\mathscr{C})$ は要素に $\mathscr{C}$ のみを持つ 1 点集合であるので，明らかに,

$$\mathcal{M}(\mathcal{R}) = \bigcup_{\mathscr{C} \in \mathcal{M}(\mathcal{R})} \mathcal{M}(\mathscr{C})$$

であり，$\mathscr{C}_i, \mathscr{C}_j \in \mathcal{M}(\mathcal{R})$ に対して,

$$\mathscr{C}_i \neq \mathscr{C}_j \ \Rightarrow \ \mathcal{M}(\mathscr{C}_i) \cap \mathcal{M}(\mathscr{C}_j) = \emptyset$$

が成り立つ．よって， $\{\mathcal{M}(\mathscr{C})\}_{\mathscr{C} \in \mathcal{M}(\mathcal{R})}$ は $\mathcal{M}(\mathcal{R})$ の分割である．つまり，公理 5 の必要条件として,

$$\Phi(N, v, \mathcal{R}) = \sum_{\mathscr{C} \in \mathcal{M}(\mathcal{R})} \alpha_{\mathscr{C}} \Phi(N, \mathscr{C}, v|_{\mathscr{C}}),$$

$\sum_{\mathscr{C} \in \mathcal{M}(\mathcal{R})} \alpha_{\mathscr{C}} = 1$ が導かれる．

定理 32 [16, 17] $\mathcal{R}$ を正則集合系とする．このとき，公理 1, 2, 3, 4, 5 を満たす解概念 $\Phi : \mathcal{G}(N, \mathcal{R}) \to \mathbb{R}^n$ がただ 1 つ存在し $\Phi = \Psi$ で与えられる．[1]

[1] [15] では，これとは異なる特徴付けが与えられている．

3.3 正則集合上のゲームに関するシャープレイ値の束上のゲームへの適用

この節では正則集合上のゲームに関するシャープレイ値の束上のゲームへ適用することを考える．

分配束 $\mathbb{L} = (L, \leq, \vee, \wedge, \top, \bot)$ は，命題 24 および定義 25 より，束同型写像 $\eta(a) := \{b \in L \mid b \leq a\}$ を通して，

$$(\eta(L), \subseteq, \cup, \cap, \mathcal{J}(L), \emptyset)$$

に対応付けられる．このとき，$\mathcal{J}(L)$ 上の集合系 $\eta(L)$ が正則であれば，$\mathbb{L}$ 上のゲーム $(N, v, \mathbb{L})$ を η を通して正則集合 $\eta(\mathbb{L})$ 上で議論することができる．そして，その結果を η^{-1} によって引き戻してやれば良いことになる．

分配束上のゲームに関するシャープレイ値の提案： $\mathbb{L} := (L, \leq, \vee, \wedge, \top, \bot)$ を分配束，$v : L \to \mathbb{R}$ を L 上のゲームとする．また，便宜上，$\mathcal{J}(L) = \{1, 2, \ldots, l\}$ とする．束 L より導かれる集合系 $(\mathcal{J}(L), \eta(L))$ が正則であるとき，v の解 $\Psi^\eta = (\Psi_1^\eta, \ldots, \Psi_l^\eta)$ を以下のように定義する．

$$\Psi_i^\eta(v) := \Psi_i(v^\eta), \quad i = 1, 2, \ldots, l. \tag{4}$$

ただし，η は (1) 式によって与えられる束同型写像，$\eta \circ v$ は (3) 式によって与えられる $(\mathcal{J}(L), \eta(L))$ 上のゲームとする．

例 33 束 L_1 (図 5 (a)) 上のゲーム v を考える．このとき，L_1 の $\vee$ 既約元全体の集合は $\mathcal{J}(L_1) = \{d, e, f\}$ となり，$\eta(a) = \{d, e, f\}$, $\eta(b) = \{d, e\}$, $\eta(c) = \{e, f\}$, $\eta(d) = \{d\}$, $\eta(e) = \{e\}$, $\eta(f) = \{f\}$, $\eta(g) = \emptyset$ となる．つまり，$L_1 = \{d, e, f, b, c, a\}$ に

$$\eta(L_1) = \{\{d\}, \{e\}, \{f\}, \{d, e\}, \{e, f\}, \{d, e, f\}\}$$

が対応する．

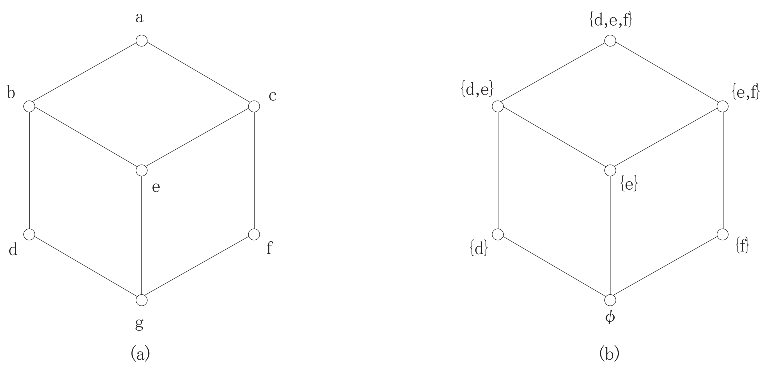

図 5: (a) 分配束 L_1 と (b) 正則集合系 $\eta(L_1)$

また，$\eta(L_1)$ の極大鎖と L の極大鎖はそれぞれ以下のように対応する．

$$\begin{aligned}
\{\emptyset, \{d\}, \{d, e\}, \{d, e, f\}\} &\leftrightarrow \{g, d, b, a\} \\
\{\emptyset, \{e\}, \{d, e\}, \{d, e, f\}\} &\leftrightarrow \{g, e, b, a\} \\
\{\emptyset, \{e\}, \{e, f\}, \{d, e, f\}\} &\leftrightarrow \{g, e, c, a\} \\
\{\emptyset, \{f\}, \{e, f\}, \{d, e, f\}\} &\leftrightarrow \{g, f, c, a\}
\end{aligned}$$

例えば $\Psi^\eta_d(v)$ は，$\Psi^\eta(v)=\Psi(v^\eta)$ より，

$$
\begin{aligned}
\Psi^\eta_d(v) &= \frac{1}{4}\left[\{v^\eta(\{d\})-v^\eta(\emptyset)\}+\{v^\eta(\{d,e\})-v^\eta(\{e\})\}\right.\\
&\quad \left.+\{v^\eta(\{d,e,f\})-v^\eta(\{e,f\})\}+\{v^\eta(\{d,e,f\})-v^\eta(\{e,f\})\}\right]\\
&= \frac{1}{4}\left[\{v(d)-v(g)\}+\{v(b)-v(e)\}+\{v(a)-v(c)\}+\{v(a)-v(c)\}\right]\\
&= \frac{1}{4}\left[v(d)+v(b)-v(e)\right]+\frac{1}{2}\left[v(a)-v(c)\right]
\end{aligned}
$$

となる．

協力ゲームの拡張概念として提案されている多選択肢ゲーム (multichoice game) や bi-cooperative game, 双容量 (bi-capacity) [21] や Faigle と Kern の半順序集合上のゲーム等は，図 3 などからも分かるように分配束上のゲームとして扱うことができる [10, 13].

多選択肢ゲームに関する解概念も，分配束から導かれる集合系上のゲームを通して導入することができる．

まず，参加プロファイル全体の集合 $\mathbb{L}=L_1\times,\cdots,\times L_n$, $L_i=\{1,2,\ldots,l_i\}$ の上に，次のような自然な順序 $\leq$:

$$\boldsymbol{a}=(a_1,\ldots,a_n),\boldsymbol{b}=(b_1,\ldots,b_n)\in\mathbb{L} \text{ に対して}$$

$$\boldsymbol{a}\leq\boldsymbol{b} \quad\Leftrightarrow\quad a_i\leq b_i \ \ \forall i\in N.$$

を導入すると，$\boldsymbol{a},\boldsymbol{b}\in\mathbb{L}$ の上限，下限は

$$\boldsymbol{a}\vee\boldsymbol{b}=(a_1\vee b_1,\ldots,a_n\vee b_n)$$
$$\boldsymbol{a}\wedge\boldsymbol{b}=(a_1\wedge b_1,\ldots,a_n\wedge b_n)$$

のように与えられ，$(\mathbb{L},\leq,\vee,\wedge,\boldsymbol{l},\mathbf{0}),\boldsymbol{l}:=(l_1,\ldots,l_n)$ は分配束をなす．よって，多選択肢ゲーム $(N,v,\mathbb{L})$ は，束上のゲーム $(\mathbb{L},\leq,\vee,\wedge,\boldsymbol{l},\mathbf{0},v)$ とみなせる．このとき，$\mathcal{J}(\mathbb{L})$ は，

$$\mathcal{J}(\mathbb{L})=\{(a_i,\mathbf{0}_{-i})\mid i\in N,a_i\in L_i\}$$

と表される．ただし，$\boldsymbol{b}\in\mathbb{L}$ と $a_i\in\mathbb{L}_i$ に対して，

$$(a_i,\boldsymbol{b}_{-i}):=(b_1,\ldots,b_{i-1},a_i,b_{i+1},\ldots,b_n)$$

と表記する．また，明らかに $\mathbb{L}=L_1\times,\cdots,L_n$ から導かれる集合系は正則 (i.e., $(\mathcal{J}(\mathbb{L}),\eta(\mathbb{L}))$ は正則) であるので，(4) 式を通して，多選択肢ゲームに１つの解概念を導くことができる．

多選択肢ゲームの解を考える際には若干の注意が必要である．束 $\mathbb{L}$ 上のゲームでは，$\mathcal{J}(\mathbb{L})$ をプレイヤーの集合とみなして議論している．ここで，多選択肢ゲームを束上のゲームとして見た場合，$\mathcal{J}(\mathbb{L})$ の要素は $(a_i,\mathbf{0}_{-i})$ なる形式で表わされる．つまり，この束上のゲームでのプレイヤーは，実際のプレイヤー $i\in N$ とそのプレイヤーの参加レベル a_i との組となっている．そのため，束上でのゲームの解 $\Psi^\eta_{(a_i,\mathbf{0}_{-i})}(v)$ は，プレイヤー $i\in N$ の参加レベルが a_i であった場合の平均的価値を表わしていることになる．

例 34 プレイヤー 1，プレイヤー 2 が参加レベルの集合 $L_1=\{0,\ 1,\ 2\}$, $L_2=\{0,\ 1,\ 2,\ 3\}$ を持つような 2-プレイヤー多選択肢ゲームを考える．このとき，$L=L_1\times L_2$ には，図 6 のような自然な束構造が入り，分配束上のゲームとみなすことができる．また，$\mathcal{J}(L)=\{(1,0),(2,0),(0,1),(0,2),(0,3)\}$ となり，束 L より導かれる集合系 $(\mathcal{J}(L),\eta(L))$ (図 6) は明らかに正則である．よって，(4) 式を用いることにより，この多選択肢ゲームに対して解を与えることができる．

