Constructive Proof of Existence and Characterization of the Farsighted Stable Set in a Price-Leadership Cartel Model under the Optimal Pricing Policy

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January 25, 2008‡

Abstract

Using a constructive algorithm, we provide an alternative proof of the existence of a unique stable set of cartels in a price-leadership cartel model à la Diamantoudi (2005, *Economic Theory* 26, pp. 907–921). With this result, we can fully characterize the stable set of cartels: it contains at least one Pareto-efficient cartel; in particular, the largest stable cartel that belongs to it is Pareto efficient. By using a simple example, we also demonstrate that there can be some stable but not Pareto-efficient cartels, and some Pareto-efficient but not stable cartels.

JEL classification: C71, D43, L13. *Keywords*: price leadership model, cartel stability, foresight, stable set

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‡Printed: April 30, 2008. 1st version: September 25, 2007.

1 Introduction

D'Aspremont et al. (1983) have examined a price-leadership cartel model in which the dominant cartel sets price at a level that maximizes the profits of the member firms; moreover, each firm is assumed to be able to freely enter or exit from the dominant cartel. They have also revealed that if the number of firms is finite, there always exists a stable cartel such that once it is formed, no member firm will wish to exit from it and no outside firm will wish to enter it.

As pointed out by Diamantoudi (2005), the analysis by d'Aspremont et al. (1983) exhibited a certain amount of inconsistency with respect to firms' attitudes or perspectives toward other firms' reactions. In the d'Aspremont et al. model, it is assumed that a cartel firm contemplating an exit from the current cartel compares its current profit (as a member of the cartel) with the prospective profit (as a fringe firm) that can be gained from a new price that is set by the new (smaller) cartel formed by the remaining cartel firms. Further, it assumes that the firm will actually deviate from the current cartel if the latter is higher than the former. A similar argument also applies to a fringe firm contemplating an entry into the current cartel. In other words, in the d'Aspremont et al. model, each firm contemplating a deviation (entry or exit) is assumed to possess the ability to recognize the reaction of readjusting price by the members of the new cartel established after its deviation; in a sense, each firm is assumed to be farsighted to some extent. Despite this farsightedness, each firm in the d'Aspremont et al. model ignores the possible reactions wherein entry/exit is contemplated by other firms subsequent to its own deviation. Thus, firms are assumed to be farsighted on one hand, but myopic on the other.

Diamantoudi (2005) has modified the d'Aspremont et al. model so that each firm is sufficiently farsighted to be able to recognize not only the reaction of readjusting price by the new cartel members but also the reactions wherein an entry-exit is contemplated by other firms subsequent to its own deviation. Moreover, by adopting (a variant of) the von Neumann-Morgenstern stable set as the solution concept, she has proved the existence of a unique set of stable cartels.[1] Her proof of the existence and uniqueness of the stable set depends heavily on a well-known theorem formulated by von Neumann and Morgenstern (1953), which states that an abstract system with a strictly acyclic relation admits a unique stable set. Her argument is essentially existential, and the characteristics of the stable set for the price-leadership cartel model have not been fully investigated.[2]

In this paper, we investigate the characteristics of the stable set for the price-leadership cartel model. To this end, we first provide an alternative, constructive proof for the existence of the stable set. The method of our proof is based on an algorithm that determines and constructs the stable

[1]Kamijo and Muto (2007) and Kamijo and Nakanishi (2007) have modified the Diamantoudi (2005) model in two different directions. Kamijo and Muto (2007) have examined the case in which *coalitional* entry/exit is allowed. Kamijo and Nakanishi (2007) have examined the case in which the dominant cartel can set price at any (nonnegative) level it desires (i.e., the dominant cartel adopts the flexible pricing policy). These studies have demonstrated that in each model, every individually rational and Pareto-efficient cartel structure constitutes a (farsighted) stable set.

[2]Diamantoudi (2005) has examined the relationship between the stable set of cartels and the myopic core of cartels and demonstrated that their intersection contains the smallest cartel belonging to the stable set (Diamantoudi (2005), Theorem 5).

set. Following this proof, we demonstrate that the stable set of cartels and the set of Pareto-efficient cartels have a nonempty intersection; in particular, we establish that the largest cartel in the stable set is Pareto efficient. Further, by using a simple example, we demonstrate that neither the stable set nor the Pareto-efficient set can include the other as its subset.

2 Model

2.1 Price-leadership cartel

We consider an industry composed of $n \geqq 2$ identical firms that produce a homogeneous good. The demand for the good is represented by a continuous, monotonically decreasing function $d(p)$, where p is the price of the good. Each firm has an identical cost function $c(q)$, where q is the output level of the firm.

