Managing Strategic Buyers*

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May 13, 2009

Abstract

We consider the problem of a monopolist who must sell her inventory before some deadline, facing n buyers with independent private values. The monopolist posts prices but has no commitment power. When there is only one unit and only a few buyers, she essentially posts unacceptable prices up to the very end, at which point prices collapse to a "reservation price" that exceeds marginal cost. When there are many buyers, the seller abandons this reservation price in order to more effectively screen buyers. Her optimal policy then replicates a Dutch auction, with prices decreasing continuously over time. Prices with many units are analogous, except that the price jumps upward after each sale.

*We thank Bruno Biais, Jeroen Swinkels and Rakesh Vohra for useful comments, and thank the National Science Foundation for financial support.

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1 Introduction

1.1 Revenue Management

The revenue management literature addresses the pricing of goods sharing three essential characteristics: (i) there is a fixed quantity of resource for sale, (ii) the resource is perishable (i.e., there is a time after which it is valueless), and (iii) consumers have heterogeneous valuations. Revenue management is practiced in a variety of industries, including airlines, apparel, entertainment, freight, hotels, pipelines and rental cars. In the standard model, each buyer must be served immediately upon arrival or forever lost, and the only relevant price from a buyer's point of view is the current one.[1] In contrast, this paper examines revenue management with buyers who strategically choose their time of purchase.

It is often argued that the problem of revenue management with strategic buyers but no commitment is equivalent to that of a durable-goods monopolist.[2] A good example of price dynamics consistent with such a claim is provided by the cruise-line industry (see Talluri and van Ryzin [28, pp. 560–561] or Coleman, Meyer and Scheffman [13]), where significant, last-minute discounts are common and customers often wait in order to purchase at deep discounts. Similarly, last-minute deals are sold, often through a variety of intermediaries, by theaters and firms in the travel industry, supposedly to clear inventory diversely known as deadwood (entertainment industry) or distressed inventory (airline industry).

However, unlike for a durable goods monopolist, a revenue-managing seller's profit does not tend to zero as price-revision opportunities become more frequent. To see why, notice that a seller with two opportunities to set prices must get a payoff at least as large as a seller with only one opportunity, since by charging a price above the choke price at the first opportunity, she can be sure that no consumer will accept it and can then duplicate the one-period payoff. Iterating this argument, the seller's payoff must be at least as large as the (discounted) static monopoly profit. In fact, we show that the seller can also secure the revenue from a Dutch auction without a reserve price (which might be greater, or lower, than the monopoly profit).

This raises many questions. Will the monopolist be able to price discriminate? Will prices be lower, and welfare higher, than under commitment? Will prices be driven down to

[1] See Talluri and van Ryzin [28] for an introduction to revenue management, and Gershkov and Moldovanu [16] for an extension to heterogeneous objects.

[2] Talluri and van Ryzin [28, p. 365], for example, argue that customers of seemingly perishable revenue-managed goods are unlikely to buy more than one unit during the life cycle of the product, making the product effectively infinitely lived. In addition, customers for such goods are exhausted over time. Most importantly, customers are aware that the revenue management monopolist finds it difficult to commit to its price.

marginal cost, justifying the analogy with the Coase conjecture? If so, will this happen in the "twinkling of an eye," and when? Once a unit is sold, what happens to the price of the remaining units?

This paper answers these questions, considering a monopolist facing a fixed, known number of strategic buyers with unit demand. The seller can set a price in each of a finite number of instants. There is a terminal date after which any remaining unit has no value, if unsold. We concentrate on the case in which the seller cannot make commitments, in the sense that the prices she posts must be sequentially rational. We allow the time between successive offers to become arbitrarily small: we believe that setting and reacting to prices takes time, but perhaps not very much time, and so follow the durable-goods literature in considering the limit.

The existence of a deadline raises a significant technical challenge, as both the residual demand and the time remaining are payoff-relevant variables.[3] The residual demand curve depends on past prices, while the buyer's optimal strategy that determines the residual demand depends on future prices. To make some progress, we shall assume that the buyers' valuations are drawn independently and uniformly.[4] We start by considering the case in which the monopolist has one unit to sell, and explicitly describe the unique equilibrium of the game, before moving to the case of several units. Our main results are as follows:

1. *There is a trade-off between high prices and price discrimination:* The seller can lower prices very slowly, in the process keeping prices relatively high up to the end of the sales horizon, but then can only imprecisely discriminate among buyers. On the other hand, she can lower prices quickly, in the process achieving more effective price discrimination, but cannot resist subsequent price reductions that make it more tempting for buyers to wait and thus erode the benefits from the price discrimination. We show how the optimal strategy balances these two forces.

2. *If there are few buyers, keeping prices high is optimal:* The seller prefers to lower her price very slowly (although she does not sit on the choke price), to mitigate the precipitous price cuts that occur close to the deadline. Sequential rationality compels the seller to lower prices in lumpy chunks, imperfectly discriminating between buyers of different valuations, but the terminal price serves as a reserve price, as it remains bounded above the lowest buyer's valuation. When time periods are short and the price can be revised very rapidly, the price does drop in the twinkling of an eye, as in the Coase conjecture, but not to the lowest valuation, and only very near the deadline. Until then, the seller charges a price very close to the choke price, and a sale is very

[3]This means that, for a fixed residual demand curve, the seller's price will depend on the time remaining, so that the optimal strategy must be analyzed with partial difference equations.

[4]It is straightforward, at the cost of additional notation, to extend the results to power distributions of the form $F(x) = x^{\alpha}$, $\alpha > 0$. We indicate how this assumption can be further relaxed for some of the more basic results.

unlikely. Because she takes into account her inability to commit, the monopolist ends up charging prices that are higher, until near the very end, than those she would set if she could commit.

3. *If there are sufficiently many buyers, price discrimination is optimal:* In that case, the seller lowers the price steadily and orderly over time, even when price revisions get arbitrarily frequent. Prices are always lower than they would be under commitment. She gives up on her ability to keep prices high, as the price at the deadline tends to the lowest valuation (as price revisions get arbitrarily frequent). In the limit, her expected profit is then equal to the revenue in a Dutch auction with a zero reserve price. The price does drop to the lowest valuation, as in the Coase conjecture, but not in the twinkling of an eye.

4. *With multiple units, prices jump up after a sale:* With sufficiently many buyers, and for a given number of units, the price decreases continuously over time, as in the previous case. However, immediately after a unit gets sold, the price jumps up again, reflecting the new balance between supply and demand. Unlike in a static environment, the seller's profit can be strictly lower with more units to sell.

Insights into the trade-off between giving up on price discrimination but insisting on a reserve price, and achieving some price discrimination without a reserve price, can be obtained by comparing the performance of a posted price and a Dutch auction without a reserve price, when values are drawn independently according to some common distribution F (admitting a bounded density). On one hand, the loss in revenue from a Dutch auction, *relative to an optimal auction*, tends to zero at an exponential rate as the number of buyers n increases: given that the optimal reserve price r is independent of n, the event that the reserve price matters, and hence the zero-reserve-price auction sacrifices revenue, is of the order $F(r)^n$. On the other hand, the loss in revenue from a posted price tends to zero at a polynomial rate: the probability that a sale occurs under the optimal posted price goes to one as n increases, and hence the expected number of bidders whose value exceeds the posted price must increase without bound. Therefore, the difference between the second-highest value and the third-highest value, which is of the order $1/n$, is a lower bound on the difference between the price that would result from an optimal auction and the posted price. Hence, while with only one bidder, it is clearly a better idea to post a price, a Dutch auction without reserve price becomes increasingly attractive, relative to a posted price, as the number of bidders increases.

1.2 The Literature

There are four related bodies of work. First, as we have noted, a large revenue management literature (e.g., Talluri and van Ryzin [28]) has examined the case of a seller who faces sequentially-arriving buyers. The standard assumption in this literature is that the buyers are myopic, i.e. they base their decision on a comparison of the prevailing price with their valuation. This removes all consideration of whether selling to all or some of the current marginal buyers has any effect on next period's optimal price, a consideration that will play a prominent role in our analysis.[5] In contrast, our buyers remain until the good is sold and are fully strategic, constantly trading off buying the good today or waiting for a chance to buy later at a lower price. As Besanko and Winston [5] argue, mistakenly treating forward-looking customers as myopic may have an important impact on revenue (in their example, which has no capacity constraint, the seller's profit is more than halved as a consequence). Comparatively few papers discuss the case of a monopolist with scarce supply and no commitment power selling to forward-looking customers. Given the technical difficulties, Aviv and Pazgal [3] and Jerath, Netessine, and Veeraraghavan [18] do so in a model with two periods.

Second, the difficulties faced by a seller who cannot make commitments lies at the center of the durable-goods monopoly problem (e.g., Ausubel and Deneckere [2] and Gul, Sonnenschein and Wilson [17]). The durable-good setting differs from ours in its infinite horizon and in the fact that there are as many goods as buyers. The scarcity of the good in our setting changes the issues surrounding price discrimination, with the impetus for buying early at a high price now arising out of the fear that another agent will snatch the good in the meantime, rather than discounting.[6] Our seller can always do at least as well as waiting until the final period and setting the monopoly price, immediately generating considerably more commitment power than that enjoyed by a typical durable-goods monopolist.

The central dilemma facing a durable-goods monopolist is the inability to commit to not lowering future prices. The seller would like buyers to purchase at the static monopoly price now, on the strength of the promise that no lower price will be forthcoming, but faces an irresistible temptation to lower prices once she has the chance. A similar phenomenon arises in the example we present in Section 2. Unlike in the durable-goods monopoly, however, this feature is not intrinsic to our problem. Cases can arise (cf. Section B.2) in which the seller's difficulty is that she would prefer that future prices be much lower (and hence future demand brisk, in order to make current buyers more anxious to buy), but cannot commit to

[5] Several papers in this literature take into account the seller's limited capacity, providing an analysis of how the option value of postponing a sale to myopic consumers affects optimal pricing (e.g., Bitran and Mondschein [6] and Gallego and van Ryzin [15]).

[6] Kahn [19] introduces an element of scarcity within a period by examining a durable-goods monopolist with increasing costs, showing that this allows the seller to escape the zero-profit conclusion of the Coase conjecture. Similarly, a sufficiently small capacity constraint (a stylized form of increased costs) introduces scarcity within a period and allows positive profits. McAfee and Wiseman [23] show that capacity constraints have this effect even if the seller can choose to increase the capacity constraint in any period at a nominal cost. Bagnoli, Salant and Swierzbinski [4] and von der Fehr and Kühn [29] clarify the circumstances under which a Coasian firm can effectively commit, while Cho [11] examines an alternative source of commitment, arising out of the assumption that the good deteriorates while held by the seller.

lowering them.

Third, our seller can be viewed as conducting a Dutch auction without commitment. McAfee and Vincent [22] and Skreta [26] examine a seller who conducts a sequence of auctions and a sequence of optimal mechanisms, respectively. As in our case, scarcity is of paramount importance. McAfee and Vincent examine an infinite horizon with discounting, focussing attention on the sequence of reserve prices set by the seller. As agents become more patient (or equivalently as the time between auctions decreases), the seller's revenue converges to that of an optimal auction with a zero reserve price. The infinite horizon thus effectively precludes commitment to a reserve price. Skreta concentrates on a two-period model with discounting, finding that if buyers are symmetric, then it is optimal for the seller to conduct an auction in each period, with a reserve price that decreases across periods.

The most important difference between our analysis and that of McAfee and Vincent [22] or Skreta [26] is that the latter papers allow their sellers to commit to a mechanism within each period. For example, in the limit as the discount factor gets large, the sequential-mechanisms problem becomes trivial in our setting—the seller should simply wait until the last period and implement an optimal mechanism. In addition, direct mechanisms make it difficult to tell just what commitment power is allowed the seller. We typically interpret direct mechanisms not as literal descriptions of the interaction between seller and buyers, but as a way of analyzing an underlying indirect mechanism. Depending on the nature of the latter, allowing the seller to commit to a direct mechanism in each period may invest her with enormous commitment powers. As a result, we consider it important to take an indirect-mechanism approach that is specific about the actions available to the seller in each period. We must expect the results to be sensitive to the particular indirect mechanism chosen, of course, as repeated bargaining may give a different result than repeated price-posting, but we see no other way of examining commitment.[7]

Finally, our paper is related to Chen [10] and Bulow and Klemperer [8]. Chen [10] considers a model that, up to its interpretation, is equivalent to ours with one unit. Besides showing the existence of an equilibrium for general distributions, she duplicates our first main result: an equilibrium must be one of two types: either the price remains bounded away from the lowest valuation, in which case there is imperfect discrimination, or prices go down to the lowest valuation, in which case prices finely discriminate among customers. Numerical simulations provide examples of such behavior. We postpone until Section 8 a discussion of how our work is related to Bulow and Klemperer [8].

[7] For example, McAdams and Schwarz [21] examine the case of a single seller facing multiple buyers over an infinite horizon, where delay is costly for the seller but not the buyers. The buyers make offers for the object in each period while the seller decides only whether to accept an offer or proceed to the next period. They find that the seller fares worse than she would in an optimal auction unless her cost of delay is very high (allowing commitment to a first-price auction in the first period) or very low (allowing an English auction to be run over a sequence of periods).

2 An Example

This section illustrates our results with a simple example, making the following points:

- The seller without commitment can always do as well as a static monopoly. The ability to commit to an unchanging price allows the seller to earn (only) the static monopoly price, and is hence of no value.

- The seller can in general do better than a static monopoly, even without commitment power. How the seller does so depends on the number of buyers. When buyers are scarce, the seller maintains relatively high prices, effectively setting a nontrivial reserve price, at the cost of imprecise price discrimination. When buyers are plentiful, the seller effectively sets no reserve price, tolerating low prices in order to practice more precise price discrimination.

- The seller could do better with the ability to commit to arbitrary price sequences. The resulting prices may be either higher or lower than the no-commitment solution.

We consider the seller of a single good facing buyers whose valuations are drawn from the set $\{v_0, v_1, v_2, v_3\}$ with respective probabilities $\{\rho_0, \rho_1, \rho_2, \rho_3\}$, where

$$\begin{array}{ll} v_3 = 1.000 & \rho_3 = 0.09 \\ v_2 = 0.520 & \rho_2 = 0.09 \\ v_1 = 0.333 & \rho_1 = 0.09 \\ v_0 = 0 & \rho_0 = 0.73 \end{array} . \tag{1}$$

The seller's payoff is the transfer she receives from the buyers (i.e., the good is valueless to the seller) while a buyer's payoff is the difference between his valuation (iff he receives the good) and the amount he pays the seller. There are three periods.[8] In each period, the seller names a price and the buyers then simultaneously accept or reject. The game ends with the good being allocated equiprobably among those accepting if there are any (with the winning buyer paying the posted price), and the process otherwise continues to the next period (if there is one). In this example, there is no discounting.

One buyer: Static monopoly. Suppose first there is only one buyer. One possibility for the seller is to set the price equal to v_3 in each period, i.e., to set the price sequence (v_3, v_3, v_3), allowing the seller to sell the object at price v_3 if the buyer is type v_3.[9] Alternatively, the

[8] With only three possible nonzero valuations, additional periods are of no value to the seller.

[9] If the seller is concerned that an indifferent buyer might reject price v_3, a price of $v_3 + \epsilon$ induces a strict preference and ensures acceptance, for arbitrarily small ϵ. There are many other price sequences that also allow the seller to sell at price v_3, in which the last price equals v_3, and all previous prices are at least as high and rejected for sure.

seller could set price sequence (v_3, v_3, v_2) (or an equivalent sequence, such as (v_3, v_2, v_2)), selling at price v_2 if the buyer is either type v_2 or v_3.[10] Finally, the buyer might set the price sequence (v_1, v_1, v_1) (or any of a number of equivalents, such as (v_3, v_2, v_1)) and sell to the buyer at price v_1 no matter what the buyer's type.[11] Letting $\pi_1(p, q, r)$ be the payoff from naming the price sequence (p, q, r) when there is one buyer, the payoffs from these various price sequences are (with the inequality following from (1))

$$\pi_1(v_2, v_2, v_2) = (1 - (\rho_0 + \rho_1))v_2 \;>\; \begin{cases} \pi_1(v_3, v_3, v_3) = (1 - (\rho_0 + \rho_1 + \rho_2))v_3 \\ \pi_1(v_1, v_1, v_1) = (1 - \rho_0)v_1 \end{cases}.$$

The seller should accordingly set price sequence (v_2, v_2, v_2). The seller is effectively a static monopoly in this case, and the corresponding monopoly price is v_2. Similarly, a seller with commitment power could do no better than to commit to the price sequence (v_2, v_2, v_2). With only one buyer, commitment power brings no advantage.

Two buyers: Optimal reserve price. Suppose now there are two buyers. We can calculate

$$\pi_2(v_3, v_3, v_3) = (1 - (\rho_0 + \rho_1 + \rho_2)^2)v_3 \;>\; \begin{cases} \pi_2(v_2, v_2, v_2) = (1 - (\rho_0 + \rho_1)^2)v_2 \\ \pi_2(v_1, v_1, v_1) = (1 - \rho_0^2)v_1 \end{cases},$$

and hence the price path (v_2, v_2, v_2) is now dominated by (v_3, v_3, v_3) (and many equivalent price paths). Equivalently, the static monopoly price is now v_3 rather than v_2. This reflects two straightforward and general results—the static monopoly price increases in the number of buyers, and the seller can always earn at least the static monopoly payoff (or, equivalently, the payoff available to a seller able to commit to constant sequences of prices).

Can the seller do better than $\pi_2(v_3, v_3, v_3)$? Perhaps. Because there are two buyers, the seller can now practice price discrimination. A buyer may purchase at a relatively high price, even knowing that the next price will be lower, if it is more likely that the buyer will obtain the good at the higher price. We see here the important role played by scarcity (in contrast to the standard Coase-conjecture formulation where the seller has as many goods as there are buyers).

One possibility is to set price v_3 in the first period, rejected by all buyers, then a price $p_3 \in (v_2, v_3)$ in the second period that is accepted by buyers of valuation v_3, and then to set

[10]The seller might hope that a type v_3 buyer would accept one of the initial v_3 prices (or some such initial price higher than v_2), with the seller then lowering the price to v_2 if the buyer rejects in order to sell at v_2 if the buyer is type v_2, but this is impossible with a single buyer. Anticipating the subsequently lower price, a buyer of type v_3 would simply wait for price v_2.

[11]Again, the fact that there is only one buyer ensures that attempts to charge higher prices to higher-type buyers would simply prompt the buyer to wait until price v_1 appears.

price v_2 in the last period if p_3 draws no acceptances.[12] What makes us think this strategy is better than simply setting price v_3? We can do the relevant calculations (presented in Section B.1), but can also appeal to another familiar result. The optimal reserve price in an auction is independent of the number of bidders (Krishna [20, pp. 25–26]). The reserve price is v_2 with only 1 bidder, and hence the optimal price sequence with any number of bidders culminates in v_2. The price sequence (v_3, p_3, v_2) is an optimal auction, and would be the strategy of a seller with full commitment power.

There is only one difficulty with the preceding paragraph's argument. Because the seller cannot commit to subsequent prices, the presumption that the seller can set the sequence of prices (v_3, p_3, v_2) requires that once the rejection of p_3 has revealed there are no v_3 buyers, price v_2 (rather than v_1) is optimal. The sequential rationality condition is

$$\left(1 - \left(\frac{\rho_0 + \rho_1}{\rho_0 + \rho_1 + \rho_2}\right)^2\right) v_2 \;\geq\; \left(1 - \left(\frac{\rho_0}{\rho_0 + \rho_1 + \rho_2}\right)^2\right) v_1, \tag{2}$$

which fails (given (1)). Should the seller screen out the v_3 buyers in the penultimate period, her last move would be to set price v_1 rather than v_2.

All is not lost. The seller can set a price $p_3' > p_3$ in the penultimate period, in response to which the v_3 buyers mix, some accepting and some rejecting. The possibility that a v_3 buyer has rejected p_3' ensures that there are more buyers in the last period willing to pay price v_2 than would otherwise be the case, and the v_3 rejection probability in the penultimate period can be set so that the counterpart of (2) holds with equality in the final period, allowing the seller to rationally set price v_2.[13]

Is this an optimal strategy for the seller? There are two obvious alternatives (as well as some other strategies that are easily shown to be suboptimal). The seller could still insist on price v_3 by choosing the price path (v_3, v_3, v_3). The result that the optimal reserve price in an optimal auction is independent of the number of buyers does not tell us that pricing sequence (v_3, p_3', v_2) dominates (v_3, v_3, v_3), since we lack the ability to commit to the optimal sequence of prices $((v_3, p_3, v_2))$ on which this result rests. However, one can calculate that (v_3, p_3', v_2) is indeed superior to (v_3, v_3, v_3). Alternatively, the seller may choose a strategy (p_3'', p_2'', v_1), inducing all of the v_3 buyers to accept in the first period, then all of the v_2 buyers in the second period, and finally all of the v_1 buyers in the final period.[14] Once again, we can calculate that price path (v_3, p_3', v_2) is superior. Notice that in this case, the lack of

[12] We use p_j to denote a price accepted by buyer types v_j and above. Why would a buyer accept p_3 rather than waiting for v_2? Because only one buyer can receive the object. A buyer accepting price p_3 faces competition only if the other buyer also has valuation v_3, while waiting for price v_2 raises the risk not only that a v_3 competitor will grab the good, but that one will have to compete with a v_2 competitor. There is then a price $p_3 \in (v_2, v_3)$ which v_3 buyers will accept.

[13] Notice that $p_3' > p_3$, because a rejecting buyer faces stiffer competition in the final period under price sequence (v_3, p_3', v_2), making rejecting less attractive.