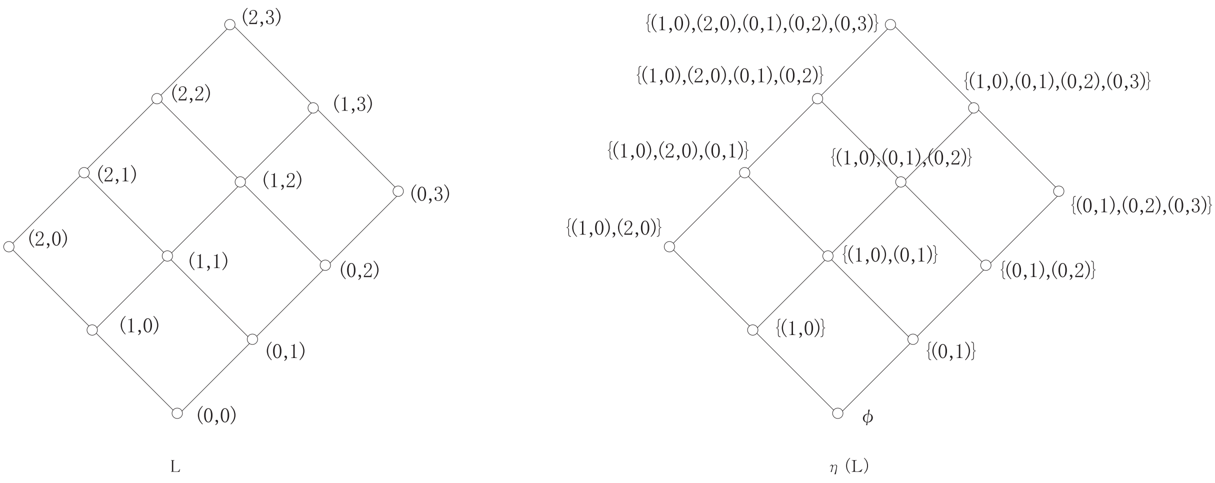

図 6: (a) 分配束 L と (b) 正則集合系 $\eta(L)$

参考文献

[1] E. Algaba, Extensión de juegos definidos en sistemas de conjuntos, Ph.D. Thesis, University of Seville, Spain, 1998.

[2] J.M. Bilbao, Cooperative games on combinatorial structures, Kluwer Academic Publisher, 2000.

[3] P. Borm, G. Owen, and S. Tijs, On the position value for communication situations, SIAM Journal on Discrete Mathematics, Vol. 5, No. 3, 305-320, 1992.

[4] A. Dukhovny, General entropy of general measures, Int. J. Uncertain. Fuzziness Knowledge-Based Syst. Vol. 10, 213–225, 2002.

[5] P. H. Edelman and R. E. Jamison, The theory of convex geometry, Geom. Dedicata, vol. 19, 247-270, 1985.

[6] U. Faigle and W. Kern, The Shapley value for cooperative games under precedence constraints, Int. J. of Game Theory, Vol. 21, 249–266, 1992.

[7] D. S. Felsenthal and M. Machover, Ternary voting games, Internationa Journal of Game Theory, Vol. 26, No. 3, 335–351, 1997.

[8] Fujimoto K, Honda A (2009) A value via posets induced by graph-restricted communication situation. In: Proc. 2009 IFSA World Congress/2009 EUSFLAT Conference, Lisbon, Portugal.

[9] M. Grabisch and Ch. Labreuche, Bi-capacities for decision making on bipolar scales, In EUROFUSE Workshop on Informations Systems, Varenna, Italy, September 2002.

[10] M. Grabisch, Capacities and Games on Lattices: A Survey of Results, Int. J. of Uncertainty, Fuzziness, and Knowledge-Based Systems, Vol 14, No. 4, 371–392, 2006.

[11] G. Hamiache. A value with incomplete communication, Games and Economic Behaviour, Vol. 26, 59–78, 1999.

[12] C.R. Hsiao and T.E.S. Raghavan, Shapley value for multi-choice cooreerative games, I, Games and Economic Behaviour, Vol. 5, No. 2, 240–256, 1993.

[13] A. Honda and M. Grabisch, Entropy of capacities on lattices. Information Sciences, 176, pp. 3472-3489, 2006.

[14] A. Honda and M. Grabisch, An axiomatization of entropy of capacities on set systems, European Journal of Operational Research, Vol. 190, 526-538, 2008.

[15] A. Honda, Y. Okazaki, Axiomatization of Shapley value of Faige and Kern type on set systems, Journal of Advanced Computational Intelligence and Intelligent Informatics, Vol. 12 No. 5, 2008, 409–415, 2008.

[16] A. Honda, Y. Okazaki, Shapley type axiomatization of a solution of game on set systems, Proceedings of the Modeling Decisions for Artificial Intelligence 2008, Spain, 2008.

[17] A. Honda, Y. Okazaki, A new characterization of a value of the generalized game, 九州工業大学情報工学テクニカルレポート, CSSE-32, 2009.

[18] R. Myerson, Graphs and cooperation in games, Mathematics of Operations Research, Vol. 2, No. 3, 225–229, 1977.

[19] L.S. Shapley, A value for n-person games, Kuhn HW, Tucker, AW (eds) Contributions to the Theory of Games Vol. II, Princeton, 307-317, 1953.

[20] M. Sugeno, Fuzzy measures and fuzzy integrals: a survey, in M.M. Gupta, G.N. Saridis, and B.R. Gains, editors, Fuzzy automata and decision processes, pp. 89-102, North Holland, Amsterdam, 1977.

[21] ミシェル・グラビシュ, クリストフ・ラブルーシュ, 室伏俊明, 双極尺度のためのファジィ測度と積分, 知能と情報：日本知能情報ファジィ学会誌, Vol.16, No.4, 2004, 311–318,

[22] G.-C. Rota, On the foundations of combinatorial theory I., Theory of Möbius functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete,

[23] H. Whitney, On the abstract properties of linear dependence, Amer. J. Math., vol. 57, 509–533, 1935.

Representations of Importance and Interaction of fuzzy measures, capacities, games and its extensions: A survey*

Katsushige FUJIMOTO

Abstract This paper gives a survey of the theory and results on representation of importance and interaction of fuzzy measures, capacities, games and its extensions: games on convex geometries, bi-capacities, bi-cooperative games, and multi-choice games, etc. All these games are regarded as games on products of distributive lattices or on regular set systems.

1 Introduction

The *measure* is one of the most important concepts in mathematics and so is the integral with respect to the measure. They have many applications in economics, engineering, and many other fields, and their main characteristics is additivity. This is very effective and convenient, but often too inflexible or too rigid. As an solution to the rigidness problem the *fuzzy measure* has been proposed [28]. It is an extension of the measure in the sense that the additivity of the measure is replaced with weaker condition, the monotonicity. The non-additivity is the main characteristic of the fuzzy measure, and can represent *interaction phenomena* among elements to be measured.

Definition 1 (fuzzy measures). Let N be a non-empty finite set. A *fuzzy measure*, also called a *capacity*, on N is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$, and $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq N$. A fuzzy measure is *normalized* if $v(N) = 1$. A *transferable utility game in characteristic form* [5], or simplicity *game*, is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$.

Katsushige Fujimoto
College of Symbiotic Systems Science, Fukushima University, 1 Kanayagawa,Fukushima 960-1296, Japan, e-mail: fujimoto@sss.fukushima-u.ac.jp

* This research was partially supported by the Ministry of Education, Culture, Sports, Science and Technology-Japan, Grant-in-Aid for Scientific Research (C), 19510136, 2007-2009.

Given a subset $S \subseteq N$, the precise meaning of the quantity $v(S)$ depends on the kind of intended application or domain [10]:

- N **is the set of states of nature.** Then $S \subseteq N$ is an *event* in decision under uncertainty or under risk, and $v(S)$ represents the degree of certainty, belief, etc.
- N **is a the set of criteria, or attributes.** Then $S \subseteq N$ is a group of criteria (or attributes) in multi-criteria (or multi-attributes) decision making, and $v(S)$ represents the degree of importance of S for making decision.
- N **is the set of voters, political parties.** Then $S \subseteq N$ is called a *coalition* in voting situations, and $v(S) = 1$ iff bill passes when coalition S votes in favor of the bill, and $v(S) = 0$ else.
- N **is the set of players, agents, companies, etc.** Then $S \subseteq N$ is also called a *coalition* in cooperative game theory, and $v(S)$ is the worth (or payoff, or income, etc.) won by S if all members in S agree to cooperate, and the other ones do not.

As mentioned above, fuzzy measures (or capacities) are a special type of games (i.e., monotone games). Throughout this paper, we will use the term “games” on behalf of fuzzy measures, capacities, and games unless monotonicity is essential in situations to be considered, and “players” on behalf of events, voters, criteria, attributes, etc.

1.1 Intuitive representations of importance and interaction [8]

In order to intuitively approach the concept of importance of player and of interaction among players, consider two players $i, j \in N$. Clearly, $v(i)$ is one of representations of importance of player $i \in N$. An inequality

$$v(\{i,j\}) > v(\{i\}) + v(\{j\}) \quad (resp. <)$$

seems to model a *positive (resp. negative) interaction* or *complementary (resp. substitutive) effect* between i and j. However, as discussed in Grabisch and Roubens [13], the intuitive concept of interaction requires a more elaborate definition. We should not only compare $v(\{i\})$, $v(\{j\})$, and $v(\{i,j\})$ but also see what happens when i, j, and $\{i,j\}$ join coalitions. That is, we should take into account all coalitions of the form $T \cup \{i\}$, $T \cup \{j\}$, and $T \cup \{i,j\}$. For a play i and a coalition $T \not\ni i$,

$$\Delta_{\{i\}} v(T) := v(T \cup \{i\}) - v(T) \tag{1}$$

seems to represent an index of importance of i in $T \cup \{i\}$. The equation (1) is called the *marginal contribution* of a player i to a coalition T in cooperative game theory. Then it seems natural to consider that if for T not containing i and j

$$\Delta_{\{i\}} v(T \cup \{j\}) > \Delta_{\{i\}} v(T) \quad (resp. \ <)$$

then i and j interact positively (resp. negatively) in the presence of T since the presence of player j increases (resp. decreases) the marginal contribution of i to coalition T. Then

$$\Delta_{\{i,j\}}v(T) := \Delta_{\{i\}}v(T \cup \{j\}) - \Delta_{\{i\}}v(T) \tag{2}$$

is called the *marginal interaction* [12] between i and j in the presence of T. Note that

$$\Delta_{\{i\}}v(T \cup \{j\}) - \Delta_{\{i\}}v(T) = \Delta_{\{j\}}v(T \cup \{i\}) - \Delta_{\{j\}}v(T).$$

For three players $i, j, k \in N$ and a coalition T not containing i, j and k, $\Delta_{\{i,j,k\}}v(T)$ can be naturally defined as

$$\Delta_{\{i,j,k\}}v(T) := \Delta_{\{i,j\}}v(T \cup \{k\}) - \Delta_{\{i,j\}}v(T).$$

Then we have $\Delta_{\{i,j\}}v(T \cup \{k\}) - \Delta_{\{i,j\}}v(T) = \Delta_{\{i,k\}}v(T \cup \{j\}) - \Delta_{\{i,k\}}v(T) = \Delta_{\{j,k\}}v(T \cup \{i\}) - \Delta_{\{j,k\}}v(T)$. Moreover, for two distinct coalitions S and $T \subseteq N \setminus S$,

$$\Delta_S v(T) := \Delta_{S\setminus\{i\}}v(T \cup \{i\}) - \Delta_{S\setminus\{i\}}v(T) \tag{3}$$

for $i \in S$. Similarly, when, for example, $\Delta_S v(T) > 0$ (resp. $<$), we shall consider that players among S interact positively (resp. negatively) in the presence of T.