When a firm does not participate in a cartel, it behaves competitively. We denote the supply function of a (competitive) fringe firm by $q_f(p)$, which is derived from the price-equal-marginal cost condition. If $k \geqq 1$ firms decide to form a cartel, then the cartel becomes empowered to determine the market price of the good; as in d'Aspremont et al. (1983) and Diamantoudi (2005), we assume that there can only be one dominant cartel. We denote the cartel consisting of k firms as $C(k)$ and the set of all possible cartels as $\mathbb{C} \equiv \{C(0), C(1), \ldots, C(n)\}$.[3] Although $C(0)$ actually represents the situation in which no cartel exists, we include $C(0)$ in the

[3] In fact, $\mathbb{C}$ is the set of all possible cartel sizes (in terms of the number of firms). We regard two different cartels of the same size as being identical. In this paper, as in d'Aspremont et al. (1983) and Diamantoudi (2005), cartels are identified by their respective sizes.

set of all possible cartels for notational convenience.

Taking account of the responses by the non-cartel firms, the firms in $C(k)$ derive the residual demand and divide it equally among them. Further, the production per firm in $C(k)$ can be written as a function of the number of firms in the cartel, k, and the price, p: For $k = 1, \ldots, n$,

$$r(k, p) \equiv \frac{\max\left\{d(p) - (n-k)q_f(p),\ 0\right\}}{k}. \tag{1}$$

The optimal price for cartel $C(k)$—in terms of maximizing the joint profit of its members—can be written as a function of k: for $k = 1, \ldots, n$,

$$p^*(k) \equiv \arg\max_{p \geqq 0}\ p \cdot r(k, p) - c(r(k, p)). \tag{2}$$

Next, the profits of a fringe firm and a cartel firm evaluated at the optimal price $p^*(k)$ can be written as functions of the cartel size k as follows:

$$\pi_f^*(k) \equiv p^*(k)q_f(p^*(k)) - c(q_f(p^*(k))), k = 1, \ldots, n-1, \quad \text{(fringe firm)} \tag{3}$$

$$\pi_c^*(k) \equiv p^*(k)r(k, p^*(k)) - c(r(k, p^*(k))), k = 1, \ldots, n. \quad \text{(cartel firm)} \tag{4}$$

We assume that competitive equilibrium prevails if no actual cartel exists (i.e., $k = 0$). We denote the competitive equilibrium price by p^{comp}, which is derived from the market clearing condition $d(p) = nq_f(p)$. Therefore, we have $\pi_f^*(0) \equiv p^{\text{comp}}q_f(p^{\text{comp}}) - c(q_f(p^{\text{comp}}))$. Under certain regularity conditions on the demand and cost functions, we can show the following results.

Proposition 1. *π_c^* and π_f^* satisfy the following properties:*

(i) *$\pi_c^*(k)$ is increasing in k;*

(ii) $\pi_c^*(k) > \pi_f^*(0)$ *for all* $k = 1, \ldots, n$*; and*

(iii) $\pi_f^*(k) > \pi_c^*(k)$ *for all* $k = 1, \ldots, n-1$.

We have omitted the proof of the above proposition (see d'Aspremont et al., 1983). Property (i) indicates that the entry of a new cartel member is beneficial to each incumbent cartel member. Property (ii) indicates that the situation wherein no cartel exists is the worst situation for every firm. Property (iii) indicates that for a given cartel size, it is more preferable for a firm to remain outside the cartel than to join it.

Given a cartel $C(k)$, we can specify (i) the members of the cartel (i.e., $i \in C(k)$), (ii) the fringe firms (i.e., $j \in N \setminus C(k)$), and (iii) the price level $p^*(k)$ set by the cartel (or the competitive price p^{comp} when $k = 0$). Therefore, we can regard the cartel $C(k)$ itself as a description of the current state of the economy. With this understanding, we can state that "$C(k)$ Pareto dominates $C(m)$" if all firms' profits under $C(k)$ are not lower than under $C(m)$ and some firms' profits are strictly higher under $C(k)$ than under $C(m)$. A cartel $C(k)$ is said to be Pareto efficient if there is no other cartel $C(m)$ that Pareto dominates $C(k)$. Let $P \subset \mathbb{C}$ be the set of Pareto-efficient cartels. The following proposition characterizes P.

Proposition 2. *The set of Pareto-efficient cartels is characterized as follows:*

$$P = \left\{ C(k) \in \mathbb{C} \;\middle|\; C(k) = C(n) \text{ or } \pi_f^*(k) > \pi_c^*(n) \right\}. \tag{5}$$

Proof. Let P' be the set appearing on the right-hand side of Eq. (5). We first demonstrate that $P \supset P'$. Let us consider $C(n) \in P'$ and take an arbitrary cartel $C(m)$; note that $m < n$ by definition. Under $C(n)$, all firms receive

$\pi_c^*(n)$. If $C(m) \neq \emptyset$, the members of the cartel $C(m)$ receive $\pi_c^*(m)$. By Proposition 1, we have $\pi_c^*(m) < \pi_c^*(n)$. If $C(m) = \emptyset$, all firms receive $\pi_f^*(0)$. Again, by Proposition 1, $\pi_f^*(0) < \pi_c^*(n)$. In this case, $C(m)$ cannot Pareto dominate $C(n)$.