[14] Why doesn't this conflict with our contention that the seller cannot commit to price v_2 once she has

commitment power leads to a *higher* sequence of prices $((v_3, p_3', v_2)$ versus (v_3, p_3, v_2), with $p_3' > p_3)$ than would appear under commitment.

With two buyers, the seller thus chooses prices (v_3, p_3', v_2). In weighing the choice between (v_3, p_3', v_2) and (p_3'', p_2'', v_1), the seller faces a trade-off. The price sequence (p_3'', p_2'', v_1) takes the seller below the optimal reserve price, diminishing her payoff in the process. On the other hand, it allows her to more precisely discriminate between buyer types v_2 and v_3 (than does (v_3, p_3', v_2)), since all the v_3 buyers are induced to buy at price p_3'' rather than some slipping through the first screen to be lumped with the v_2 buyers. With two buyers, this trade-off between high prices and being able to more finely discriminate between buyers is resolved in terms of the former.

Three buyers: Price discrimination. Now let us go one more step to consider the case of three buyers. The static monopoly price is still v_3 (as expected, since the static monopoly price can only increase in the number of bidders), and the price sequence (v_3, v_3, v_3) gives this payoff. This would be the sequence set by a seller able to commit only to unchanging prices. The optimal reserve price remains v_2, ensuring that the price sequence (v_3, p_3, v_2), with p_3 set so that all type-v_3 buyers attempt to purchase at p_3, is superior to (v_3, v_3, v_2). This would be the price sequence set by a seller with full commitment power. However, reproducing the counterpart of (2) for three buyers shows that we again have a commitment problem. The price v_2 is no longer optimal once buyers of type v_3 have been screened out, ensuring that (v_3, p_3, v_2) is not feasible for a seller without commitment power. The seller has two remaining choices. She can set the price sequence (v_3, p_3', v_2), with p_3' calculated so that buyers of type v_3 purchase with just the right probability required to make price v_2 optimal should p_3' be rejected.[15] This sequence preserves the optimal reserve price but lumps some v_3 buyers together with v_2 buyers. Alternatively, the seller can set prices (p_3'', p_2'', v_1), with all buyers of type v_3 purchasing at price p_3'', sacrificing high prices in order to perfectly screen v_2 and v_3 buyers. Calculations analogous to those for the two-buyer case show that price schedule (p_3'', p_2'', v_1) yields a higher payoff than does either (v_3, p_3', v_2) or (v_3, v_3, v_3). When $n = 3$, higher buyer types are more likely, making it less likely that the reserve price is relevant and more important to finely screen high-type buyers. These contending forces thus now tip in favor of better price discrimination and hence price sequence (p_3'', p_2'', v_1). In this case, we have $p_3'' < p_3'$ and $p_2'' < v_2$, so that the seller sets lower prices without commitment than with commitment.

learned there are no v_3 buyers? Because the price p_2'' in this case falls short of v_2 and occurs in the penultimate period (rather than equalling v_2 and being set in the final period), and is chosen to make a buyer of type v_2 just indifferent between accepting p_2'' and rejecting to take his chances on getting the good at price v_1.

[15] The price p_3' required to induce some type-v_3 buyers to purchase at price p_3' and some to wait for price v_2 will be different when there are only two buyers and when there are three (as will prices p_3'' and p_2''), though we do not distinguish them with our notation.

Our general model allows a continuum of possible buyer valuations. The seller will attempt to screen these buyers as finely as possible by setting each period's price lower than its predecessor. If the terminal price is allowed to approach zero (as the length or a pricing period decreases, and hence the price sequence lengthens), these prices can be set arbitrarily close together, allowing very fine price discrimination among the buyers at the cost of relatively low prices. If the terminal price is positive, there is necessarily some lumpiness in the price discrimination—the final period poses a monopoly pricing problem leading to a terminal price that is necessarily bounded below the penultimate price (if the former is positive), with similar calculations applying to preceding periods. High prices are thus purchased only at the cost of lumpy price discrimination. High prices are relatively important when there are few buyers and price discrimination relatively important when there are many buyers.

3 The Model

We consider a dynamic game between a single seller, with one unit for sale (until Section 8), and n buyers. The good is to be consumed at a fixed future date that we normalize to 1, and is valueless thereafter. The seller has the interval $[0, 1]$ of time in which to make an agreement with a buyer on the purchase of the good. The buyer and seller discount at the same rate r, and there is then no loss of generality in taking this interest rate to be zero.[16]

The seller can post a price at each time $\{\Delta, 2\Delta, \ldots, 1\}$ (restricting attention throughout to values of Δ that divide 1 without remainder). We can thus think of the seller as facing a finite horizon of length $T_\Delta = \frac{1}{\Delta}$. Since our arguments will typically involve solving backwards from the final period and the number of periods will vary with Δ, we find it convenient to let $t = 1, \ldots, T_\Delta$ index the number of *remaining* periods, so that T_Δ is the first and 1 the last period. At each period t, the seller posts a price $p_t \in \mathbb{R}$. After observing the price, buyers simultaneously and independently accept or reject. If the price is accepted by at least one buyer, the game ends with a transaction at this posted price between the seller and a buyer randomly selected from among the accepting buyers. If the offer is rejected, the game moves on to the next period.[17]

Each buyer has a private valuation v, independently drawn from a uniform distribution on $[0, 1]$ and constant throughout the game. We do not require uniformity to establish the

[16] One possibility is that the buyer pays for the good at time 1, in which case we can take all prices to be time-1 prices and discounting is irrelevant. Alternatively, the buyer may pay upon purchase, but the commonality of the discount factor ensures that the equilibrium is unchanged, with a time-τ price p equivalent for both agents to a time-1 price of $pe^{-r(1-\tau)}$. It then simplifies the notation to present the analysis in terms of time-1 prices.

[17] An outcome of the game is a vector $(\mathbf{v}, t, p_t, i)$, $i = 1, \ldots, n$, or $(\mathbf{v}, 0, \varnothing)$; with the interpretation that the realized profile of valuations is $\mathbf{v} = (v_1, \ldots, v_n)$ and the price p_t is accepted in period t by buyer i if the outcome is $(\mathbf{v}, t, p_t, i)$, and that no buyer ever accepts in case $(\mathbf{v}, 0, \varnothing)$.

existence of an equilibrium (cf. Chen [10, Proposition 1]), but it is useful in obtaining characterization results. The assumption of uniformity is stronger than we need in this respect, but clears away some technical clutter. We discuss as we proceed where the uniformity of buyer valuations is used, and the extent to which we can generalize this assumption. A buyer of valuation v who receives the object at price p garners payoff $v - p$. The seller has a zero reservation value, with her payoff being the price at which she sells the good.[18]

A *nontrivial* history $h^t \in H^t$ is a history after which the game is not effectively over, i.e. a sequence $(p_{T_\Delta}, \ldots, p_{t+1})$ of prices that were posted by the seller and rejected by all buyers (we set $H^{T_\Delta} = \varnothing$). A behavior strategy for the seller is a finite sequence $\{\sigma_S^t\}_{t=1}^{T_\Delta}$, where σ_S^t is a probability transition from H^t into $\mathbb{R}$, mapping the history of prices h^t into a probability distribution over prices. A behavior strategy for buyer i is a finite sequence $\{\sigma_i^t\}_{t=1}^{T_\Delta}$, where σ_i^t is a probability transition from $[0,1] \times H^t \times \mathbb{R}$ into $\{0,1\}$, mapping buyer i's type, the history of prices, and the current price into a probability of acceptance.[19]

We will focus our attention on perfect Bayesian equilibria.[20] The seller then has no commitment power—each price must be sequentially rational, given the history of previous play and anticipations of optimal continuation play. "Real world" sellers may often work hard at making price commitments, perhaps by offering a guarantee that they will refund the difference should a consumer subsequently discover a lower price. Such devices appear most likely to allow commitments to constant price paths. Such commitments are worthless in the current setting, in the sense that a seller without commitment can always (if there are at least two buyers) do better than committing to constant prices. The full-commitment solution, which we also calculate as a benchmark, involves a descending sequence of prices that may sometimes lie above and sometimes below the corresponding no-commitment sequence. Commitments of this complexity may be sufficiently demanding in some circumstances as to make the no-commitment solution interesting.

We call the seller's price *serious* if it is accepted by some buyer with positive probability, and *losing* otherwise. Clearly, the specification of losing prices in an equilibrium is to a large extent arbitrary. Therefore, statements about uniqueness are understood to be made up to the specification of the losing prices.

As a useful benchmark, let $\pi^D(n)$ denote the expected revenue from a Dutch auction with n bidders and a zero reserve price. This is the seller's revenue in the equilibrium considered by Bulow and Klemperer [8] in an infinite-horizon, continuous time game in which the seller has no commitment power.

[18]The seller's von Neumann-Morgenstern utility function over outcomes is simply p_t if the outcome is $(\mathbf{v}, t, p_t, i)$, and zero otherwise. Buyer i's utility is $v_i - p_t$ if the outcome is $(\mathbf{v}, t, p_t, i)$ and zero otherwise. We define the seller's and buyers' expected utilities over lotteries of outcomes in the standard fashion.

[19]That is, for each $h^t \in H^t$, $\sigma_S^t(h^t)$ is a probability distribution over $\mathbb{R}$, and the probability $\sigma_S^t(\cdot)[A]$ assigned to any Borel set $A \subset \mathbb{R}$ is a measurable function of h^t, and similarly for σ_i^t.

[20]The generalization of Fudenberg and Tirole's [14, Definition 8.2]) definition to our infinite game is immediate.

4 Lower Bounds on Payoffs

We begin with a pair of preliminary results, designed to illustrate the commitment power the firm achieves as a result of facing a deadline. In the process, we see both that commitments to a constant price sequence would be of no value, and that our seller confronts a quite different situation than that facing a durable-goods monopolist.

Let $\pi_\Delta(n)$ denote the seller's payoff from the perfect Bayesian equilibrium of the game with n buyers and period length Δ. (Proposition 3 below establishes the uniqueness of perfect Bayesian equilibrium, ensuring this is well defined.) Notice that $\pi_1(n)$ is the static monopoly payoff with n buyers, being uniquely defined by

$$\pi_1(n) = \max_{p\in[0,1]} p(1-p)^n. \tag{3}$$

Proposition 1. *If $\Delta < \Delta'$ and hence $T_\Delta > T_{\Delta'}$, then*

$$\pi_\Delta(n) \geq \pi_{\Delta'}(n).$$

In particular, $\pi_\Delta(n) \geq \pi_1(n)$. The opportunity to revise prices more quickly increases what the seller can guarantee, and the seller can always do at least as well as the static monopoly payoff.

A game with more rapid price revision thus gives at least as high a payoff as the equilibrium in a game with more sluggish price revision. Since $\pi_1(n)$ is the best that can be achieved by a seller committed to an optimal constant price scheme, such commitments are not valuable.

We need the uniformity of the buyers's valuations only to ensure the equilibrium payoff is unique (via Proposition 3). Section A.1 states and proves a more general version of this proposition, establishing a monotonicity result for the (potentially nonsingleton) *set* of equilibrium payoffs, with no assumptions about the buyers' valuations. This allows us to obtain the proposition via elementary and particularly insightful arguments, without deriving the equilibrium.[21] The basic idea is that a seller facing k remaining periods can always set a price of 1 and thereafter duplicate the equilibrium behavior of a seller facing $k-1$ remaining periods, ensuring that an additional period can only increase the seller's payoff.

A price sequence concluding with the static monopoly price, or indeed any positive price, carries a cost. Let $v_{\Delta k}$ be the valuation of the buyer who is just indifferent between accepting and rejecting the period-k price (cf. Lemma 2 below), given period Δ. Section A.2 proves:

Lemma 1. *If* $\lim_{\Delta\to 0} v_{\Delta 1} > 0$*, then for all* k*,* $\lim_{\Delta\to 0} v_{\Delta k+1} > \lim_{\Delta\to 0} v_{\Delta k}$.

[21] Indeed, if we restricted attention to the case of a uniform distribution and simply calculated the equilibria, we could strengthen the weak inequalities in Propositions 1 to strict ones, and could establish (5) below for all Δ, rather than simply in the limit as $\Delta \to 0$.

If the terminal price is to be positive, and hence the terminal indifferent buyer valuation positive, then in any previous period k the seller must be charging the same price to all buyers in a nonnegligible interval of valuations. The seller is thus imprecisely discriminating in the prices it induces the buyers to accept. This result extends beyond the uniform to any distribution over buyer types with a positive density. The idea is that the final marginal buyer's valuation is determined by setting the monopoly price given the posterior distribution, and hence can be nonzero only if it is smaller than the upper bound on this distribution, and hence smaller than the previous-period marginal buyer's valuation. A similar argument allows us *to work backward through the chain of marginal buyers.

The seller can practice more effective price discrimination if she allows the terminal price to drop to zero. In that case, we have an alternative lower bound on payoffs:

Proposition 2. *The seller can always achieve at least the payoff of an optimal auction with zero reserve price:*

$$\pi_n \equiv \lim_{\Delta \to 0} \pi_\Delta(n) \geq \pi^D(n). \tag{4}$$

More buyers are better for the seller, i. e., for every n,

$$\pi_{n+1} \geq \pi_n. \tag{5}$$

Section A.3 again establishes a more general version of this result, allowing the possibility of sets of equilibrium payoffs for a given n, with weaker assumptions on the buyers' valuations, and relying on elementary arguments rather than equilibrium constructions. The straightforward part of the proof (requiring no assumptions on f) is to show that if the seller could commit to working through a succession of sufficiently close and equally spaced prices, then the optimal buyer response would give her a payoff approaching (as Δ gets small and the price gaps shrink to zero) that of an auction. The more difficult step (for which we need $f' \leq 0$, as in the case of a uniform distribution) shows that the seller has available an alternative strategy that is at least as lucrative, and hence maintains the payoff bound, but which is sequentially rational and so requires no commitment.[22] Having established (4), the argument behind (5) is then immediate. Bulow and Klemperer [9] show that under the standard increasing-virtual-valuations assumption (satisfied by the uniform distribution), the payoff from a zero-reserve-price English auction with $n+1$ bidders exceeds the payoff from an optimal mechanism with n bidders. Since the former is a lower bound on the (lowest) equilibrium payoff $\pi_\Delta(n+1)$ for Δ small enough and the latter by definition an upper bound on the (highest) equilibrium payoff $\pi_\Delta(n)$, the result follows.

The static monopoly payoff is the more stringent of the lower bounds established in Propositions 1–2 if and only if there are one or two buyers.

[22]The sequentially rational strategy again calls for price gaps that shrink to zero as does Δ, a property we verify via explicit calculation for the uniform distribution in Section 7, and which Chen [10, Theorem 2] shows holds more generally.

5 Equilibrium

This section provides our primary existence and uniqueness result, including a complete statement of the equilibrium strategies. While this is an important step in the analysis, the strategy specifications are complicated and not particularly intuitive. As a result, one may prefer to skip to Sections 6–7, which derive implications of these strategies.

5.1 One Buyer

We warm up by confirming that when there is only one buyer, the argument developed in the example of Section 2 is general—the seller is effectively a static monopolist.

It follows from Samuelson [25] that among all mechanisms, the optimal ones are equivalent to having the seller make a take-it-or-leave-it offer to the buyer. As the seller can always do so by posting a price of 1 in every period but the last, every equilibrium must then yield this maximal payoff to the seller. In particular, in every equilibrium, she must charge the optimal take-it-or-leave-it offer (say p^*) on the equilibrium path at some point. Since all buyers with values above p^* must accept it, no lower price can ever be assigned positive probability, while all higher offers must always be rejected with probability one. Observe now that, if the buyer accepted with positive probability a price (namely, p^*) before the last period, then at least one of the subsequent prices would be strictly lower (reflecting the adverse information about the buyer's valuation conveyed by a rejection), and therefore the buyer would not have been willing to accept the earlier price. Therefore, all prices but that posted in the last period must be losing prices, and the price in the last period must be p^*.

To summarize, with $n = 1$ and for any Δ, all equilibria are such that all equilibrium prices are at least p^*, and the last one is p^*. All prices are rejected except the last one, which is accepted by buyer of type v if and only if $v \geq p^*$. Hence, when $n = 1$, a deadline is an effective way for the seller to commit. Independently of the period length, she does as well as in the optimal mechanism.

5.2 Many Buyers

We hereafter assume that there are at least two buyers, so that we indeed have the potential competition between buyers that is the hallmark of revenue management with strategic buyers. Our first lemma establishes that the perfect Bayesian equilibrium concept has enough power in our setting to determine beliefs, with these beliefs satisfying a skimming property. Any equilibrium or nonequilibrium history must cause the seller to update her beliefs about buyers' types by truncating the distribution from above. Section A.4 proves:

Lemma 2. *Fix Δ and an equilibrium $\left\{\{\sigma_S^t\}_{t=1}^{T_\Delta}, \{\sigma_i^t\}_{t=1}^{T_\Delta}\right\}_{i=1}^{n}$, and suppose period t has been reached via a nontrivial history h^t. Then the seller's posterior belief is that the buyers'* *valuations are independently drawn from a uniform distribution over* $[0, v_{t+1}]$*, for some* $v_{t+1} \in (0, 1]$.

It follows from the definition of perfect Bayesian equilibrium that these beliefs are always common knowledge (on- and off-path).

We find the following sequence helpful in specifying equilibrium strategies. Let $\{q_t\}_{t=0}^{\infty}$ be an increasing solution to the second-order difference equation

$$q_{t+1}^n - \frac{nq_t^n - 1}{q_t} q_{t+1} + n(q_t^{n-1} q_{t-1} - 1) - q_{t-1}^n = 0, \tag{6}$$

with the boundary conditions

$$\begin{aligned} q_0 &= 1, \\ q_1 &= (n+1)^{\frac{1}{n}}. \end{aligned}$$

As is shown in Section A.5, this sequence exists and is unique.

Proposition 3. *Suppose the seller's period-t posterior belief has support* $[0, v_{t+1}]$ *after some nontrivial history. (If* $t = T_\Delta$*, take* $v_{T_\Delta+1} = 1$*.) Then there is a unique perfect Bayesian equilibrium in the resulting continuation game. In particular, the seller sets the period-t price,*

$$p_t = \left[1 - \frac{q_t - q_{t-1}}{q_t^n - q_{t-1}^n} \left(q_{t-1}^{n-1} - q_{t-1}^{-1}\right)\right] \frac{q_t}{q_{t-1}} v_{t+1}, \tag{7}$$

and given an arbitrary *price* $\hat{p}_t$*, each buyer* i *with valuation* $v_i \geq v(\hat{p}_t, v_{t+1}, t)$ *accepts the price and each buyer* i *with* $v_i < v(\hat{p}_t, v_{t+1}, t)$ *rejects it, where* $v(\hat{p}_t, v_{t+1}, t)$ *is the unique value* v *solving*[23]

$$1 - \frac{\hat{p}_t}{v} = \frac{v_{t+1} - v}{v_{t+1}^n - v^n} v_{t+1}^{n-1} \left(1 - q_{t+1}^{-n}\right). \tag{8}$$

The path of equilibrium behavior is straightforward to trace, even if the statement of the strategies is somewhat formidable. A sufficient statistic for continuation play in each period t is the upper bound v_{t+1} on the buyers' valuations. Given this bound, the seller can calculate the optimal price p_t according to (7), and this price will be accepted by buyers in the interval $[v_t, v_{t+1}]$, where the cutoffs v_t evolve (in equilibrium) according to

$$v_t = \frac{q_t}{q_{t-1}} v_{t+1}. \tag{9}$$

[23] A buyer with valuation $v_i = v(\hat{p}, v_{t+1}, t)$ will be indifferent between accepting and rejecting this price, and the equilibrium is unique only up to the the behavior of this measure-zero set of buyers.

Should a deviation to an out-of-equilibrium price on the part of the seller push us off the equilibrium path in period t, we will depart from this progression of marginal buyer valuations, with next period's value of v_t now defined by (8) rather than (9). Once we have obtained this new value, however, we face a continuation game with a unique perfect Bayesian equilibrium, defined by (7)–(8). Buyer deviations have no effect on continuation strategies.

We can see here the simplifications allowed by the uniform distribution, and the extent to which we can generalize it. One typically looks for some stationary structure when solving dynamic optimization problems. Our finite horizon complicates the problem by introducing one nonstationarity, with each period's continuation game differing from the last by the sheer fact that there is one less period to go. As mentioned in footnote 4, our analysis could be extended to the broader class of power densities having this property (albeit with more notation), but different techniques are required to go beyond this class.[24]

6 Price Discrimination vs. Reserve Prices

Our first step in characterizing the equilibrium is to ask when and how the seller can do better than an optimal auction with zero reserve price. Section 5.1 gives a complete description of the equilibrium with only one buyer, and we assume there are at least two buyers.

Let $p_{\Delta t}$ denote the equilibrium price set by the seller when there are t periods to go (including the current one), given period length Δ. Given $p_{\Delta t}$, let $v_{\Delta t} \in [0,1]$ denote the valuation of the "critical" buyer, who is indifferent between accepting and rejecting in period t, characterized in Proposition 3.[25] (Set $v_{\Delta t} = 1$ if every buyer rejects, and $v_{\Delta t} = 0$ if every buyer accepts.)

Proposition 4. *Fix a period length Δ and a number of buyers $n > 1$. The sequences $\{p_{\Delta t}\}_{t=1}^{T_\Delta}$ of equilibrium prices and $\{v_{\Delta t}\}_{t=1}^{T_\Delta}$ of equilibrium critical buyer valuations take values in $(0,1)$ and are strictly increasing in t (i.e., decreasing over time). Further:*

(4.1) For $n < 6$, $\lim_{\Delta\to 0} v_{\Delta 1} > 0$ *and* $\lim_{\Delta\to 0} \pi_\Delta(n) > \pi^D(n)$.