1.2 Generalizations of domains of games [***10***]

In ordinary cooperative game theory, and decision problems described through the use of fuzzy measures and/or capacities, it is implicitly assumed that all subsets S of N can be formed; however, this is generally not the case. Let us elaborate on this, and distinguish several cases.

- **Some subsets of N may be not meaningful.** When N is the set of political parties, it means that some coalitions of parties are unlikely to occur, or even impossible (coalition mixing left and right parties). When N is the set of players, for players in order to coordinate their actions, they must be able to communicate [27].
- **Subsets of N may be not "black and white",** which means that the membership of an element to N may be not simply resume to a matter of member or nonmember. This is the case with multi-criteria decision making when underlying scales are bipolar, which is a demarcation between values considered as "good", and as "bad", the central value being neutral [11]. In voting situation, it is convenient to consider that players may also abstain, hence each voter has three possibilities [7]. When N is the set of players, one may consider that each player can play at different level of participation [16].

2 Fuzzy measures, capacities, games and its extensions

Definition 2 (lattices). Let L be a non empty set and $\leq$ a partial order on L (i.e., $(L, \leq)$ is a poset). $(L, \leq)$ is said to be a *lattice* if for $x, y \in L$, the supremum $x \vee y$ and the infimum $x \wedge y$ always exist. $\top$ and $\perp$ are the greatest and least elements of L, if they exists. An element $j \in L$ is *join-irreducible* if it is not $\perp$ and cannot be express as a supremum of other elements (i.e., there are no $i,k < j$ such that $j = i \vee k$). The set of all join-irreducible elements of L is denoted by $J(L)$.

Proposition 1. [3] Let L be a distributive lattice. Any element $x \in L$ can be written as an irredundant supremum of join-irreducible elements in a unique way. That is, for any $x \in L$ there uniquely exists $\{j_1,\ldots,j_m\} \subseteq J(L)$ such that

$$x = \bigvee_{i=1}^{m} j_i \tag{4}$$

and that if there exists $M \subseteq J(L)$ such that $x = \bigvee_{j\in M} j$, then $\{j_1,\ldots,j_m\} \subseteq M$. The equation (4) is called *minimal decomposition* of x and the $\{j_1,\ldots,j_m\}$ is denoted by $\eta^*(x)$. For any x, we denote by $\eta(x) := \{j \in J(L) \mid j \le x\}$, then $x = \bigvee_{j\in\eta(x)} j$. For example, in Fig. 1 (b), $\eta(23,1) = \{(\emptyset,13),(2,13),(\emptyset,12),(3,12)\}$ and $\eta^*(23,1) = \{(2,13),(3,12)\}$.

Definition 3 (fuzzy measures and games on lattices). A *game on a lattice* L is a function $v : L \to \mathbb{R}$ such that $v(\bot) = 0$. A *fuzzy measure*, also called a *capacity*, on a lattice L is a function $v : L \to \mathbb{R}$ such that $v(\bot) = 0$, and $v(A) \le v(B)$ whenever $A \le B \le \top$. A fuzzy measure on a lattice is *normalized* if $v(\top) = 1$.

2^N can be coincided with the Boolean lattice $B(|N|)$. Therefore, ordinary fuzzy measures and games on N are regarded as fuzzy measures and games on lattices.

2.1 Examples of generalizations of games [10]

Definition 4 (games on convex geometries[2]). Let N be a set of players. A collection $\mathscr{C}$ of subsets of N is a *convex geometry* if (i) it contains the empty set, (ii) is closed under intersection, and $S \in \mathscr{C}$, $S \ne N$ implies that it exists $j \in N \setminus S$ such that $S \cup \{j\} \in \mathscr{C}$. A *game on a convex geometry* $\mathscr{C}$ is a function $v : \mathscr{C} \to \mathbb{R}$ such that $v(\emptyset) = 0$. Similar approaches on other restricted domains, games on *union stable systems* and on *matroids*, also have been studied by Bilbao [2].

Definition 5 (bi-cooperative games and bi-capacities[11]). Let $\mathscr{Q}(N) := \{(S_1,S_2) \mid S_1,S_2 \subseteq N,\ S_1 \cap S_2 = \emptyset\}$. A *bi-cooperative game* on N is a function $v : \mathscr{Q}(N) \to \mathbb{R}$ such that $v(\emptyset,\emptyset) = 0$ and a *bi-capacity* on N is a bi-cooperative game on N such that $v(A,\cdot) \le v(B,\cdot)$ and $v(\cdot,A) \ge v(\cdot,B)$ whenever $A \subseteq B \subseteq N$. A bi-capacity is *normalized* if $v(N,\emptyset) = 1$ and $v(\emptyset,N) = -1$.

Definition 6 (multi-choice games[16]). Let N be a set of players. Each player $i \in N$ has a finite number of feasible participation levels whose set we denote by $M_i = \{0,1,\ldots,m_i\}$ and $\mathbb{M} = \prod_{i\in N} M_i$. Each element $\mathbf{s} = (s_1,s_2,\ldots,s_n) \in \mathbb{M}$ specifies a *participation profile* for players and is referred to as a *multi-choice coalition*. So, a multi-choice coalition indicates the participation level of each player. A *multi-choice game* is a function $v : \mathbb{M} \to \mathbb{R}$ such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0,0,\ldots,0) \in \mathbb{M}$.

Definition 7 (games on product lattices[18]). Let $\mathbb{L} := L_1 \times \cdots \times L_n$ be a product of distributive lattices (i.e., $\mathbb{L}$ is also a distributive lattice with product order), where $L_1, \ldots, L_n$ are finite distributive lattices. A *game on a product lattice* $\mathbb{L}$ is a function $v : \mathbb{L} \to \mathbb{R}$ such that $v(\perp) = 0$, where $\perp = (\perp_1, \ldots, \perp_n)$.

Here, we consider some examples of games on $\mathbb{L}$. If $L_i := \{\perp, \top\}$ for all players $i \in N$, then we get ordinary games on 2^N. If $L_i := \{\perp, x, \top\}$, $\perp < x < \top$ (e.g., $\{-1, 0, 1\}$) $\forall i \in N$, then we have bi-cooperative games. If $L_i := \{0, 1, \ldots, m_i\}$ $\forall i \in N$, we obtain multi-choice games.

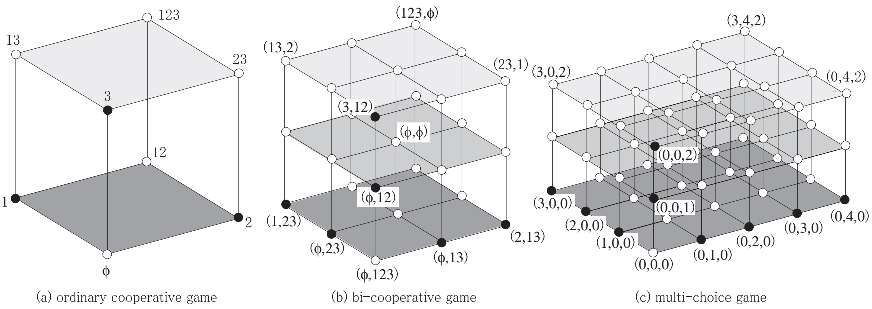

Fig. 1 Examples of games on lattices: elements indicated by black circles are join-irreducible.

3 The Möbius transforms and Derivatives

Definition 8 (the Möbius transform [24]). The *Möbius transform* of a game $v : 2^N \to \mathbb{R}$ is a game on N denoted by $\Delta^v : 2^N \to \mathbb{R}$ and is defined by

$$\Delta^v(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T) \quad \text{for each } S \in 2^N.$$

Equivalently, we have that

$$v(S) = \sum_{T \subseteq S} \Delta^v(T) \quad \forall S \in 2^N.$$

Thus, the worth $v(S)$ of a coalition S is equal to the sum of the Möbius transforms of all its subcoalitions. This gives a recursive definition of the Möbius transform. The Möbius transform of every singleton is equal to its worth, while recursively, the Möbius transform of every coalition of at least two players is equal to its worth minus the sum of the Möbius transforms of all its proper subcoalitions. In this sense, the Möbius transform of a coalition S can be interpreted as the extra contribution of the cooperation among the players in S that they did not already achieve by smaller coalitions. The Möbius transform is also called the *Harsanyi dividends* [14].

