Nash Equilibrium Allocations of Multiple Public Goods

Tomoyuki Kamo *

First Version: December 2008,
This Version: September 2009

Abstract

This paper studies welfare properties of allocations of multiple public goods attainable through Nash equilibrium of voluntary contribution. We characterize Nash equilibrium allocation and establish a version of the fundamental theorem of welfare economics in an economy with multiple public goods. At first, we investigate optimality of Nash equilibrium allocation when private resource is non-consumable. Next, we discuss an extension of our results to more general case with private resource consumption.

JEL classification: C72, D61, H41

Keywords: multiple public goods, voluntary contribution, Nash equilibrium, welfare theorem

*Faculty of Economics, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto, 603-8555 JAPAN. E-mail: kamo@cc.kyoto-su.ac.jp.

1 Introduction

Resource allocation in a public goods economy is central in public economics. When there are multiple public goods, the problem is quite complex. We have to examine how to allocate scarce resources not only between private consumption and public goods, but also among the public goods.

In this paper, we consider a resource allocation problem in an economy with multiple public goods. The allocation mechanism is simple voluntary contribution (private provision of resources). As a specific feature of our model, we assume private resources are non-consumable. This is because we focus on resource allocation among the public goods. If there is only one public good, then the problem is trivial: All agents contribute all their resources to public good, so an optimal allocation is always achieved. But if there are multiple public goods, the problem becomes nontrivial even if private resources are non-consumable. The basic problem, such as "whether or not is Nash equilibrium allocation Pareto optimal?", is unsolved in this framework.

This paper investigates the following classical problems of welfare economics in a multiple public goods economy.

(A) Are Nash equilibrium allocations Pareto optimal? If the answer is "no" in general, when is it "yes"?

(B) Are any Pareto optimal allocations attainable through Nash equilibrium when redistribution of private resources is possible?

We provide answers to the problems by establishing a version of fundamental welfare theorems. With respect to (A), we derive several sufficient conditions for Nash equilibrium allocations to be Pareto optimal (Proposition 1,2, and 3). To deal with (B), we first give a characterization of allocations attainable through Nash equilibrium with transfaer (Theorem 2). This result enables us to derive a necessary and sufficient condition for any Pareto optimal allocation to be achieved by Nash equilibrium with transfer (Proposition 4).

Next, we discuss an extension of our results to the economy with private consumption. It is well known that when private resource is consumable, Nash equilibrium allocation is not Pareto optimal even if there is only one public good. This fact shows difficulty in achieving optimal allocation between private and public goods. Since we focus on resource allocation among the public goods, we will introduce the concept of "constraint Pareto optimality" as another criterion of optimality to evaluate an allocation of public goods. Under this optimality concept, we examine welfare properties of Nash equilibrium allocations with or without transfer. In similar to the non-consumable resource case, we show that Nash equilibrium allocation is constrained Pareto optimal if all the agents' preferences are identical, and the necessary conditions for Nash equilibrium allocation is also sufficient for Nash equilibrium with transfer in an economy with private consumption.

Our model with non-consumable resources may be regarded as income redistribution game formulated by Nakayama (1980) if the number of public goods is equal to the one of agents in the economy [1]. Nakayama (1980) provide sufficient conditions for Nash equilibrium of income redistribution games to be Pareto optimal. We extend his results to the public goods economy. Our results admit the possibity of productions, and are independent of the number of the agents in the economy.

The paper is organized as follows. In section 2, the basic model is introduced, and the definition of Pareto optimality in our model is provided. Section 3 contains the formal description of our games. Section 4 introduces allocations corresponding to Nash equilibrium of the game, and gives a necessary condition for Nash equilibrium allocation. Section 5 contains several sufficient conditions for the first welfare theorem. In section 6, we investigate the possibility of the second welfare theorem in our economy. Section 7 extends our model to the case of consumable private resources, and discuss the results.

2 The Basic Economy

Our description of the economy with public goods follows Aumann, Kurz, and Neyman [1], [2]. There are one non-consumable resources and m public goods. Public goods are produced from resources. The production technology is represented by a production function $F : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ such that for any $x = (x_1, \dots, x_m) \in \mathbb{R}^m_+$

$$F(x) = \begin{pmatrix} f_1(x_1) \\ \vdots \\ f_m(x_m) \end{pmatrix},$$

where x_i is resource input and f_i is production function for public good i ($i = 1, \dots, m$). Let H be a set of agents whose cardinality is n, i.e., $|H| = n$. The agent h is characterized by the pair (u_h, e_h) of utility function $u_h : \mathbb{R}^m_+ \to \mathbb{R}$ and initial endowment of resources e_h.

Definition 1. A *public goods economy* $\mathcal{E}$ is a list of the set of agents H, the production

[1] Then the public good i should be interpreted as the redistributed income of agent i.

technology F, and the agents' characteristics (u_h, e_h):

$$\mathcal{E} = \Big(H, F, (u_h, e_h)_{h\in H}\Big).$$

We shall assume the standard convex environments.

Assumption 1.

(i) f_i is increasing, continuous, and concave for $i = 1, \ldots, m$.

(ii) u_h is increasing, continuous and strictly quasi concave for every $h \in H$.

(iii) $e_h \geq 0$ for all $h \in H$.