(4.2) For $n \geq 6$, $\lim_{\Delta\to 0} v_{\Delta 1} = 0$ *and* $\lim_{\Delta\to 0} \pi_\Delta(n) = \pi^D(n)$.

(4.3) $\lim_{\Delta\to 0} v_{\Delta 1}$ *is decreasing in n.*

[24] Restrictions to such distributions in analyzing dynamic mechanisms under incomplete information are common, beginning at least with Sobel and Takahashi [27, Section 4].

[25] If type $v_{\Delta t}$ is indifferent between accepting and rejecting price $p_{\Delta t}$, then it must be that higher types accept and lower types reject, consistent with Lemma 2. This skimming property holds despite the absence of discounting because there are more buyers than objects (it is here that we use the fact that $n > 1$) and an acceptance ends the game.

This proposition tells us that the seller's limiting price and payoff depend on the number of buyers. The larger is the number of buyers, the lower does the seller allow the ultimate price to drop (Proposition 4.3). If there are more than six buyers (the specific number is an artefact of the uniform distribution of buyer values), then the seller's payoff matches that of a descending auction and her asking price approaches zero (Proposition 4.2). In this case the seller's lack of commitment power poses no difficulties in discriminating between buyers, but she abandons all hope of maintaining a reservation price. With five or fewer buyers, the finite horizon allows the seller to commit to a positive reservation price, no matter how long the horizon, resulting in a payoff higher than that of the Dutch auction and a limiting price (equal to $v_{\Delta 1}$) larger than zero (Proposition 4.1).

What lies behind these results? The proof in Section A.6 makes clear that, with $n \geq 6$, the (closure of) the set of prices that are charged at some point or another is the entire unit interval, so not only the revenue, but the entire outcome of the game mimics the outcome of the Dutch auction. With fewer buyers instead, this set has only one accumulation point, with the corresponding critical valuation being the highest possible buyer valuation, reflecting the fact that only the very few last prices have a reasonable chance of being accepted.

Sequential rationality forces the seller to set a series of prices that decline over time, in each period skimming off an upper interval of high-valuation buyers. As Δ shrinks and price-revision opportunities come more frequently, the seller sets a higher and higher initial price $p_{\Delta T_\Delta}$, using her frequent price revisions to skim off smaller intervals in each period and hence more effectively price discriminate among the buyers. If $p_{\Delta T_\Delta}$ increases sufficiently rapidly as Δ shrinks, the higher starting price and smaller skimming intervals will counteract the more frequent price revisions and the terminal price $p_{\Delta 1}$ will never fall to zero—the seller commits to a reserve price. If $p_{\Delta T_\Delta}$ increases more slowly as Δ shrinks, the more frequent price revisions will more than make up for the higher initial price and smaller intervals, and $p_{\Delta 1}$ will approach zero—no commitment.

Which is optimal? At one extreme, with only one buyer, the seller finds it optimal to commit by setting a serious price $p_{\Delta 1}$ (equal to the static monopoly price) only in the last period, no matter how many previous prices she can set. Suppose there are more buyers and the seller chooses a price path culminating in $p_{\Delta 1} = v_{\Delta 1} > 0$. This path has the advantage (over a smaller terminal price) of increasing revenue in the event that the highest and second-highest buyer valuations straddle $v_{\Delta 1}$. This benefit, overwhelming for small n, evaporates exponentially fast as the number of buyers gets large—a static monopolist owning one unit sets a price at which she is arbitrarily likely to sell as the number of buyers gets arbitrarily large. Setting $p_{\Delta 1} > 0$ has the cost that the seller loses if all valuations fall short of $p_{\Delta 1}$, and while this cost also evaporates as the number of buyers grows, it only evaporates at a polynomial rate. Finally, fixing $p_{\Delta 1} = v_{\Delta 1} > 0$ (along with sequential rationality) fixes $v_{\Delta 2} > v_{\Delta 1}$ and $v_{\Delta 3} > v_{\Delta 2}$ and so on, imposing constraints on the seller's prices that impede her ability to discriminate among buyers of higher valuations. This cost goes to zero relatively slowly in the number of buyers, ensuring that the seller prefers to abandon the attempt to commit and to let $p_{\Delta 1}$ and $v_{\Delta 1}$ approach zero when there are enough buyers.

7 Pricing Dynamics

The previous section has described which prices the seller sets. We are interested here in characterizing the timing of these prices. To do so, we study the limiting path of prices and indifferent types, as the period length Δ goes to zero.

7.1 Commitment

As a benchmark, we first consider the case in which the monopolist can commit to prices. Let $v_{\Delta t}$ denote the indifferent type of buyer with t instants to go. Given any period length Δ and given the sequence of indifferent types $\{v_{\Delta T_\Delta}, \ldots, v_{\Delta 1}\}$ maximizing the payoff of a seller with commitment, define the step function

$$v_\Delta(x) = v_{\Delta t} \text{ for all } x \in \left[\frac{t-1}{T_\Delta}, \frac{t}{T_\Delta}\right), \qquad v_\Delta(1) = 1,$$

where, in keeping with our use of t to identify the number of remaining pricing opportunities, we think of x as the time remaining before hitting the terminal horizon. Our purpose is to prove that the (continuous extension of the) limit

$$v(x) = \lim_{\Delta \to 0} v_\Delta(x) \tag{10}$$

exists, and determine this limit, as well as the corresponding limit $p(x)$ of the analogously defined price function $p_\Delta(x)$. Clearly, with only one buyer, only the last posted price matters, and we accordingly assume $n > 1$.

Proposition 5. *Let $n \geq 2$. The limiting function v (cf. (10)) describing the path of indifferent buyers induced by a seller who can commit to prices is well-defined, and equal to*

$$v(x) = \frac{1}{2}\left(\left(2^{\frac{n+1}{3}} - 1\right)x + 1\right)^{\frac{3}{n+1}}, \tag{11}$$

while the corresponding price function is given by

$$p(x) = \frac{(n-1)v(x)^n + 2^{-n}}{nv(x)^{n-1}}. \tag{12}$$

See Figure 1. The limiting indifferent buyer's type $v(x)$ and the seller's price $p(x)$ thus both decline as the terminal point approaches (x decreases). As expected, $v(1) = 1$ and $v(0) = 1/2$, so that the seller begins (at $x = 1$) slicing off the highest-type buyers, moving downward to a valuation of $1/2$ (at $x = 0$). The function v is concave in x (it is affine in x for $n = 2$), so that the seller runs through buyers more rapidly as time goes on, and is increasing in n. Prices are also increasing in n, and of course increasing in x—prices decline over time—but they are not concave in x. Rather, they are convex for x low enough, and concave for high enough values of x (this higher interval being empty if and only if $n \leq 3$). Prices decrease relatively slowly at the beginning and end of the interval, progressing somewhat more rapidly in the middle.

7.2 Noncommitment

We now turn to the non-commitment case, and define the limiting functions $v(x)$ and $p(x)$ exactly as before, but without commitment and hence with a sequence of indifferent types $\{v_{\Delta T_\Delta}, \ldots, v_{\Delta 1}\}$ maximizing the payoff of a seller without commitment

Proposition 6.

[6.1] For $n \geq 6$, the limiting function v describing the path of indifferent buyers induced by a seller who cannot commit to prices is well-defined, and equal to

$$v(x) = x^{\frac{3}{n+1}}, \tag{13}$$

while the corresponding price function is given by

$$p(x) = \frac{n-1}{n} x^{\frac{3}{n+1}}. \tag{14}$$

[6.2] For $n \leq 5$, the functions v and p both converge to $v(x) = 1$ and $p(x) = 1$ on $[0, 1)$. Hence, losing prices are set and no sales made throughout the interval, with all serious offers being made in the last instants.

When $n < 6$ (and in the limit as Δ gets small), all of the action occurs at the very end of the horizon. The seller sets the choke price until the last instants, at which point the price p and the marginal buyer v cascade in chunks to nonzero terminal values. For larger values of n, marginal valuations and prices both decline as time passes (v and p both increase in x). Both functions are concave, so that the seller moves through buyers and prices more rapidly as the endpoint approaches. Marginal valuations and prices are both increasing in the number of buyers, and both are smaller than their counterparts without commitment.

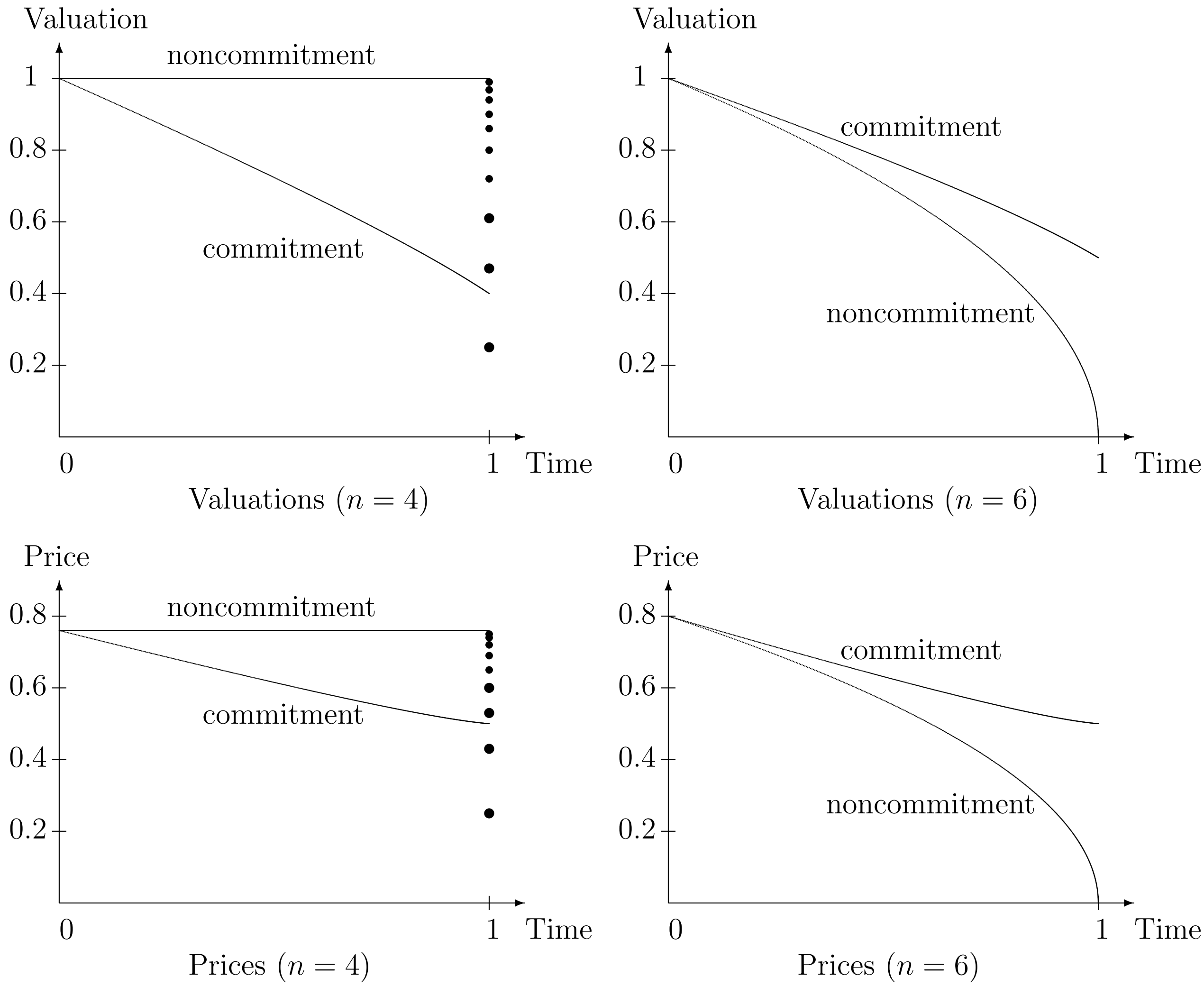

Figure 1. Limiting (as $\Delta \to 0$) critical valuations v and prices p, as a function of the time that has elapsed.

Figure 1 illustrates these results. While Proposition 6 calculates the type of the marginal buyer and the price as a function of the time remaining, we make Figure 1 more intuitive by translating these into functions that give marginal valuations and prices as a function of the *time that has elapsed*. Notice that the function v initially picks out marginal buyers whose types are arbitrarily close to 1, while the prices that make these buyers indifferent are quite a bit lower. Notice also that there is some arbitrariness in the price path under noncommitment and few buyers. The price over the interval $[0, 1)$ must be high enough that there are no sales, and many price paths will have this effect. Combining (13)–(14), we find that when $n \geq 6$, a buyer of valuation v purchases the object (if a competitor does not snatch it first) at price[26]

$$p(v) = \frac{n-1}{n} v.$$

The price is thus a linear function of the buyer's valuation. As the number of buyers increases, the seller gains from the fact that the likelihood of a high-valuation buyer increases, but also from the fact that increased competition among buyers pushes each buyer to pay a price closer to his valuation.

8 Multiple units

We now turn to the case of more than one unit for sale. Letting k be the number of units, we assume $n \geq k+5$, which suffices to ensure that the seller's price eventually declines to zero. Let $p_{kn}(v)$ be the price paid by a buyer of valuation v, in the limiting case of arbitrarily short time periods, when there are k objects and n buyers. Let π_{kn} be the seller's payoff when selling k objects to n buyers. Arguments analogous to those of the case $k=1$ give[27]

$$p_{kn}(v) = \frac{n-k}{n} v, \tag{15}$$

$$\pi_{kn} = k \frac{n-k}{n+1}. \tag{16}$$

A buyer of valuation v thus pays more for the object when facing more competitors, but pays less when there are more objects for sale. The seller's payoff is increasing in the number of buyers, and is increasing in the number of objects as long as there are at least twice as many buyers as objects. If the seller has too many objects for sale, she would be better off

[26] As a check on this result, we can then calculate that the seller's expected payoff when facing n buyers is

$$\int_0^1 p(v) n v^{n-1} dv = \int_0^1 (n-1) v^n dv = \frac{n-1}{n+1} = \pi^D(n),$$

in keeping with Proposition 4.2. In this calculation, $p(v)$ is the price paid by a buyer of type v and nv^{n-1} is the density of the highest bidders' valuation, obtained by noting there are n candidates for the highest bidder and for each valuation v the probability that it is higher than the other valuations is v^{n-1}.

[27] The derivations are much more complicated with multiple objects and do not yield closed-form solutions for the functions $v(x)$ and $p(x)$. Section B.4 sketches the arguments. To provide some insight into these functions, one can verify both that π_{kn} is the expected value of a $k+1$-st price auction with n bidders, and that π_{kn} and p_{kn} satisfy the recursion

$$\pi_{kn} = \int_0^1 n v^{n-1} [p_{kn}(v) + v \pi_{k-1,n-1}] dv,$$

where $v\pi_{k-1,n-1}$ is the continuation value of selling $k-1$ objects to $n-1$ buyers with valuations uniformly distributed on $[0, v]$.

destroying some of them before offering the remainder for sale to the buyers. Notice that the seller could do just as well by withholding the surplus objects from the market, but would then face an irresistible urge to sell the reserved objects once her intended sales quota had been met. Destroying the objects beforehand provides the requisite commitment to limit sales.

Suppose now that the seller begins with k objects and n buyers, and consider the limiting case of vanishingly small period lengths Δ. The price drops until some buyer of type v buys the first object at price $\frac{n-k}{n}v$. At this point, the price jumps upward to $\frac{(n-1)-(k-1)}{n-1}v = \frac{n-k}{n-1}v$, as the seller now continues with the optimal strategy given one less object and one less buyer, with the remaining buyers' valuations distributed on $[0, v]$. The price continues to fall until another buyer of type v' purchases at $\frac{n-k}{n-1}v'$, at which point the price jumps to $\frac{n-k}{n-2}$. This continues until a single object is left, to be eventually sold to a buyer of type v'' at price $\frac{n-k}{n-(k-1)}v''$.

Figure 2 illustrates these dynamics. The seller begins with two objects and lets the price fall, decreasing the indifferent buyer type, until the first purchase occurs. The price now jumps upward as the seller switches to the appropriate single-object price path, while the identity of the indifferent buyer continues to decline from the valuation of the buyer who purchased.

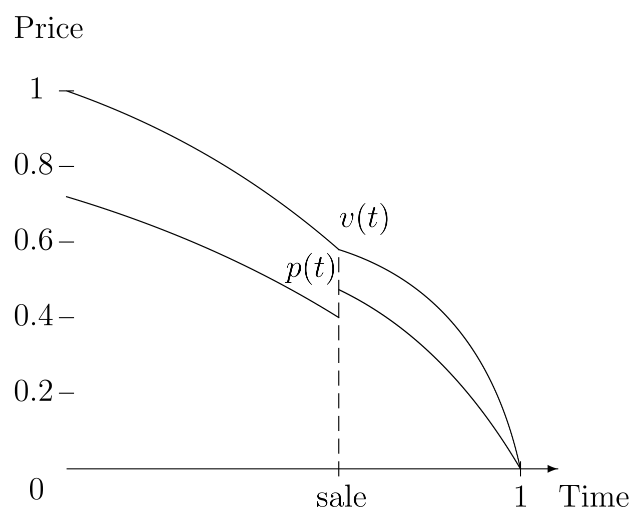

Figure 2: Prices and marginal valuations for $n = 7$ (with a sale at time $t = .6$)

The price jumps in this progression are reminiscent of the frenzies in Bulow and Klemperer [8]. Each sale in their model raises the possibility of a frenzy, in which additional buyers purchase at the price of the most recent sale, or even a price increase, in the event that more buyers than there are remaining objects attempt to purchase at the most recent sale price. The revenue earned by our seller (for sufficiently large n) matches that of Bulow and Klemperer's. Bulow and Klemperer work directly in continuous time and impose conditions on the path of prices set by the seller, including that price must decline continuously to zero in the absence of a sale, that a sale must be followed by repeated opportunities for additional buyers to purchase at the sale price, and that the price must jump upward if these opportunities for additional purchases lead to excess demand for the good. The result is one of the many continuous-time price paths that maximize the seller's revenue. Instead, our analysis begins in discrete time and places no restrictions beyond sequential rationality on the seller's prices, in the process uniquely selecting one of the optimal continuous-time price paths as the limit of the optimal pricing scheme with very short, discrete pricing periods.

9 Discussion

More buyers or more prices? Section 2 illustrated our results via an example with a discrete set of buyer valuations, while our analysis is conducted for the case of a continuum of valuations. Connecting these two requires us to recognize an order-of-limits question. In particular, the essence of the example is that the seller is unable to commit to price v_2 in a final period, after having screened out type v_3 buyers. With two buyers, the seller optimally resolved this conflict by holding the line at a reserve price of 2 while imperfectly screening buyers, while with three buyers the seller sacrifices the reserve price in order to more effectively screen.

If the number of buyers were sufficiently large, the seller in our example could commit to setting price v_2 after having learned that there are no v_3 buyers.[28] This phenomenon is general. For any valuation drawn from a finite set, there is a sufficiently large number of buyers for which that allocation will be the static monopoly price, conditional on having learned there are no higher-valuation buyers. Hence, numerous buyers banish commitment problems.

How do we reconcile this with our observation that commitment problems are pervasive with a continuum of valuations, no matter how numerous the buyers? The closer are the valuations in the previous paragraph's finite set, the larger the number of buyers required to make commitment feasible. Suppose then we think of a series of finite models to which we add ever more possible buyer valuations and ever more buyers. If the number of buyers grows rapidly relative to the set of valuations, then commitment problems will vanish. If the set of possible valuations grows rapidly relative to the number of buyers, we obtain our model in which commitment problems are endemic. Which is the relevant case? This may depend on the setting, though some intuition can be gained by asking whether sellers are more likely to fret over having too few buyers, or over having too few possible buyer valuations.

[28] A sufficiently large n will reverse the inequality in (2).

Generalizations. We have already mentioned that it would make for more difficult computations but raise no new conceptual issues to expand the analysis to the class of "scalable" distributions of buyer values $F(v) = v^{\alpha}$. What is the difficulty in extending the analysis beyond the set of scalable distributions?

The proof of Proposition 4 adduces an induction argument, asking how the seller's behavior changes as the number of periods increases. After formulating the problem, the first observation is that every additional period translates into another uniquely defined, nontrivial price offer on the part of the seller. We show that this is the case by considering the first- and second-order conditions for the seller's maximization problem of choosing a price in the first period, finding that the solution is interior. When the distribution of buyer types is uniform, we can obtain a recursive but explicit characterization of the seller's maximization problem that can be differentiated to obtain the required result. Without a scalable distribution, we could at best hope to replace this step with an envelope argument. The argument is straightforward to write and appears to work flawlessly, until we ask how we can be assured we have the requisite absolute continuity to appeal to the envelope theorem. Once we recognize such difficulties, there appears to be little hope for general or analytical solutions for more general cases. Observe that, compared to the literature on durable goods, there is an additional state variable in our environment, namely the number of periods to go.

Unknown number of buyers. We have assumed that our seller knows how many buyers she faces. What if this is not the case? The obvious alternative is to consider a model in which the number of sellers is determined by a Poisson process.[29] In this case, the seller's optimal strategy always entails a positive terminal price. As the price falls without a purchase in our model, the seller draws the inference that all of the buyers happen to have low valuations, while remaining convinced of the number of buyers. The importance of price discrimination remains unaltered, and (when there are sufficiently many buyers) the seller's decision to sacrifice the reserve price in the interests of price discrimination remains unaltered.