Definition 9 (the Möbius transforms on posets). Let $\mathbb{P} := (P, \leq)$ be a poset. For a function $f : P \to \mathbb{R}$, the *Möbius transform* Δ^f of f is the unique solution of the equation:

$$f(x) = \sum_{y \leq x} \Delta^f(y) \quad \forall x \in P,$$

given by

$$\Delta^f(x) = \sum_{y \leq x} \mu(y,x) f(y), \quad x \in P,$$

where μ is the so-called Möbius function on P and given by

$$\mu(y,x) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{y \leq z \leq x} \mu(y,z) & \text{if } y < x, \\ 0 & \text{otherwise.} \end{cases}$$

As noted in section 2.1, any bi-cooperative game on N is regarded as a game on a lattice (i.e., a poset). Therefore, the Möbius transform of a bi-cooperative game is obtained as follows:

Definition 10 (the Möbius transforms of bi-cooperative games). The Möbius transform of a bi-cooperative game $v : \mathscr{Q}(N) \to \mathbb{R}$ is defined by

$$\Delta^v(A_1,A_2) := \sum_{\substack{(B_1,\ B_2) \sqsubseteq (A_1,\ A_2) \\ B_2 \cap A_1 = \emptyset}} (-1)^{|A_1 \setminus B_1| + |B_2 \setminus A_2|} v(B_1,B_2) \quad \text{for each } (A_1,A_2) \in \mathscr{Q}(N),$$

where $(B_1,B_2) \sqsubseteq (A_1,A_2)$ means that $B_1 \subseteq A_1$ and $A_2 \subseteq B_2$. Equivalently, we have that

$$v(A_1,A_2) = \sum_{(B_1,B_2) \sqsubseteq (A_1,A_2)} \Delta^v(B_1,B_2).$$

Definition 11 (derivatives). The *first order derivative* of $v : 2^N \to \mathbb{R}$ w.r.t. $i \in N$ at $S \subseteq N \setminus \{i\}$ is given by

$$\Delta_{\{i\}} v(S) := v(S \cup \{i\}) - v(S).$$

It is also called the *marginal contribution* of i to $S \cup \{i\}$ in cooperative game theory. The *derivative* of v w.r.t. $T \subseteq N$ at $S \subseteq N \setminus T$ is iteratively defined by

$$\Delta_T v(S) := \Delta_{\{i\}}[\Delta_{T \setminus \{i\}} v(S)] \quad i \in T$$

with convention $\Delta_\emptyset v(S) = v(S)$. It is the *marginal interaction*, discussed in section 1.1, among players in S in the presence of T. The explicit formula is:

$$\Delta_T v(S) = \sum_{U \subseteq T} (-1)^{|T \setminus U|} v(S \cup U).$$

Equivalently, we have that

$$\Delta_T v(S) = \sum_{U \subseteq S} \Delta^v(T \cup U).$$

In particular, the Möbius transform $\Delta^v(T)$ can be represented as follows:

$$\Delta^v(T) = \Delta_T v(\emptyset) \quad \forall T \subseteq N.$$

Definition 12 (k-monotonicity of games (capacities)). Let $k \geq 2$ be an integer. A game v on N is said to be *k-monotone* (see e.g., [4, §2]) if, for any k coalitions $A_1, A_2, \ldots, A_k \subseteq N$, we have

$$v\Big(\bigcup_{i=1}^{k} A_i\Big) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{j+1} v\Big(\bigcap_{i \in J} A_i\Big). \tag{5}$$

It is easy to verify [4, §2] that k-monotonicity, with any $k \geq 2$, implies l-monotonicity for all $l \in \{2, \ldots, k\}$. By extension, 1-monotonicity (which does not correspond to $k = 1$ in Eq. (5)) is defined as standard monotonicity, i.e.,

$$v(S) \leq v(T) \quad \text{whenever} \quad S \subseteq T.$$

The notion of derivatives and of k-monotonicity are closely linked to each other.

Proposition 2. [8] *Let $k \geq 1$. A game v is k-monotone if and only if, for all $S \subseteq N$ such that $1 \leq |S| \leq k$ and all $T \subseteq N \setminus S$, we have $\Delta_S v(T) \geq 0$.*

Definition 13 (derivatives on distributive lattices). Let L be a distributive lattice. The *first order derivative* of $f : L \to \mathbb{R}$ w.r.t. $i \in J(L)$ at $x \in L$ is given by

$$\Delta_i f(x) := f(x \vee i) - f(x).$$

The *derivative* of f w.r.t. $y \in L$ at $x \in L$ is iteratively defined by

$$\Delta_y f(x) := \Delta_{j_1}[\Delta_{j_2}[\cdots \Delta_{j_{m-1}}[\Delta_{j_m} f(x)]\cdots]] \quad \forall x \in L,$$

where $\eta^*(y) = \{j_1, j_2, \ldots, j_m\}$. Note that if $j_k \leq x$ for some k, the derivative is null. Also, $\Delta_y f(x)$ does not depend on the order of the j_k's. The explicit formula is:

$$\Delta_y f(x) = \sum_{S \subseteq \{1,\ldots,m\}} (-1)^{m-|S|} f(x \vee \bigvee_{k \in S} j_k)$$

Equivalently,

$$\Delta_y f(x) = \sum_{y \leq z \leq x \vee y} \Delta^f(z).$$

In particular,

$$\Delta_y f(\bot) = \Delta^f(y) \quad \forall y \in L.$$

Similarly, the derivative of a bi-cooperative game is obtained as follows.

Definition 14 (derivatives of bi-cooperative games). The *first order derivative* of bi-cooperative game $v : \mathscr{Q} \to \mathbb{R}$ w.r.t. $(\{i\}, \emptyset)$ at $(S_1, S_2) \in \mathscr{Q}(N \setminus \{i\})$ (resp. $(\emptyset, \{i\})$ at $(S_1, S_2) \in \mathscr{Q}(N), S_2 \ni i$) is given by

$$\Delta_{(\{i\},\emptyset)}(S_1, S_2) := v(S_1 \cup \{i\}, S_2) - v(S_1, S_2)$$

$$(resp. \quad \Delta_{(\emptyset,\{i\})}(S_1, S_2) := v(S_1, S_2 \setminus \{i\}) - v(S_1, S_2) \ \ )$$

The *derivative* of v w.r.t. (S_1, S_2) at $(T_1, T_2) \in \mathscr{Q}(N \setminus S_1)$, $T_2 \supseteq S_2$ is defined by

$$\Delta_{(S_1,S_2)} v(T_1,T_2) := \sum_{\substack{L_1 \subseteq S_1 \\ L_2 \subseteq S_2}} (-1)^{|S_1 \setminus L_1| + |S_2 \setminus L_2|} v(T_1 \cup L_1, T_2 \setminus L_2).$$

4 Importance and interaction indices

The study of the notion of importance of each player has been one of the most important topics in cooperative game theory and been studied as *values*, or *allocation rules*, or *power indices* in a game [1, 6, 14, 26, 29].

Definition 15 (the Shapley value). The Shapley value $\phi^v(i)$ w.r.t. any player $i \in N$ in a game v is defined by

$$\phi^v(i) := \sum_{T \subseteq N \setminus \{i\}} \frac{(|N| - |T| - 1)!\,|T|!}{|N|!} \Delta_{\{i\}} v(T).$$

Equivalently, we have that

$$\phi^v(i) = \sum_{T \ni i} \frac{1}{|T|} \Delta^v(T).$$

On the other hand, the study of the notion of interaction among players is relatively recent in the framework of cooperative game theory. The first attempt is due to Owen [23, §5] for superadditive games. More recent developments are due to Murofushi and Soneda [21], Roubens [25], Marichal and Roubens [20], and Fujimoto et.al. [8] and led successively to the concepts of *interaction index*. The concept of *interaction index*, which can be seen as an extension of the notion of *value*, is fundamental for it enables to measure the interaction phenomena modelled by a game on a set of players.

4.1 Interaction indices for ordinary games

Grabisch and Roubens have proposed an axiomatic characterization of the *interaction index* $I(v,S)$ [13, §3] as the unique index satisfying the following axioms [2]:

- *Linearity axiom (L)* : I is a linear function with respect to its first argument.
- *Dummy player axiom (D)* : If $i \in N$ is a dummy player in a game v (i.e., $v(S \cup \{i\}) = v(S)\ \forall S \subseteq N$), then
  (i) $I(v,\{i\}) = v(\{i\})$,
  (ii) $I(v,S \cup \{i\}) = 0 \quad \forall S \subseteq N \setminus i,\ S \neq \emptyset$.
- *Symmetry axiom (S)* : For any permutation π on N, and any v,
  $I(v,S) = I(\pi v, \pi(S)) \quad \forall S \subseteq N, S \neq \emptyset$.

[2] Lately, Fujimoto et.al. [8] have provided more intuitive axioms.

- *Recursive axiom (R)* : For all finite N, $|N| \geq 2$, for all v on N,
  $I(v,S) = I(v_{\cup j}^{N\setminus\{j\}},\ S\setminus\{j\}) - I(v^{N\setminus\{j\}},\ S\setminus\{j\}), \quad \forall S \subseteq N, |S| \geq 2, \forall j \in S,$
  where $v^{N\setminus\{j\}}$ is the restriction of v to $N\setminus\{j\}$ and $v_{\cup j}^{N\setminus\{j\}}(S) := v(S\cup\{j\}) - v(\{j\})$ $\forall S \subseteq N\setminus\{j\}$.
- *Efficiency (E)* : $\sum_{i\in N} I(v,\{i\}) = v(N)$.

Definition 16 (interaction indices). The *interaction index* w.r.t. $S \subseteq N$ of v is defined by

$$I(v,S) := \sum_{T\subseteq N\setminus S} \frac{(|N|-|T|-|S|)!\ |T|!}{(|N|-|S|+1)!} \Delta_S v(T). \tag{6}$$

Equivalently,

$$I(v,S) = \sum_{T\supseteq S} \frac{1}{|T|-|S|+1} \Delta^v(T).$$

This index is an extension of the Shapley value in the sense that $I(v,\{i\})$ coincides with the Shapley value $\phi^v(i)$ of any player i.

4.2 Interaction indices for games on product lattices

From now on, we discuss on a specific type of game on lattice, where the lattice is a product of distributive lattices. Let $N := \{1,\dots,n\}$ and $\mathbb{L} := L_1 \times \cdots \times L_n$, where $L_1,\dots,L_n$ are finite distributive lattices. Then, $\mathbb{L}$ is also a distributive lattice and all join-irreducible elements of $\mathbb{L}$ are of the form $(\perp_1,\dots,\perp_{i-1}, j_i, \perp_{i+1},\dots,\perp_n)$ for some $i \in N$ and some $j_i \in J(L_i)$. A *vertex* of $\mathbb{L}$ is any element whose components are either top or bottom. Vertices of $\mathbb{L}$ will be denoted by $\top_Y$, $Y \subseteq N$, whose coordinates are $\top_k$ if $k \in Y$, $\perp_k$ otherwise. Each lattice L_i represents the poset of action, choice, participation level of player $i \in N$ to the game.

Definition 17 (antecessors). The *antecessor* $\underline{x}$ of $x \in \mathbb{L}$ is defined as

$$\underline{x} = \bigvee\{j \in \eta(x) \mid j \notin \eta^*(x)\}$$

with convention $\underline{\perp} = \perp$.