Let $\bar{e}$ be the total resources of $\mathcal{E}$; $\bar{e} := \sum_h e_h$. The set of feasible resource input vectors is denoted by $C(\bar{e})$:

$$C(\bar{e}) := \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+ \;\middle|\; \sum_{i=1}^{m} x_i \leq \bar{e} \right\}$$

The feasible set of public goods bundle in $\mathcal{E}$ is denoted by $\mathcal{A}(\mathcal{E})$:

$$\mathcal{A}(\mathcal{E}) := \{ g \in \mathbb{R}^m_+ \mid g \leq F(x) \text{ for some } x \in C(\bar{e}) \}.$$

Definition 2. A feasible allocation g is *Pareto optimal* if there exists no $g' \in \mathcal{A}(\mathcal{E})$ such that $u_h(g') \geq u_h(g)$ for any $h \in H$ and $u_h(g') > u_h(g)$ for some $h \in H$.

Remark 1. Suppose all of the utility functions u_h and the production functions f_i are smooth and concave. Then a feasible allocation $g^* \in \mathbb{R}^m_{++}$ is Pareto optimal if and only if there exists $\lambda \in \mathbb{R}^n_+$ such that

$$\frac{1}{\nabla F(x^*)} = \sum_{h\in H} \lambda_h \nabla u_h(g^*)$$

where $\frac{1}{\nabla F(x^*)} := \left(\frac{1}{f_1'(x_1^*)}, \ldots, \frac{1}{f_1'(x_m^*)} \right)$ and $g^* = F(x^*)$.

3 The Public Goods Games

In this section, we formulate voluntary contribution to public goods as a strategic form game. Let x_i^h be a contribution (resource input) to public good i of agent h, and $x^h = (x_1^h, \dots, x_m^h)$ be a contribution vector of agent h.

Given total resources $\bar{e}$, the set of feasible resource allocation vectors is denoted by $T(\bar{e})$;

$$T(\bar{e}) := \left\{ t = (t_h)_{h \in H} \in \mathbb{R}_+^n \;\middle|\; \sum_{h \in H} t_h = \bar{e} \right\}.$$

Given the resource allocation $t \in T(\bar{e})$, a contribution x^h of agent h is feasible if $\sum_i x_i^h \leq t_h$. Let $X_h(t_h)$ be a set of feasible contribution of agent h;

$$X_h(t_h) := \left\{ x^h = (x_1^h, \dots, x_m^h) \in \mathbb{R}_+^m \;\middle|\; \sum_{i=1}^m x_i^h \leq t_h \right\}.$$

Let $X(t)$ be a set of profiles of feasible contributions; $X(t) := \prod_{h \in H} X_h(t_h)$. For any $x \in X(t)$, let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ be a total contribution (aggregate resource inputs); $\bar{x} := \sum_{h \in H} x^h$. For any $x = (x^h)_{h \in H} \in X(t)$, let x^{-h} be the list $(x^k)_{k \in H \setminus \{h\}}$.

The payoff functions are defined as follows:

$$U_h(x) := u_h \circ F(\bar{x}) = u_h\left(f_1\left(\sum_{j \in H} x_1^j \right), \dots, f_m\left(\sum_{j \in H} x_m^j \right) \right) \quad \text{for } h \in H$$

$$U(x) := \prod_{h \in H} U_h(x).$$

Definition 3. Given $t \in T(\bar{e})$, a *public goods game* $\mathcal{G}_t(\mathcal{E})$ consists of the set of player H, the set of strategy profiles $X(t)$, and the payoff function U:

$$\mathcal{G}_t(\mathcal{E}) = (H, X(t), U).$$

Definition 4. A strategy profile $x \in X(t)$ is *Nash equilibrium* of $\mathcal{G}_t(\mathcal{E})$ if for any $h \in H$ and any $y^h \in X_h(t_h)$

$$U_h(x) \geq U_h(y^h, x^{-h})$$

For consistency, we shall prove the existence of Nash equilibrium of our games via a standard fixed point argument.

Theorem 1. *There exists a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$ for any $t \in T(\bar{e})$.*

Proof . Note that $U_h(x)$ is continuous and strictly quasi concave. Followig standard arguments, we define the best reply functions such that for any $x \in X(t)$,

$$b_h(x^{-h}) := \arg\max\{\ U_h(y^h, x^{-h}) \mid y^h \in X_h(t_h)\ \}.$$
$$B(x) := \prod_{h \in H} b_h(x^{-h}).$$

$B : X(t) \to X(t)$ is continuous and $X(t)$ is compact and convex. By Brouwer's fixed point theorem, there exists $x \in X(t)$ such that $x = B(x)$. Then x is Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. $\square$

Remark 2. In general, a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$ is *not* unique for $t \in T(\bar{e})$.

4 Nash Equilibrium Allocation

The goal of this section is to provide the necessary condition for an allocation to be attainable through Nash equilibrium given a resource distribution (Lemma 1). We first introuduce the concept of *Nash equilibrium allocation* which is the one corresponding to Nash equilibrium of the public goods game.

Definition 5. Given $t \in T(\bar{e})$, a public goods bundle $g = (g_1, \ldots, g_m)$ is *Nash equilibrium allocation* of $\mathcal{G}_t(\mathcal{E})$ if $g = F(\bar{x})$ for some Nash equilbrium x of $\mathcal{G}_t(\mathcal{E})$ where $\bar{x} = \sum_{h \in H} x^h$.