Next, let us consider a cartel $C(k) \in P'$ that satisfies $\pi_f^*(k) > \pi_c^*(n)$. Take an arbitrary cartel $C(m)$ with $m \neq k$. Similar to the above paragraph, if $m < k$, $C(m)$ cannot Pareto dominate $C(k)$. Suppose $m > k$; in this case, there must exist a firm that is a fringe firm under $C(k)$ but a cartel firm under $C(m)$. Such a firm receives $\pi_f^*(k)$ under $C(k)$ and $\pi_c^*(m)$ under $C(m)$. By assumption and by the monotonicity of π_c^*, we have $\pi_f^*(k) > \pi_c^*(n) \geqq \pi_c^*(m)$. Consequently, $C(m)$ cannot Pareto dominate $C(k)$; hence, $P \supset P'$.

Lastly, we show that $P \subset P'$. Take an arbitrary cartel $C(m)$ that does not belong to P'; that is, $C(m)$ satisfies $\pi_f^*(m) \leqq \pi_c^*(n)$. By Proposition 1, we have $\pi_c^*(m) < \pi_f^*(m)$ if $m \neq 0$ and $\pi_f^*(0) < \pi_c^*(n)$ if $m = 0$. Thus, $C(n)$ Pareto dominates $C(m)$ and $C(m) \notin P$; hence, $P \subset P'$. $\square$

2.2 Diamantoudi model

A cartel is considered to be stable if once it is established, no member firm wishes to exit from it and no fringe firm wishes to enter it. For example, let us consider a cartel firm i belonging to $C(m)$. As a member of $C(m)$, firm i receives profit $\pi_c^*(m)$. If firm i exits from $C(m)$, then the cartel changes to $C(m-1)$ and firm i receives profit $\pi_f^*(m-1)$ as a fringe firm. A myopic firm will actually exit from $C(m)$ if $\pi_f^*(m-1) > \pi_c^*(m)$. On the other hand, a farsighted firm, anticipating the reactions by the other firms subsequent to its own exit, may decide not to exit from $C(m)$ even if $\pi_f^*(m-1) > \pi_c^*(m)$. A farsighted firm looks forward, and it decides to deviate from the current state only if the ultimate outcome can give rise to a higher profit.

To incorporate the farsightedness of firms into the model, Diamantoudi (2005) has defined the following dominance relation, denoted by $\gg$, on $\mathbb{C}$.

Definition 1 (Indirect domination). *For $C(k), C(m) \in \mathbb{C}$, we have $C(k) \gg C(m)$ if*

$$\begin{cases} k > m \text{ and } \pi_c^*(k) > \pi_f^*(m+j) \ \forall j = 0, \ldots, k-1-m, \text{ or} \\ k < m \text{ and } \pi_f^*(k) > \pi_c^*(m-j) \ \forall j = 0, \ldots, -k-1+m. \end{cases} \tag{6}$$

The first line indicates that there is an increasing sequence of cartels—$C(m)$, $C(m+1)$,..., $C(k-1)$, $C(k)$—such that each entering firm will be better-off as a member of the final cartel $C(k)$. The second line indicates that there is a decreasing sequence of cartels—$C(m)$, $C(m-1)$,..., $C(k+1)$, $C(k)$—such that each exiting firm will be better-off as a fringe firm outside the final cartel $C(k)$. When $C' \gg C$ for $C, C' \in \mathbb{C}$, we can simply state that "C' indirectly dominates C." The pair comprising the set of all possible cartels $\mathbb{C}$ and the binary relation $\gg$ defines an abstract system that is associated with the price-leadership cartel under the optimal pricing policy: $(\mathbb{C}, \gg)$. The dominance relation $\gg$ on $\mathbb{C}$ is said to be strictly acyclic if there is no infinite sequence of elements $C, C', C'', \cdots \in \mathbb{C}$ such that $C \ll C' \ll C'' \ll \cdots$ (*ad infinitum*).[4]

The von Neumann-Morgenstern stable set (hereafter, the stable set) for $(\mathbb{C}, \gg)$ is defined as follows.

[4]For $C, C' \in \mathbb{C}$, we use $C \gg C'$ and $C' \ll C$ interchangeably.

Definition 2 (The stable set). *A set $K \subset \mathbb{C}$ is said to be a stable set for $(\mathbb{C}, \gtrdot)$ if it satisfies the following two conditions:*

- *for any $C \in K$, there does not exist $C' \in K$ such that $C' \gtrdot C$,*
- *for any $C \in \mathbb{C} \setminus K$, there exists $C' \in K$ such that $C' \gtrdot C$.*

These conditions are called "internal stability" and "external stability," respectively.