Definition 18 (interaction indices on product lattices[18]). Let f be a game on a product lattice $\mathbb{L}$, $x \in \mathbb{L}$, and $X := \{i \in N \mid x_i \neq \perp_i\}$. The *interaction index* w.r.t. x of f is defined by

$$I^f(x) := \sum_{Y\subseteq N\setminus X} \alpha_{|Y|}^{|X|} \Delta_x f(\underline{x} \vee \top_Y), \tag{7}$$

where $\alpha_k^j := \dfrac{k!(n-j-k)!}{(n-j+1)!}$, for all $j = 0,\dots,n$ and $k = 0,\dots,n-j$. Equivalently,

$$I^f(x) := \sum_{z\in[x,x^\perp]} \frac{1}{k(z)-k(x)+1} \Delta^f(z), \tag{8}$$

where $x^\perp := \top_i$ if $x_i = \perp_i$ and $x^\perp := x_i$ if $x_i \neq \perp_i$, and $k(y) = |\{i \in N \mid y_i \neq \perp_i\}|$.

Each interaction index of ordinary games, bi-cooperative games, and multi-choice games is obtained as a special case of this interaction index.

Definition 19 (interaction indices of bi-cooperative games). The interaction index $I^v(S_1,S_2)$ w.r.t. $(S_1,S_2)\in\mathscr{Q}(N)$ of a bi-cooperative game v is defined by

$$I^v(S_1,S_2) := \sum_{T\subseteq N\setminus(S_1\cup S_2)} \frac{(|N|-|S_1\cup S_2|-|T|)!\,|T|!}{(|N|-|S_1\cup S_2|+1)!}\Delta_{(S_1,S_2)}(T, N\setminus(T\cup S_1)).$$

5 Concluding remarks

This paper gave a survey of representations of importance and interaction of fuzzy measures and adjacent fields. However, this survey shows only indices based on the Shapley values on products of distributive lattices. Some indices based on other values and the Shapley values on non-distributive lattices can be seen in [8, 9].

6 Appendix

6.1 Another interaction index of bi-cooperative games

A probabilistic interpretation of the Shapley value in the framework of aggregation by the Choquet integral C_v w.r.t. v is due to Marichal [19]. Given a vector $x\in\mathbb{R}^{|N|}$ and $a\in\mathbb{R}$, we denote by $(x \mid x_i = a)$ the vector of $\mathbb{R}^{|N|}$ that differ from x only in its i-th component which is equal to a. Furthermore, let

$$\delta_i C_v(x) := C_v(x \mid x_i = 1) - C_v(x \mid x_i = 0).$$

Marichal [19] then showed that

$$\int_{[0,1]^{|N|}} \delta_i C_v(x)\; dx = \phi^v(i). \tag{9}$$

Kojadinovic [17] has proposed another interaction index of a bi-cooperative game as a generalization of the equation (9) through the Choquet integral w.r.t. bi-capacities and the recursive axiom (R).

Definition 20 (Kojadinovic's interaction indices of bi-cooperative games). Kojadinovic's interaction index $\mathscr{I}^v(S_1,S_2)$ w.r.t. $(S_1,S_2)\in\mathscr{Q}(N)$ of a bi-cooperative game v on N is defined by

$$\mathscr{I}^v(S_1,S_2) := \sum_{(T_1,T_2)\in\mathscr{Q}(N\setminus(S_1\cup S_2))} \frac{1}{2^{|T|}}\frac{(|N|-|S|-|T|+1)!\,|T|!}{(|N|-|S|+1)!}\Delta_{(S_1,S_2)}(T_1, T_2\cup S_2),$$

where $T := T_1\cup T_2$ and $S := S_1\cup S_2$.

6.2 Importance indices of games on regular set systems

Honda and Fujimoto [15] have proposed another importance index of a game on a *regular set system* as a generalization of importance indices of all ordinary games, games on convex geometries, bi-cooperative games, and multi-choice games.

Definition 21 (regular set systems). Let $\mathfrak{N} \subseteq 2^N$ and $A, B \in \mathfrak{N}$. We say that A is *covered* by B if $A \subsetneq B$ and that there is no $C \in \mathfrak{N}$ such that $A \subsetneq C \subsetneq B$. Then we denote $A \prec B$. We say that $\mathfrak{N}$ is a *regular set system* if the following conditions hold:
(i) $\emptyset, N \in \mathfrak{N}$,
(ii) $A, B \in \mathfrak{N}, A \prec B \implies |B \setminus A| = 1$.

Definition 22 (games on regular set systems). A *game on a regular set system* $\mathfrak{N}$ is a function $v : \mathfrak{N} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

Definition 23 (maximal chains of regular set systems). Let $\mathfrak{N} \subseteq 2^N$ be a regular set system. If $\mathscr{C} = (C_0, \ldots, C_n)$ satisfies that $\{C_i\}_{i \in \{0,\ldots,n\}} \subseteq \mathfrak{N}$ and $\emptyset = C_0 \prec C_1 \prec \cdots \prec C_n = N$, then $\mathscr{C}$ is called a *maximal chain* of $\mathfrak{N}$. The set of all maximal chains of $\mathfrak{N}$ is denoted by $\mathsf{M}(\mathfrak{N})$.

Definition 24 (importance indices on regular set systems). The *importance index* $\Psi^v(i)$ w.r.t. $i \in N$ of a game v on a regular set system $\mathfrak{N}$ is defined by

$$\Psi^v(i) := \frac{1}{|\mathsf{M}(\mathfrak{N})|} \sum_{\mathscr{C} \in \mathsf{M}(\mathfrak{N})} [v(C_{i^*}^{\mathscr{C}} \cup \{i\}) - v(C_{i^*}^{\mathscr{C}})], \tag{10}$$

where $C_{i^*}^{\mathscr{C}}$ is the component C_k of $\mathscr{C}$ such that $i \notin C_k$ and $i \in C_{k+1}$.

Let $(L, \leq, \vee, \wedge, \top, \bot)$ be a distributive lattice. Then $(L, \leq, \vee, \wedge, \top, \bot) \cong (\eta(L), \subseteq, \cup, \cap, J(L), \emptyset)$ with lattice-isomorphism η [3].

Definition 25 (set systems induced by lattices). Let $(L, \leq)$ be a distributive lattice. Then $(J(L), \eta(L))$ is called the *set system* induced by $(L, \leq)$.

As discussed in section 2.1, all ordinary games, bi-cooperative games, and multi-choice games are regarded as games on lattices. All the set systems induced by these lattices become regular. Therefore, we have another importance index of these games via lattice-isomorphism η:

$$I^f(\{i\}) := \Psi^{f\eta^{-1}}(\eta(i)) \quad \forall i \in J(L).$$

References

1. Banzhaf JF (1965) Weighted voting doesn't work: A mathematical analysis. Rutgers Law Review, 19:317–343
2. Bilbao JM (2000) Cooperative games on combinatorial structures, Kluwer Acadmic Publ., Boston

3. Birkhoff G (1967) Lattice Theory. American Mathematical Society, 3rd edition
4. Chateauneuf A, Jaffray JY (1989) Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. Mathematical social sciences 17(3):263–283
5. Curiel I (1997) Cooperative game theory and applications. Kluwer Acadmic Publ., Boston
6. Dubey P, Neyman A, Weber RJ (1981) Value theory without efficiency. Mathematics of Operations Research, 6:122–128
7. Felsenthal D, Machover M (1997) Ternary voting games. International journal of game theory 26:335–351
8. Fujimoto K, Kojadinovic I, Marichal JL (2006) Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. Games and Economic Behavior, 55:72-99
9. Fujimoto K, Honda A (2009) A value via posets induced by graph-restricted communication situation. In: Proc. 2009 IFSA World Congress/2009 EUSFLAT Conference, Lisbon, Portugal.
10. Grabisch M (2006) Capacities and Games on Lattices: A Survey of Results. International journal of uncertainty, fuzziness, and knowledge-based systems 14(4):371–392
11. Grabisch M, Labreuche C (2005) Bi-capacities — I: definition, Möbius transform and interaction. Fuzzy sets and systems 151:211-236
12. Grabish M, Marichal JL, Roubens M (2000) Equivalent representations of set functions. Mathematics of Operations Research, 25(2):157–178
13. Grabisch M,Roubens M (1999) An axiomatic approach to the concept of interaction among players in cooperative games. International Journal of Game Theory, 28(4):547–565
14. Harsanyi J (1963) A simplified bargaining model for the n-person cooperative game. International Economic Review, 4:59–78
15. Honda A, Fujimoto K (2009) A generalization of cooperative games and its solution concept (in Japanese). Journal of Japan society for fuzzy theory and intelligent informatics, 21(4):491–499
16. Hsiao CR, Raghavan TES (1993) Shapley value for multi-choice cooperative game. Games and economic behavior 5:240–256
17. Kojadinovic I (2007) A weigh-based approach to the measurement of the interaction among criteria in the framework of aggregation by the bipolar Choquet integral. European journal of operational research, 179:498–517
18. Lange F, Grabisch M (2009) The interaction transform for functions on lattices. Discrete Mathematics, 309:4037–4048
19. Marichal JL (1998) Aggregation operators for multicriteria decision aid, Ph.D. thesis, University of Liège.
20. Marichal JL, Roubens M (1999) The chaining interaction index among players in cooperative games. In: Meskens N and Roubens M (eds) Advances in decision analysis Kluwer Acad. Publ., Dordrecht
21. Murofushi T, Soneda S (1993) Techniques for reading fuzzy measures (iii): interaction index (in Japanese). In: 9th Fuzzy System Symposium, 693–696, Sapporo, Japan
22. Murofushi T, Sugeno M (2000) Fuzzy measures and fuzzy integrals, In: Grabisch et.al. (eds) Fuzzy Measures and Integrals. Physica-Verlag, Heidelberg New York
23. Owen G (1972) Multilinear extensions of games. Management Science, 18:64–79
24. Rota GC (1964) On the foundations of combinatorial theory I. Theory of Möbius functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 2:340–368
25. Roubens M (1996) Interaction between criteria and definition of weights in MCDA problems. In: Proc. 44th Meeting of the European Working Group “Multiple Criteria Decision Aiding”
26. Shapley L (1953) A value for *n*-person games. In Contrib. Theory of Games, II, Annals of Mathematics Studies, 28:307–317
27. Slikker M, Van Den Nouweland A (2001) Social and economic networks in cooperative game theory. Kluwer Academic Publ., Boston
28. Sugeno M (1974) Theory of fuzzy integrals and its applications. Doctoral Thesis, Tokyo Institute of Technology.
29. Weber RJ (1989) Probabilistic values for games. In: Roth AE (ed) The Shapley value. Essays in honor of Lloyd S. Shapley, Cambridge University Press, Cambridge

A value via posets induced by graph-restricted communication situations

Katsushige Fujimoto[1] Aoi Honda[2]
1.College of Symbiotic Systems Science, Fukushima University
1 Kanayagawa Fukushima 960-1296, Japan
2.Department of Systems Design and Informatics, Kyushu Institute of Technology
680-4 Kawazu Iizuka Fukuoka 820-8502, Japan
Email: fujimoto@sss.fukushima-u.ac.jp, aoi@ces.kyutech.ac.jp

Abstract— *This paper provides a new value (solution concept or allocation rule) of cooperative games via posets induced by graphs. Several values in a graph-restricted communication situation have been proposed or introduced by Myerson, Borm, and Hamiache... However, these values have been subjected to some criticisms in certain types of games. The value proposed in this paper withstands these criticisms. Moreover, these existing values have been defined only in situations represented by undirected graphs, while the notion of the value proposed in this paper can be extended to situations represented by directed graphs.*

Keywords— graph-restricted situations, communication situations, values, posets, cooperative games.