Given $g \in \mathcal{A}(\mathcal{E})$, we define constrained feasible set $\mathcal{A}_g(\mathcal{E})$ as follows:

$$\mathcal{A}_g(\mathcal{E}) := \{g' \in \mathcal{A}(\mathcal{E}) \mid g'_i \geq g_i \text{ for all } i\ \}.$$

Note that $\mathcal{A}(\mathcal{E})$ is compact in $\mathbb{R}^m$ and so is $\mathcal{A}_g(\mathcal{E})$ for any $g \in \mathcal{A}(\mathcal{E})$. Given $g \in \mathbb{R}^m_+$ and $J \subset \{1, \ldots, m\}$, we define $g(J) = (g_1(J), \ldots, g_m(J)) \in \mathbb{R}^m_+$ as follows;

$$g_i(J) := \begin{cases} g_i & (i \notin J) \\ 0 & (i \in J). \end{cases}$$

For notational simplicity, we write $g(i)$ instead of $g(\{i\})$.

Let us define the key concep in the following discussion.

Definition 6. Let $g \in \mathcal{A}(\mathcal{E})$ and $I := \{i \mid g_i > 0\}$. g satisfies *condition* **(M)** if for any $i \in I$, there exists $h \in H$ such that $u_h(g) \geq u_h(g')$ for any $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$.

If an allocation g satisfies (M), then g may be regarded as a common solution of several utility maximization problem over some constrained feasible set $\mathcal{A}_{g(i)}(\mathcal{E})$.

The next lemma states that (M) is the necessary condition for Nash equilibrium allocation.

Lemma 1. *Given $t \in T(\bar{e})$, let g be a Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$. Then g satisfies condition (M).*

Proof . If g is a Nash equilibrium allocation, then $u_h(g) \geq u_h(g')$ for any $g' \in \mathcal{A}_{g^h}(\mathcal{E})$ and for all $h \in H$, where $g^h := F(\bar{x} - x^h)$ and $x \in X(t)$ is Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. Let us define $I_h := \{i \mid g_i > g_i^h\}$.

We will show that g is a maximizer of u_h over $\mathcal{A}_{g(I_h)}(\mathcal{E})$. Suppose $u_h(g') > u_h(g)$ for some $g' \in \mathcal{A}_{g(I_h)}(\mathcal{E})$. Then $g(\varepsilon) := \varepsilon \cdot g' + (1-\varepsilon) \cdot g \in \mathcal{A}_{g^h}(\mathcal{E})$ for suffiently small ε and $u_h(g(\varepsilon)) > u_h(g)$ by quasi concavity of u_h, which is contradiction.

Since $g_i = 0$ for $i \notin \bigcup_h I_h$, $I = \bigcup_h I_h$. Therefore for any $i \in I$, there exists $h \in H$ such that $i \in I_h$, which implies the result. □

5 Optimality of Nash equilibrium Allocations

In this section, we investigate optimality of Nash equilibrium allocations. Unfortunately, Nash equlibrium allocations are not Pareto optimal in general even if private resources are non-consumable.

Example 1 (The case of $m = 3, n = 2$)**.** Suppose $H = \{a, b\}$. The initial endowments of resources are $e_a = 2, e_b = 1$. The utility functions of each agent are $u_a(g_1, g_2, x_3) = -(g_1-6)^2-(g_2-6)^2-(g_3-3)^2$ and $u_b(g_1, g_2, g_3) = -(g_1-6)^2-(g_2-3)^2-(g_3-6)^2$. The production functions are $f_1(x) = f_2(x) = f_3(x) = x$ (identity mapping). Note that the utility functions are smooth, strictly concave, and increasing over $\mathcal{A}(\mathcal{E})$. Then $g^* = (g_1^*, g_2^*, g_3^*) = (1, 1, 1)$ is Nash equilibrium allocation, which is Pareto-dominated by $(1 + 2\varepsilon, 1 - \varepsilon, 1 - \varepsilon)$ for sufficiently small $\varepsilon > 0$. Note that (1) both agents have different preferences, (2) do not contribute their resources to all the public goods in the equilibrium, [2] and (3) the number of public goods is 3, which is greater than 2. □

When is a Nash equilibrium allocation Pareto optimal? We provide several sufficient conditions for Nash equilibrium allocation to be Pareto optimal.

[2]The equilibrium strategies of each agent at g^* are $x_a^* = (1, 1, 0), x_b^* = (0, 0, 1)$, respectively.

Proposition 1. *If all the agents' preferences for public goods bundle are identical, then a (unique) Nash equilibrium allocation of* $\mathcal{G}_t(\mathcal{E})$ *for any* $t \in T(\bar{e})$ *is Pareto optimal.*

Proof . Suppose all the agents' preferences are identical. Then there exists the (representative) utility function $u(g)$ such that $u(g) = u_h(g)$ for all $h \in H$. Let g^* be Nash equilibrium allocation and $I := \{i \mid g_i^* > 0\}$. By Lemma 1, g^* satisfies condition (M). Under the identical preference assumption, (M) is equivalent to that $u(g^*) \geq u(g)$ for any $g \in \mathcal{A}_{g^*(i)}(\mathcal{E})$ and any $i \in I$. We will show that $u(g^*) \geq u(g)$ for any $g \in \mathcal{A}(\mathcal{E})$. Suppose the contrary. Then there exists $g' \in \mathcal{A}(\mathcal{E})$ such that $u(g') > u(g^*)$. It follows from (M) that $g' \notin \mathcal{A}_{g^*(i)}(\mathcal{E})$ for any $i \in I$, which implies $g'_j < g^*_j$ for some $j \notin I$. It follows from the fact that $g^*_j = 0$ for $j \notin I$ that $g' \notin \mathcal{A}(\mathcal{E})$, which is contradiction. Therefore $g^* \in \arg\max\{u(g) \mid g \in \mathcal{A}(\mathcal{E})\ \}$. The maximizer g^* must be unique by strict quasi concavity of $u(g)$. □

The next two results are extensions of the sufficient conditions for Pareto optimality in Nakayama (1980) to the public goods economy. Note that our results are independent of the number of agents.