As the price falls without a purchase in a model with a Poisson-distributed number of buyers, the seller draws the inference not only that the buyers have low valuations, but also that there are simply not many buyers there. Eventually, the seller becomes very pessimistic about the number of buyers, and a reasoning analogous to the one applying to the case of a low, but known number of buyers implies here as well that the optimal continuation path of play entails a positive terminal price.

[29] The Poisson process allows especially convenient calculations, allowing the problem to take on a recursive structure much like that induced by the uniform distribution of valuations in our fixed-number-of-buyers model.

A Appendix: Proofs

A.1 Proof of Proposition 1

We state a more general version of the proposition. Let F be the cumulative distribution function of buyer valuations, let $\underline{\pi}_\Delta(n)$ (resp. $\overline{\pi}_\Delta(n)$) denote the infimum (resp., supremum) of this payoff over all equilibria, and define $\pi_1(n) = \max_{p\in[0,1]} p(1-(F(p))^n)$.

Proposition 7. *If $\Delta < \Delta'$ and hence $T_\Delta > T_{\Delta'}$, then*

$$\underline{\pi}_\Delta(n) \geq \underline{\pi}_{\Delta'}(n), \text{ and } \overline{\pi}_\Delta(n) \geq \overline{\pi}_{\Delta'}(n).$$

In particular, $\underline{\pi}_\Delta(n) \geq \pi_1(n)$.

Proof. If $\Delta = 1$, the seller literally gets only one chance to set a price, and is a static monopoly, ensuring that she earns at least $\pi_1(n)$. We now proceed by induction. Fix a period length Δ' and corresponding $T_{\Delta'}$, and let period-length Δ permit one additional pricing period. One possibility open to the seller under period length Δ is to set a losing offer in the first period, in which case play will continue with some equilibrium of the resulting continuation game, which must also be an equilibrium in the game with period length Δ'. Setting a losing price in the first period thus ensures a payoff of at least $\underline{\pi}_{\Delta'}(n)$, and hence any equilibrium with period length Δ must bring at least this payoff (or $\underline{\pi}_\Delta(n) \geq \underline{\pi}_{\Delta'}(n)$). Similarly, considering strategies in which the seller makes a losing offer in the first period followed by an equilibrium giving payoff $\overline{\pi}_{\Delta'}(n)$ gives $\overline{\pi}_\Delta(n) \geq \overline{\pi}_{\Delta'}(n)$. ■

A.2 Proof of Lemma 1

Suppose $v_{\Delta 1} > 0$. Then $v_{\Delta 2}$ must be such that

$$v_{\Delta 1} = \arg\max v(1-(F_2(v))^n),$$

where F_2 is the posterior cumulative distribution over the buyers' types, given that these types are contained in the interval $v_{\Delta 2}$. If F_2 has a strictly positive density, then this ensures that $v_{\Delta_2} > v_{\Delta 1}$ and, along with $\lim_{\Delta\to 0} v_{\Delta 1} > 0$, that $\lim_{\Delta\to 0} v_{\Delta 2} > \lim_{\Delta\to 0} v_{\Delta 1}$. An induction argument then establishes the corresponding inequality for every k. ■

A.3 Proof of Proposition 2

We state a more general form of the proposition.

Proposition 8. *If the density f of buyer valuations is differentiable and satisfies $f' < 0$, then the seller can always achieve at least the payoffs of an optimal auction with zero reserve price:*

$$\underline{\pi}(n) \equiv \lim_{\Delta \to 0} \underline{\pi}_\Delta(n) \geq \pi^D(n).$$

If in addition the virtual valuation $v - \frac{1-F(v)}{f(v)}$ is increasing in n, then more buyers are better for the seller, i. e., for every n,

$$\underline{\pi}_{n+1} \geq \overline{\pi}_n \equiv \lim_{\Delta \to 0} \overline{\pi}_\Delta(n).$$

The second result follows immediately from Bulow and Klemperer [9]. The proof of the first begins by supposing the seller could space her prices uniformly throughout the unit interval, i.e., could set prices $(1-\Delta, 1-2\Delta, \ldots, \Delta, 0)$. In particular, for any $v \in (0,1]$, let R_v^K denote the lowest revenue among all equilibria of a Dutch auction in which the K prices $\{vk/K, k = 0, \ldots, K-1\}$ are quoted in descending order, with n buyers whose valuations are independently drawn from the distribution $F(\cdot)/F(v)$ on $[0, v]$. (If multiple bidders accept the same price, the unit is randomly allocated among them.) Observe that, fixing v, as $K \to \infty$, the revenue and buyers' strategies in this K-price Dutch auction converge to the revenue R_v^D and equilibrium strategies of the standard Dutch auction with no reserve price.[30]

It may appear that this first step already gives the desired result, but we must next deal with the fact that such a sequence of prices may not be sequentially rational. Moreover, sequential rationality is difficult to characterize because at each stage of the auction, the seller's optimal action depends upon the buyers' behavior, which in turn depends upon the continuation equilibrium, about which we know very little. The proof exploits the following insight, formalized in Lemma 3 below: Given any remaining interval of possible buyer types $[0, v]$, there is no incentive-compatible mechanism (and hence no continuation equilibrium) that gives a buyer of type v a higher payoff than a zero-reserve-price Dutch auction. Bearing this in mind, suppose the seller is in the middle of a sequence of prices and considering her next move. She could always set a new price Δ lower than her previous one. It would be easy to identify the buyers that will accept this price *if* the seller would continue shaving her price by Δ each period. She might not choose to do so, of course, but Lemma 3 ensures that if the seller contemplates doing anything else, then the marginal buyer will have an even bleaker future, and hence will be all the more willing to accept the current price. This in turn ensures that in every period, the seller can garner an incremental payoff at least as high as she could from cutting prices by Δ each period, and hence can altogether ensure a payoff approaching (as $\Delta \to 0$) the payoff of an optimal auction with zero reserve price.

[30]This follows from Athey [1, Theorem 6, proof]: incentive compatibility implies that equilibrium strategies are increasing in types, so that any sequence of such strategies, indexed by K, must have a convergent subsequence, and its limit must be an equilibrium of the standard Dutch auction. But the latter admits a unique equilibrium.

To make this precise, given v and K, let v_{vk}^K denote the buyer type that is indifferent between accepting and rejecting the price vk/K. Convergence implies that for any $\varepsilon > 0$, there exists K_ε such that for all $K > K_\varepsilon$ and all $k = 0, \ldots, K$, we have

$$\begin{aligned} |R_v^K - R_v^D| &< \varepsilon, \quad (17) \\ |vk/K - p^D(v_{vk}^K)| &< \varepsilon, \quad (18) \end{aligned}$$

where $p^D(v_{vk}^K)$ is the price at which v_{vk}^K is indifferent between accepting and rejecting in the standard Dutch auction. Further, since R_v^K is a continuous function of v and $v \in [0,1]$, a compact set, we may choose K_ε independently of v.

Let R_v^T denote the lowest revenue among all equilibria in our pricing game with T periods to go, and residual demand on $[0,v]$ with distribution $F(\cdot)/F(v)$. Since R_v^T is increasing in T (because waiting one more period is always an option), $R_v \equiv \lim_{T\to\infty} R_v^T$ is well-defined.

Assume, for the sake of contradiction, that $R_1^D - R_1 > 2\varepsilon > 0$. Then because R_v^D and R_v are continuous functions of v with $R_0^R = R_0 = 0$, (17) ensures that we can find an infinite sequence of values of $K \geq K_\varepsilon$, with a corresponding value v_{1k}^K, for some $k = 1, \ldots, K$, such that

$$R_{v_{1k}^K} < R_{v_{1k}^K}^K - \varepsilon \text{ and } R_{v_{1,k-1}^K} > R_{v_{1,k-1}^K}^K - \varepsilon. \quad (19)$$

Since both revenues are less than ε when $v \leq \varepsilon$, we have $v_{1k}^K \geq \varepsilon$. (We can always ensure that both inequalities are strict, at least for a subsequence of the original sequence, by considering a slightly lower or larger value of ε if need be.) Pick some $K \geq K_{\frac{\varepsilon}{2}}$. Given this K and corresponding v_{1k}^K, (19) ensures that we can pick T large enough so that

$$R_{v_{1k}^K}^T < R_{v_{1k}^K}^K - \frac{\varepsilon}{2} \text{ and } R_{v_{1,k-1}^K}^{T-1} \geq R_{v_{1,k-1}^K}^K - \frac{\varepsilon}{2}. \quad (20)$$

Observe that a strategy available to the seller, given v_{1k}^K and T periods to go, is to offer the price that makes type $v_{1,k-1}^K$ indifferent between accepting and rejecting. Lemma 3 below shows that this price is at least as large as the corresponding price in the standard Dutch auction (since his utility is lower in the continuation than it would be in the standard Dutch auction). That is (using $K \geq K_{\frac{\varepsilon}{2}}$ and (18) as well as the second inequality in (20) for the second inequality below),

$$\begin{aligned} R_{v_{1k}^K}^T &\geq (1 - F^n(v_{v_{1k}^K,k-1}^K))p^D(v_{1k}^K) + F^n(v_{v_{1k}^K,k-1}^K) R_{v_{1,k-1}^K}^{T-1} \\ &\geq (1 - F^n(v_{v_{1k}^K,k-1}^K))(k/K - \frac{\varepsilon}{2}) + F^n(v_{v_{1k}^K,k-1}^K)(R_{v_{1,k-1}^K}^K - \frac{\varepsilon}{2}) = R_{v_{1k}^K}^K - \frac{\varepsilon}{2}, \end{aligned}$$

contradicting the first inequality in (20). ■

Lemma 3. *If F has differentiable density f with $f' \leq 0$, then an efficient auction maximizes the expected utility of the highest type buyer (i.e, a buyer with valuation 1) among all feasible and incentive-compatible mechanisms.*

It is then immediate that if the game reaches a period with remaining buyer types $[0, v]$, then continuing with an efficient auction maximizes (over the set of incentive-compatible mechanisms) the payoff of buyer type v.[31]

This lemma is the only place in the proof of Proposition 2 we use the assumption that $f' \leq 0$. The argument for the lemma begins with Myerson's [24] characterization of incentive compatibility. Myerson shows that in any incentive compatible mechanism, the payoff of the highest-type buyer is given by $\int_0^1 q(v)dv$, where $q(v)$ is the probability that a buyer of type v is allocated the object (conditional on being of type v). Then how do we raise the highest-type buyer's payoff? By setting every buyer's probability of receiving the object as high as possible, with incentive compatibility ensuring that this spills over into a higher payoff for the highest type. Unfortunately, there are feasibility constraints on the extent to which buyers can be promised the object—they cannot all receive it with probability one. The most effective way to boost the overall acceptance probability $\int_0^1 q(v)dv$ without running afoul of these constraints is to make $q(v)$ high when $f(v)$ is small, effectively making promises that affect incentive compatibility, thereby increasing the highest type's payoff, but that are unlikely to have to be kept, thus also preserving feasibility. When $f' < 0$, this means that we should make $q(v)$ large when v is large, doing our utmost to award the object to a high-valuation buyer. But nothing does this more effectively than an efficient auction, opening the door to the result.

More formally, let $q : [0, 1] \to [0, 1]$ be a measurable function representing the allocation from some incentive-compatible mechanism, so that $q(v)$ can be interpreted as the probability that a buyer of valuation v receives the object. From Border [7], we know that feasibility requires

$$\forall v : \int_v^1 q(s) f(s) ds \leq \frac{1 - F(v)^n}{n},$$

or equivalently,

$$\forall v : \int_v^1 \left(q(s) - F(s)^{n-1}\right) f(s) ds \leq 0.$$

From Myerson [24], the expected utility of the highest type given q can be written as $\int_0^1 q(s)\, ds$, and so the difference between his expected utility given q and given the efficient auction is then

$$\int_0^1 \left(q(s) - F(s)^{n-1}\right) ds \leq 0.$$

[31] Lemma 3 does not hold without the assumption that $f' \leq 0$. Suppose, for example, that the cumulative distribution function of bidder valuations is given by $F(v) = (e^v - 1)/(e - 1)$, with support $[0, 1]$, and with two bidders. We have $f' > 0$ in this case, though this distribution satisfies the assumption of increasing virtual valuation. The utility of the highest type in the efficient auction is $(e - 2)/(e - 1) \approx .4$, which is less than what he gets if the good is just given away, namely $(1/2)$. Hence, it is not the case that the efficient auction maximizes the payoff of the highest type buyer, over the set of incentive-compatible mechanisms.

Let $\theta(s) \equiv q(s) - F(s)^{n-1}$ and consider then the problem

$$\max_{\theta} \int_0^1 \theta(s)\, ds \quad \text{such that} \quad \forall v : \int_v^1 \theta(s) f(s)\, ds \leq 0.$$

Our task is to show that $\theta(s) = q(s) - F(s)^{n-1} \equiv 0$ solves this problem.

Because θ is measurable, it can be approximated by step functions, and so we are led to consider the discrete problem, for $K \in \mathbb{N}$, and some nonincreasing positive sequence $\{f_k : k = 0, \ldots, K\}$,

$$\max_{\{a_k\}_{k=0}^K} \sum_{k=0}^{K} a_k \quad \text{such that} \quad \forall j = 0, \ldots, K : \sum_{k=j}^{K} a_k f_k \leq 0.$$

We claim that $a_k = 0\ \forall k$ is a solution to this program. Suppose that $\{a_k : k = 0, \ldots, K\}$ is a solution, and that for some j, $\sum_{k=j}^K a_k f_k < 0$. Then we may as well assume that $a_{j-1} = 0$. If indeed $a_{j-1} > 0$, then by increasing a_j and lowering a_{j-1} by some small $\varepsilon > 0$, all the constraints remain satisfied, and the objective function cannot decrease. By induction, we may as well assume that $a_k = 0$ for all $k = 0, \ldots, j-1$. This, however, implies that $\sum_{k=0}^K a_k f_k < 0$, which is impossible at an optimum, as it would then be feasible to increase a_0 and so increase the objective without violating the constraints. It follows that, at any optimum, $\sum_{k=j}^K a_k f_k = 0$ for all j, and so setting $a_k = 0$ for all k is a solution to the finite program. Hence, setting $\theta = 0$ is a solution to the infinite program, so that the efficient auction maximizes the highest type's expected utility. ■

A.4 Proof of Lemma 2

Fix a candidate equilibrium. Suppose that a buyer with valuation v finds it optimal to accept price p in some period t. Then it must be the case that

$$\sum_{k=0}^{n-1} q(k) \frac{1}{1+k}(v-p) \geq q(0) \sum_{p_\ell \in P} \rho(p_\ell)(v - p_\ell),$$

where $q(k)$ is the probability that k other buyers accept the price p, P is the finite set of prices the seller will set in the remaining $t-1$ periods under the candidate equilibrium, and $\rho(p_\ell)$ is the probability the buyer will purchase the good at such a price. Notice that $\rho(p_\ell)$ combines the buyer's subsequent decisions of when to say yes to a price, as well as the probability that other buyers will accept either p_ℓ or an earlier price. We have $\sum_{p_\ell \in P} \rho(p_\ell) \leq 1$, and hence the derivative in v of the left side of this inequality is at least as large as that of the right. The derivative of the left side is strictly larger than that of the right, and hence all buyers with valuations $v' > v$ find it strictly optimal to also accept the offer (which would suffice for the result), with the possible exception of a case in which $q(0) = 1$ and $\rho(p_\ell)$ attaches probability 1 to prices equal to p. Here, all buyers are indifferent between accepting and rejecting the current price. In this case, there is no loss of generality in taking the accepting set of buyers to be an upper interval of buyer types. ■

A.5 Proof of Proposition 3

We fix Δ (and hence the number of periods T_Δ) and use an induction argument on the number of remaining periods to show that, with t periods to go and beliefs about buyers' types that are uniform over a set $[0, v_{t+1}]$,

(i) the perfect Bayesian equilibrium in the continuation game is unique,

(ii) the seller's payoff equals $\mu_t v_{t+1}$ for some μ_t that is independent of v_{t+1}, and

(iii) the period-t price is such that buyers accept if and only if their valuation exceeds that of an indifferent type v_t given by some $\gamma_t v_{t+1}$, where $\gamma_t \in (0, 1)$ is independent of v_{t+1}.

To show this, we use the seller's first-order conditions to determine a recursion (and initial values) that characterize the sequences γ_t and μ_t. We show that these define a unique sequence, with the property that $\gamma_t < 1$ for all t. We then show that these values achieve a maximum of the seller's objective function.

A.4.1 The Last Period

Consider the last period ($t = 1$) and let the seller's posterior belief be that the buyers' valuations are uniformly distributed on $[0, v_2]$. Then buyer v_i accepts the price p_1 if and only if $p_1 \leq v_i$, and the seller chooses $p_1 = v_1$ to maximize

$$\left(1 - \left(\frac{v_1}{v_2}\right)^n\right) v_1 = \left(\left(1 - \left(\frac{v_1}{v_2}\right)^n\right) \frac{v_1}{v_2}\right) v_2,$$

so indeed $v_1 = \gamma_1 v_2$ is linear in v_2, where γ_1 maximizes

$$(1 - \gamma_1^n)\, \gamma_1, \qquad \text{and hence} \qquad \gamma_1 = (n+1)^{-1/n}.$$

The value of the problem, $V_1(v_2)$, is then

$$V_1(v_1) = \mu_1 v_2, \quad \text{where } \mu_1 = \frac{n}{n+1}\gamma_1,$$

and so V_1 is indeed linear in v_2 as well. This solution is obviously unique.

A.4.2 The Induction Step

Now fix t and assume that for any $\tau < t$ periods to go, and for every uniform distribution of buyer valuations on $[0, v_{\tau+1}]$, the equilibrium is unique and characterized by values μ_τ and $\gamma_\tau < 1$ such that the seller sets a price accepted by all buyers with types above $\gamma_\tau v_{\tau+1}$, for an expected continuation revenue of $\mu_\tau v_{\tau+1}$. Consider the game with t periods to go, and beliefs that are uniform over $[0, v_{t+1}]$.

The buyer's indifference condition. To characterize the buyers' reaction to the seller's prices, suppose that type v_t is indifferent between accepting price p_t and rejecting in order to accept p_{t-1}. If buyer v_t accepts, his payoff is

$$\sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} \left(1 - \left(\frac{v_t}{v_{t+1}}\right)\right)^j \left(\frac{v_t}{v_{t+1}}\right)^{n-1-j} (v_t - p_t) \;=\; \frac{1 - (v_t/v_{t+1})^n}{n\,(1 - (v_t/v_{t+1}))} (v_t - p_t)\,.$$

The first term in the summation is the probability that he is awarded the good if j other buyers accept the posted price, the binomial expression is the probability that j such buyers accept this price, and $v_t - p_t$ is the resulting payoff. By waiting one more period instead, buyer v_t gets

$$\begin{aligned} &\left(\frac{v_t}{v_{t+1}}\right)^{n-1} \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} \left(1 - \left(\frac{v_{t-1}}{v_t}\right)\right)^j \left(\frac{v_{t-1}}{v_t}\right)^{n-1-j} (v_t - p_{t-1}) \\ = \;& \left(\frac{v_t}{v_{t+1}}\right)^{n-1} \frac{1 - (v_{t-1}/v_t)^n}{n\,(1 - (v_{t-1}/v_t))} (v_t - p_{t-1})\,. \end{aligned}$$

Letting $\gamma_t \equiv v_t/v_{t+1}$, and setting these expressions equal, we obtain the indifference condition

$$\frac{1 - \gamma_t^n}{1 - \gamma_t} (v_t - p_t) = \gamma_t^{n-1} \frac{1 - \gamma_{t-1}^n}{1 - \gamma_{t-1}} (v_t - p_{t-1})\,.$$

Hence

$$\begin{aligned} \frac{1 - \gamma_t^n}{1 - \gamma_t} (v_t - p_t) \;&=\; \gamma_t^{n-1} \frac{1 - \gamma_{t-1}^n}{1 - \gamma_{t-1}} (v_t - v_{t-1}) + \gamma_t^{n-1} \frac{1 - \gamma_{t-1}^n}{1 - \gamma_{t-1}} (v_{t-1} - p_{t-1}) \\ &=\; \gamma_t^{n-1} \frac{1 - \gamma_{t-1}^n}{1 - \gamma_{t-1}} (1 - \gamma_{t-1})\, v_t + \gamma_t^{n-1} \left[\gamma_{t-1}^{n-1} \frac{1 - \gamma_{t-2}^n}{1 - \gamma_{t-2}} (v_{t-1} - p_{t-2})\right] \\ &=\; \gamma_t^{n-1} \left(1 - \gamma_{t-1}^n\right) v_t + \gamma_t^{n-1} \gamma_{t-1}^{n-1} \left(1 - \gamma_{t-2}^n\right) v_{t-1} + \cdots\,. \end{aligned}$$

That is,

$$\frac{1 - \gamma_t^n}{1 - \gamma_t} (v_t - p_t) = \sum_{\tau=1}^{t-1} (1 - \gamma_\tau^n) \left(\prod_{l=\tau+1}^{t} \gamma_l^{n-1}\right) v_{\tau+1}. \tag{21}$$

This calculation identifies the critical buyer, given an arbitrary price p_t. Higher-valuation buyers will accept the price and lower-valuation buyers reject it. Note that, dividing both sides by v_t, the right-hand side becomes a telescoping sum, so that

$$\frac{1-\gamma_t^n}{1-\gamma_t}\left(1-\frac{p_t}{v_t}\right)=\gamma_t^{n-1}\left(1-\prod_{\tau=1}^{t-1}\gamma_\tau^n\right).$$

Using $\gamma_t=\frac{v_t}{v_{t+1}}$ and our subsequently introduced convention $q_t\equiv\frac{v_{t+1}}{v_1}$ to simplify (21), we obtain the characterization of buyer behavior given by (8).