This is a well-known theorem formulated by von Neumann and Morgenstern (1953), which states that an abstract system with a strictly acyclic dominance relation admits a unique stable set. Therefore, to prove the existence and uniqueness of the stable set for $(\mathbb{C}, \gtrdot)$, it suffices to demonstrate that $\gtrdot$ is strictly acyclic. Diamantoudi (2005) has already established this fact as follows.[5]

Fact 1 (Diamantoudi, 2005). *The binary relation $\gtrdot$ on $\mathbb{C}$ is strictly acyclic.*

Taken together, Fact 1 and the von Neumann-Morgenstern theorem directly imply the following theorem:

Theorem 1 (Diamantoudi, 2005). *There exists a unique, non empty stable set of cartels for $(\mathbb{C}, \gtrdot)$.*

Although the stable set for $(\mathbb{C}, \gtrdot)$ is determined uniquely, this does not imply that the stable set itself is a singleton; that is, the unique stable set may contain several different cartel sizes. In this paper, we call a cartel in the stable set for $(\mathbb{C}, \gtrdot)$ as a "stable cartel under the optimal pricing."

[5]In Diamantoudi (2005), Fact 1 is mentioned only in the proof of Theorem 3.

Theorem 1 is essentially existential. Unfortunately, it does not provide us with much information about the shape or characteristics of the stable set. Since the von Neumann-Morgenstern theorem is very general, we are unable to extract useful information from any specific model. In the next section, we provide an alternative, constructive proof of Theorem 1. Based on this proof, we can fully characterize the unique stable set of cartels under the optimal pricing.

3 Results

Our alternative proof of Theorem 1 is both elementary and constructive; it relies neither on Fact 1 nor on the von Neumann-Morgenstern theorem. To prove the theorem, we first define an algorithm that determines a certain subset of cartels, which becomes a candidate for the stable set; subsequently, we demonstrate that this subset is actually the unique stable set for $(\mathbb{C}, \gtrdot)$.

Before stating our main results, we present a useful lemma.

Lemma 1. *For $C(k), C(m) \in C$ such that $C(k) \gtrdot C(m)$, we obtain the following results:*

(i) *if $k > m$, then $C(k) \gtrdot C(\ell)$ for any ℓ with $k > \ell \geqq m$; and*

(ii) *if $k < m$, then $C(k) \gtrdot C(\ell)$ for any ℓ with $k < \ell \leqq m$.*

Proof. Since a similar argument can also be applied to (ii), we only prove (i). By definition, $C(k) \gtrdot C(m)$ implies $\pi_f^*(m') < \pi_c^*(k)$ for all $m' = m, m+1, \ldots, k-1$. In particular, because $k > \ell \geqq m$, we have $\pi_f^*(m') < \pi_c^*(k)$ for all $m' = \ell, \ell+1, \ldots, k-1$. Therefore, $C(k) \gtrdot C(\ell)$. $\square$

3.1 Algorithm for constructing the stable set

Let us define a sequence of integers, $h_1, h_2, \ldots$, recursively as follows:[6]

$$\begin{aligned} h_1 &\equiv 1, \\ h_{j+1} &\equiv \min \left\{ k \in Z \;\middle|\; \pi_c^*(k) \geqq \pi_f^*(h_j) \right\},\; j = 1, 2, \ldots, \end{aligned} \tag{7}$$

where $Z \equiv \{1, 2, \ldots, n\}$. Since $\pi_f^*(k) > \pi_c^*(k)$ for all $k = 1, \ldots, n-1$ and π_c^* is increasing, the above recursive procedure is defined clearly, and it determines a finite sequence of integers: $h_1, h_2, \ldots, h_J$. Let $H \equiv \{h_1, h_2, \ldots, h_J\}$ be the set of such integers. We can easily verify that $h_j < h_{j+1}$ for all $j = 1, 2, \ldots, J-1$. Further, let us define a subset of cartels as follows:

$$D \equiv \{ C(k) \in \mathbb{C} \mid k \in H \}. \tag{8}$$

By construction, we have $C(h_j) \not\gg C(h_{j+1})$ for all $j = 1, 2, \ldots, J-1$. This implies $C(h_r) \not\gg C(h_s)$ for any $h_r, h_s \in H$, with $r < s$. However, it is possible to have $C(h_j) \lessdot C(h_{j+1})$ for some values of j.

Next, we construct a new subset of cartels by deleting certain elements from D according to the following recursive deletion procedure:

- Let $D^{(0)} \equiv D$.
- From $D^{(0)}$, delete all the cartels that are indirectly dominated by $C(h_J)$—which is the largest cartel in $D^{(0)}$—and let $D^{(1)}$ be the resulting set of cartels;
- From $D^{(1)}$, delete all the cartels that are indirectly dominated by the second largest cartel in $D^{(1)}$ (the largest one is $C(h_J)$), and let $D^{(2)}$ be the resulting set of cartels;

[6] An analogous technique has been utilized in Nakanishi (2007) to prove the existence of a purely noncooperative farsighted stable set for an n-player prisoners' dilemma.

- In general, from $D^{(\ell-1)}$, delete all the cartels that are indirectly dominated by the ℓth largest cartel in $D^{(\ell-1)}$, and let $D^{(\ell)}$ be the resulting set of cartels.