Proposition 2. *Suppose* x *is Nash equilibrium of* $\mathcal{G}_t(\mathcal{E})$ *and* g *is the corresponding Nash equilibrium allocation. If there exists* $h \in H$ *such that* $x_i^h > 0$ *for all* $i \in I = \{i \mid g_i > 0\ \}$*, then* g *is Pareto optimal.*

Proof . Let a be the agent such that $x_i^a > 0$ for all $i \in I$. By the proof of Lemma 1, $u_a(g) \geq u_a(g')$ for any $g' \in \mathcal{A}_{g(I)}(\mathcal{E})$. It follows from $\mathcal{A}_{g(I)}(\mathcal{E}) = \mathcal{A}(\mathcal{E})$ that there is no $g' \in \mathcal{A}(\mathcal{E})$ that Pareto dominates g. □

Proposition 3. *In the case of* $m = 2$*, any Nash equilibrium allocation of* $\mathcal{G}_t(\mathcal{E})$ *is Pareto optimal for any* $t \in T(\bar{e})$*.*

Proof . Let g be Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$ which is not Pareto optimal. Then there exists g' such that $u_h(g') > u_h(g)$ for all $h \in H$. Suppose $g \in \mathbb{R}^2_{++}$. Without loss of generality, we assume $g' \in \mathcal{A}_{g(2)}(\mathcal{E})$. This implies $g \notin \arg\max u_h(g'')$ subject to $g'' \in \mathcal{A}_{g(2)}(\mathcal{E})$ for any $h \in H$, which contradicts Lemma 1. If $g_1 = 0$ and $g_2 > 0$, then $g' \in \mathcal{A}_{g(2)}(\mathcal{E})$. So $g \notin \arg\max u_h(g'')$ subject to $g'' \in \mathcal{A}_{g(2)}(\mathcal{E})$ for any $h \in H$, which contradicts Lemma 1. If $g_1 > 0$ and $g_2 = 0$, then $g' \in \mathcal{A}_{g(1)}(\mathcal{E})$. So $g \notin \arg\max u_h(g'')$ subject to $g'' \in \mathcal{A}_{g(1)}(\mathcal{E})$ for any $h \in H$, which contradicts Lemma 1. □

6 Decentralization of Pareto Optimal Allocation

In this section, we consider implementability of Pareto optimal allocations via Nash equilibrium under resource redistribution policy (transfer). At first, we define the concept of Nash equilibrium allocation with transfer.

Definition 7. A public goods bundle g is *Nash equilibrium allocation with transfer* in $\mathcal{E}$ if there exists $t \in T(\bar{e})$ such that g is Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$.

In the following theorem, we provide characterization of the allocations attainable through Nash equilibrium with transfer.

Theorem 2. *Let g be a feasible allocation of public goods in $\mathcal{E}$. Then g is Nash equilibrium allocation with transfer if and only if g satisfies condition (M).*

Proof . The "only if" part of the theorem is direct consequence of Lemma 1. We will show the converse. Suppose g is a feasible allocation in $\mathcal{E}$. Then there exists resource inputs $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in C(\bar{e})$ such that $g = F(\bar{x})$. Suppose that for every i there exists $h_i \in H$ such that $u_{h_i}(g) \geq u_{h_i}(g)$ for any $g \in \mathcal{A}_{g(i)}(\mathcal{E})$. We define the resource allocation vector $t = (t^h)_{h \in H}$ in the following way;

$$t_h = \sum_{h=h_i} \bar{x}_i$$

where $t_h = 0$ if $h \neq h_i$ for all i. Given t, consider the strategy profile $x = (x^h)_{h \in H} \in X(t)$ such that

$$x_j^h = \begin{cases} \bar{x}_i & \text{if } j = i \text{ and } h = h_i \\ 0 & \text{otherwise.} \end{cases}$$

We show that the strategy profile x is a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. For this, it is sufficient to show that x^h is best reply to x^{-h} for $h \in H$. Fix $h \in H$ arbitrarily. If $h = h_i$ for some i, then the feasible set of public goods by agent h with his resources t_h is $\mathcal{A}_{g(i)}(\mathcal{E})$, given the contributions x^{-h} of other agents. By the assumption, $u_h(g) \geq u_h(g')$ for $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$. Therefore x^h maximize $U_h(x^h, x^{-h})$; x^h is a best reply to x^{-h}. If $h \neq h_i$ for all i, then $x^h = (0, \ldots, 0)$, $t_h = 0$ and $X_h(t_h) = \{(0, \ldots, 0)\}$. It is clear that x^h is best reply to x^{-h}. So x is a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. Therefore g is a Nash equilibrium allocation with transfer in $\mathcal{E}$. □

Note that (M) is the necessary condition for Nash equilibrium allocation by Lemma 1. Theorem 2 states that (M) is also sufficient condition for Nash equilibrium allocation with transfer.

We consider the following question,"When is any Pareto optimal allocation Nash equilibrium allocation with transfer?" By Theorem 2, the answer is as follows: "If any Pareto optimal allocation satisfies (M), then the second welfare theorem holds". But, in general, this property does not hold in this economy: some Pareto optimal allocation may not be Nash equilibrium allocation with transfer. The next example shows this fact.