1 Introduction and Preliminaries

Throughout the paper, N denotes the universal set of n elements. For convenience, we often number the elements such that the universal set is $N = \{1, 2, \ldots, n\}$. A real-valued function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game*. A monotone game (i.e., $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq N$) is called a *capacity* or *a fuzzy measure*. We often call the pair (N, v), rather than v, a game or a capacity. The set of all games on N is denoted by $\mathcal{G}^N$. A real vector-valued function $\Phi : \mathcal{G}^N \to \mathbb{R}^{|N|}$ is called a *value*. In cooperative game theory, N is considered to be the set of all players. For every subset S of N, often called a *coalition*, $v(S)$ represents the (transferable) utility/profits that players in S can obtain if they decide to cooperate. For every game (N, v), the value $\Phi(N, v)$ represents an allocation rule, which provides an assessment of the benefits for each player from participating in a game v. For the sake of simplicity, we mainly discuss games in terms of various set functions (e.g., games, capacities, fuzzy measures, and so forth.) on N.

To avoid cumbersome notations, we often omit braces for singletons, e.g., by writing $v(i)$, $U \setminus i$ instead of $v(\{i\})$, $U \setminus \{i\}$. Similarly, for pairs, we write ij instead of $\{i, j\}$. Furthermore, cardinalities of subsets $S, T, \ldots$, are often denoted by the corresponding lower case letters $s, t, \ldots$, otherwise by the standard notation $|S|, |T|$,...

1.1 Games and capacities with graph restricted situations

In ordinary cooperative game theory it is implicitly assumed that all coalitions of N can be formed; however, this is generally not the case. For players to coordinate their actions, they must be able to communicate. The bilateral communication channels between players in N are described by a *communication network*. Such a network can be represented by an *undirected graph* (N, L), which has the set of players as its *nodes* $S \subseteq N$ and in which the players are connected by the set of *links* $L \subseteq \{ij \mid i, j \in N, i \neq j\}$; i.e., players i and j can communicate (directly) with each other if $ij \in L$. This paper will deals with only situations induced by communication networks described by undirected graphs. Many other approaches to the situations can be seen via the literatures [1, 2].

Definition 1.1 (communication situation)
The triple (N, v, L), which reflects a situation consisting of a game v on N and a communication network (N, L), is called a *communication situation*. We denote the set consisting of all communication situations on N by CS^N. For a coalition $T \subseteq N$, the restriction of (N, L) to T is denoted by $(T, L(T))$ and defined by $L(T) := \{ij \in L \mid ij \subseteq T\}$.

Definition 1.2 (feasible coalition) We say that players j and k are *connected in* $S \subseteq N$ if $j = k$ or there exists a subset $\{i_1, \cdots, i_m\} \subseteq S$ such that $j = i_1$, $k = i_m$, and $\{i_t, i_{t+1}\} \in L$ for all $t \in \{1, \cdots, m-1\}$. Then we denote $j \sim_S k$. Clearly, this relation $\sim_S$ is an equivalence relation. Hence, the notion of connectedness in S induces a partition $S/L := S/\sim_S$ of S. A coalition $S \subseteq N$ is said to be *feasible* in the communication network (N, L) if any two players, $j \in S$ and $k \in S$, are connected in S (i.e., $S/L = \{S\}$).

Example 1.1
Consider the communication situation (N_1, v, L_1) with $N_1 = \{1, 2, 3, 4, 5, 6, 7\}$ and $L_1 = \{12, 15, 26, 37, 47, 56\}$ (Fig.1). Then, all the players in $\{1, 2, 6\}$ can communicate with other;

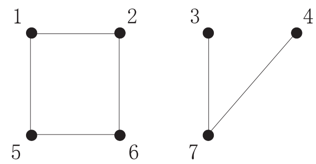

Figure 1: Communication network(N, L_1).

i.e., the coalition $\{1, 2, 6\}$ is feasible. Hence, they can fully coordinate their actions and obtain the value $v(\{1, 2, 6\})$. On the other hand, in the coalition $\{1, 2, 3, 4\}$, players 1 and 2 can communicate with each other, but players 3 and 4 cannot communicate with any other players in $\{1, 2, 3, 4\}$. Thus, feasible subcoalitions of $\{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{3\}$, and $\{4\}$ (i.e., forming the coalition $\{1, 2, 3, 4\}$ is unfeasible). Hence, the value attainable by the players in $\{1, 2, 3, 4\}$ should be $v(\{1, 2\}) + v(\{3\}) + v(\{4\})$. In general, the value attainable by the players in $S \in N$ under a communication situation (N, v, L) is represented by

$$\sum_{T \in S/L} v(T). \tag{1}$$

Definition 1.3 (network-restricted game [3]) The *network-restricted game* (N, v^L) associated with (N, v, L) is defined as

$$v^L(S) := \sum_{T \in S/L} v(T) \quad \text{for each } S \subseteq N. \tag{2}$$

Note that if (N, L) is the complete graph (i.e., $L = \{ij \mid i, j \in N, i \neq j\}$), the network-restricted game v^L is equal to the original game v.

The network-restricted game evaluates the possible gains from cooperation in a communication situation from the viewpoint of the players. The next example focuses on the importance of communication channels and links in a communication situation.

Example 1.2 In the communication situation L_1 depicted in Fig.1, the value obtainable by the players in the grand coalition N is

$$v^{L_1}(N) = v(\{1, 2, 5, 6\}) + v(\{3, 4, 7\}), \tag{3}$$

since $N/L_1 = \{\{1, 2, 5, 6\}, \{3, 4, 7\}\}$. If for some reason the communication link between players 4 and 7 is lost, the communication network L_1 becomes the new communication network $L_2 = \{12, 15, 26, 37, 56\}$. Then, $N/L_2 = \{\{1, 2, 5, 6\}, \{4\}, \{3, 7\}\}$ and the value obtainable by the players in the grand coalition N becomes

$$v^{L_2}(N) = v(\{1, 2, 5, 6\}) + v(\{4\}) + v(\{3, 7\}). \tag{4}$$

Then,

$$v^{L_1}(N) - v^{L_2}(N) \tag{5}$$

can be interpreted as a type of marginal contribution of the link $\{4, 7\} \in L_1$ to the communication network L_1.

Definition 1.4 (link game [4]) The *link game* (L, γ^v) associated with (N, v, L) consisting of a zero-normalized game v is a game on L defined by

$$\gamma^v(M) := v^M(N) = \sum_{T \in N/M} v(T) \quad \text{for each } M \subseteq L. \tag{6}$$

Note that, for an ordinary game v, γ^v is not a game on L since $\gamma^v(\emptyset) = \sum_{T \in N/\emptyset} v(T) = \sum_{i \in N} v(\{i\}) \neq 0$.

The link game $\gamma^v(M)$ represents the worth of the communication network $M \subseteq L$ as the worth of the grand coalition in the communication situation (N, v, M) through the network-restricted game v^M.

Definition 1.5 (Möbius Transform [5]**)** The *Möbius transform* of a game $v : 2^N \to \mathbb{R}$ (resp. $\gamma : 2^L \to \mathbb{R}$) is a game on N (resp. L) denoted by $\Delta^v : 2^N \to \mathbb{R}$ (resp. $\Delta^\gamma : 2^L \to \mathbb{R}$) and is defined by

$$\Delta^v(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T) \quad \text{for each } S \in 2^N. \tag{7}$$

$$(resp.\ \Delta^\gamma(M) := \sum_{K \subseteq M} (-1)^{|M \setminus K|} \gamma(K) \quad \text{for each } M \in 2^L). \tag{8}$$

Equivalently, we have that

$$v(S) = \sum_{T \subseteq S} \Delta^v(T) \quad \forall S \in 2^N. \tag{9}$$

$$(resp.\ \gamma(M) = \sum_{K \subseteq M} \Delta^\gamma(K) \quad \forall M \in 2^L). \tag{10}$$

Thus, the worth $v(S)$ (resp. $\gamma(M)$) of a coalition S (resp. communication network M) is equal to the sum of the Möbius transform of all its subcoalitions (subnetworks). This gives a recursive definition of the Möbius transform. The Möbius transform of every singleton is equal to its worth, while recursively, the Möbius transform of every coalition (resp. communication network) of at least two players (resp. links) is equal to its worth minus the sum of the Möbius transform of all its proper subcoalitions (resp. subnetworks). In this sense, the Möbius transform of a coalition S (resp. communication network M) can be interpreted as the extra contribution of the cooperation/synergy among the players in S (resp. links in M) that they did not already achieve by smaller coalitions (resp. networks). In fact, in the context of *interaction indices* (e.g.,[6, 7]), the Möbius transform $\Delta^v(S)$ is called the *internal interaction index* of S, which represents the magnitude of a type of interaction among the elements in S. The Möbius transform is also occasionally called the *Harsanyi dividends*[8].

Definition 1.6 (unanimity game) The *unanimity game* for a non-empty coalition $T \subseteq N$ is denoted by u_T and defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

For any game $v : 2^N \to \mathbb{R}$, v can be represented as

$$v(S) = \sum_{T(\neq \emptyset) \in 2^N} \Delta^v(T) \cdot u_T(S) \quad \forall S(\neq \emptyset) \in 2^N. \tag{12}$$

2 Values for communication situations

In this section, we briefly introduce the Shapley value for ordinary cooperative games and three existing values for communication situations that appear in the literatures [3, 4, 9], *the Myerson value, the position value, and the Hamiache value*.

Definition 2.1 (the Shapley value [10]) The *Shapley value* $\Phi : \mathcal{G}^N \to \mathbb{R}^{|N|}$ for a game $(N, v) \in \mathcal{G}^N$ is defined by

$$\Phi_i(N, v) := \sum_{T \ni i} \frac{1}{|T|} \Delta^v(T) \quad \text{for each } i \in N. \tag{13}$$

Definition 2.2 (the Myerson value [3]) The *Myerson value* $\Psi : CS^N \to \mathbb{R}^{|N|}$ for a communication situation $(N, v, L) \in CS^N$ is defined by

$$\Psi(N, v, L) := \Phi(N, v^L). \tag{14}$$

The Myerson value is the allocation rule that assigns to every communication situation (N, v, L) the Shapley value of the network-restricted game (N, v^L). Note that $\Psi(N, v, L) = \Phi(N, v)$ if (N, L) is the complete graph.