Since the set of cartels is finite, the above procedure will end after a finite number of steps. We denote the final set obtained from the above procedure as D^*. By construction, $C(h_J)$ is never deleted and must remain in D^*. Accordingly, D^* is non empty. We represent the subset of H that corresponds to D^* as $H^* \equiv \{h_1^*, \ldots, h_t^*, \ldots, h_T^*\}$. Without loss of generality, we can set $h_1^* < h_2^* < \cdots < h_T^*$. By definition, we have $h_J = h_T^*$, $T \leqq J$, and $h_1 \leqq h_1^*$ (it is possible to have $h_1 < h_1^*$; this implies that h_1 has been deleted in the above deletion procedure). In the following, we prove that D^* is the unique stable set for $(\mathbb{C}, \gg)$.

3.2 Alternative proof of Theorem 1

[External Stability]: Take an arbitrary cartel $C(k) \in \mathbb{C} \setminus D^*$. Consider the case where $k < h_1^*$. We show that $C(k)$ is dominated by $C(h_1^*)$. If $k = 0$, we have $\pi_f^*(k) = \pi_f^*(0) < \pi_c^*(1) \leqq \pi_c^*(h_1^*)$ by Property 1; moreover, if $h_1 = 1 \leqq k < h_1^*$, we have $C(h_1) \ll C(h_1^*)$ from the construction of h_1^*, which implies $C(k) \ll C(h_1^*)$ by Lemma 1. In any case, we obtain $C(k) \ll C(h_1^*) \in D^*$.

Next, let us consider the case where $h_t^* < k < h_{t+1}^*$ for some $h_t^* \in H^*$ (or $h_T^* < k$). Note that $h_t^* = h_r$ for some $h_r \in H$. We distinguish the following two subcases: (a) $h_t^* = h_r < k < h_{r+1}$ and (b) $h_{r+1} \leqq k < h_{t+1}^*$. In subcase (a), we have $\pi_f^*(h_t^*) = \pi_f^*(h_r) > \pi_c^*(k)$ from the definition of h_{r+1}. Since π_c^* is increasing, we have $\pi_f^*(h_t) > \pi_c^*(k) > \pi_c^*(k-1) > \cdots > \pi_c^*(h_t + 1)$.

Therefore, $C(k) \lessdot C(h_t^*) \in D^*$. In subcase (b), we should have $C(h_{r+1}) \lessdot C(h_s^*)$ for some s, with $s \geqq t+1$ by definition (note that $C(h_{r+1})$ had been deleted before $C(h_t^*)$ was reached in the recursive deletion procedure). If $s > t+1$, $C(h_{r+1}) \lessdot C(h_s^*)$ implies $C(h_{t+1}^*) \lessdot C(h_s^*)$ by Lemma 1. This contradicts the definition of $C(h_{t+1}^*)$. Thus, we have $s = t+1$. In turn, $C(h_{r+1}) \lessdot C(h_{t+1}^*)$ implies $C(k) \lessdot C(h_{t+1}^*) \in D^*$ by Lemma 1. Hence, D^* is externally stable.

[Internal Stability]: Take arbitrary $h_t^*, h_s^* \in H^*$, with $t < s$. Note that we have $h_t^* = h_r$ for some $h_r \in H$. However, by the definition of h_{r+1}, we cannot have $C(h_r) \gtrdot C(h_{r+1})$. Thus, *a fortiori*, we cannot have $C(h_t^*) = C(h_r) \gtrdot C(h_s^*)$; otherwise, Lemma 1 will be violated. Further, by definition, we cannot have $C(h_t^*) \lessdot C(h_s^*)$. Hence, D^* is internally stable.

[Uniqueness]: We first demonstrate that there is no cartel that indirectly dominates $C(h_1^*)$; this means that $C(h_1^*)$ is in the core of $(\mathbb{C}, \gtrdot)$.[7] Suppose, in negation, that there exists a cartel $C(k)$ that indirectly dominates $C(h_1^*)$. Since we have $k \notin H^*$ due to the internal stability of D^*, we are only required to distinguish the following three cases: case 1, where $k < h_1^*$; case 2, where $h_s^* < k$ and $C(h_s^*) \gtrdot C(k)$ for some $s \geqq 1$; and case 3, where $k < h_s^*$ and $C(k) \lessdot C(h_s^*)$ for some $s \geqq 2$.

Case 1. Similar to the first part of the proof for external stability, we have $C(k) \lessdot C(h_1^*)$, which implies $\pi_f^*(k) < \pi_c^*(h_1^*)$. Therefore, $C(k) \gtrdot C(h_1^*)$ cannot be true.

[7]Note that $C(h_1^*)$ is the smallest cartel in D^*; hence, this result corresponds to Theorem 5 in Diamantoudi (2005).