Example 2 (The case of $m = n = 3$)**.** Suppose $H = \{a, b, c\}$. The aggregate initial resources is $\bar{e} = 3$. We assume identical linear production functions, $f_1(x) = f_2(x) = f_3(x) = x$. The utility functions of each agents are as follows:

$$\begin{aligned} u_a(g_1, g_2, x_3) &= -\frac{1}{2}\{(g_1 - 6)^2 + (g_2 - 6)^2 + (g_4 - 3)^2\} \\ u_b(g_1, g_2, x_3) &= -\frac{1}{2}\{(g_1 - 3)^2 + (g_2 - 5)^2 + (g_3 - 5)^2\} \\ u_c(g_1, g_2, x_3) &= -\frac{1}{2}\{(g_1 - 5)^2 + (g_2 - 3)^2 + (g_3 - 5)^2\}. \end{aligned}$$

Note that the utility functions are smooth, strictly concave, and increasing over $\mathcal{A}(\mathcal{E})$.

Then $g^* = (1, 1, 1)$ is Pareto optimal. For $\nabla u_a(g^*) = (5, 5, 2), \nabla u_b(g^*) = (2, 4, 4)$, $\nabla u_c(g^*) = (4, 2, 4)$, and $\nabla F(g^*) = (1, 1, 1)$, so $\frac{1}{\nabla F(g^*)} = \frac{1}{14}\nabla u_a(g^*) + \frac{3}{28}\nabla u_b(g^*) + \frac{3}{28}\nabla u_c(g^*)$ (See Remark 1).

On the other hand, g^* does not satisfy the conodition (M). Let $\alpha = (\varepsilon, \varepsilon, -2\varepsilon)$, $\beta = (-\varepsilon, 2\varepsilon, -\varepsilon)$, and $\gamma = (2\varepsilon, -\varepsilon, -\varepsilon)$ for sufficiently small $\varepsilon > 0$. Then the allocations $g^* + \alpha, g^* + \beta$, and $g^* + \gamma$ belong to $\mathcal{A}_{g^*(3)}$. Furthermore $u_a(g^* + \alpha) > u_a(g^*)$, $u_b(g^* + \beta) > u_b(g^*)$, and $u_c(g^* + \gamma) > u_c(g^*)$, because $\nabla u_a(g^*) \cdot \alpha > 0$, $\nabla u_b(g^*) \cdot \beta > 0$, and $\nabla u_c(g^*) \cdot \gamma > 0$. That is, $g^* \notin \arg\max\{u_h(g) \mid g \in \mathcal{A}_{g^*(3)}\}$ for $h = a, b, c$. By Theorem 2, it is impossible for g^* to be supported as Nash equilibrium allocation. $\square$

To understand the situation more deeply, we look at (M) from a slightly different angle. In Example 2, all agents agree to reduce the public good 3 ($\alpha_3, \beta_3, \gamma_3 < 0$). But there is no common plan which all the agents agree, because g^* is Pareto optimal. Thus the preferences among the agents may be diverse on the local area $\mathcal{A}_{g^*(3)}$. Based on this idea, we paraphrase (M) in terms of diversity of preferences of agents.

Definition 8. Suppose g is Pareto optimal. The economy $\mathcal{E}$ satisfies *Local Non-Diversity condition at g (g-LND)* if there exists no public good i such that for every $h \in H$, there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$. The economy $\mathcal{E}$ satisfies *Local Non-Diversity condition (LND)* if $\mathcal{E}$ satisfies g-LND for all Pareto optimal allocation g.

Remark 3. In the above definition, g' may be different from every agent (g' may depend on the index of the agents).

The following theorem states that the second welfare theorem holds in the economy such that the agents' prefereces for public goods are not so diverse at Pareto optimal allocations.

Theorem 3. *Any Pareto optimal allocation is Nash equilibrium allocation with transfer in the economy $\mathcal{E}$ if and only if $\mathcal{E}$ satisfies LND.*

Proof . Suppose $\mathcal{E}$ satisfies LND. Let g be a Pareto optimal allocation. If g is not Nash equilibrium allocation with transfer, then it follows from Theorem 2 that for every $h \in H$ there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$ for some i, which contradicts LND.

Suppose $\mathcal{E}$ does not satisfy LND. Then for every $h \in H$, there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$ at some Pareto optimal g and for some i. This implies that g is not maximizer of $u_h(g)$ over $\mathcal{A}_{g(i)}(\mathcal{E})$ for all $h \in H$. It follows from Theorem 2 that g is not Nash equilibrium allocation with transfer. □

Corollary 1. *Suppose that one of the following conditions holds:*

(i) all the agents' preferences for public goods bundle are identical,

(ii) $m = 2$.

Then any Pareto optimal allocation in $\mathcal{E}$ is Nash equilibrium allocation with transfer in $\mathcal{E}$.

7 An Extension

In this section, we extend our model to the general one including consumable private resources. The resource-consumption of agent h is denoted by z_h. To avoid the boundary problem, we assume $z_h \in \mathbb{R}$ (negative consumption is permitted). Let denote $z = (z_h)_{h \in H}$. The extended utility function of agent h is denoted by $\tilde{u}_h(z_h, g)$. In this section, we assume quasi linearity of preferences and differentiability of utility and production functions.

Assumption 2.

(i) The extended utility function of every agent is quasi linear, i.e., $\tilde{u}_h(z_h, g) = z_h + v_h(g)$ where v_h is increasing, concave, and C^1-function.

