The Buy Price in Auctions with Discrete Type Distributions *

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Abstract

This paper considers second-price, sealed-bid auctions with a buy price where bidders' types are discretely distributed. We characterize all equilibria, restricting our attention to equilibria where bidders whose types are not greater than a buy price bid their own valuations. Budish and Takeyama (2001) analyzed the two-bidder, two-type framework, and showed that if bidders are risk-averse, a seller can obtain a higher expected revenue from the auction with a certain buy price than the auction without a buy price. We extend their revenue improvement result to the n-bidder, two-type framework. However, in case of three or more types, bidders' risk aversion is not a sufficient condition for the revenue improvement. Our example illustrates that even if bidders are risk-averse, a seller cannot always obtain a higher expected revenue from the auction with a buy price.

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1 Introduction

We commonly observe that sellers set buy prices in Internet auctions. Since the winning bid is not above a buy price, it seems that the seller loses by determining an upper bound of the winning bid. Several papers, however, show that it may be reasonable for a seller to set a buy price. If a bidder wins by bidding a buy price, he certainly obtains some surplus. In contrast, if he wins by bidding the amount except a buy price, his surplus is random and depends on other bidders' reservation values. That is, a buy price plays a role of insurance for a risk-averse bidder. Therefore, a seller can extract a risk premium from risk-averse bidders by introducing a buy price, and then obtain a higher expected revenue. [1]

Budish and Takeyama [1] first considered a second-price, sealed-bid auction with a buy price. They analyzed a simple model—the two-bidder, two-type framework. Their main result is that if bidders are risk-averse, a seller can obtain a higher expected revenue from the auction with a certain buy price. Hidvégi, Wang, and Whinston [2] and Reynolds and Wooders [7] extended the analysis in two directions. One is that bidders' types are continuously distributed. The other is that the auction is an open format. These two papers showed that if bidders exhibit constant absolute risk aversion (CARA), a seller can obtain a higher expected revenue by properly setting a buy price. The above results indicate that bidders' risk-aversion is a sufficient condition for a revenue improvement.

This paper extends the analysis in a different direction. We consider a second-price, sealed-bid auction with a buy price where bidders' types are discretely distributed. Specifically, we analyze a general model—the n-bidder, m-type framework. In general, there are a lot of equilibria. To limit our attention to equilibria by reasonable strategies, we introduce the notion of *partial truth-telling*: a bidder whose type is not greater than a buy price bids his own valuation. Under this reasonable restriction, we characterize all equilibria. We show that there are only two kinds of equilibria where all bidders play partially truth-telling strategies. One is the symmetric equilibrium in which all bidders whose types are greater than a buy price actually bid it. The other is the asymmetric equilibrium where $n-1$ bidders play the strategy that any types above the buy price bid it and only one bidder plays a different strategy. For each class of equilibrium, we derive a necessary and sufficient condition for existence. This condition implies that there always exists a buy price between the lowest valuation and the second lowest valuation under which we have an symmetric equilibrium. We also show that there is no asymmetric equilibrium in the two-bidder framework.

We analyze whether a seller can improve her payoffs by introducing a buy price. We assume not only that bidders are risk-neutral or risk-averse, but also that a seller is risk-neutral or risk-averse. We show that if we can find a buy price under which the symmetric equilibrium exists between the highest valuation and the second highest valuation, a seller can obtain a higher expected utility at the highest buy price under which we have the symmetric equilibrium, unless both the seller and the bidders are risk-neutral. In any two-type framework, this condition always holds because of the above result on existence. Thus, we can obtain a generalization of the result of Budish and Takeyama [1] in the two-type framework with any numbers of bidders. Our utility improvement results allow a seller's risk-aversion and do not depend on a CARA utility function unlike Hidvégi, Wang, and Whinston [2] and Reynolds and Wooders [7].

To investigate possibilities of further improvements, we take account of buy prices under which asymmetric equilibria exist and the utilities. We show that a seller cannot obtain a higher expected utility from the asymmetric equilibria than from the symmetric equilibrium.