Case 2. $C(k) \gtrdot C(h_1^*)$ implies $\pi_f^*(m) < \pi_c^*(k)$ for all $m = h_1^*, h_1^* + 1, \ldots, k$. In particular, we have $\pi_f^*(h_s^*) < \pi_c^*(k)$. This contradicts $C(h_s^*) \gtrdot C(k)$. Therefore, we cannot have $C(k) \gtrdot C(h_1^*)$.

Case 3. $C(k) \lessdot C(h_s^*)$ implies $\pi_f^*(m) < \pi_c^*(h_s^*)$ for all $m = k, k+1, \ldots, h_s^* - 1$. If $C(h_1^*) \lessdot C(k)$, then $\pi_f^*(m) < \pi_c^*(k)$ for all $m = h_1^*, h_1^* + 1, \ldots, k-1$. These facts together imply $C(h_1^*) \lessdot C(h_s^*)$. This, however, contradicts the construction of h_1^*. Again, $C(k) \gtrdot C(h_1^*)$ cannot hold. Since none of the above three cases can be true, we obtain the desired result.

Next, let K be an arbitrary stable set for $(\mathbb{C}, \gtrdot)$. In order to prove its uniqueness, it suffices to show that $K = D^*$. Note that based on the above result, it is necessary to have $C(h_1^*) \in K$; otherwise, the external stability of K will be violated. Moreover, note that by applying an argument similar to that in case 1, we have $C(k) \notin K$ for all $k < h_1^*$; otherwise, the internal stability of K will be violated. The remainder of the proof is divided into several steps. In step 1, we demonstrate that any cartel $C(k)$ such that $h_1^* < k < h_2^*$ cannot belong to K; in step 2, we prove that $C(h_2^*) \in K$; finally, in step 3, by repeatedly applying the same arguments as those in steps 1 and 2, we demonstrate that any cartel $C(k)$ such that $h_j^* < k < h_{j+1}^*$ or $h_T^* < k$ cannot belong to K and that $C(h_j^*) \in K$ for $j = 1, \ldots, T$.

Step 1. Note that $h_1^* = h_r$ for some r. We can distinguish the following two cases: case 1, where $h_1^* = h_r < k < h_{r+1} \leqq h_2^*$, and case 2, where $h_{r+1} \leqq k < h_2^*$. Suppose, in negation, that $C(k) \in K$.

Case 1. Since by the definitions of h_r and h_{r+1}, $C(h_1^*) \in K$ indirectly dominates $C(k)$, the internal stability of K is violated. Therefore, case 1 is not possible.

Case 2. Since $C(h_2^*) \gtrdot C(h_{r+1})$ by definition, we have $C(h_2^*) \gtrdot C(k)$ by Lemma 1. Due to the internal stability of K, $C(h_2^*)$ cannot belong to K. Furthermore, due to the external stability of K, there exists a cartel $C(m) \in K$ that indirectly dominates $C(h_2^*)$. We consider the following three subcases: (i) $h_1^* < m < k$, (ii) $h_{r+1} \leqq k < m < h_2^*$, and (iii) $h_2^* < m$.

In case 2-(i), $C(m) \gtrdot C(h_2^*)$ implies $C(m) \gtrdot C(k)$ by Lemma 1; however, this violates the internal stability of K.

In case 2-(ii), because $C(h_2^*) \gtrdot C(h_{r+1})$ by definition, we have $C(h_2^*) \gtrdot C(m)$ by Lemma 1. Further, we have $\pi_f^*(m) < \pi_c^*(h_2^*)$. This contradicts $C(m) \gtrdot C(h_2^*)$.

In case 2-(iii), $C(m)$ not only indirectly dominates $C(h_2^*)$ but also Pareto dominates $C(h_2^*)$. Moreover, by simply linking the sequence realizing $C(m) \gtrdot C(h_2^*)$ with the one realizing $C(h_2^*) \gtrdot C(k)$, we obtain an appropriate sequence that realizes $C(m) \gtrdot C(k)$, which contradicts the internal stability of K. Therefore, case 2 is also not possible. As a result, we can conclude that $C(k) \notin K$ for any k with $h_1^* < k < h_2^*$.

Step 2. Suppose, in negation, that $C(h_2^*) \notin K$. Consequently, due to the external stability of K, there should exist a cartel $C(m) \in K$ that indirectly dominates $C(h_2^*)$. Based on the results obtained in step 1, it is necessary to have $m > h_2^*$.[8] We can distinguish the following four cases: case 1, where $m = h_j^* > h_2^*$ for some j; case 2, where $h_2^* \leqq h_j^* \leqq h_s < m < h_{s+1} \leqq h_{j+1}^*$ for some j and s; case 3, where $h_2^* \leqq h_j^* < h_s = m < h_{j+1}^*$ for some j and s; and

[8]Recall that by definition, $C(h_1^*)$ cannot indirectly dominate $C(h_2^*)$ and that $C(k) \notin K$ for any k with $k < h_1^*$ or $h_1^* < k < h_2^*$.

case 4, where $h_2^* \leqq h_T^* < m$.