(ii) f_i is increasing, concave, and C^1 function, and $f_i(0) = 0$ for $i = 1, \ldots, m$

An extended public goods economy $\tilde{\mathcal{E}}$ is a list $(H, (\tilde{u}_h, e_h)_{h\in H})$. An extended public goods game is defined as $(H, X, \tilde{U})$ where $\tilde{U}(x) := \prod_{h\in H} \tilde{U}_h(x)$ and $\tilde{U}_h(x) := \tilde{u}_h(t_h - \sum_{i=1}^m x_i^h, F(\bar{x}))$.

An allocation $(z, g) \in \mathbb{R}^n \times \mathbb{R}^m_+$ is feasible if $\sum_h z_h + x \leq \bar{e}$ and $g \leq F(x)$ for some $x \in C(\bar{e})$. The set of feasible allocations in $\tilde{\mathcal{E}}$ is denoted by $\mathcal{A}(\tilde{\mathcal{E}})$. The concepts of Pareto optimality and Nash equilibrium allocation are defined similarly.

Lemma 2. *Let $x^* = (x^{h*})_{h\in H}$ be a Nash equilibrium of $\mathcal{G}_t(\tilde{\mathcal{E}})$, (z^*, g^*) be a Nash equilibrium allocation corresponding to x^*, and $\bar{x}^*$ be the aggregation of x^*. Then (z^*, g^*) satisfies the following conditions;*

(i) $\frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) \leq 1$ for all $h \in H$ and all $i = 1, \ldots, m$,

(ii) for any $i \in I$ there exists h_i such that

$$\frac{\partial v_{h_i}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) = 1$$

where $I := \{i | g_i^* > 0\}$.

Proof . The Nash equilibrium $x^* = (x^{h*})_{h\in H}$ is a solutions of the following maximization problems:

$$\max_{x^h \in \mathbb{R}^m_+} \tilde{U}_h(x^h, x^{-h*}).$$

The Karush-Kuhn-Tucker conditions implies

$$\frac{\partial \tilde{U}_h}{\partial x_i^h}(x^*) \leq 0 \ \ \forall i,$$

with equality for $x_i^{h*} > 0$. That is equivalent to

$$\frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) \leq 1 \ \ \forall i,$$

with equality for $g_i^* > 0$. Furthermore $g_i^* > 0$ implies that there exists h_i who contributes the public good i, i.e., $x_i^{h_i*} > 0$. Thus $\frac{\partial \tilde{U}_{h_i}}{\partial x_i^h}(x^*) = 0$, which is equivalent to $\frac{\partial v_{h_i}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i) = 1$. □

It is well known that when private resource is consumable, Nash equilibrium allocation is not Pareto optimal even if there is only one public good. This fact shows difficulty in achieving optimal allocation between private and public goods. Since we focus on resource allocation among the public goods, we will introduce another criterion of optimality to evaluate a resource allocation among public goods. Given an aggregate resource of inputs for public goods, w, define a constrained feasible set with respect to w, $\mathcal{A}^w(\mathcal{E})$, such as

$$\mathcal{A}^w(\tilde{\mathcal{E}}) := \left\{ (z, g) \in \mathbb{R}^n \times \mathbb{R}^m_+ \;\middle|\; \sum_{h \in H} z_h + w \leq \bar{e} \text{ and } g \leq F(y) \text{ for some } y \in C(w) \right\}.$$

Definition 9. Let (z, g) be a feasible allocation, x be an input vector such that $g = F(x)$, and w be an aggregation of x, i.e., $w = \sum_i x_i$. Then (z, g) is *constrained Pareto optimal* if there exists no $(z', g') \in \mathcal{A}^w(\tilde{\mathcal{E}})$ such that $\tilde{u}_h(z'_h, g') \geq \tilde{u}_h(z_h, g)$ for all $h \in H$ with at least one strict inequality.

The next lemma states that constrained optimality may be formulated as some maximization problem under the assumption of quasi linearity of preferences.

Lemma 3. *Let (z^*, g^*) be a feasible allocation such that $g^* = F(x^*)$. Then (z^*, g^*) is constrained Pareto optimal if and only if*

$$g^* \in \arg\max \left\{ \sum_{h \in H} v_h(g) \;\middle|\; (z^*, g) \in \mathcal{A}^{w^*}(\tilde{\mathcal{E}}) \right\}$$

where $w^* := \sum_{i=1}^m x_i^*$.

Proof . Under the assumption of quasi linearity of utilities, the feasible allocation (z^*, g^*) is constrained Pareto optimal if and only if

$$(z^*, g^*) \in \arg\max \left\{ \sum_{h \in H} \tilde{u}_h(z_h, g) \;\middle|\; (z, g) \in \mathcal{A}^{w^*}(\tilde{\mathcal{E}}) \right\}.$$

Since $\sum_h z_h = \bar{e} - \bar{x}^*$ for any $(z, g) \in \mathcal{A}^{w^*}$, that is equivalent to

$$g^* \in \arg\max \left\{ \sum_{h \in H} v_h(g) \;\middle|\; (z^*, g) \in \mathcal{A}^{w^*}(\tilde{\mathcal{E}}) \right\}. \quad \square$$

Note that this maximization problem is concave programming. If $x^* \neq 0$, then the Karush-Kuhn-Tucker conditions for this problem is neccesary and sufficient for the maximization. That is, $g^* \in \arg\max\{\sum_{h\in H} v_h(g) \mid (z^*, g) \in \mathcal{A}^{\bar{x}^*}(\tilde{\mathcal{E}})\}$ if and only if there exists $\mu^* > 0$ such that (g^*, μ^*) satisfies $g^* = F(x^*)$, $\sum_i x_i^* = \bar{x}^*$, and

$$\sum_{h\in H} \frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(x_i^*) \leq \mu^* \quad \forall i = 1, \ldots, m$$

with equality for $g_i^* > 0$.