If the condition for utility improvements does not hold, we need to consider the auction with a buy price that is less than the second highest valuation. We show two examples. In the first example, the seller cannot improve her expected revenue by introducing such a buy

[1] For a similar reason, a seller can obtain a higher expected revenue from a first-price, sealed-bid auction than from a second-price, sealed-bid auction. See Maskin and Riley [4] or Matthews [6] for details.

price. In the second example, the seller can improve her expected revenue by introducing such a buy price. The former concludes that even if bidders are risk-averse, a seller cannot always obtain a higher expected revenue from the auction with a buy price. In case of three or more types, bidders' risk aversion is not a sufficient condition for revenue improvements.

We introduce other results in related literature. Budish and Takeyama [1] showed that if bidders are risk-averse, then a seller can obtain a higher expected revenue by properly setting a buy price from the second-price, sealed-bid auction with a buy price than from the first-price, sealed-bid auction without a buy price. In general, it is complicated to analyze the first-price, sealed-bid auction with three (or more) bidders where bidders' types are discretely distributed. Thus, we make a comparison between the second-price, sealed-bid auctions with and without a buy price. In Internet auctions, two kinds of buy out prices are practically used. One is a buy price on Yahoo!. The other is a Buy It Now price on eBay. [2] Reynolds and Wooders [7] compared English auctions with these two buy out prices. They showed that a seller can obtain a higher expected revenue from the auction with a certain buy price than from the auction with the same Buy It Now price if bidders have CARA utility functions. Since we consider a static model, we cannot compare a buy price with a Buy It Now price. Some papers analyze auctions with a buy out price from a seller's point of view. Hidvégi, Wang, and Whinston [2] showed that a risk-averse seller can obtain a higher expected utility from the auction with a buy price. Mathews and Katzman [5] obtained a similar utility improvement result in the auction with a Buy It Now price. We also show that a risk-averse seller can obtain a higher expected utility from the second-price, sealed-bid auction with a buy price.

The remainder of this paper is organized as follows. Section 2 describes the model. In Section 3, we characterize all equilibria. Section 4 examines whether a seller can improve her expected utility by introducing a buy price. And in Section 5, we conclude.

2 The model

We consider a second-price, sealed-bid auction with a buy price. Bidders' types are discretely distributed, and are drawn independently from an identical distribution. Bidders' valuations of an item depend only on their types. This auction consists of two stages: (i) a seller sets a buy price $B \in [0, +\infty)$, and (ii) an item is up for auction. We analyze mainly this auction given a buy price B and then argue which buy price B a seller should choose.

Let $N = \{1, \ldots, n\}$ denote the set of bidders. The set of types for each bidder i is $T_i = \{v^1, \ldots, v^m\}$ with $v^1 < \cdots < v^m$. We denote by f_μ the probability that a bidder's type is v^μ. We assume that for all μ, $f_\mu > 0$, and define $F_\mu = f_1 + \cdots + f_\mu$. Note that $F_m = 1$. In addition, let $F_0 = 0$. Each bidder's von-Neumann-Morgenstern utility function is $U : \mathbb{R} \to \mathbb{R}$ with $U(0) = 0$. We assume that $U(\cdot)$ is strictly increasing and concave (possibly linear).

We assume that bidders cannot bid above a buy price B. Thus, the set of actions for each bidder i is $A_i = [0, B]$. Bidder i's payoff function is $u_i : A \times T_i \to \mathbb{R}$, where $A = \times_{i=1}^n A_i$. Given $t_i \in T_i$ and $a \in A$, bidder i's utility is:

$$u_i(a; t_i) = \begin{cases} U(t_i - \max_{j \neq i} a_j) & \text{if } a_i \neq B \text{ and } a_i > \max_{j \neq i} a_j, \\ U(t_i - B) & \text{if } a_i = B \text{ and } a_i > \max_{j \neq i} a_j, \\ \frac{1}{M} U(t_i - a_i) & \text{if } a_i = \max_{j \neq i} a_j \text{ and } M = \mathfrak{card}\{j | a_j = a_i\}, \text{ and} \\ 0 & \text{if } a_i < \max_{j \neq i} a_j. \end{cases}$$

If no one bids a buy price B, this auction is an ordinary second-price, sealed-bid auction. Thus, the highest bidder obtains the item and pays the second highest bid. If only one bidder bids a buy price B, then he immediately obtains the item but must pay it to the

[2] In the case of Yahoo!, bidders can always bid a buy price throughout the auction. In the case of eBay, bidders can bid a Buy It Now price only before the bidding process starts.

seller. If there are two or more bidders who submit the highest bid (it might be a buy price B.), we adopt a tie-breaking rule that a winner is determined with equal probability.