The seller's maximization problem. The seller's value V_{t+1} in period t is given by

$$V_{t+1}\left(v_{t+1}\right)=\max_{v_t}\left[\left(1-\left(\frac{v_t}{v_{t+1}}\right)^n\right)p_t+\left(\frac{v_t}{v_{t+1}}\right)^n V_t\left(v_t\right)\right]$$

where p_t is given by (21). We can use (21) to rewrite this as

$$\begin{aligned}
V_{t+1}\left(v_{t+1}\right) &= \max_{\gamma_t}\left[\left(1-\gamma_t^n\right)\left(p_t-v_t\right)+\left(1-\gamma_t^n\right)v_t+\gamma_t^n V_t\left(v_t\right)\right]\\
&= \max_{\gamma_t}\left[-\left(1-\gamma_t\right)\sum_{\tau=1}^{t-1}\left(1-\gamma_\tau^n\right)\left(\prod_{l=\tau+1}^{t}\gamma_l^{n-1}\right)v_{\tau+1}+\left(1-\gamma_t^n\right)v_t+\gamma_t^n V_t\left(v_t\right)\right].
\end{aligned}$$

Dividing by v_{t+1}, we have

$$\begin{aligned}
\frac{V_{t+1}(v_{t+1})}{v_{t+1}} &= \max_{\gamma_t}\left[-\left(1-\gamma_t\right)\sum_{\tau=1}^{t-1}\left(1-\gamma_\tau^n\right)\left(\prod_{l=\tau+1}^{t}\gamma_l^{n-1}\right)\frac{v_{\tau+1}}{v_{t+1}}+\left(1-\gamma_t^n\right)\frac{v_t}{v_{t+1}}+\gamma_t^n\mu_{t-1}\frac{v_t}{v_{t+1}}\right]\\
&= \max_{\gamma_t}\left[-\left(1-\gamma_t\right)\sum_{\tau=1}^{t-1}\left(1-\gamma_\tau^n\right)\left(\prod_{l=\tau+1}^{t}\gamma_l^{n}\right)+\gamma_t\left(1-\gamma_t^n\right)+\gamma_t^{n+1}\mu_{t-1}\right]\\
&= \max_{\gamma_t}\left[\left(1-\gamma_t\right)\prod_{\tau=1}^{t}\gamma_\tau^n+\gamma_t\left(1-\gamma_t^{n-1}\right)+\gamma_t^{n+1}\mu_{t-1}\right],
\end{aligned}\tag{22}$$

which is an expression that is independent of v_{t+1}, and we may thus define $\mu_t\equiv\frac{V_{t+1}(v_{t+1})}{v_{t+1}}$.

The seller's maximization. The first and second derivatives of the seller's objective (22) are

$$(n\gamma_t^{n-1}-(n+1)\gamma_t^n)\prod_{\tau=1}^{t-1}\gamma_\tau^n+1-n\gamma_t^{n-1}+(n+1)\gamma_t^n\mu_{t-1},\tag{23}$$

and

$$((n-1)n\gamma_t^{n-2} - n(n+1)\gamma_t^{n-1}) \prod_{\tau=1}^{t-1} \gamma_\tau^n - (n-1)n\gamma_t^{n-2} + n(n+1)\gamma_t^{n-1}\mu_{t-1},$$

respectively. Together with (21), this allows us to obtain the characterization of prices given in (7). The second derivative can be rewritten as

$$\frac{n}{\gamma_t}\left((n\gamma_t^{n-1} - (n+1)\gamma_t^n) \prod_{\tau=1}^{t-1} \gamma_\tau^n - n\gamma_t^{n-1} + (n+1)\gamma_t^n\mu_{t-1}\right) - n\gamma_t^{n-2} \prod_{\tau=1}^{t-1} \gamma_\tau^n + n\gamma_t^{n-2}.$$

When the first derivative equals zero, the terms in parentheses in this second derivative equal negative one, giving a second derivative of

$$n(-\gamma_t^{-1} - \gamma_t^{n-2} \prod_{\tau=1}^{t-1} \gamma_\tau^n + \gamma_t^{n-2}),$$

which is negative if $\gamma_t \in (0, 1]$. Hence, whenever the first derivative has an interior solution, the second (evaluated at that solution) is negative. This in turn ensures that if the first-order condition induced by (23) has an interior solution, that solution is unique and is a global maximizer.

Uniqueness. We must now show that the first-order condition induced by (23) has a unique, interior solution. Hence, we must show that (23) determines a sequence $\{\gamma_t\}$ with each $\gamma_t \in (0, 1)$. Let $q_t = (\prod_{\tau=1}^t \gamma_\tau)^{-1}$, so $\gamma_t = q_{t-1}/q_t$. We can then rewrite the first-order condition (23) as

$$(n+1)\left(q_{t-1}^{-n} - \mu_{t-1}\right)\left(\frac{q_{t-1}}{q_t}\right)^n + n\left(1 - q_{t-1}^{-n}\right)\left(\frac{q_{t-1}}{q_t}\right)^{n-1} = 1, \tag{24}$$

and the seller's maximization problem given by (22) to get

$$\mu_t = \left(1 - \frac{q_{t-1}}{q_t}\right) q_t^{-n} + \frac{q_{t-1}}{q_t}\left(1 - \left(\frac{q_{t-1}}{q_t}\right)^{n-1}\right) + (\frac{q_{t-1}}{q_t})^{n+1}\mu_{t-1},$$

that is,

$$\mu_t q_t^{n+1} - q_t = q_t q_{t-1}\left(q_t^{n-1} - q_{t-1}^{n-1}\right) + \mu_{t-1}q_{t-1}^{n+1} - q_{t-1}. \tag{25}$$

Now let $\xi_t \equiv \mu_t q_t^{n+1} - q_t$. Notice that the first definition give (9). Then we can rewrite (24) and (25) as

$$(n+1)\,\xi_{t-1} = n\left(q_{t-1}^n - 1\right) q_t - q_t^n q_{t-1}, \tag{26}$$

and

$$\xi_t = \xi_{t-1} + q_t q_{t-1}\left(q_t^{n-1} - q_{t-1}^{n-1}\right). \tag{27}$$

We then combine (26) and (27) to get

$$n\left(q_t^n - 1\right)q_{t+1} - q_{t+1}^n q_t = n\left(q_{t-1}^n - 1\right)q_t - q_t^n q_{t-1} + (n+1)\,q_t q_{t-1}\left(q_t^{n-1} - q_{t-1}^{n-1}\right),$$

or rearranging,

$$q_{t+1}^n - n\frac{q_t^n - 1}{q_t}q_{t+1} + n\left(q_t^{n-1}q_{t-1} - 1\right) - q_{t-1}^n = 0, \tag{28}$$

which holds for $t \geq 1$ provided we adopt the convention $q_0 = 1$ and recall that $q_1 = (n+1)^{1/n}$. This gives us the difference equation given by (6).

Observe now that the sequence $\{\gamma_t\}$ is in $(0,1)$ if and only if the sequence $\{q_t\}$ is strictly increasing. The following lemma establishes that this is the case:

Lemma 4. *Consider the polynomial P defined by*

$$P(x) \equiv x^n - n\frac{q_t^n - 1}{q_t}x + n\left(q_t^{n-1}q_{t-1} - 1\right) - q_{t-1}^n. \tag{29}$$

For each $q_{t-1} < q_t$ with $q_t > 1$, P admits a unique real root strictly larger than q_t.

Proof. Assume throughout that $q_{t-1} < q_t$. The polynomial P has two real roots if n is even, and three if n is odd. To see this, observe that for n even, it is a convex function that is negative for $x = q_t$, since

$$\begin{aligned} P\left(q_t\right) &= q_t^n - n\frac{q_t^n - 1}{q_t}q_t + n\left(q_t^{n-1}q_{t-1} - 1\right) - q_{t-1}^n \leq 0 \\ &\Leftrightarrow q_t^n - q_{t-1}^n \leq n q_t^{n-1}\left(q_t - q_{t-1}\right), \end{aligned}$$

which is the case since the function $x \mapsto x^n$ is convex for $n \geq 2$. Observe that this also establishes that P admits a real root larger than q_t. If n is odd, then P is concave on $\mathbb{R}_-$ and convex on $\mathbb{R}_+$. Further, $P(0) = n\left(q_t^{n-1}q_{t-1} - 1\right) - q_{t-1}^n \geq 0$, and (as noted) $P\left(q_t\right) \leq 0$. So, in all cases, P uniquely admits a real root x that is strictly larger than q_t. □

■

A.6 Proof of Proposition 4

A.5.1 Characterizing v_t

We investigate the sequence of critical valuations $\{v_t\}$, leading to the demonstration of Proposition 4.1 and 4.2. The heart of the argument is contained in the following three lemmas. Let $x(q_t, q_{t-1})$ denote the unique root larger than q_t solving (29).

Lemma 5.

(5.1) The root $x(q_t, q_{t-1})$ is contained in $(q_t, q_t + (q_t - q_{t-1}))$.

(5.2) For $q_{t-1} < q_t$, $x(q_t, q_{t-1})$ is strictly decreasing in q_{t-1}, and holding q_t/q_{t-1} fixed, the ratio $x(q_t, q_{t-1})/q_t$ is an increasing function of q_t.

Recall that $q_t = \frac{v_t}{v_1}$. Hence, Lemma 5.1 indicates that as the seller moves up the interval of possible buyer valuations (i.e., moves earlier in the sequence of periods $(T_\Delta, T_\Delta - 1, \ldots, 1)$), she slices off smaller and smaller intervals of buyer valuations to which to sell: $v_t - v_{t-1}$ is decreasing in t. Intuitively, the seller discriminates more finely among higher-valuation buyers. Lemma 5.2 assembles some technical results to be used in proving Lemma 6.

Proof. For (5.1), let $q_{t-1} = q(1-\alpha)$, for some $\alpha \in (0,1)$ and $q \geq 1$, $q_t = q$ and consider $P(q(1+\alpha))$. Now,

$$P(q(1+\alpha)) = (1+\alpha)^n q^n - (1-\alpha)nq^n + n\alpha(1-2q^n) > 0,$$

because

$$(1+\alpha)^n - (1-\alpha)^n > 2n\alpha,$$

as the left-hand side is convex in α with derivative equal to $2n$ at $\alpha = 0$. Therefore, it must be that $q(1+\alpha) > x$ and so $x - q_t \leq q_t - q_{t-1}$.

The first part of (5.2) is immediate, since $dP/dq_{t-1} > 0$. As for the second part, observe that we can rewrite (28) as

$$r_t^n - r_{t-1}^{-n} - n(r_t - r_{t-1}^{-1}) - \frac{n}{q_t^n}(1 - r_t) = 0,$$

where $r_t \equiv q_{t+1}/q_t$ for all t. Fixing r_{t-1}, it follows that r_t is increasing in q_t, since the left-hand side is increasing in r_t (note that $r_t > 1$) and decreasing in q_t. □

Lemma 6. *Consider a sequence u_t with $q_0 = u_0$, $q_1 \geq u_1$, and for every $t \geq 2$, $u_{t+1} \leq x(u_t, u_{t-1})$. Then $q_t \geq u_t$ for all t.*

Proof. The proof is by induction on t. Observe that, for $t = 1$, by construction both $q_1 \geq u_1$ and $q_1/q_0 \geq u_1/u_0$. Assume now that, for some $t \geq 1$, both $q_\tau \geq u_\tau$ and $q_\tau/q_{\tau-1} \geq u_\tau/u_{\tau-1}$ for all $\tau \leq t$. It follows that

$$\frac{q_{t+1}}{q_t} = \frac{x(q_t, q_{t-1})}{q_t} \geq \frac{x(u_t, \frac{u_t}{q_t} q_{t-1})}{u_t} \geq \frac{x(u_t, u_{t-1})}{u_t} \geq \frac{u_{t+1}}{u_t}.$$

The first inequality follows from the second part of Lemma 5.2, given that $u_t \leq q_t$. The second inequality follows from the facts that $\frac{u_t}{q_t} q_{t-1} \leq u_{t-1}$ (by the induction hypothesis) and $x(q_t, q_{t-1})$ is decreasing in its second argument (the first part of Lemma 5.2). The final inequality follows the fact that $x(u_t, u_{t-1})$ is an upper bound on u_{t+1}. Since $q_0 = u_0$, the conclusion that $q_{t+1} \geq u_{t+1}$ follows from this inequality and the induction hypothesis. □

Lemma 7. *Consider the sequence* $\{u_t\}_{t=0}^{\infty}$ *defined by* $u_t = (1 + n(t-1)t/2)^{1/n}$*, for all* $t \geq 0$*. The sequence* u_t *diverges and, for all* $t \geq 1$ *and all* $n \geq 6$*,* $u_t \leq q_t$*.*

Proof. Divergence is immediate from the definition of u_t. We can calculate that $u_0 = 1 = q_0$ and $u_1 = 1 < (n+1)^{\frac{1}{n}} = q_1$. The result then follows from Lemma 6 and the fact that, for every $t \geq 2$, $u_{t+1} \leq x(u_t, u_{t-1})$. This last inequality is established via a tedious calculation. Details are presented in Section B.3. □

Establishing statements (4.1) and (4.2) of Proposition 4 is now straightforward. Recall that, in an optimal auction with zero reserve price, the expected revenue is given by

$$\pi^D(n) = \frac{n-1}{n+1}.$$

This value is therefore an upper bound on the expected revenue that the seller can hope for in the dynamic game as $\Delta \to 0$, *if* $\lim_{\Delta \to 0} v_{\Delta 1} = 0$, or equivalently $\lim_{t \to \infty} q_t = \infty$. For $n \geq 6$, it follows from Lemma 6 that $\lim_{t \to \infty} q_t = \infty$ and hence $\lim_{\Delta \to 0} v_{\Delta 1} = 0$. The best the seller can hope for, as $\Delta \to 0$, is therefore $\pi^D(n)$. Because $q_t - q_{t-1}$ is decreasing in t (Lemma 5.1), it is bounded, and therefore $\lim_{\Delta \to 0} \max_{t \leq T_\Delta} v_{\Delta t} - v_{\Delta, t-1} = 0$, and so also $\lim_{\Delta \to 0} \max_{t \leq T_\Delta} p_{\Delta t} - p_{\Delta, t-1} = 0$, where $p_{\Delta t}$ is the price charged with t periods to go in the game with period Δ and hence T_Δ stages. It then follows from Proposition 1 in Chwe [12] that the expected revenue converges to $\pi^D(n)$. This gives the second conclusion of Proposition 4.

What if $n < 6$? We can explicitly compute the first terms of μ_t for $n \in \{2, \ldots, 5\}$, and observe that $\mu_t > \pi^D(n)$ for $t = 1$ if $n = 2, 3$, $t = 4$ if $n = 4$, and $t = 36$ if $n = 5$. Since one feasible strategy for the seller is to set $p_\tau = 1$ until period $t = 1$ (if $n = 2, 3$), $t = 4$ (if $n = 4$) or $t = 36$ (if $n = 5$) and then obtain value μ_t, the seller's optimal strategy must give a payoff exceeding $\pi^D(n)$, and hence $\lim_{\Delta \to 0} \mu_{T_\Delta} > \pi^D(n)$. The preceding argument establishes that a necessary condition for such a limiting payoff is that $\lim_{\Delta \to 0} v_{\Delta 1} > 0$. This establishes the first part of Proposition 4.

A.5.3 Declining Terminal Prices

We prove here that $\lim_{\Delta \to 0} v_{\Delta 1}$ is decreasing in n, giving Proposition 4.3. In particular, $\lim_{\Delta \to 0} v_{\Delta 1}$ is then lower than the last price quoted in an optimal auction with commitment, as the reserve price (which is the limit of the lowest price in the dynamic game with commitment) equals 1/2, which is $\lim_{\Delta \to 0} v_{\Delta 1}$ when $n = 1$.

The result is proved in several steps. First, recall that $v_1 = \gamma_1 v_2$, where $\gamma_1 = (n+1)^{-n}$. Now consider the following auction, parameterized by v. First, the auctioneer continuously lowers the price until the indifferent type is v. At this stage, if the unit is still not accepted, he offers the price $w = \gamma_1 v$, i.e. the monopoly price on the residual demand. If it is also rejected, the auction is over. We may compute the revenue from such an auction by first computing the probability $q(x)$ that a buyer of type x wins the object. This equals 0 if $x < w$, $(v^n - w^n)/(n(v-w))$ if $x \in [w, v)$, and x^{n-1} for $x \geq v$. The price that type x accepts is as usual $p(x) = q(x) - \int_0^x q(t)dt/q(x)$, and expected revenue $R_n(w)$, which equals $\int_0^1 p(x)dF^n(x)$, is then

$$R_n(w) = \frac{n-1}{n+1} - (n((n+1)^{1/n} - 1)w - 1)w^n,$$

which is a function of w that is increasing up to $((n+1)^{1+1/n} - n - 1)^{-1}$, and then decreasing.

Consider $n = 2, \ldots, 5$. We first claim that, given $w = \lim_{\Delta \to 0} v_{\Delta 1}$, the revenue $R_n(w)$ exceeds the limiting revenue from the equilibrium of the dynamic game (as $\Delta \to 0$). Indeed, consider the two allocations corresponding to each mechanism, the auction described above, and the allocation from the limit. In both cases, buyers' types below w do not get the unit; types in $[w, v)$ get it only if there is no type above v, with the same probability in both cases ($v = \lim_{\Delta \to 0} v_{\Delta 2}$, since the price in the last period is the monopoly price on the residual demand). So the difference originates from types above v. However, for such types, the auction described above achieves an efficient allocation, while this is not necessarily true in the other case. Since with a uniform distribution, the virtual valuation is strictly increasing in types, it follows that $R_n(w)$ exceeds the revenue from the limit of the dynamic game, and hence from the dynamic game, independently of the length of the horizon (since the seller's payoff increases with T).

By considering the first terms of the sequences μ_t (recall that it is a non-decreasing sequence) we obtain that $\lim_{\Delta \to 0} \mu_{T_\Delta} > 4/10$ for $n = 2$, $\lim_{\Delta \to 0} \mu_{T_\Delta} > .515$ for $n = 3$, and $\lim_{\Delta \to 0} \mu_{T_\Delta} > .6019$ for $n = 4$. Yet $R_n(w)$ exceeds those values only if $w > 4/10$ (for $n = 2$), $w > .32$ (for $n = 3$) and $w > .24$ (for $n = 4$). Since the sequence $1/q_t$ is decreasing, with $\lim_{t \to \infty} 1/q_t = \lim_{\Delta \to 0} v_{\Delta 1}$, it is now easy to verify that, after computing the first few terms, $\lim_{t \to 0} 1/q_t$ is less than .4 for $n = 3$, less than .32 for $n = 4$ and less than .2 for $n = 5$. It follows that $\lim_{\Delta \to 0} v_{\Delta 1}$ is decreasing in n for $n = 3, 4, 5$. Since this limit is 0 for $n \geq 6$, the same holds for all $n > 2$, and clearly the conclusion also holds for $n = 1$ and $n = 2$ (in the former case, the only price accepted with positive probability is 1/2, while in the latter case, by computing the first few terms, it is verified that $\lim_{\Delta \to 0} v_{\Delta 1} < 1/2$.) ■

A.7 Proof of Proposition 5

A.6.1 The Seller's Payoff with Commitment

We first express the seller's payoff in terms of the indifferent buyers' valuations. Fix a period length Δ and hence number of periods T_Δ, and then suppress Δ in the notation. The seller's payoff with commitment can be written as

$$\Pi = (1 - v_T^n)\, p_T + \left(v_T^n - v_{T-1}^n\right) p_{T-1} + \cdots + (v_2^n - v_1^n)\, p_1, \tag{30}$$

where

$$\begin{aligned}
\frac{v_{t+1}^n - v_t^n}{v_{t+1} - v_t}(v_t - p_t) &= \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_t - p_{t-1}) = v_t^n - v_{t-1}^n + \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_{t-1} - p_{t-1}) \\
&= v_t^n - v_{t-2}^n + \frac{v_{t-1}^n - v_{t-2}^n}{v_{t-1} - v_{t-2}}(v_{t-2} - p_{t-2}) = \cdots = v_t^n - v_1^n,
\end{aligned}$$

so that

$$\left(v_{t+1}^n - v_t^n\right) p_t = \left(v_{t+1}^n - v_t^n\right) v_t - (v_{t+1} - v_t)(v_t^n - v_1^n). \tag{31}$$

Substituting (31) into (30), we have

$$\begin{aligned}
\Pi &= (1 - v_T^n) v_T + \left(v_T^n - v_{T-1}^n\right) v_{T-1} + \cdots + (v_2^n - v_1^n) v_1 \\
&\quad - (1 - v_T)(v_T^n - v_1^n) - (v_{T-1} - v_T)\left(v_{T-1}^n - v_1^n\right) - \cdots - (v_3 - v_2)(v_2^n - v_1^n) \\
&= v_T - (1 - v_{T-1}) v_T^n - (v_T - v_{T-2}) v_{T-1}^n - \cdots - (v_3 - v_1) v_2^n + (1 - v_2 - v_1) v_1^n.
\end{aligned}$$

We can think of the seller as choosing the identities of the indifferent buyers in order to maximize this payoff. Taking derivatives with respect to these valuations (and setting $v_{T+1} = 1$), we obtain the first-order conditions

$$n v_t^{n-1} = \frac{v_{t+1}^n - v_{t-1}^n}{v_{t+1} - v_{t-1}} \;\; (t = 2, \ldots, T), \tag{32}$$

$$n v_1^{n-1}(1 - v_2 - v_1) = v_2^n - v_1^n. \tag{33}$$

The first formula can be re-written as

$$\sigma_t^n - n\sigma_t = \sigma_{t-1}^{-n} - n\sigma_{t-1}^{-1},$$

where $\sigma_t \equiv v_{t+1}/v_t$.