Proposition 1 can be extended to the case of consumable resource.

Proposition 4. *If all the agents' preferences for public goods bundle are identical, then Nash equilibrium allocation is constrained Pareto optimal.*

Proof . When all the preferences of the agents are identical, it follows from Lemma 2 that $\frac{\partial v_{h_i}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i) \leq 1$ for all $h \in H$ with equality for $i \in I$. Then the Karush-Kuhn-Tucker conditions for the maximization problem in Lemma 3 are satisfied when we set the multiplier $\mu^* = n$. □

We extend Proposition 2 to the consumable resource case under the more stringent condition. It is insufficient for constrained optimality of Nash equilibrium allocation that there exists an agent who contributes their resources to all public goods. Nash equilibrium allocation is constrained optimal if all agents contribute their resources to all public goods.

Proposition 5. *Suppose x^* is Nash equilibrium of $\mathcal{G}_t(\tilde{\mathcal{E}})$ and (z^*, g^*) is the corresponding Nash equilibrium allocation. If $x_i^{*h} > 0$ for all $i \in I = \{i \mid g_i^* > 0\}$ and all $h \in H$, then g^* is constrained Pareto optimal.*

Proof . Suppose $x_i^{*h} > 0$ for all $i \in I = \{i \mid g_i^* > 0\}$ and all $h \in H$. Then for all $h \in H$,

$$\frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i) \leq 1 \ \ \forall i$$

with equaility for $i \in I$. Thus

$$\sum_{h\in H} \frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(x_i^*) \leq n \quad \forall i,$$

with equality for $i \in I$. This implies constrained optimality of (z^*, g^*). □

What happen when some agent does not contribute to all public goods? Then the Nash equilibrium allocation is not constrained optimal in many cases: It is "dense" property that a Nash equilibrium allocation is constrained Pareto suboptimal. See Appendix for a formal definition of the space of the economy and its topology.

Proposition 6. *Let (z^*, g^*) be Nash equilibrium allocation in $\tilde{\mathcal{E}}$ and $\bar{x}^*$ be the aggregate inputs for g^*. Suppose that there exists $h_i \in H$ such that $\frac{\partial v_{h_i}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) < 1$ for some $i \in I$ and $|I| \geq 2$ where $I := \{i | g_i^* > 0\}$. Then the set of the economy in which (z^*, g^*) is constrained Pareto suboptimal is dense in the space of the economy.*

Proof . Suppose the Nash equilibrium allocation (z^*, g^*) is constrained Pareto optimal and $\frac{\partial v_{h_i}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) < 1$ for some agent h_i and some $i \in I$ in the economy $\tilde{\mathcal{E}}$. We will show that any sufficiently small perturbation of the utility function v_{h_i} makes (z^*, g^*) still Nash equilibrium allocation, but constrained suboptimal. We consider the following perturbation of v_{h_i}:

$$v_{h_i}^{\varepsilon}(g) := v_{h_i}(g) + \varepsilon \cdot g_i \quad \text{for sufficiently small } \varepsilon > 0.$$

Then $\frac{\partial v_{h_i}^{\varepsilon}}{\partial g_i}(g) \neq \frac{\partial v_{h_i}}{\partial g_i}(g)$ and $\frac{\partial v_{h_i}^{\varepsilon}}{\partial g_j}(g) = \frac{\partial v_{h_i}}{\partial g_j}(g)$ for $j \neq i$. It follows from $\frac{\partial v_{h_i}^{\varepsilon}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) < 1$ for sufficiently small ε that (z^*, g^*) is a Nash equilibrium allocation under the perturbed utility $v_{h_i}^{\varepsilon}$. On the other hand, because (z^*, g^*) is constrained optimal under the unperturbed utilty v_{h_i},

$$\sum_{h \in H} \frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(x_i^*) = \sum_{h \in H} \frac{\partial v_h}{\partial g_j}(g^*) \cdot f_j'(x_j^*) \quad \text{for } j \in I, j \neq i.$$

It follows from $\frac{\partial v_{h_i}^{\varepsilon}}{\partial g_i}(g^*) \neq \frac{\partial v_{h_i}}{\partial g_i}(g^*)$ that (z^*, g^*) is not constrained Pareto optimal under the perturbed utility $v_{h_i}^{\varepsilon}$. □

Next, we discuss implementability of constrained optimal allocation as Nash equilibrium allocation with transfer. We begin with extension of Theorem 2 to the consumable resource case. In similar to Theorem 2, necessary conditions for Nash equilibrium allocation in $\tilde{\mathcal{E}}$ are also sufficient for Nash equilibrium with transfer. The proof strategy is similar to that of Theorem 3 in essence.

Theorem 4. *Let (z^*, g^*) be a feasible allocation and $\bar{x}^*$ be an input vector for g^*, i.e., $g^* = F(\bar{x}^*)$. Then (z^*, g^*) is a Nash equilibrium allocation with transfer if and only if*

(i) $\frac{\partial v_h}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) \leq 1$ *for all $h \in H$ and all $i = 1, \ldots, m$,*

(ii) for any $i \in I$ there exists h_i such that

$$\frac{\partial v_{h_i}}{\partial g_i}(g^*) \cdot f_i'(\bar{x}_i^*) = 1$$

where $I := \{i | g_i^* > 0\}$.