Definition 2.3 (position value [4]) The *position value* $\pi : CS^N \to \mathbb{R}^{|N|}$ for a communication situation $(N, v, L) \in CS^N$ is defined by

$$\pi_i(N, v, L) := \frac{1}{2} \sum_{\substack{l \in L \\ l \ni i}} \Phi_l(L, \gamma^v) \quad \text{for each } i \in N. \tag{15}$$

The Shapley value $\Phi_l(L, \gamma^v)$ of a link $l \in L$, which is induced via (13) for the link game (L, γ^v), can be interpreted as a type of *expected marginal contribution* of the link l to all communication networks containing l. Then, the value is divided equally between the two players at the ends of the considered link $l \in L$. The position value of a given player $i \in N$ is obtained as the sum of all these shares.

We focus to a third value for communication situations, introduced by Hamiache [9]. Given a communication situation (N, v, L) and $S \subseteq N$, we denote by S^* the set of all nodes of the communication network (N, L) that are adjacent to at least one of the nodes of S,

$$S^* := \{i \in N \mid \exists j \in S \text{ such that } ij \in L\}. \tag{16}$$

Definition 2.4 (associated game [9]) For a value ϕ on CS^N (i.e., $\phi : CS^N \to \mathbb{R}^{|N|}$), the *associated game* v^*_ϕ of v with respect to ϕ is defined for $S \subseteq N$, by

$$v^*_\phi(S) := \begin{cases} v(S) + \displaystyle\sum_{j \in S^* \setminus S} \left(\phi_i(S^{+j}, v|_{S^{+j}}, L(S^{+j})) - v(j)\right) & \text{if } |S/L| = 1, \\ \displaystyle\sum_{T \in S/L} v^*_\phi(T) & \text{otherwise,} \end{cases} \tag{17}$$

where $S^{+j} := S \cup \{j\}$ and $v|_{S^{+j}}$ is the restriction of v to S^{+j}.

Hamiache [9] claims that there is a unique value ϕ, the so-called *Hamiache value*, for communication situations satisfying the following five properties, *component-efficiency*, *linearity w.r.t. games*, *independence of irrelevant players*, *positivity*, and *associated consistency*:

Component-efficiency :
For any (N, v, L) and any $S \in N/L$,

$$\sum_{i \in S} \phi_i(N, v, L) = v(S). \tag{18}$$

Linearity w.r.t. games :
For any $\alpha, \beta \in \mathbb{R}$ and $(N, v, L), (N, w, L) \in CS^N$,

$$\phi(N, \alpha v + \beta w, L) = \alpha\phi(N, v, L) + \beta\phi(N, w, L). \tag{19}$$

Independence of irrelevant players :
For any (N, L) and for any two feasible coalitions $R \subseteq T$,

$$\phi_i(N, u_R, L) = \phi_i(T, u_R, L(T)) \ \ \forall i \in T. \tag{20}$$

Positivity :
For any feasible coalition $T \subseteq N$,

$$\phi_i(T, u_T, L(T)) \geq 0 \ \ \forall i \in T. \tag{21}$$

Associated consistency:
For any $(N, v, L) \in CS^N$,

$$\phi(N, v, L) = \phi(N, v^*_\phi, L). \tag{22}$$

Note that $\phi(N, v, L) = \Phi(N, v)$ if (N, L) is the complete graph.

3 Posets induced by communication networks

3.1 Communication networks and posets

In this subsection, we consider and introduce a subposet of $B(n) := (2^N, \subseteq)$ induced by a communication network (N, L).

For a communication network (N, L), the set of all feasible coalitions in (N, L) is denoted by $P(N, L)$. i.e.,

$$P(N, L) := \{S \subseteq N \mid |S/L| = 1\}. \tag{23}$$

The set $P(N, L)$, together with set inclusion $\subseteq$ as an order on $P(N, L)$, is called the *poset induced by the communication network* (N, L).

Example 3.1 Let $N = \{1, 2, 3\}$, $L_a = \{12, 13, 23\}$, $L_b = \{13, 23\}$, and $L_c = \{12\}$. Then the posets induced by communication networks (N, L_a), (N, L_b), and (N, L_c), as shown in (a) – (c) in Fig. 2, are represented as shown in (a) – (c) in Fig. 3, respectively.

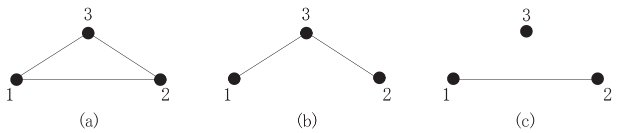

Figure 2: Communication networks on $N = \{1, 2, 3\}$.

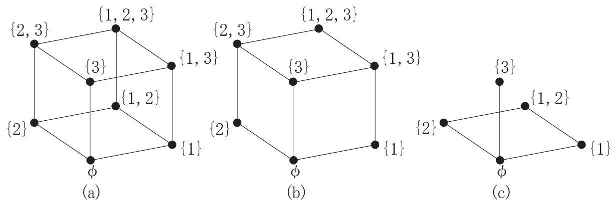

Figure 3: Posets corresponding to networks in Fig. 2.

Definition 3.1 (Möbius transform on posets)
Let $P := (N, \leq)$ be a poset. For a function $v : P \to \mathbb{R}$, the *Möbius transform* Δ^v of v is a function on P satisfying the following equation:

$$v(x) = \sum_{y \leq x} \Delta^v(y) \quad \forall x \in P. \tag{24}$$

Definition 3.2 (representation functions)
The *representation function of a communication situation* (N, v, L) is a function v^P on the poset $P(N, L)$ defined by

$$v^P(S) = v(S) \quad \text{for each } \ S \in P(N, L). \tag{25}$$

Then, the Möbius transform Δ^{v^P} of v^P is represented as

$$\Delta^{v^P}(S) := \sum_{\substack{T \in P(N,L) \\ T \subseteq S}} (-1)^{|S \setminus T|} v^P(T) \quad \forall S \in P(N, L). \tag{26}$$

Conversely,

$$v^P(S) := \sum_{\substack{T \in P(N,L) \\ T \subseteq S}} \Delta^{v^P}(T) \quad \forall S \in P(N, L). \tag{27}$$

Definition 3.3 (poset representation) The *poset representation of a communication situation* (N, v, L) is the pair $(P(N, L), \Delta^{v^P})$ of the poset induced by (N, v, L) and the Möbius transform Δ^{v^P} of representation function v^P of (N, v, L).

4 A new value in communication situations

In this section, we introduce a new value for communication situations.

4.1 An interpretation of the Shapley value

Now, we consider the case $N = \{1, 2\}$; the Shapley value $\Phi_1(N, v)$ of player 1 in a game v is obtained, from (13), as

$$\Phi_1(N, v) = \frac{1}{1}\Delta^v(\{1\}) + \frac{1}{2}\Delta^v(\{1, 2\}). \tag{28}$$

This can be interpreted as an allocation rule of *Harsanyi dividends* (i.e., the Möbius transform) described as follows:

Allocation rule of Harsanyi dividends : We consider a process to form the coalition $\{1, 2\}$. Then, there are two shortest paths from $\emptyset$ to $\{1, 2\}$ in Fig. 2. One is the path $\emptyset \to \{1\} \to \{1, 2\}$; another is the path $\emptyset \to \{2\} \to \{1, 2\}$. The path $\emptyset \to \{1\} \to \{1, 2\}$ can be interpreted as follows: *Player 1 makes an offer to player 2 for forming the coalition* $\{1, 2\}$. *Player 2 accepts the offer and adds to the coalition* $\{1\}$ *to form the new coalition* $\{1, 2\}$. Among these two paths, the only path that passes through $\{1\}$ is $\emptyset \to \{1\} \to \{1, 2\}$. That is, the number of paths from $\emptyset$ to $\{1, 2\}$ is 2, while of the number of paths via $\{1\}$ is 1. Then player 1 obtains $\frac{1\ path}{2\ paths}$ of the amount of the Harsanyi dividend $\Delta^v(\{1, 2\})$ (i.e., $\frac{1}{2}\Delta^v(\{1, 2\})$). In the same way, player 1 obtains $\frac{1}{1}\Delta^v(\{1\})$ and $\frac{0}{1}\Delta^v(\{2\})$. The Shapley value of player 1 is obtained as the sum of all these shares.

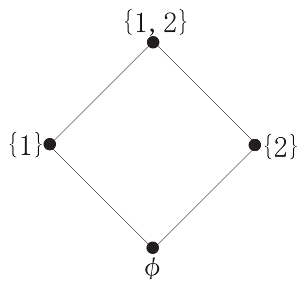

Figure 4: The Boolean lattice $B(2)$ on $N = \{1, 2\}$.

This allocation rule can be extended to the case $N = \{1, 2, 3\}$ (Fig. 5).

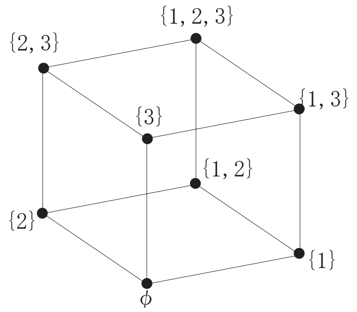

Figure 5: The Boolean lattice on $N = \{1, 2, 3\}$.

Indeed,

$$\begin{aligned}\Phi_1(N, v) &= \frac{1}{1}\,\Delta^v(\{1\}) + \frac{1}{2}\,\Delta^v(\{1, 2\}) + \frac{1}{2}\,\Delta^v(\{1, 3\}) \\ &+ \frac{0}{2}\,\Delta^v(\{2, 3\}) + \frac{2}{6}\,\Delta^v(\{1, 2, 3\}).\end{aligned} \tag{29}$$

For instance, there are six shortest paths from $\emptyset$ to $\{1, 2, 3\}$. Among them, two paths pass through $\{1\}$, as shown in Fig. 6.

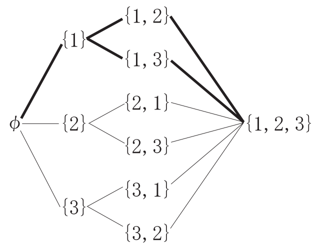

Figure 6: Shortest paths from $\emptyset$ to $\{1, 2, 3\}$.