A.6.2 Two Preliminary Inequalities

This section collects two useful technical results.

Lemma 8. *Let $h(x) \equiv x^n - nx$. Then, for $n \geq 2$,*

$$h(2 - x) \geq h(x) \;\; (x \in [0, 1]), \; \text{and} \; \lim_{x \uparrow 1} \frac{h^{-1} \circ h(x) - 1}{x - 1} = -1, \tag{34}$$

where h^{-1} is the inverse of $h : [0, \infty) \to \mathbb{R}$.

Proof. Because the function $y \mapsto y^n$ is convex,

$$(1+y)^n - (1-y)^n \geq 2ny,$$

for $y \in [0,1]$, so that, for $x = 1-y$,

$$(2-x)^n - n(2-x) \geq x^n - nx,$$

i.e. $h(2-x) \geq h(x)$. Now, observe that the limit is simply the derivative of $h^{-1} \circ h(x)$ at 1. Because $h'(1) = 0$,

$$h(1-\varepsilon) - h(1) = \frac{h''(1)}{2}\varepsilon^2 + o\left(\varepsilon^3\right), h(1+\delta) - h(1) = \frac{h''(1)}{2}\delta^2 + o\left(\delta^3\right),$$

and so, if $h(1-\varepsilon) = h(1+\delta) \to h(1)$, it follows that $\varepsilon/\delta \to 1$, so that $\left(h^{-1} \circ h\right)'(1) = 1$. $\square$

Lemma 9. *For all $n \geq 2$, there exists $K > 0$ such that, for all $t \geq 1$,*

$$h\left(\left(1 + \frac{1}{t+K}\right)^{-\frac{3}{n+1}}\right) \geq h\left(\left(1 + \frac{1}{t+1+K}\right)^{\frac{3}{n+1}}\right). \tag{35}$$

Proof. For $n = 2$, it is easy to verify that the two sides are equal, independently of the value of K. Consider $n > 2$. Taking a Taylor expansion, we have that

$$h\left((1+y)^{-\frac{3}{n+1}}\right) - h\left(\left(1 + \frac{y}{1+y}\right)^{\frac{3}{n+1}}\right) = \frac{3n(n-1)(n-2)(2n-1)}{5}y^5 + o\left(y^6\right),$$

so that there exists $\bar{y}$ such that, for all $y \in [0, \bar{y}]$,

$$h\left((1+y)^{-\frac{3}{n+1}}\right) \geq h\left(\left(1 + \frac{y}{1+y}\right)^{\frac{3}{n+1}}\right).$$

Letting $K \equiv \bar{y}^{-1} - 1$, the result follows. $\square$

A.6.3 Properties of the Commitment Solution

We now use these inequalities to characterize the sequence $\{v_t\}_{t=1}^{\infty}$ of critical buyer types.[32] Fix $v_1 \in (0,1)$ and $\sigma_1 > 1$ and consider the sequence $\{v_t\}_{t=1}^{\infty}$ defined by v_1, σ_1 and

$$\sigma_t^n - n\sigma_t = \sigma_{t-1}^{-n} - n\sigma_{t-1}^{-1}, \text{ i.e. } h(\sigma_t) = h\left(\sigma_{t-1}^{-1}\right),$$

for $h(x) = x^n - nx$. Observe that, since h is decreasing on $[0,1]$, and increasing on $[1,\infty)$, $\sigma_t \geq 1$ for all t. Further, because $h(x) \geq h\left(x^{-1}\right)$ for all $x \geq 1$, it is strictly decreasing in t, with limit given by 1.

[32]For a fixed Δ, only the first T_Δ terms in the infinite sequence we study will be relevant, but the entire infinite sequence will come into play as $\Delta \to 0$.

Lemma 10. *1. For all n, the sequence $\{v_t\}$ is concave, with*

$$\lim_{t\to\infty}\frac{v_{t+1}-v_t}{v_t-v_{t-1}}=1.$$

2. For all n, there exists $K>0$ such that

$$\sigma_t \geq \left(1+\frac{1}{t+K}\right)^{\frac{3}{n+1}}.$$

3. For all n, and $m\in\mathbb{N}$,

$$\underline{\lim}_{t\to\infty}\frac{v_{mt}}{v_t}\geq m^{\frac{3}{n+1}}.$$

We use Lemma 10.2 in the proof of Lemma 10.3, and use Lemmas 10.1 and Lemma 10.3 in Section A.4.3.

Proof. First, observe that

$$v_{t+1}-v_t\leq v_t-v_{t-1}\Leftrightarrow \sigma_t\leq 2-\sigma_{t-1}^{-1},$$

for $\sigma_t=v_{t+1}/v_t$. Now

$$h(\sigma_t)=h\left(\sigma_{t-1}^{-1}\right)\leq h\left(2-\sigma_{t-1}^{-1}\right),$$

where the last inequality follows from (34), given that $\sigma_{t-1}^{-1}\leq 1$. Since h is increasing for $x\geq 1$, and both $\sigma_t\geq 1$ and $2-\sigma_{t-1}^{-1}\geq 1$, it follows that indeed $\sigma_t\leq 2-\sigma_{t-1}^{-1}$, so that the sequence v_t is concave. Further, since

$$\frac{v_{t+1}-v_t}{v_t-v_{t-1}}=\frac{\sigma_t-1}{1-\sigma_{t-1}^{-1}}=\frac{h^{-1}\circ h\left(\sigma_{t-1}^{-1}\right)-1}{1-\sigma_{t-1}^{-1}},$$

and $\lim_t\sigma_t=1$, it follows from $\lim_{x\uparrow 1}\left(h^{-1}\circ h(x)-1\right)/(1-x)=1$ that

$$\lim_t\left(v_{t+1}-v_t\right)/\left(v_t-v_{t-1}\right)=1.$$

Given σ_1, fix K such that both $\sigma_1\geq\left(1+\frac{1}{1+K}\right)^{\frac{3}{n+1}}$ and (35) is satisfied. Let

$$\nu_t\equiv\left(1+\frac{1}{t+K}\right)^{\frac{3}{n+1}}.$$

By induction, we show that $\sigma_t\geq\nu_t$. By definition of K, $\sigma_1\geq\nu_1$. Suppose now that $\sigma_{t-1}\geq\nu_{t-1}$. Since h is decreasing on $[0,1]$, and given (35),

$$h(\sigma_t)=h\left(\sigma_{t-1}^{-1}\right)\geq h\left(\nu_{t-1}^{-1}\right)\geq h(\nu_t),$$

and since h is increasing on $[1,\infty)$,

$$\sigma_t \geq \nu_t.$$

Observe that

$$\frac{v_{mt}}{v_t} = \Pi_{\tau=t}^{mt-1}\sigma_\tau \geq \Pi_{\tau=t}^{mt-1}v_\tau = \left(\frac{mt+K}{t+K}\right)^{\frac{3}{n+1}},$$

so that

$$\underline{\lim}_t \frac{v_{mt}}{v_t} \geq m^{\frac{3}{n+1}}.$$

□

Lemma 10 tells us about the sequence $\{v_t\}_{t=1}^{\infty}$ given a value v_1. We must next identify the appropriate value v_1. One strategy available to the seller is to set a price with t periods to go equal to $\frac{1+t/T}{2}$, causing v_1 to converge to $\frac{1}{2}$ as Δ gets small (and hence T_Δ large). It follows from standard results (Athey [1]) that her revenue then converges to the revenue of the optimal auction. Conversely, her revenue converges to the revenue of the optimal auction only if $p_1 = v_1$ converges to $1/2$ as Δ gets small, allowing us to take $v_1 = \frac{1}{2}$. It follows from the first-order conditions (32)–(33) that v_2 then converges to v_1, so that asymptotically the entire sequence $\{v_t\}_{t=1}^{\infty}$ is contained with $[0,1]$.

A.6.4 The Limit $\Delta \to 0$

We now consider the limit $\Delta \to 0$. Consider the sequence of functions $v_\Delta(x)$ on $[0,1]$ defined as follows. For any period length Δ, define the step function

$$v_\Delta(x) = v_{\Delta t} \text{ for all } x \in \left[\frac{t-1}{T_\Delta}, \frac{t}{T_\Delta}\right), \; v_\Delta(1) = 1.$$

Pick a subsequence of functions $\{v_\Delta(x)\}$ that converges on the rationals, to some limit function. Because each sequence is non-decreasing, so must be the limit, and let $x \mapsto v(x)$ denote the right-continuous extension of this limit. Since the sequence $\{v_t\}$ is concave (Lemma 10.1), the function v must be concave, and it is therefore continuous on $(0,1)$, and admits left- and right-derivatives everywhere on $(0,1)$.

Because the sequence σ_t defined by a value of σ_1 and the recursion $h(\sigma_t) = h\left(\sigma_{t-1}^{-1}\right)$ is increasing in σ_1, and given that $v_{T_\Delta} = 1$, it follows that the value of σ_1 solving the commitment problem for fixed v_1 is decreasing in v_1. Since $\lim_{\Delta\to 0} v_1 = 1/2$, σ_1 is bounded above in Δ, so that, since for a fixed σ_1,

$$\lim_{t\to\infty} \frac{v_{t+1}-v_t}{v_t - v_{t-1}} = 1,$$

it follows also that, for all values $k > 0$ such that $kT \in \mathbb{N}$,

$$\lim_{T\to\infty} \frac{v_{kT+1} - v_{kT}}{v_{kT} - v_{kT-1}} = 1.$$

It follows that the left- and right derivatives of v agree everywhere, so that v is differentiable on $(0, 1)$. Therefore, considering the equation

$$nv\left(x\right)^{n-1}\left(v\left(x+\delta\right) - v\left(x-\delta\right)\right) = \left(v\left(x+\delta\right)^n - v\left(x-\delta\right)^n\right),$$

we might use a Taylor expansion to the third degree as $\delta \to 0$, to obtain

$$n\left(n-1\right)v\left(x\right)^{n-3}v'\left(x\right)\left[v\left(x\right)v''\left(x\right) + \frac{\left(n-2\right)}{3}v'\left(x\right)^2\right]\delta^3 + o\left(\delta^4\right).$$

Because $\underline{\lim}_{t\to\infty} \frac{v_{mt}}{v_t} \geq m^{\frac{3}{n+1}}$ for all m (Lemma 10.3), $v'\left(x\right) > 0$. Hence it must be that

$$v\left(x\right)v''\left(x\right) + \frac{\left(n-2\right)}{3}v'\left(x\right)^2 = 0.$$

This differential equation has as general solution

$$v\left(x\right) = K_1\left(x + K_2\right)^{\frac{3}{n+1}},$$

for constants K_1, K_2, and our boundary conditions $v\left(1\right) = 1/2$, $v\left(1\right) = 1$ allow us to identify these constants, giving (11):

$$v\left(x\right) = \frac{1}{2}\left(\left(2^{\frac{n+1}{3}} - 1\right)x + 1\right)^{\frac{3}{n+1}}.$$

Since

$$\frac{v_{t+1}^n - v_t^n}{v_{t+1} - v_t}\left(v_t - p_t\right) = v_t^n - v_1^n,$$

and $\lim_{\varepsilon\to 0} \frac{v(x+\varepsilon)^n - v(x)^n}{v(x+\varepsilon) - v(x)} = nv(x)^{n-1}$, it follows that $nv(x)^{n-1}(v(x) - p(x)) = v(x)^n - v(0)^n$, and the expression (12) for $p(x)$ follows. ■

A.8 Proof of Proposition 6

Our characterization of the non-commitment solution builds on our proof of Proposition 4. We first derive an asymptotic estimate of the sequence q_t/q_{t+1} (from (6). The polynomial (28) that defines q_t can be rewritten as

$$q_{t+1}^n - q_{t-1}^n = \frac{n}{q_t}(q_t^n(q_{t+1} - q_{t-1}) - (q_{t+1} - q_t)).$$

Since the sequence q_t diverges, we may ignore the second term from the right-hand side, and so, defining $s_t = q_t/q_{t+1}$ (i.e., in terms of the notation of Section A.3.2, $s_t = r_t^{-1}$), we have, for large t,

$$s_t^{-n} - s_{t-1}^n - n(s_t^{-1} - s_{t-1}) \approx 0.$$

As we also know that $s_t \to 1$, we let $s_t = 1 - \varepsilon_t$, and, so using Taylor expansions to the third order,

$$3\varepsilon_t^2 + (n+4)\varepsilon_t^3 - 3\varepsilon_{t-1}^2 + (n-2)\varepsilon_{t-1}^3 \approx 0.$$

Since $\varepsilon_t \to 0$, this implies that $\lambda_t \equiv \varepsilon_t/\varepsilon_{t-1} \to 1$. Rewriting this equation, we have

$$3(\varepsilon_t - \varepsilon_{t-1})(1+\lambda_t)\varepsilon_{t-1} + ((n+4)\lambda_t^2 + (n-2)\lambda_t^{-1})\varepsilon_t\varepsilon_{t-1}^2 = 0,$$

so, approximately,

$$\varepsilon_t - \varepsilon_{t-1} + \frac{n+1}{3}\varepsilon_{t-1}\varepsilon_t = 0.$$

If we let $\mu_t = (n+1)\varepsilon_t/3$, this gives

$$\mu_{t-1} - \mu_t = \mu_t\mu_{t-1},$$

or

$$1/\mu_t - 1/\mu_{t-1} = 1,$$

so we obtain that $\mu_t = (t+C)^{-1}$, for a constant C (possibly infinite). That is, for large t, either $\varepsilon_t = 0$ or $\varepsilon_t = \frac{3}{n+1}t^{-1}$. However, recall that we already know (cf. Lemma 6) that

$$\frac{q_t}{q_{t+1}} \le \frac{u_t}{u_{t+1}} = \left(1 - \frac{nt}{1 + \frac{nt(t+1)}{2}}\right)^{1/n} < 1 - \frac{n}{t},$$

and so the possibility that $\varepsilon_t = 0$ could be ruled out. We conclude that $s_t = 1 - \frac{3}{(n+1)t}$ asymptotically.[33]

It also follows that

$$\lim_t \frac{q_{t+1} - q_t}{q_t - q_{t-1}} = \lim_t \frac{s_t^{-1} - 1}{1 - s_{t-1}} = \lim \frac{t}{t-1} = 1.$$

[33] Observe that we made approximations sequentially in the process of deriving this solution. If we plug in our solution into the recursion involving s_t and s_{t-1}, we find that the second approximation is of the order $o(t^{-3})$, and the term that was ignored in the initial polynomial is of the same order, so the order of approximations is irrelevant. Also, observe that, since the term that is being ignored is of order $o(t^{-3})$, yet the slope of the function $s_t \mapsto s_t^{-n} - s_{t-1}^n - n(s_t^{-1} - s_{t-1})$ at 1 is equal to $o(t^{-1})$, the impact of the approximation is of the order $o(t^{-2})$, so that even the cumulative impact of the approximations is negligible, justifying the approximation.

Therefore, if we define, as in the case with commitment, the sequence of functions $v_\Delta(x)$ on $[0,1]$ as the step function

$$v_\Delta(x) = v_t \text{ for all } x \in \left[\frac{t-1}{T_\Delta}, \frac{t}{T_\Delta}\right), \; v_\Delta(1)(1) = 1,$$

and, following what has been done with commitment, we pick a subsequence of functions $\left\{v^T(x)\right\}$ that converges on the rationals, to some limit function (which, because each sequence is non-decreasing, is non-decreasing as well, as well as concave since the sequence q_t is), and we let $x \mapsto v(x)$ denote the right-continuous extension of this limit, it follows that the left- and right-derivatives coincide everywhere on $(0,1)$. Now,

$$v'(x) = \lim_{\Delta\to 0} \frac{v_{t+1} - v_t}{\Delta} = \lim_{\Delta\to 0} T\frac{q_t - q_{t-1}}{q_T} = \lim_{\Delta\to 0} \frac{T}{t}\frac{q_t}{q_T}t\frac{q_t - q_{t-1}}{q_t} = \frac{3}{n+1}\frac{v(x)}{x},$$

with boundary condition $v(1) = 1$. This gives $v(x) = x^{\frac{3}{n+1}}$, or (13). Since

$$\frac{v_{t+1}^n - v_t^n}{v_{t+1} - v_t}(v_t - p_t) = v_t^n - v_0^n,$$

the solution (14) for $p(x)$ follows.

B Appendix: Not for Publication

B.1 Calculations, Section 2

In performing the calculations behind the example in Section 2, it is helpful to define the seller's payoff not in terms of the prices set in each period, but the type of buyer who purchases in each period. Hence, let $\Pi(3,2,1)$ be the payoff to a pricing scheme that induces type v_3 buyers to purchase in the first period, type v_2 buyers to purchase in the second, and type v_1 to purchase in the third. We use an "x" to denote a period in which no buyers purchase, so that $\Pi(x,x,3)$ is the payoff of waiting until the final period and then selling to buyers of type v_3. We can assume that all buyers of type higher than that indicated in a period purchase if they have not already done so, so that $\Pi(x,2,1)$ is the payoff of selling to no buyers in the first period, to buyers v_2 and v_3 in the second, and to buyers of type v_1 in the final period. In each case, the corresponding prices are the solution to the problem of maximizing the seller's payoff subject to the pattern of buyer purchases.

For the case of one buyer, we have

$$\begin{aligned}
\Pi_1(x,x,3) &= (1-(\rho_0+\rho_1+\rho_2))v_3 \\
\Pi_1(x,x,2) &= (1-(\rho_0+\rho_1))v_2 \\
\Pi_1(x,x,1) &= (1-(\rho_0))v_1.
\end{aligned}$$

There are many other strategies available to the seller, but all of them give payoffs equivalent to one of these. It will be useful to define the following, which we can then calculate:

$$\begin{aligned}
\Delta_{32} &= (1-(\rho_0+\rho_1+\rho_2))v_3 - (1-(\rho_0+\rho_1))v_2 = -.0036 \\
\Delta_{21} &= (1-(\rho_0+\rho_1))v_2 - (1-(\rho_0))v_1 = .00369.
\end{aligned}$$

With one buyer, the optimal strategy is to thus sell to buyer types v_2 and v_3 at price v_2.

Now suppose there are $n > 1$ buyers. In the example, we concentrate on the cases $n = 2$ and $n = 3$. We need to consider the following possible payoffs:

$$\begin{gathered}
\Pi_n(x,x,3) \\
\Pi_n(x,x,2) \\
\Pi_n(x,x,1) \\
\Pi_n(x,2,1) \\
\Pi_n(x,3,1) \\
\Pi_n(x,3,2) \\
\Pi_n(3,2,1) \\
\Pi_n(x,32,2),
\end{gathered}$$

where $\Pi_n(x,32,2)$ is the payoff from a pricing sequence that induces some v_3 buyers to purchase in the second period and some to delay purchase to the final period, at which point type v_2 and v_3 buyers purchase. Some of these strategies are obviously suboptimal. For example, $\Pi_n(3,2,1) \geq \Pi_n(x,x,1)$, as it can only improve the seller's payoff to sell to higher buyer types (at higher prices) before offering the object for sale at price v_1. Similarly, we have $\Pi_n(3,2,1) \geq \Pi_n(x,2,1)$ and $\Pi_n(x,32,2) \geq \Pi_n(x,x,2)$. Similar reasoning appears to give $\Pi_n(3,2,1) \geq \Pi_n(x,3,1)$, and we will proceed as if this is the case, though it is not completely obvious and we will verify it at the end of the calculations.

Now consider $\Pi(x,3,2)$. This calls for type v_3 buyers to purchase in the second period and type v_2 buyers to purchase at price v_2 in the final period. The difficulty here is that this strategy is not sequentially rational. If the object is not sold in the second period, then the buyers in the third period are known to be of types v_0, v_1, or v_2. The seller can now set price v_2 and sell to only type v_2 buyers, or set price v_1 and sell to both v_1 and v_2 buyers. We can calculate

$$\begin{aligned}
\left(1-\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^2\right)v_2 &< \left(1-\left(\frac{\rho_0}{\rho_0+\rho_1+\rho_2}\right)^2\right)v_1, \\
\left(1-\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^3\right)v_2 &< \left(1-\left(\frac{\rho_0}{\rho_0+\rho_1+\rho_2}\right)^3\right)v_1.
\end{aligned}$$

This ensures that when there are either two of three buyers, once it has been revealed that there are no v_3 buyers, then the static monopoly price is v_1 rather than v_2. As a result, there is no way for the seller to first sell to type v_3 buyers in the penultimate period and then sell at price v_2 to type v_2 buyers in the final period.