Case 1. $C(m) = C(h_j^*) \gtrdot C(h_2^*)$ contradicts the definition of H^*.

Case 2. $C(m) \gtrdot C(h_2^*)$ implies $C(m) \gtrdot C(h_s)$ by Lemma 1. On the other hand, by the definition of h_{s+1} and the monotonicity of π_c^*, we have $\pi_f^*(h_s) > \pi_c^*(m)$; therefore, $C(m)$ cannot indirectly dominate $C(h_s)$. This is a contradiction.

Case 3. From the construction of H^*, $C(h_{j+1}^*)$ indirectly dominates $C(m)$ $(= C(h_s))$. In addition, by construction, $C(h_{j+1}^*)$ Pareto dominates $C(m)$. Thus, similar to case 2-(iii) in step 1, we have $C(h_2^*) \lessdot C(h_{j+1}^*)$. This contradicts the definition of H^*.

Case 4. $C(m) \gtrdot C(h_2^*)$ implies $C(m) \gtrdot C(h_T^*)$ by Lemma 1. However, by the definition of h_T^* $(= h_J)$, we have $\pi_f^*(h_T^*) > \pi_c^*(m)$; therefore, $C(m)$ cannot indirectly dominate $C(h_T^*)$. This is a contradiction. Since all the above four cases in step 2 are not possible, we have $C(h_2^*) \in K$.

Step 3. By repeatedly applying the same arguments as those in steps 1 and 2, we can prove that that any cartel $C(k)$ wherein $h_{j-1}^* < k < h_j^*$ or $h_T^* < k$ cannot belong to K and that $C(h_j^*) \in K$ for $j = 1, \ldots, T$. Hence, we can conclude that $K = D^*$. □

3.3 Characterization of the stable set

The stable set for $(\mathbb{C}, \gtrdot)$ is in fact D^*, which is obtained through the algorithm we have presented. With this result, we can effectively conduct comparisons between the set of stable cartels under the optimal pricing and the set of Pareto-efficient cartels.

Theorem 2. *The set of stable cartels under the optimal pricing and the set of Pareto-efficient cartels have a non empty intersection. In particular, the largest cartel in the set of stable cartels under the optimal pricing is Pareto efficient, that is,*

$$C(h_T^*) \in D^* \cap P.$$

Proof. From the definition of the recursive deletion procedure, we have $C(h_T^*) = C(h_J) \in D^*$. However, it must still be proven that $C(h_T^*) = C(h_J) \in P$. Recall that h_J is the last and the largest integer to be generated from the recursive equation (Eq. (7)). If $h_J = n$, then the proof ends. Suppose that $h_J < n$ and, in negation, that $C(h_J) \notin P$. From Proposition 2 and the properties of π_f^* and π_c^*, we have $\pi_c^*(n) \geqq \pi_f^*(h_J) > \pi_c^*(h_J)$. Thus, another integer, h_{J+1}, must be generated after h_J by Eq. (7). This is a contradiction. Hence, $C(h_J) \in P$. □

4 Example

Theorem 2 includes all the characteristics of the relationship between the set of stable cartels under the optimal pricing and the set of Pareto-efficient cartels that can be specified in a general setting. To gain further insight, we construct a concrete example. Through this example, we demonstrate that neither D^* nor P can contain the other as a subset.

We specify the demand and cost functions as follows:

$$d(p) \equiv a - bp, \quad a, b > 0, \tag{9}$$

$$c(q) \equiv \frac{q^2}{2}. \tag{10}$$

Based on the above, the supply function of a fringe firm becomes $q_f(p) \equiv p$. The competitive equilibrium price p^{comp} can be derived from $d(p) = nq_f(p)$. Next, the per-firm residual demand for a cartel firm becomes the following:

$$r(k,p) \equiv \frac{a-(b+n-k)p}{k}. \tag{11}$$

Further, the profits of a fringe firm and a cartel firm can be written as functions of p and k as follows:

$$\pi_f(p) \equiv \frac{p^2}{2}, \tag{12}$$

$$\pi_c(k,p) \equiv pr(k,p) - \frac{1}{2}\left\{r(k,p)\right\}^2. \tag{13}$$

By applying a usual procedure, we obtain the following optimal price for a cartel of size k:

$$p^*(k) \equiv \frac{a(b+n)}{(b+n)^2-k^2}, \quad k = 1,2,\ldots,n. \tag{14}$$

By substituting $p^*(k)$ and p^{comp} into π_f and π_c, we obtain π_f^* and π_c^* as follows:

$$\pi_f^*(k) \equiv \frac{a^2(b+n)^2}{2\left\{(b+n)^2-k^2\right\}^2}, \quad k = 0,1,\ldots,n-1, \tag{15}$$

$$\pi_c^*(k) \equiv \frac{a^2}{2\left\{(b+n)^2-k^2\right\}}, \quad k = 1,2,\ldots,n. \tag{16}$$