A bidder i's strategy is $\sigma_i : T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over A_i. [3] A solution concept is Bayesian Nash equilibrium: a strategy profile $\sigma = (\sigma_i)_{n=1}^n$ is a Bayesian Nash equilibrium if for all i, all $t_i \in T_i$, and all $a_i' \in A_i$,

$$E[u_i(a;t_i)|\sigma_i, \sigma_{-i}, \rho(t_{-i})] \geq E[u_i(a_i', a_{-i}; t_i)|\sigma_{-i}, \rho(t_{-i})],$$

where σ_{-i} is the vector of the other $n-1$ bidders' strategies, and $\rho(\cdot)$ is the probability distribution of the other $n-1$ bidders' types.

3 Characterization of equilibria

We consider the auction with a buy price $B \in (v^k, v^{k+1}]$ $(k = 1, \ldots, m-1)$. Indeed, we have a lot of equilibria. To restrict our attention to equilibria by reasonable strategies, we propose the notion of partial truth-telling.

Definition 1. *A strategy $\sigma_i(\cdot)$ is partially truth-telling if $\sigma_i(t_i) = t_i$ for all $t_i < B$.*

When a bidder plays a partially truth-telling strategy, his type that is not greater than a buy price B bids his own valuation. That is, a bidder whose type is not greater than a buy price B takes a weakly dominant action. In this paper, we only consider equilibria where all bidders play partially truth-telling strategies. [4] In particular, we pay much attention to the following partially truth-telling strategy:

$$\sigma_i^*(t_i) = \begin{cases} B & \text{if } t_i \geq v^{k+1}, \\ t_i & \text{otherwise.} \end{cases}$$

The strategy $\sigma_i^*(\cdot)$ is reasonable, because it is a weakly dominated action for the bidder whose type is greater than a buy price B to bid an amount $b \neq B$.

The strategy $\sigma_i^*(\cdot)$ plays an important role. Indeed, at least $n-1$ bidders play the strategy $\sigma_i^*(\cdot)$ in any equilibria.

Proposition 1. *Any strategy profiles where at least two bidders play strategies except the strategy $\sigma_i^*(\cdot)$ do not become an equilibrium.*

Proof. See Appendix.

By Proposition 1, it suffices to consider only two kinds of strategy profiles. One is the symmetric strategy profile where all bidders play the strategy $\sigma_i^*(\cdot)$. The other is asymmetric strategy profiles where only one bidder does not play the strategy $\sigma_i^*(\cdot)$, while all other bidders play it.

3.1 Symmetric equilibrium

In this subsection, we consider symmetric equilibrium. By Proposition 1, it suffices to consider the symmetric strategy profile $\sigma^* = (\sigma_i^*)_{i=1}^n$. The symmetric strategy profile σ^* is

[3] When $\sigma_i(\cdot)$ is a pure strategy, we often regard the range of $\sigma_i(\cdot)$ as A_i.

[4] If we analyze all equilibria, as Inami [3] shows, we cannot accurately compare seller's expected revenues. Indeed, a seller can obtain a higher expected revenue from the equilibrium by weakly dominated strategies than from the equilibrium by partially truth-telling strategies. Thus, our restriction on equilibria by partially truth-telling strategies would be a reasonable solution concept for study of buy prices.