Proof . ("if" part) We define the resource allocation vector $t = (t_h)_{h\in H}$ as

$$t_h = \begin{cases} z_h^* + \bar{x}_i^* & \text{if } h = h_i \text{ for some } i \\ z_h^* & \text{otherwise.} \end{cases}$$

Construct a strategy profile $(x^{*h})_{h\in H}$ in the following way:

$$x_j^{*h} = \begin{cases} \bar{x}_i^* & \text{if } h = h_i \text{ and } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x^{*h})_{h\in H}$ is Nash equilibrium of the extended public goods game whose outcome is (z^*, g^*).

("only if" part) If $\frac{\partial v_h}{\partial g_i}(g^*)\cdot f_i'(\bar{x}_i^*) > 1$ for some h and i, then z_h^* is not supported as a best reply. If $\frac{\partial v_h}{\partial g_i}(g^*)\cdot f_i'(\bar{x}_i^*) < 1$ for any $h \in H$ and some $i \in I$, then g_i^* is not supported as Nash equilibrium allocation. □

In similar to Proposition 6, a constrained optimal allocation is not Nash equilibrium allocation with transfer in many cases.

Proposition 7. *The set of the economy in which a constrained Pareto optimal allocation is not Nash equilibrium allocation with transfer is dense in the space of the economy.*

Proof . Suppose the constrained optimal allocation (z^*, g^*) is Nash equilibrium allocation with transfer. Then we consider the perturbation of the production functions as follows: for sufficiently small $\varepsilon_i > 0$,

$$f_i^{\varepsilon}(x) := f_i(x) + \varepsilon_i \cdot x \quad \text{for all } i \in I := \{i | g_i^* > 0\}.$$

If we set $(\varepsilon_i)_{i\in I}$ such that

$$\frac{\varepsilon_i}{\varepsilon_j} = \frac{\sum_{h\in H} \frac{\partial v_h}{\partial g_j}(g^*)\cdot f_j'(x_j^*)}{\sum_{h\in H} \frac{\partial v_h}{\partial g_i}(g^*)\cdot f_i'(x_i^*)} \quad \forall i, j \in I,$$

then

$$\sum_{h\in H} \frac{\partial v_h}{\partial g_i}(g^*)\cdot f_i^{\varepsilon_i \prime}(x_i^*) = \sum_{h\in H} \frac{\partial v_h}{\partial g_j}(g^*)\cdot f_j^{\varepsilon_j \prime}(x_j^*) \quad \forall i, j \in I.$$

Thus (z^*, g^*) is constrained Pareto optimal in the perturbed economy.

On the other hand, since (z^*, g^*) is also Nash equilibrium allocation with transfer, for any $i \in I$, $\frac{\partial v_h}{\partial g_i}(g^*)\cdot f_i'(x_i^*) = 1$ for some agents by Theorem 4. It follows from $\varepsilon_i > 0$ that $\frac{\partial v_h}{\partial g_i}(g^*)\cdot f_i^{\varepsilon_i \prime}(x_i^*) > 1$ for the agents. Thus (z^*, g^*) is not Nash equilibrium with transfer under the perturbation of production functions. □

8 Concluding Remarks

We study Nash equilibrium allocations of voluntary contribution of non-consumable private resources in an economy with multiple public goods. We give us the several sufficient conditions for Nash equilibrium allocations to be Pareto optimal (the first welfare theorem). We provide us with also the necessary and sufficient condition for Pareto optimal allocations to be Nash equilibrium allocations with transfer (the second welfare theorem). We point out that both the first and the second welfare theorems always hold when all the agents' preferences for public goods bundle are identical, or the number of public goods is two (Proposition 1, 3 and Corollary 1). The economy with identical preferences or with two public goods may be a special case. We cannot assume these settings without loss of generality.

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A Appendix: the space of the economy

Let $\mathscr{C}^1(K_\alpha)$ be a set of C^1-functions on a compact set K_α which is a subset of Euclidean space $\mathbb{R}^\alpha$, and $\|\cdot\|$ be a norm on $\mathscr{C}^1(K_\alpha)$;

$$\|f\| := \sup_{x\in K} |f(x)| + \sup_{x\in K} |\nabla f(x)|_\alpha$$

where $|\nabla f(x)|_\alpha := \max\Big\{\dfrac{\partial f}{\partial x_1}(x), \ldots, \dfrac{\partial f}{\partial x_\alpha}(x)\Big\}$. Let $\mathscr{U} \subset \mathscr{C}^1(K_m)$ be a set of utility functions;

$$\mathscr{U} := \big\{v \in \mathscr{C}^1(K_m) \mid v \text{ is increasing and concave.}\big\}$$

Let $\mathscr{F} \subset \mathscr{C}^1(K_1)$ be also a set of production functions;

$$\mathscr{F} := \big\{f \in \mathscr{C}^1(K_1) \mid f \text{ is increasing, concave, and } f(0) = 0.\big\}$$

Then we define the space of economies as $\mathscr{U}^n \times \mathscr{F}^m$ where $\mathscr{U}^n$(resp.$\mathscr{F}^m$) is n (resp. m) fold of $\mathscr{U}$ (resp. $\mathscr{F}$).