Both π_f^* and π_c^* are monotonically increasing in k. Clearly, we have

$$\pi_f^*(k) = \frac{(b+n)^2}{(b+n)^2-k^2} \cdot \pi_c^*(k), \quad k = 1,\ldots,n. \tag{17}$$

For all $k \geqq 1$, the multiplier of π_c^* on the right-hand side of the above equation is greater than unity. Therefore, we have $\pi_f^*(k) > \pi_c^*(k)$ for all

$k \geqq 1$. By simple calculation, we can show that

$$\pi_c^*(k+1) \geqq \pi_f^*(k) \quad \Leftrightarrow \quad k^4 \geqq (b+n)^2(k^2-2k-1). \tag{18}$$

In addition, assuming that $(b+n)$ is sufficiently large, we can obtain the following facts:

$$\pi_c^*(k+1) > \pi_f^*(k), \quad k = 0,1,2, \tag{19}$$

$$\pi_c^*(k+1) < \pi_f^*(k), \quad k = 3,4,\ldots,n. \tag{20}$$

These facts imply that $C(k) \lessdot C(k+1)$ for $k = 0,1,2$ and that $C(k-1) \gtrdot C(k)$ for $k = 4,5,\ldots,n$.

Now, let us construct the set D. For more concrete results, we henceforth assume that $b = 1$ and $n = 20$. Given h_j, the integer h_{j+1} is the minimum integer k satisfying $\pi_f^*(h_j) \leqq \pi_c^*(k)$. By simple calculation, we can show that

$$\pi_f^*(h_j) \leqq \pi_c^*(k) \quad \Leftrightarrow \quad \left[\frac{2(b+n)^2(h_j)^2 - (h_j)^4}{(b+n)^2}\right]^{1/2} \leqq k. \tag{21}$$

By definition, $h_1 = 1$. Further, by repeatedly applying the above equation, we obtain the following results:

$$h_1 = 1,\ h_2 = 2,\ h_3 = 3,\ h_4 = 5,\ h_5 = 7,\ h_6 = 10,\ h_7 = 14,\ h_8 = 18. \tag{22}$$

That is, $J = 8$. Hence,

$$D = \{C(1), C(2), C(3), C(5), C(7), C(10), C(14), C(18)\}. \tag{23}$$

Next, let us consider the recursive deletion procedure and construct D^*. Let $D^{(0)} = D$. The largest cartel in $D^{(0)}$ is $C(18)$; however, it does not indirectly dominate any other cartel in $D^{(0)}$, and therefore, $D^{(1)} = D^{(0)}$. The second largest cartel in $D^{(1)}$ is $C(14)$; however, it does not indirectly dominate any other cartel in $D^{(1)}$, and therefore, $D^{(2)} = D^{(1)}$. Similarly, we have $D^{(0)} = D^{(1)} = \cdots = D^{(5)}$. The sixth largest cartel in $D^{(5)}$ is $C(3)$. In this case, $C(3)$ indirectly dominates both $C(1)$ and $C(2)$. By deleting $C(1)$ and $C(2)$ from $D^{(5)}$, we obtain $D^{(6)} = D^{(5)} \setminus \{C(1), C(2)\}$, following which the deletion procedure ends. Consequently, we have

$$D^* = D^{(6)} = \{C(3), C(5), C(7), C(10), C(14), C(18)\}. \tag{24}$$

In this case, we have $h_1^* = h_3 = 3$, $h_2^* = h_4 = 5$, $h_3^* = h_5 = 7$, $h_4^* = h_6 = 10$, $h_5^* = h_7 = 14$, and $h_6^* = h_8 = 18$ (i.e., $T = 6$).

Let us examine the set P of Pareto-efficient cartels. Clearly, we have $C(n) = C(20) \in P$. For $C(k)$ with $k \neq n$ to be included in P, it is necessary to have $\pi_f^*(k) > \pi_c^*(n)$. By simple calculation, we obtain the following necessary-sufficient condition for this inequality:

$$k > \left[(b+n)^2 - \left[(b+n)^2 \left\{ (b+n)^2 - n^2 \right\} \right]^{1/2} \right]^{1/2}. \tag{25}$$

By substituting $b = 1$ and $n = 20$, we obtain $k > 17.508\ldots$ Hence,

$$P = \{C(18), C(19), C(20)\}. \tag{26}$$

As Theorem 2 indicates, we have $C(18) \in D^* \cap P$. In fact, in this specific example, $C(18)$ is the only stable cartel belonging to both D^* and P. Furthermore, we have both $D^* \setminus P \neq \emptyset$ and $P \setminus D^* \neq \emptyset$. Therefore, neither $D^* \subset P$ nor $P \subset D^*$ can be true.

Acknowledgement

An earlier version of this note was written while Nakanishi was at McGill University as a visiting scholar. Nakanishi expresses his gratitude to the Department of Economics at McGill University for its hospitality and to the *Rokkodai-Kouenkai* Foundation (Kobe University) for its financial support to Nakanishi's stay at McGill.

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