a Bayesian Nash equilibrium if and only if

$$\left\{\sum_{\nu=0}^{n-1}\frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^{\nu}\right\}U(v^{\kappa}-B) \\ \geq \sum_{\mu=1}^{k}\left\{(F_{\mu})^{n-1}-(F_{\mu-1})^{n-1}\right\}U(v^{\kappa}-v^{\mu}) \tag{1}$$

for all $\kappa = k+1,\dots,m$. The LHS of (1) is the expected payoff that a bidder's v^{κ}-type obtains when he bids a buy price B. ${}_{n-1}C_{\nu}$ $(1-F_k)^{n-1-\nu}(F_k)^{\nu}$ is the probability that there are ν other bidders whose types are below v^k. In this case, his winning probability is $1/\{n-\nu\}$ because the winner is determined among the bidders who bid a buy price B with equal probability. The RHS of (1) is the expected payoff that a bidder's v^{κ}-type obtains when he bids $b \in (v^k, B)$. $(F_{\mu})^{n-1}-(F_{\mu-1})^{n-1}$ is the probability that the second highest bid among $n-1$ other bidders is v^{μ}. In this case, since this auction is a second-price, sealed-bid auction, he wins and pays v^{μ}. If he bid $b' \leq v^k$, then he could not (surely) win against opponents' v^k-type because they bid v^k. In other words, he cannot maximize his winning probability by bidding b'. Thus, the RHS of (1) is the maximum expected payoff by bidding the amount except a buy price B.

Indeed, we do not need to consider all those inequalities.

Proposition 2. *The strategy profile σ^* is a Bayesian Nash equilibrium if and only if* (1) *holds for $\kappa = k+1$.*

Proof. The necessity part is straightforward. We only prove the sufficiency part.

Fix $\kappa \geq k+1$. For all $\mu \in \{1,\dots,k\}$,

$$U(v^{\kappa}-B)-U(v^{k+1}-B) \geq U(v^{\kappa}-v^{\mu})-U(v^{k+1}-v^{\mu}),$$

where the inequality follows because $U(\cdot)$ is concave. In addition,

$$\begin{aligned}\sum_{\nu=0}^{n-1}\frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^{\nu} &= \frac{1-(F_k)^n}{n(1-F_k)} \\ &= \frac{1+F_k+\cdots+(F_k)^{n-2}+(F_k)^{n-1}}{n} \\ &> (F_k)^{n-1} \\ &= \sum_{\mu=1}^{k}\left\{(F_{\mu})^{n-1}-(F_{\mu-1})^{n-1}\right\}.\end{aligned}$$

As a result, we have

$$\left\{\sum_{\nu=0}^{n-1}\frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^{\nu}\right\}\left\{U(v^{\kappa}-B)-U(v^{k+1}-B)\right\} \\ \geq \sum_{\mu=1}^{k}\left\{(F_{\mu})^{n-1}-(F_{\mu-1})^{n-1}\right\}\left\{U(v^{\kappa}-v^{\mu})-U(v^{k+1}-v^{\mu})\right\}.$$

Thus, if (1) holds for $\kappa = k+1$, then (1) also holds for $\kappa \geq k+2$. Hence, the strategy profile σ^* is a Bayesian Nash equilibrium. *Q.E.D.*

We consider when there exists a buy price $B \in (v^k, v^{k+1}]$ such that (1) for $\kappa = k+1$ holds. The following condition ensures the existence of such buy price B:

$$U(v^{k+1}-v^k) > \frac{\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^{k+1}-v^\mu)}{\sum_{\nu=0}^{n-1}\frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^\nu} \tag{2}$$

$$= \frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^{k+1}-v^\mu)}{1-(F_k)^n}.$$

(2) can be interpreted in a way of certainty equivalent. Consider the lottery: probability is that in a second-price, sealed-bid auction without a buy price, the second highest bid is v^μ conditioning that a bidder's v^{k+1}-type wins; outcome is monetary payoff obtained from the case in which he wins and pays v^μ. [5] Thus, the RHS of (2) is the expected utility of this lottery. (2) requires that the certainty equivalent of this lottery is not grater than $v^{k+1}-v^k$.