What the seller can do is set a price in the second period that makes type v_3 buyers indifferent between accepting and rejecting, with these buyers mixing in their accept/reject decisions in such a way as to make price v_2 in the final period just optimal for the seller. Intuitively, just enough type v_3 buyers now slip through to the final period to make v_2 the static monopoly price. Let $\overline{\rho}_3(n)$ denote the probability that a buyer is type v_3 and purchases in the second period (under this pricing strategy, and when there are n buyers), and $\underline{\rho}_3(n)$ the probability that the buyer is type v_3 and waits until the final period to purchase. Clearly, $\underline{\rho}_3(n) + \overline{\rho}_3(n) = \rho_3$. The condition that the seller be just willing to set price v_2 in the final period is equivalent to

$$\left(1 - \left(\frac{\rho_0 + \rho_1}{\rho_0 + \rho_1 + \rho_2 + \underline{\rho}_3(2)}\right)^2\right) v_2 \quad = \quad \left(1 - \left(\frac{\rho_0}{\rho_0 + \rho_1 + \rho_2 + \underline{\rho}_3(2)}\right)^2\right) v_1, \tag{36}$$

$$\left(1 - \left(\frac{\rho_0 + \rho_1}{\rho_0 + \rho_1 + \rho_2 + \underline{\rho}_3(3)}\right)^3\right) v_2 \quad = \quad \left(1 - \left(\frac{\rho_0}{\rho_0 + \rho_1 + \rho_2 + \underline{\rho}_3(2)}\right)^3\right) v_1. \tag{37}$$

Combining these arguments, we can restrict attention to payoffs $\Pi_n(x, x, 3)$, $\Pi_n(x, 32, 2)$, and $\Pi_n(3, 2, 1)$. Our task is to show that given the values we have chosen, we have

$$\begin{array}{ll} \Pi_2(x, 32, 2) > \Pi_2(x, x, 3) & \qquad \Pi_3(3, 2, 1) > \Pi_3(x, x, 3) \\ \Pi_2(x, 32, 2) > \Pi_2(3, 2, 1) & \qquad \Pi_3(3, 2, 1) > \Pi_3(x, 32, 2). \end{array}$$

A preliminary result is helpful. First, it will simplify subsequent notation to let

$$\begin{aligned} \alpha^n &= \frac{(\rho_0 + \rho_1 + \rho_2)^n - (\rho_0 + \rho_1)^n}{\rho_2}, \\ \beta^n &= \frac{(\rho_0 + \rho_1)^n - \rho_0^n}{\rho_1}. \end{aligned}$$

We will then calculate $\Pi_n(x, 3, 2)$ ignoring the fact that commitment constraints render this strategy unattainable (or, equivalently, assuming temporarily that the seller can commit). We have

$$\begin{aligned} \Pi_n(x, 3, 2) &= [1 - (\rho_0 + \rho_1 + \rho_2)^n]p + \left[(\rho_0 + \rho_1 + \rho_2)^n \left(1 - \left(\frac{\rho_0 + \rho_1}{\rho_0 + \rho_1 + \rho_2}\right)^n\right)\right] v_2 \\ &= [1 - (\rho_0 + \rho_1 + \rho_2)^n]p + [(\rho_0 + \rho_1 + \rho_2)^n - (\rho_0 + \rho_1)^n]v_2, \end{aligned}$$

where p is a price that makes v_3 buyers indifferent between purchasing now and waiting a period to purchase, or

$$(v_3 - p)\frac{1-(\rho_0+\rho_1+\rho_2)^n}{1-(\rho_0+\rho_1+\rho_2)} = (\rho_0+\rho_1+\rho_2)^{n-1}\frac{1-\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^n}{1-\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}}(v_3-v_2),$$

and hence

$$\begin{aligned}(v_3 - p)\frac{1-(\rho_0+\rho_1+\rho_2)^n}{1-(\rho_0+\rho_1+\rho_2)} &= \frac{(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n}{\rho_2}(v_3-v_2).\\ &= \alpha^n(v_3-v_2)\end{aligned}$$

Using this for the second inequality in the following, we now calculate

$$\begin{aligned}\Pi_n(x,x,3)-\Pi_n(x,3,2) &= [1-(\rho_0+\rho_1+\rho_2)^n](v_3-p)-[(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n]v_2\\ &= [1-(\rho_0+\rho_1+\rho_2)^n]\left(\frac{1-(\rho_0+\rho_1+\rho_2)}{1-(\rho_0+\rho_1+\rho_2)^n}\alpha^n(v_3-v_2)\right)\\ &\qquad -[(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n]v_2\\ &= \alpha^n[(1-(\rho_0+\rho_1+\rho_2))v_3-(1-(\rho_0+\rho_1+\rho_2))v_2-\rho_2 v_2]\\ &= \alpha^n\Delta_{32}.\end{aligned}$$

We next calculate

$$\begin{aligned}\Pi_n(3,2,1) &= [1-(\rho_0+\rho_1+\rho_2)^n]p_3+(\rho_0+\rho_1+\rho_2)^n\left(1-\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^n\right)p_2\\ &\qquad +(\rho_0+\rho_1+\rho_2)^n\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^n\left(1-\left(\frac{\rho_0}{\rho_0+\rho_1}\right)^n\right)v_1\\ &= [1-(\rho_0+\rho_1+\rho_2)^n]p_3+[(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n]p_2+[(\rho_0+\rho_1)^n-\rho_0^n]v_1,\end{aligned}$$

where p_3 is the price at which v_3 buyers purchase in the first period and p_2 the price at which v_2 buyers purchase in the second, and hence

$$\begin{aligned}(v_3 - p_3)\frac{1-(\rho_0+\rho_1+\rho_2)^n}{1-(\rho_0+\rho_1+\rho_2)} &= (\rho_0+\rho_1+\rho_2)^{n-1}\frac{1-\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^n}{1-\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}}(v_3-p_2)\\ &= \frac{(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n}{\rho_2}(v_3-p_2),\\ &= \alpha^n(v_3-p_2)\end{aligned}$$

and

$$(v_2-p_2)\left(\frac{1-\left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^n}{1-\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}}\right) = \left(\frac{\rho_0+\rho_1}{\rho_0+\rho_1+\rho_2}\right)^{n-1}\frac{1-\left(\frac{\rho_0}{\rho_0+\rho_1}\right)^n}{1-\frac{\rho_0}{\rho_0+\rho_1}}(v_2-v_1)$$

or

$$(v_2 - p_2)\alpha^n = \beta^n(v_2 - v_2).$$

We will find it helpful to solve this for

$$p_2 = v_2 - (v_2 - v_1)\frac{\beta^n}{\alpha^n}.$$

Now we calculate

$$\begin{aligned}
\Pi_n(x,x,3) - \pi_n(3,2,1) &= [1-(\rho_0+\rho_1+\rho_2)^n](v_3 - p_3) \\
&\qquad - [(\rho_0+\rho_1+\rho_2)^n - (\rho_0+\rho_1)^n]p_2 - [(\rho_0+\rho_1)^n - \rho_0^n]v_1 \\
&= [1-(\rho_0+\rho_1+\rho_2)^n]\alpha^n \frac{1-(\rho_0+\rho_1+\rho_2)}{1-(\rho_0+\rho_1+\rho_2)^n}(v_3 - p_2) \\
&\qquad - [(\rho_0+\rho_1+\rho_2)^n - (\rho_0+\rho_1)^n]p_2 - ((\rho_0+\rho_1)^n - \rho_0^n)v_1 \\
&= \alpha^n[1-(\rho_0+\rho_1+\rho_2)]v_3 - \alpha^n(1-(\rho_0+\rho_1+\rho_2)+\rho_2)p_2 - \beta^n\rho_1 v_1 \\
&= \alpha^n[1-(\rho_0+\rho_1+\rho_2)]v_3 - \alpha^n(1-(\rho_0+\rho_1))p_2 - \beta^n\rho_1 v_1 \\
&= \alpha^n[1-(\rho_0+\rho_1+\rho_2)]v_3 - \alpha^n(1-(\rho_0+\rho_1))\left(v_2 - (v_2-v_1)\frac{\beta^n}{\alpha^n}\right) - \beta^n\rho_1 v_1 \\
&= \alpha^n\Delta_{32} + \alpha^n(1-(\rho_0+\rho_1))\left((v_2-v_1)\frac{\beta^n}{\alpha^n}\right) - \beta^n\rho_1 v_1 \\
&= \alpha^n\Delta_{32} + \beta^n\Delta_{21}.
\end{aligned}$$

The inequalities we need to show are thus:

$$\begin{aligned}
\Pi_3(x,x,3) - \Pi_3(3,2,1) &= \alpha^3\Delta_{32} + \beta^3\Delta_{21} \quad < 0 &(38)\\
\Pi_3(x,32,2) - \pi_3(3,2,1) &= \Pi_3(x,32,2) - \Pi_3(x,3,2) \\
&\qquad + \Pi_3(x,3,2) - \Pi_3(x,x,3) + \Pi_3(x,x,3) - \Pi_3(3,2,1) \\
&= \Pi_3(x,32,2) - \Pi_3(x,3,2) - \alpha^3\Delta_{32} + \alpha^3\Delta_{32} + \beta^3\Delta_{21} \\
&= \Pi_3(x,32,2) - \Pi_3(x,3,2) + \beta^3\Delta_{21} \quad < 0 &(39)\\
\Pi_2(x,32,2) - \pi_2(3,2,1) &= \Pi_2(x,32,2) - \Pi_2(x,3,2) + \beta^2\Delta_{21} \quad > 0 &(40)\\
\Pi_2(x,32,2) - \Pi_2(x,x,3) &= \Pi_2(x,32,2) - \Pi_2(x,3,2) + \Pi_2(x,3,2) - \Pi_2(x,x,3) \\
&= \Pi_2(x,32,2) - \Pi_2(x,3,2) + \beta^2\Delta_{21} - \alpha^2\Delta_{32} - \beta^2\Delta_{21} \\
&= \Pi_2(x,32,2) - \Pi_2(x,3,2) - \alpha^2\Delta_{32} \quad > 0. &(41)
\end{aligned}$$

Attention thus turns to calculating $\Pi_n(x,32,2) - \Pi_n(x,3,2)$. We have

$$\Pi_n(x,32,2) = [1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n]p + [(\rho_0+\rho_1+\rho_2+\underline{\rho}_3)^n - (\rho_) + \rho_1)^n]v_2,$$

where the price p is now set so that

$$(v_3 - p)\frac{1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n}{1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))} = \frac{(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n-(\rho_0+\rho_1)^n}{\rho_2+\underline{\rho}_3(n)}(v_3-v_2)$$

and hence

$$p = v_3 - (v_3-v_2)\frac{(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n-(\rho_0+\rho_1)^n}{\rho_2+\underline{\rho}_3(n)}\frac{1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))}{1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n}.$$

We can thus write

$$\begin{aligned}\Pi_n(x,32,2) &= [1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n]v_3 \\ &\quad -(v_3-v_2)\frac{(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n-(\rho_0+\rho_1)^n}{\rho_2+\underline{\rho}_3(n)}(1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n)) \\ &\quad +[(\rho_0+\rho_1+\rho_2+\underline{\rho}_3)^n-(\rho_)+\rho_1)^n]v_2.\end{aligned}$$

When $\underline{\rho}_3(n)=0$, we have

$$\begin{aligned}\Pi_n(x,3,2) &= [1-(\rho_0+\rho_1+\rho_2)^n]v_3 \\ &\quad -(v_3-v_2)\frac{(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n}{\rho_2}(1-(\rho_0+\rho_1+\rho_2) \\ &\quad +[(\rho_0+\rho_1+\rho_2)^n-(\rho_)+\rho_1)^n]v_2.\end{aligned}$$

Our interest is now in the difference

$$\begin{aligned}& \Pi_n(x,32,2)-\Pi_n(x,3,2) \\ = \; & [(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n]v_3 \\ & -(v_3-v_2)\left[\frac{(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n-(\rho_0+\rho_1)^n}{\rho_2+\underline{\rho}_3(n)}(1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n)))\right. \\ & \left.-\frac{(\rho_0+\rho_1+\rho_2)^n-(\rho_0+p+1)^n}{\rho_2}(1-(\rho_0+\rho_1+\rho_2)))\right] \\ & +v_2[(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n-(\rho_0+\rho_1+\rho_2)^n] \\ = \; & (v_3-v_2)\left[(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n\right. \\ & -\frac{(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))^n-(\rho_0+\rho_1)^n}{\rho_2+\underline{\rho}_3(n)}(1-(\rho_0+\rho_1+\rho_2+\underline{\rho}_3(n))) \\ & \left.+\frac{(\rho_0+\rho_1+\rho_2)^n-(\rho_0+\rho_1)^n}{\rho_2}(1-(\rho_0+\rho_1+\rho_2))\right].\end{aligned}$$

Our calculation then proceeds as follows. First, we fix the values of v_1, v_2, v_3, and ρ_0, ρ_1, ρ_2 and ρ_3 as specified in Section 2. Next, we calculate $\underline{\rho}_3(2)$ and $\underline{\rho}_3(3)$ as the solutions to (36)–(37). These solutions must be calculated numerically, giving

$$\begin{aligned}\underline{\rho}_3(2) &= .0495908\\ \underline{\rho}_3(3) &= .0337164.\end{aligned}$$

We can then verify that for each of these values, the left side of the appropriate equation in (36)–(37) exceeds the right side (with the difference on the order of 10^{-8}). We then consider the slightly smaller values

$$\begin{aligned}\underline{\rho}_3'(2) &= .0495907\\ \underline{\rho}_3'(3) &= .03371639,\end{aligned}$$

verifying that now the left side of the appropriate equation in (36)–(37) falls short of the right side, again with a difference on the order of 10^{-8}. The actual values of $\underline{\rho}_3(2)$ and $\underline{\rho}_3(3)$ thus fall within the intervals defined by these two pairs. We now verify numerically that, throughout this interval, the inequalities (38)–(41) hold.

Finally, we return to our commitment to verify the seemingly intuitive inequality that, for $n = 2, 3$,

$$\begin{aligned}\Pi_n(3,2,1) - \Pi_n(x,3,1) &= \Pi_n(3,2,1) - \Pi_n(x,x,3) + \Pi_n(x,x,3) - \Pi_n(x,3,1)\\ &= -(\alpha^n\Delta_{32} + \beta^n\Delta_{21}) + \Pi_n(x,x,3) - \Pi_n(x,3,1) > 0.\end{aligned}$$

We have

$$\Pi_3(x,3,1) = [1 - (\rho_0+\rho_1+\rho_2)^n]p + [(\rho_0+\rho_1+\rho_2)^n - (\rho_0)^n]v_1$$

where the price p now satisfies

$$(v_3 - p)\frac{1-(\rho_0+\rho_1+\rho_2)^n}{1-(\rho_0+\rho_1+\rho_2)} = (\rho_0+\rho_1+\rho_2)^{n-1}\left(\frac{1-\left(\frac{\rho_0}{\rho_0+\rho_1+\rho_2}\right)^n}{1-\frac{\rho_0}{\rho_0+\rho_1+\rho_2}}\right)(v_3 - v_1),$$

and hence

$$v_3 - p = \frac{1-(\rho_0+\rho_1+\rho_2)}{1-(\rho_0+\rho_1+\rho_2)^n}\,\frac{(\rho_0+\rho_1+\rho_2)^n - \rho_0^n}{\rho_1+\rho_2}(v_3 - v_1).$$

This allows us to obtain

$$\begin{aligned}\Pi_n(x,x,3) - \Pi_n(x,3,1) &= \left[(1-(\rho_0+\rho_1+\rho_2))\frac{v_3 - v_1}{\rho_1+\rho_2} - v_1\right][(\rho_0+\rho_1+\rho_2)^n - (\rho_0)^n]\\ &= [(1-(\rho_0+\rho_1+\rho_2))v_3 - (1-\rho_0)v_1]\,\frac{(\rho_0+\rho_1+\rho_2)^n - (\rho_0)^n}{\rho_0+\rho_1}\\ &= \Delta_{31}\frac{(\rho_0+\rho_1+\rho_2)^n - (\rho_0)^n}{\rho_0+\rho_1}.\end{aligned}$$

Putting these pieces together, we have

$$\Pi_n(3,2,1) - \Pi_n(x,3,1) = -\alpha^n \Delta_{32} - \beta^n \Delta_{21} + \frac{(\rho_0 + \rho_1 + \rho_2)^n - (\rho_0)^n}{\rho_0 + \rho_1} \Delta_{31}.$$

We again verify this numerically.

B.2 Committing to Lower Prices

This section provides an example in which the seller cannot commit to charging a low enough price in the second stage, and an example in which the seller cannot commit to charging a high enough price.

A.9.1 The Model

Assume that there are two buyers and two periods. Buyers have one of three possible valuations, v_1, v_2, or $v_3 = 1$, with $v_1 < v_2 < 1$. A buyer has valuation v_i with probability ρ_i, where

$$\rho_1 = \frac{1}{8}, \qquad \rho_2 = \frac{1}{4}, \qquad \rho_3 = \frac{5}{8}.$$

Conditional on all buyers being of type v_1 or v_2 in the last period, the seller's choice is obviously between charging v_1 or v_2. She chooses the latter, higher price if and only if

$$\Delta = \left(1 - \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)^n\right) v_2 - v_1 > 0.$$

There are four obvious pure strategies in the two-period game: selling to type v_3 first, and then to type v_2; selling to type v_3, and then to v_1; selling to types v_3 and v_2 first, and then to v_1; and finally, selling to no one first, and then to type v_3. The seller could also wait and sell to some larger subset of types in the second period, but it is clear that this is worse than some strategy in which type v_3 accepts in the first period. (Of course, the latter strategy may not satisfy sequential rationality). We consider these strategies are in turn.

Selling to type v_3, and then to type v_2. Denote the price charged in the first period by p_{32} (the second price is v_2), and the expected payoff by V_{32}. The price p_{32} must satisfy

$$\frac{1 - (\rho_1 + \rho_2)^n}{1 - (\rho_1 + \rho_2)} (v_3 - p_{32}) = (\rho_1 + \rho_2)^{n-1} \frac{1 - (\rho_1/(\rho_1 + \rho_2))^n}{1 - \rho_1/(\rho_1 + \rho_2)} (v_3 - v_2),$$

and the payoff V_{32} must satisfy

$$V_{32} = (1 - (\rho_1 + \rho_2)^n) p_{32} + (\rho_1 + \rho_2)^n \left(1 - \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)^n\right) v_2.$$

Solving, we find that

$$V_{32} = (1-\rho_1^n)v_3 - \frac{1-\rho_1}{\rho_2}((\rho_1+\rho_2)^n - \rho_1^n)(v_3 - v_2).$$

Selling to type v_3, and then to type v_1. Denote the price charged in the first period by p_{31} (the second price is v_1), and the expected payoff by V_{31}. The price p_{31} must satisfy

$$\frac{1-(\rho_1+\rho_2)^n}{1-(\rho_1+\rho_2)}(v_3 - p_{31}) = (\rho_1+\rho_2)^{n-1}(v_3 - v_1),$$

and the payoff V_{31} must satisfy

$$V_{31} = (1-(\rho_1+\rho_2)^n)p_{32} + (\rho_1+\rho_2)^n v_1.$$

Solving, we find that

$$V_{31} = v_3 - (\rho_1+\rho_2)^{n-1}(v_3 - v_1).$$

Selling to type v_2, and then to type v_1. Denote the price charged in the first period by p_{21} (the second price is v_1), and the expected payoff by V_{21}. The price p_{21} must satisfy

$$\frac{1-\rho_1^n}{1-\rho_1}(v_2 - p_{21}) = \rho_1^{n-1}(v_2 - v_1),$$

and

$$V_{21} = (1-\rho_1^n)p_{21} + \rho_1^n v_1.$$

Solving, we find that

$$V_{21} = v_2 - \rho_1^{n-1}(v_2 - v_1).$$

Selling to type v_3 in the second period. Clearly, this yields a payoff of $V_3 = (1-(\rho_1+\rho_2)^n)v_3$.

A.9.2 Case 1: $(v_1, v_2) = (1/8, 1/4)$. The seller cannot commit to a low price.

It is easy to check that $\Delta = 7/72 > 0$—conditional on the buyers not being of type v_3, it is optimal to set the price to v_2 in the one-stage game. However, we have that

$$\frac{43}{64} = V_{31} > \begin{cases} V_{32} = 21/32 \\ V_{21} = 15/64 \\ V_3 = 5/8 \end{cases}.$$

That is, the optimal two-stage strategy is to sell to high types first, and then to all types.

This also dominates all schemes involving mixing (since if type v_2 or type v_3 is supposed to randomize in the first period, this lowers the probability of acceptance, relative to the same type accepting with probability one in the first period, as well as the price paid in the first period, and it does not affect the price in the second).

But since $\Delta > 0$, the seller cannot achieve this payoff, because in the second stage, he cannot help but charge a high price. Therefore, this is an example in which the seller cannot commit to charge low enough a price in the second stage.

A.9.3 Case 2: $(v_1, v_2) = (4/5, 8/9)$. The seller cannot commit to a high price.

It is easy to check that $\Delta = -4/405 > 0$. Conditional on the buyers not being of type v_3, it is optimal to set the price to v_1 in the one-stage game. However, we have

$$\frac{539}{576} = V_{32} > \left\{ \begin{array}{l} V_{31} = 37/40 \\ V_{21} = 79/90 \\ V_3 = 5/8 \end{array} \right. .$$

This also dominates all schemes involving mixing (for the same reasons as before).

But since $\Delta < 0$, the seller cannot achieve this payoff, since in the second stage, he cannot help but charge a low price. This is therefore an example in which the seller cannot commit to charge high enough a price in the second stage.

B.3 Details, Proof of Lemma 7

Our purpose is to prove that, for all $t \geq 2$ and $n \geq 6$,

$$(1 + nt(t+1)/2)^{\frac{1}{n}} \leq x((1 + n(t-1)t/2)^{\frac{1}{n}}, (1 + n(t-2)(t-1)/2)^{\frac{1}{n}}).$$

or, equivalently, for all $t \geq 1$ and $n \geq 6$,

$$(1 + n(t+1)(t+2)/2)^{\frac{1}{n}} \leq x((1 + nt(t+1)/2)^{\frac{1}{n}}, (1 + n(t-1)(t-2)/2)^{\frac{1}{n}}).$$

(At this point, letting $x = 1/t$ and $y = 1/n$, one can rewrite this inequality as a function on the unit square and then gain some confidence in its veracity by using a program such as *Mathematica* to plot it.) Upon manipulation, this is equivalent to showing that, for all $t \geq 1$, $n \geq 6$,

$$4t\left(2 + nt\left(t+1\right)\right)^{1/n} + \left(2 + nt\left(t+1\right)\right)\left(2 + n\left(t-1\right)t\right)^{1/n} - nt\left(t+1\right)\left(2 + n\left(t+1\right)\left(t+2\right)\right)^{1/n} \leq 0,$$

or

$$\left(1 + \frac{n\left(t+1\right)}{1 + \frac{nt(t+1)}{2}}\right)^{1/n} - \left(\frac{2}{nt\left(t+1\right)} + 1\right)\left(1 - \frac{nt}{1 + \frac{nt(t+1)}{2}}\right)^{1/n} - \frac{4}{n\left(t+1\right)} \geq 0. \tag{42}$$

This will be done in two steps.