4.2 An interpretation of the Myerson value

The Myerson value of (N, v, L) is the Shapley value of the network-restricted game (v^L, N). That is, the Myerson value is obtained by applying the above allocation rule to Harsanyi dividends $\{\Delta^{v^L}\}$ of v^L. Then Δ^{v^L} is given as follows:

Proposition 4.1 Let $(N, v, L) \in CS^N$ be a communication situation and $(B(n), \Delta^{v^L})$ the poset representation of the network-restricted game (N, v^L) associated with (N, v, L). Then,

$$\Delta^{v^L}(S) = \begin{cases} \Delta^v(S) & \text{if } S \in P(N, L), \\ \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

4.3 Criticisms of existing values

Each of the existing values for communication situations, the Myerson value, the position value, and the Hamiache value, has been subject to criticisms, as follows.

The Myerson value :

$$\Psi_i(N, u_S, L) = \Psi_i(N, u_S, M) = \frac{1}{|S|} \quad \forall i \in N \tag{31}$$

whenever S is a feasible coalition in both (N, L) and (N, M). For example, consider the communication situation with $L = \{ij \subseteq N \mid j \in N \setminus i\}$ (i.e., L is a star with a central player i); then every player receives the same value (see Example $\Psi(N, v, L_e)$ in Example 5.3).

The position value :

Irrelevant null players often have positive values (see Example 5.2), where a null player $i \in N$ of the game (N, v) is a player satisfying $v(S \cup i) = v(S)$ for any $S \subseteq N$.

The Hamiache value :

It is very complex to compute the Hamiache value. Not only that, associated consistency is rather technical.

4.4 A new value in communication situations

In this section, we propose a new value for communication situations that withstands all these criticisms.

Definition 4.1 (chain, saturated chain) A *chain* (or a *totally ordered set* or *linear ordered set*) is a poset in which any two elements are comparative. That is, a subset C of $P(N,L)$ is called a chain if $S \subseteq T$ or $T \subseteq S$ for any $S,T \in C$. The chain C of $P(N,L)$ is *saturated* (or *unrefinable*) if there does not exist $W \in P(N,L) \setminus C$ such that $S \subsetneq W \subsetneq T$ for some $S,T \in C$ and that $C \cup W$ is a chain.

Definition 4.2 (shortest path) For two feasible coalitions S, $T \in P(N,L)$, a saturated chain $\mathcal{P}$ of $P(N,L)$ is called a *shortest path* from S to T if $S,T \in \mathcal{P}$ and $S \subseteq W \subseteq T$ for any $W \in \mathcal{P}$. Then, we denote the set of all shortest paths from S to T by $\{S \to T\}$.

In the following, we propose a new value in communication situations, based on the interpretation of the Shapley value mentioned in Subsection 4.1.

Definition 4.3 We now propose a new value $\sigma(N,v,L)$ of a communication situation (N,v,L), as follows.

$$\sigma_i(N,v,L) := \sum_{S\in P(N,L)} \frac{|\{i \to S\}|}{|\{\emptyset \to S\}|}\, \Delta^{v^P}(S) \quad \text{for each } i \in N. \tag{32}$$

The number $|\{\emptyset \to S\}|$ of all shortest paths from $\emptyset$ to S indicates the number of all processes in which the feasible coalition S is formed. Also, $|\{i \to S\}|$ indicates the number of all processes in which the feasible coalition S is formed by the initiator $i \in N$. Then, the player $i \in N$ obtains $\frac{|\{i \to S\}|}{|\{\emptyset \to S\}|}$ of the amount of $\Delta^{v^P}(S)$ if $\Delta^{v^P}(S)$ is allocated in proportion to the frequency with which the player i initiates the formation of the feasible coalition S. The value proposed here of a given player $i \in N$ is obtained as the sum of all these shares.

Now we show an example that supports the naturalness of the definition of this value.

Example 4.1 We consider the communication situation (N,v,L) with $N = \{1,2,3\}$, $L = \{13,23\}$, and $\Delta^{v^P}(S) \geq 0$ for any $S \in P(N,L)$. The value $\sigma_i(N,v,L)$ proposed here of player $i \in N$ is represented as the values of the ammeters A_i in the electric circuit with current sources $I_S = \Delta^{v^P}(S)$, as shown in Fig.7.

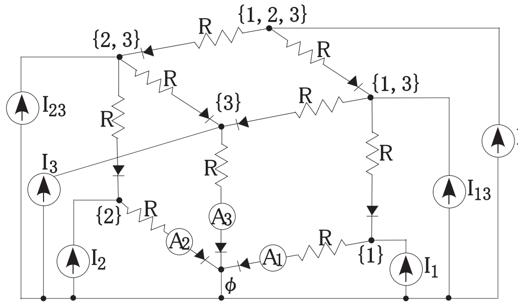

Figure 7: Electric circuit representing (N,v,L).

Property 1 The value σ proposed here satisfies *component-efficiency, linearity w.r.t. games, independence of irrelevant players*, and *positivity*.

Property 2 Let (N,u_N,L_c^*) be a communication situation with $L_c^* = \{cj \mid j \in N \setminus c\}$, $c \in N$. Then,

$$\sigma_i(N,u_N,L_c^*) = \begin{cases} \dfrac{1}{2} & \text{if } i = c, \\ \dfrac{1}{2(n-1)} & \text{otherwise.} \end{cases} \tag{33}$$

That is, if the communication network (N,L) is a star-graph with central player $c \in N$, in the unanimity game u_N, the central player obtains a half of the total amount of $u_N(N) = 1$ and the rest of the amount are shared out equally among the other players (see L_b, L_e in example 5.3).

However, we have not found any axiomatic characterization of the value proposed in this paper yet.

5 Comparison of existing values

In this section, we compare the existing four values (the Shapley, Myerson, position, and Hamiache values) and the value proposed in this paper. Examples 5.1 and 5.2 not only compare them but also illustrate the criticisms against the Shapley, Myerson, and position values, respectively.

Example 5.1 Consider the communication situation (N,v,L) with $N = \{1,2,3\}$, $L = \{13,23\}$ ((b) in Fig. 2), and

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ 30 & \text{if } |S| = 2 \\ 36 & \text{if } S = N. \end{cases} \tag{34}$$

Then,

$$\Phi(N,v) = (12,12,12), \quad \Psi(N,v,L) = (7,7,22),$$
$$\pi(N,v,L) = (9,9,18), \quad \phi(N,v,L) = (9,9,18),$$
$$\sigma(N,v,L) = (9,9,18).$$

Example 5.2 Consider the communication situation (N,v,L) with $N = \{1,2,3\}$, $L = \{12,13,23\}$ (L_d in Fig. 8), and

$$v(S) = \begin{cases} 12 & \text{if } S \supseteq \{1,2\} \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Then,

$$\Phi(N,v) = (6,6,0), \quad \Psi(N,v,L) = (6,6,0),$$
$$\pi(N,v,L) = (5,5,2), \quad \phi(N,v,L) = (6,6,0),$$
$$\sigma(N,v,L) = (6,6,0).$$

Example 5.3 Consider communication situations (N,u_N,L) with $2 \leq |N| \leq 4$, $|N/L| = 1$ (i.e., (N,L) is connected). Fig.8 shows all connected graphs (up to isomorphism) with $2 \leq n \leq 4$ nodes. Then, for any such communication situations (N,u_N,L),

$$\Phi_i(N,u_N,L) = \Psi_i(N,u_N,L) = \frac{1}{|N|} \quad \forall i \in N. \tag{36}$$

Table 1 shows comparisons of the remaining values (i.e., the position value π, the Hamiache value ϕ, and the value σ proposed in this paper), and illustrates that the value σ does not always coincide with the Hamiache value ϕ.

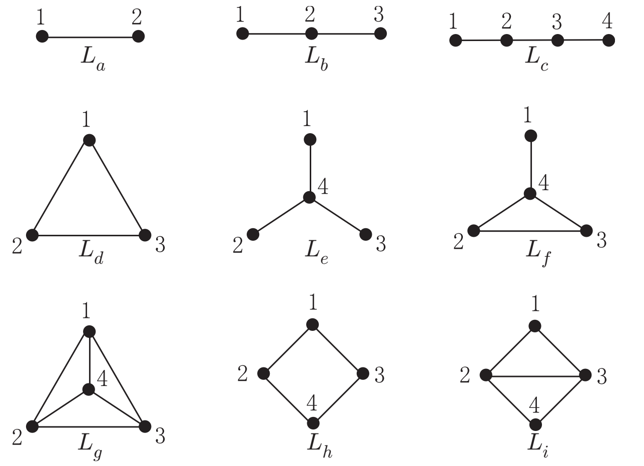

Figure 8: Graphs with at most four nodes.

Table 1: Comparison of existing values.

	π	ϕ	σ
L_a	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
L_b	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$
L_c	$(\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})$	$(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$	$(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$
L_d	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
L_e	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{6})$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{6})$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{6})$
L_f	$(\frac{3}{12}, \frac{2}{12}, \frac{2}{12}, \frac{5}{12})$	$(0.172, 0.190, 0.190, 0.448)$	$(\frac{2}{14}, \frac{3}{14}, \frac{3}{14}, \frac{6}{14})$
L_g	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
L_h	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
L_i	$(\frac{13}{60}, \frac{17}{60}, \frac{17}{60}, \frac{13}{60})$	$(\frac{3}{14}, \frac{4}{14}, \frac{4}{14}, \frac{3}{14})$	$(\frac{2}{10}, \frac{3}{10}, \frac{3}{10}, \frac{2}{10})$

Acknowledgment

This research was partially supported by the Ministry of Education, Culture, Sports, Science and Technology-Japan, Grant-in-Aid for Scientific Research (C), 19510136, 2007-2009.

References

[1] M. Grabisch and F. Lange. Games on latices, multichoice games and the shapley value: a new approach. *mathematical Methods of Operations Research*, 65(1):153–167, 2007.

[2] J. M. .Bilbao. *Cooperative games on combinatorial syructures*. Kluwer academic publishers, Boston, 2000.

[3] R. Myerson. Graphs and cooperation in games. *Mathematics of Operations Research*, 2:225–229, 1977.

[4] P. Borm, G. Owen, and S. Tijs. On the position value for communication situuations. *SIAM Journal on Discrete Mathematics*, 3:305–320, 1992.

[5] G.-C. Rota. On the foundations of combinatorial theory I. theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368, 1964.

[6] K. Fujimoto. Interaction indices with respect to fuzzy measures (in japanese). *Journal of Japan Society of Fuzzy Theory and Intelligent Informatics*, 16(4):303–310, 2004.

[7] K. Fujimoto, I. Kojadinovic, and J.-L. Marichal. Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. *Games and Economic Behavior*, 55:72–99, 2006.

[8] J. Harsanyi. A simplified bargaining model for the n-person cooperative game. *International Economic Review*, 4:194–220, 1963.

[9] G. Hamiache. A value with incomplete information. *Games and Economic Behaviour*, 26:59–78, 1999.

[10] L. Shapley. A value for *n*-person games. *Contribution Theory of Games, II. Annals of Mathematics Studies*, 28:307–317, 1953.