If (2) holds, then we can also find the highest buy price B under which the symmetric strategy profile σ^* is a Bayesian Nash equilibrium. Consider (1) for $\kappa = k+1$:

$$U(v^{k+1}-B) \geq \frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^{k+1}-v^\mu)}{1-(F_k)^n}. \tag{3}$$

Arranging (3), we have

$$B \leq v^{k+1} - U^{-1}\left(\frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^{k+1}-v^\mu)}{1-(F_k)^n}\right).$$

Here let

$$B^*_{k+1} := v^{k+1} - U^{-1}\left(\frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^{k+1}-v^\mu)}{1-(F_k)^n}\right).$$

Thus, when we consider the auction with a buy price $B \in (v^k, B^*_{k+1}]$, the symmetric strategy profile σ^* is a Bayesian Nash equilibrium. On the contrary, when we consider the auction with a buy price $B \in (B^*_{k+1}, v^{k+1}]$, the symmetric strategy profile σ^* is not an equilibrium. In this case, by Proposition 1, there is no symmetric equilibrium.

Proposition 3. *Suppose that* (2) *holds. Then,*
(i) when we consider the auction with a buy price $B \in (v^k, B^*_{k+1}]$*, the strategy profile* σ^* *is a unique symmetric equilibrium, and*
(ii) when we consider the auction with a buy price $B \in (B^*_{k+1}, v^{k+1}]$*, there is no symmetric equilibrium.*

Indeed, (2) always holds in the auction with a buy price B between certain valuations. We consider the auction with a buy price B between the lowest valuation and the second lowest valuation.

Proposition 4. (2) *for* $k = 1$ *always holds.*

Proof. We show that

$$U(v^2-v^1) > \frac{n(1-F_1)(F_1)^{n-1}U(v^2-v^1)}{1-(F_1)^n}.$$

Since

$$1 > \frac{n(F_1)^{n-1}}{1+F_1+\cdots+(F_1)^{n-2}+(F_1)^{n-1}} = \frac{n(1-F_1)(F_1)^{n-1}}{1-(F_1)^n},$$

we immediately have the result. *Q.E.D.*

[5] Note that as the sum of probability is unity, the rest is assigned to the outcome 0 ($= v^{k+1} - v^{k+1}$).

Figure 1: Example 1

$(v^k, v^{k+1}]$	The LHS of (2) for k	The RHS of (2) for k
$(v^3, v^4]$ $(k = 3)$	2.00	1.31
$(v^2, v^3]$ $(k = 2)$	0.71	0.77
$(v^1, v^2]$ $(k = 1)$	5.05	0.98

From the view point of certainty equivalent, the result in Proposition 4 is obvious. Because the certainty equivalent is not greater than $v^2 - v^1$ once the positive probability is assigned to the outcome 0 $(= v^2 - v^2)$. By Proposition 4, (2) always holds in the two-type framework. Thus, (2) always holds in the two-bidder, two-type framework of Budish and Takeyama [1].

In general, there is no regularity, e.g. monotonicity, as to whether (2) holds. We show an example.

Example 1. *Consider a three-bidder, four-type framework. Let* $U(x) = \sqrt{x}$, $f_1 = \frac{3}{10}$, $f_2 = \frac{1}{10}$, $f_3 = \frac{1}{10}$, $v^1 = 10$, $v^2 = 35.5$, $v^3 = 36$ *and* $v^4 = 40$.

In this example, both (2) for $k = 1$ and (2) for $k = 3$ hold. On the contrary, (2) for $k = 2$ does not hold. (See Figure 1.) Thus, there is no monotonicity as to whether (2) holds. [6]

We consider the auction with a buy price B between other valuations, $(v^k, v^{k+1}]$ $(k \neq 1)$. In general, (2) does not always hold. However, even in this case, (2) for $k \neq 1$ holds in a large number of bidders rather than in a small number of bidders.

Proposition 5. *Fix* $U(\cdot)$, f_μ, v^μ *for all* $\mu = 1, \cdots, k$ *and* v^{k+1}. *There exists* n_0 *such that for all* $n \geq n_0$,

$$U(v^{k+1} - v^k) > \frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^{k+1} - v^\mu)}{1-(F_k)^n}.$$

Proof. Let

$$G_\mu(n) := \frac{n(F_\mu)^{n-1}}{1-(F_k)^n}$$

for all $\mu = 1, \ldots, k$. Then, we have

$$\begin{aligned} & U(v^{k+1} - v^k) - \frac{n(1-F_k)\sum_{\mu=1}^{k}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^{k+1} - v^\mu)}{1-(F_k)^n} \\ &= U(v^{k+1} - v^k) - (1-F_k)\sum_{\mu=1}^{k} G_\mu(n)\{U(v^{k+1} - v^\mu) - U(v^{k+1} - v^{\mu+1})\}. \end{aligned} \tag{4}$$