A.10.1 The Case $t = 1$

In that case, we must show that

$$g_L(n) := n\left((1+3n)^{1/n} - 1\right) \geq 2(1+n)^{1/n} + 1 =: g_R(n).$$

Observe that, for $x > 0$,

$$\frac{d}{dx}\left(x \ln\left(1 + x^{-1}\right)\right) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{1+x} \geq 0,$$

where the last step follows from the standard inequality $\ln x \geq (1+x)^{-1}$ applied to $1/x$. It follows that g_R is decreasing in n.

Consider now the function g_L. Its second derivative with respect to n is

$$\frac{(1+3n)^{\frac{1}{n}-2}}{n^3} \lambda(n),$$

where

$$\lambda(n) = (1+3n)\ln(1+3n)\left((1+3n)\ln(1+3n) - 6n\right) - 9(n-1)n^2.$$

We claim that λ is negative $\forall n \geq 1$. To see this, observe first that

$$\frac{d^3\lambda}{dn^3} = \frac{-54}{(3n+1)^2}\left(1 + 3n + 9n^2 - 2(1+3n)\ln(1+3n)\right) < 0,$$

because

$$1 + 3n + 9n^2 \geq 2(1+3n)\ln(1+3n),$$

which is because, from the standard inequality $\ln\left(1 + \frac{1}{x}\right) \leq \frac{1}{\sqrt{x^2+x}}$, it follows that $\ln(1+3n) \leq 3n/\sqrt{1+3n}$. Taking squares in the resulting inequality and collecting terms yield the desired result.

Therefore,

$$\frac{d^2\lambda}{dn^2} = 18\left(\frac{1}{1+3n} + \ln(1+3n) + \ln^2(1+3n) - (1+3n)\right)$$

is decreasing, and it is negative for $n = 1$, so it is negative for all $n \geq 1$.

In turn, this implies that

$$\frac{d\lambda}{dn} = 3\left(2(1+3n)\ln^2(1+3n) - 9n^2 - 6n\ln(1+3n)\right)$$

is decreasing, and it is negative for $n = 1$, so it is negative for all $n \geq 1$. Repeating once more the argument, this establishes that λ is decreasing, and again it is negative for $n = 1$, and therefore for all $n \geq 1$.

We have now established that $d^2 g_L/dn^2 \leq 0$ for all $n \geq 1$. Thus, dg^L/dn is decreasing in n. However, $\lim_{n\to\infty} dg^L/dn = 0$, and so $dg^L/dn \geq 0$. This proves that g^L is an increasing function.

This part of the proof is concluded by observing that $g^L(6) > g^R(6)$. Since g^L is increasing, while g^R is decreasing, the inequality follows for all $n \geq 6$.

A.10.2 The General Case, $t > 1$

A.10.2a A Sufficient Inequality Recall that, from Taylor's theorem,

$$(1+x)^{1/n} \geq 1 + \frac{x}{n} - \frac{n-1}{2n^2}x^2 + \frac{(n-1)(2n-1)}{6n^3}x^3 - \frac{(n-1)(2n-1)(3n-1)}{24n^4}x^4$$
$$+\frac{(n-1)(2n-1)(3n-1)(4n-1)}{120n^5}x^5 - \frac{(n-1)(2n-1)(3n-1)(4n-1)(5n-1)}{720n^6}x^6,$$

and similarly,

$$(1-x)^{1/n} \leq 1 - \frac{x}{n} - \frac{n-1}{2n^2}x^2 - \frac{(n-1)(2n-1)}{6n^3}x^3 - \frac{(n-1)(2n-1)(3n-1)}{24n^4}x^4$$
$$-\frac{(n-1)(2n-1)(3n-1)(4n-1)}{120n^5}x^5 - \frac{(n-1)(2n-1)(3n-1)(4n-1)(5n-1)}{720n^6}x^6.$$

We now apply these two bounds to the left side of (42), inserting $x = \frac{n(t+1)}{1+\frac{nt(t+1)}{2}}$ and $x = nt/\left(1+\frac{nt(t+1)}{2}\right)$ respectively. We obtain a rational function whose denominator is positive (being a square) and whose numerator is twice the following polynomial in n of degree 6:

$$a_6 n^6 + a_5 n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0,$$

with

$$\begin{aligned}
a_0 &= -4t^6 + 24t^5 - 120t^4 + 480t^3 - 1440t^2 - 2880t - 2880,\\
a_1 &= 48t^7 + 78t^6 + 1330t^5 + 670t^4 - 7818t^3 - 9454t^2 - 6166t,\\
a_2 &= 12t^9 + 24t^8 + 852t^7 + 890t^6 - 9240t^5 - 23184t^4 - 21588t^3 - 13104t^2 - 1950t,\\
a_3 &= 360t^9 + 990t^8 - 3030t^7 - 12645t^6 - 15635t^5 - 4805t^4 + 3285t^3 + 5990t^2 + 1730t,\\
a_4 &= 60t^{11} + 240t^{10} - 930t^9 - 5370t^8 - 11580t^7 - 15376t^6\\
&\quad -16824t^5 - 19620t^4 - 14730t^3 - 6960t^2 - 1410t,\\
a_5 &= -180t^{11} - 945t^{10} - 2115t^9 - 2610t^8 - 168t^7 + 5322t^6\\
&\quad +11830t^5 + 16105t^4 + 12093t^3 + 4904t^2 + 836t,\\
a_6 &= 3t(1+t)(1+2t)(-80 + t(1+t)(-272 + t(1+t)(-126 + t(1+t)(8 + 5t(1+t))))).
\end{aligned}$$

We must show that this polynomial is positive.

A.10.2b Preliminary Observations Observe first that a_6 is positive for $t \geq 2$. Indeed, the last factor is a polynomial of degree 4 in $x = t\,(1+t)$, namely

$$-80 - 272x - 126x^2 + 8x^3 + 5x^4.$$

Since the coefficients change signs only once, Descartes' rule implies that there is at most one strictly positive root. Since this polynomial is negative when evaluated at $x = 0$, and positive when evaluated at $x = 6$ (i.e. $t = 2$), the root must be in $(0, 2)$, and so the polynomial is positive for all $t \geq 2$.

Observe that, by Descartes' rule, a_1 can have at most one strictly positive root. The coefficient a_1 is negative for $t = 2$ and positive for $t = 3$, so that the unique root is in $(2, 3)$, and so a_1 is negative for $t \geq 2$. Similarly, a_2 can have at most one strictly positive root. The coefficient a_2 is negative for $t = 3$ and positive for $t = 4$, so the unique root is in $(3, 4)$, and so $a_2 > 0$ for $t \geq 4$. Similarly, a_3 can have at most two strictly positive roots. Further, the signs of a_3 at $t = 1/2$, $t = 1$ and $t = 4$ alternate, so that here again, there is no root for $t \geq 4$, and so $a_3 > 0$ for $t \geq 4$. By the same method, a_4 can have at most one strictly positive root, and a_4 is negative for $t = 4$ and positive for $t = 5$, so $a_4 > 0$ for $t \geq 5$. Finally, a_5 can have at most one strictly positive root, and it is positive at $t = 1$ and negative at $t = 2$, so it is strictly negative for $t \geq 2$.

We need two further facts. First, $-a_5 > -a_0$ for $t \geq 2$. To see this, let us compute the difference

$$\begin{aligned} a_5 - a_0 \;\; = \;\; & -180t^{11} - 945t^{10} - 2115t^9 - 1610t^8 - 168t^7 + 5326t^6 + 11806t^5 \\ & +16225t^4 + 11613t^3 + 6344t^2 + 3716t + 2880, \end{aligned}$$

so, again by Descartes rule, there can be at most one positive root of the difference, and the difference is positive for $t = 1$ and negative for $t = 2$, and so this difference is negative for $t \geq 2$.

Second, we claim that $-a_5/a_6$ is increasing for $t \geq 4$. To see this, observe that the derivative of the ratio a_5/a_6 is equal to the ratio of the following numerator, over a denominator which is positive since it is a square,

$$\begin{aligned} & -150t^{17} - 2820t^{16} - 19500t^{15} - 363160t^{14} + 129880t^{13} + 933852t^{12} + 2769050t^{11} \\ & +53161174t^{10} + 6507696t^9 + 5494474t^8 + 3239750t^7 + 2186194t^6 + 2877454t^6 \\ & +3504246t^5 + 2839892t^4 + 1532112t^3 + 550712t^2 + 120816t + 11904, \end{aligned}$$

so it has at most one strictly positive root, and it is positive for $t = 3$ and negative for $t = 4$. So the ratio $-a_5/a_6$ is increasing for $t \geq 4$ and so always less than its limit, which equals

$$\lim_{t\to\infty} -\frac{a_5}{a_6} = 6.$$

A.10.2c The Result For $n > 6$ We are now ready to get our result, at least in the case $n > 6$ for now. We use the Lagrange-McLaurin theorem.[34] Given some polynomial of degree n, with real coefficients $\{a_i\}$, let $m = \sup\{i | a_i < 0\}$, and $B = \sup\{-a_i | a_i < 0\}$. Then any real root r of the polynomial satisfies

$$r < 1 + \left(\frac{B}{a_n}\right)^{\frac{1}{n-m}}.$$

Given our previous analysis, it follows that, applying the theorem to the polynomial in n for $t \geq 5$, any real root is less than

$$1 - \frac{a_5}{a_6} < 7.$$

This establishes the inequality (*) for the case $n > 6$ and $t \geq 5$. For $n > 6$ but for each $t = 2, 3, 4$, we can compute

$$1 - \max\left\{-\frac{a_0}{a_6}, -\frac{a_1}{a_6}, -\frac{a_2}{a_6}, -\frac{a_3}{a_6}, -\frac{a_4}{a_6}, -\frac{a_5}{a_6}\right\},$$

which of course is independent of n. It is still less than 7 for both $t = 3, 4$. In both cases, the maximum is achieved by $-a_5/a_6$. In the case $t = 2$, the maximum is achieved by $-a_4/a_6$, and in that case the bound on the root is only $n < 30$. However, we can directly verify that for $t = 2$ and each value $n = 7, \ldots, 30$, the polynomial is positive.

A.10.2d The result for $n = 6$ We are left with proving the result for the case $n = 6$. Plugging into the polynomial in n, we obtain the following polynomial in t,

$$\begin{gathered}155520t^{11} + 1321920t^{10} + 4324320t^9 + 8065440t^8 - 40593600t^7 - 168237440t^6 \\ -321927240t^5 - 358960440t^4 - 234969960t^3 - 83572680t^2 - 12520920t - 5760.\end{gathered}$$

Once more, by Descartes' rule, there can be at most one strictly positive root, and since this polynomial is negative for $t = 2$, and positive for $t = 3$, we are done —except for the case $n = 6$ and $t = 2$. Evaluating the original inequality for that one case concludes the proof.

B.4 Multiple Objects

This section derives the price function (15) and payoff function (15). We provide the preliminary analysis for any number of objects, and then specialize to the case of two objects.

[34] Riccardo Benedetti and Jean-Jacques Risler, *Real algebraic and semi-algebraic sets* (Hermann, Paris, 1990), Theorem 1.2.2.).

The buyer's indifference condition. As in the single-object argument, we begin by identifying indifferent buyers. Suppose that there are k units left. Define

$$\phi_t = k \sum_{j=0}^{n-1} \frac{\binom{n-1}{j}}{j+1} \gamma_t^{n-1-j} (1-\gamma_t)^j + \sum_{j=0}^{k-2} \binom{n-1}{j} \left(1 - \frac{k}{j+1}\right) \gamma_t^{n-1-j} (1-\gamma_t)^j ,$$

where, as usual, $\gamma_t = v_t / v_{t+1}$. By accepting now, the buyer with valuation v_t gets

$$\phi_t (v_t - p_t) .$$

By waiting one period instead, he gets

$$\gamma_t^{n-1} \phi_{t-1} (v_t - p_{t-1}) + \sum_{j=1}^{k-1} \binom{n-1}{j} (1-\gamma_t)^j \gamma_t^{n-1-j} W_{k-j,t} v_t,$$

where $W_{k-j,t}$ is the normalized expected payoff when only $k-j$ units are left (and the number of bidders has gone down to $n-j$) and t periods to go. Indifference requires the two to be equal. Observe that, defining

$$\phi_t (v_t - p_t) = M_t v_{t+1}, \tag{43}$$

the buyer's indifference condition becomes

$$M_t v_{t+1} = \gamma_t^{n-1} \phi_{t-1} (v_t - v_{t-1}) + \sum_{j=1}^{k-1} \binom{n-1}{j} (1-\gamma_t)^j \gamma_t^{n-1-j} W_{k-j,t} v_t + \gamma_t^{n-1} M_{t-1} v_t. \tag{44}$$

The sellers's maximization problem. The seller's payoff is

$$\begin{aligned} S_{t+1} v_{t+1} &= \max \left\{ \gamma_t^n S_t v_t + \sum_{j=1}^{k-1} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} (j p_t + Y_{k-j,t} v_t) + k \sum_{j=k}^{n} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} p_t \right\} \\ &= \max \left\{ \begin{array}{c} \gamma_t^n S_t v_t - \left[\sum_{j=1}^{k-1} j \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} + k \sum_{j=k}^{n} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} \right] (v_t - p_t) \\ + \sum_{j=1}^{k-1} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} (Y_{k-j,t} - j) v_t - k \sum_{j=k}^{n} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} v_t \end{array} \right\}, \end{aligned}$$

where $Y_{k-j,t}$ is the seller's normalized continuation payoff when only $k-j$ units are left, with t periods to go. Observe now that

$$\sum_{j=1}^{k-1} j \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} + k \sum_{j=k}^{n} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n-j} = n (1-\gamma_t) \phi_t,$$

so we may re-write the seller's payoff as

$$S_{t+1} = \max \left\{ \begin{array}{c} \gamma_t^{n+1} S_t - n\,(1-\gamma_t) \left[\gamma_t^n \phi_{t-1}\,(1-\gamma_{t-1}) + \sum_{j=1}^{k-1} \binom{n-1}{j} (1-\gamma_t)^j \gamma_t^{n-j} W_{k-j,t} + \gamma_t^n M_{t-1} \right] \\ + \sum_{j=1}^{k-1} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n+1-j} \left(Y_{k-j,t} - j\right) - k \sum_{j=k}^{n} \binom{n}{j} (1-\gamma_t)^j \gamma_t^{n+1-j} \end{array} \right\},$$

which is a function to be maximized over γ_t. This can be written more compactly as

$$S_{t+1} = \max \left\{ \gamma_t^{n+1} S_t + h\,(\gamma_t) \right\}. \tag{45}$$

The seller's maximization. Taking derivatives of (45) with respect to the γ_t, the seller's first-order conditions are

$$S_t = -h'\,(\gamma_t) \,/\, \left((n+1)\,\gamma_t^n\right), \tag{46}$$

and therefore, using (46 in (45),

$$h'\,(\gamma_{t+1}) = \gamma_{t+1}^n \left(\gamma_t h'\,(\gamma_t) - (n+1)\,h\,(\gamma_t) \right). \tag{47}$$

Writing h as

$$h\,(\gamma_t) = g\,(\gamma_t) - n\,(1-\gamma_t)\,\gamma_t^n M_{t-1} \tag{48}$$

and using this expression to substitute for h in (47) gives

$$g'\,(\gamma_{t+1}) - n\,(n - (n+1)\,\gamma_{t+1})\,\gamma_{t+1}^{n-1} M_t = \gamma_{t+1}^n \left(\gamma_t g'\,(\gamma_t) - (n+1)\,g\,(\gamma_t) + n\gamma_t^n M_{t-1} \right). \tag{49}$$

We further have, from the price recursion (44)

$$M_t = A_t + \gamma_t^n M_{t-1}, \tag{50}$$

with

$$A_t = \gamma_t^n \phi_{t-1}\,(1-\gamma_{t-1}) + \sum_{j=1}^{k-1} \binom{n-1}{j} (1-\gamma_t)^j \gamma_t^{n-j} W_{k-j,t}. \tag{51}$$

Using (51) in (50) to eliminate M_{t-1}, we solve for

$$M_t = \frac{g'(\gamma_{t+1}) - \gamma_{t+1}^n \left(\gamma_t g'(\gamma_t) - (n+1) g(\gamma_t) - n A_t \right)}{n^2 \gamma_{t+1}^{n-1} (1-\gamma_{t+1})}. \tag{52}$$

Therefore, inserting in (49),

$$\frac{g'(\gamma_{t+1}) - \gamma_{t+1}^n \left(\gamma_t g'(\gamma_t) - (n+1) g(\gamma_t) - n A_t \right)}{\gamma_{t+1}^{n-1} (1-\gamma_{t+1})} - n^2 A_t =$$
$$\gamma_t \frac{g'(\gamma_t) - \gamma_t^n \left(\gamma_{t-1} g'(\gamma_{t-1}) - (n+1) g(\gamma_{t-1}) - n A_{t-1} \right)}{(1-\gamma_t)}.$$

The expression $\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) - nA_t$ can be further simplified. Indeed,

$$\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) - nA_t = \sum_{j=1}^{k-1} j(1-\gamma_t)^{j-1}\gamma_t^{n-j}\left(\binom{n-1}{j}n(1-\gamma_t)W_{k-j,t} - \binom{n}{j}\gamma_t Y_{k-j,t}\right)$$
$$+\sum_{j=1}^{k-1}\binom{n}{j}j^2(1-\gamma_t)^{j-1}\gamma_t^{n+1-j} + k\sum_{j=k}^{n}\binom{n}{j}j(1-\gamma_t)^{j-1}\gamma_t^{n+1-j}.$$

The function $v(x)$. We now let $k=2$ and seek the function $v(x)$, giving the identity of the indifferent buyer given that there are two units for sale and the length of time to the deadline is x. Given $k=2$, we have

$$Y_{1,t} \approx \frac{n\frac{q_{t-1}}{q_t} - \left(\frac{q_{t-1}}{q_t}\right)^n}{n+1}, \text{ and } W_{1,t} \approx \frac{1}{n}\left(\frac{v_t}{v_{t+1}}\right)^{n-1}.$$

Observe that

$$\frac{1}{\gamma(x)} - 1 \approx \frac{v'(x)}{v(x)}.$$

If we let $t+1 = x+\varepsilon$, $t = x$ and $t-1 = x-\varepsilon$, we can approximate $Y_{1,t}$ by

$$\frac{1}{n}\left((n-1)\left(1+\frac{3\varepsilon}{nx}\right)^{-1} - \left(1+\frac{3\varepsilon}{nx}\right)^{1-n}\right),$$

(recall that there is one fewer buyer) and $W_{1,t}$ by

$$\frac{1}{n-1}\left(1+\frac{3\varepsilon}{nx}\right)^{1-(n-1)}.$$

Finally, we can approximate γ_t as follows:

$$\gamma_{t+1} = \left(1+\frac{v'(x)}{v(x)}\varepsilon + \left(\frac{v''(x)}{v(x)} - \left(\frac{v'(x)}{v(x)}\right)^2\right)\varepsilon^2\right)^{-1},$$
$$\gamma_t = \left(1+\frac{v'(x)}{v(x)}\varepsilon\right)^{-1},$$
$$\gamma_{t-1} = \left(1+\frac{v'(x)}{v(x)}\varepsilon - \left(\frac{v''(x)}{v(x)} - \left(\frac{v'(x)}{v(x)}\right)^2\right)\varepsilon^2\right)^{-1},$$

and do an asymptotic expansion in ε around 0, obtaining

$$(n^2(n+1)w(x)^4 - 2nw'(x)^2 + w(x)^2(3+n(3n+1)w'(x)))\varepsilon^3 + o(\varepsilon^4) = 0,$$

where $w(x) \equiv v'(x)/v(x)$. We also know that $v(0) = 0, v(1) = 1$. Calculating the valuations $v(x)$ is thus a matter of solving the ordinary differential equation.

$$n^2(n+1)w(x)^4 - 2nw'(x)^2 + w(x)^2(3 + n(3n+1)w'(x)) = 0. \tag{53}$$

The price function $p(x)$ and payoff π. Turning now to the price $p(x)$, from $\phi_t(v_t - p_t) = M_t v_{t+1}$ (cf. (43)), it follows that

$$p_t = v_t - \frac{M_t}{\phi_t} v_{t+1} = v_{t+1}\left(\gamma_t - \frac{M_t}{\phi_t}\right).$$

We have expression (52) for M_t, and thus attention turns to computing

$$\gamma_t - \frac{M_t}{\phi_t}.$$

Using our approximations W, X and γ, it is straightforward to verify that, in the case $k = 2$,

$$\lim_{\varepsilon \to 0} \gamma_t - \frac{M_t}{\phi_t} = \frac{n-2}{n}.$$

This in turn gives the price function

$$p(x) = \frac{n-2}{n} v(x).$$

It is then straightforward that the seller's payoff is given by $2\frac{n-2}{n-1}$.

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