For all μ, $(1-F_k)G_\mu(n) \to 0$ as $n \to 0$. Thus, there exists n_0 such that $(4) > 0$. Hence, we have the result. *Q.E.D.*

Since a bidder's v^{k+1}-type confronts more other bidders who have the same type, the probability of winning decreases as the number of bidders increases. This means that more probability is assigned to the outcome 0 $(= v^{k+1} - v^{k+1})$. Thus, the certainty equivalent of the lottery decreases. Also, by Proposition 5, there exists n^* such that in the auctions with a buy price B between all valuations, each corresponding strategy profile σ^* is a Bayesian Nash equilibrium. Since in Internet auctions, a seller faces many bidders, (2) might hold.

[6] Strictly speaking, we have a problem that bidders' utility function $U(\cdot) = \sqrt{x}$ is not defined on $(-\infty, 0)$. To resolve this problem, we modify the utility function $U(\cdot)$ as follows: at some point $\tilde{x}$ close to 0, $U(\cdot) = \sqrt{x}$ if $x \geq \tilde{x}$ and $U(\cdot) = x/\sqrt{\tilde{x}}$ if $x < \tilde{x}$. We make the same modification in Example 3.

3.2 Asymmetric equilibrium

In this subsection, we consider asymmetric equilibrium. By Proposition 1, it suffices to consider asymmetric strategy profiles where only one bidder, say, bidder 1, does not play the strategy $\sigma_i^*(\cdot)$ and all other bidders play it.

Consider the incentive of bidder 1. Since bidder 1 does not play the strategy $\sigma_1^*(\cdot)$, there exists $k+1 \le \kappa \le m$ such that

$$\begin{aligned}&\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^\kappa - v^\mu)\\ &\quad\ge \left\{\sum_{\nu=0}^{n-1}\frac{1}{n-\nu}\binom{n-1}{\nu}(1-F_k)^{n-1-\nu}(F_k)^\nu\right\}U(v^\kappa - B).\end{aligned} \tag{5}$$

The LHS of (5) is the expected payoff that bidder 1's v^κ-type obtains by bidding the amount except a buy price B. The RHS of (5) is the expected payoff that he obtains by bidding a buy price B. If (5) does not hold, bidder 1's v^κ-type obtains a higher expected payoff by bidding a buy price B.

Suppose that such κ exists. By a similar argument to that of Proposition 2, we can find k^* such that (5) holds if $\kappa \le k^*$ and (5) does not hold if $\kappa \ge k^*+1$. We fix $b \in (v^k, B)$. We then define the following strategy:

$$\overline{\sigma}_1(t_1; b) = \begin{cases} B & \text{if } t_1 \ge v^{k^*+1}, \\ b & \text{if } v^{k+1} \le t_1 \le v^{k^*}, \\ t_1 & \text{otherwise.}\end{cases}$$

We examine whether the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma_{-1}^*)$ is an equilibrium. First, consider the incentive of bidder 1. By the definition of k^*, for any b, bidder 1's strategy $\overline{\sigma}_1(\cdot; b)$ is a best response to other bidders' strategies $\sigma_{-1}^* = (\sigma_j^*)_{j\neq 1}$. Thus, it suffices to consider the incentive of bidder j ($j = 2, \ldots, n$). To be an equilibrium, for all j and for all $\kappa \in \{k+1, \ldots, m\}$,

$$\begin{aligned}&\left\{\sum_{\nu=0}^{n-2}\binom{n-2}{\nu}\left(\frac{1-F_{k^*}}{n-\nu}+\frac{F_{k^*}}{n-1-\nu}\right)(1-F_k)^{n-2-\nu}(F_k)^\nu\right\}U(v^\kappa - B)\\ &\quad\ge \sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(v^\kappa - v^\mu) + (F_{k^*}-F_k)(F_k)^{n-2}U(v^\kappa - b)\end{aligned} \tag{6}$$

must hold. The LHS of (6) is the expected payoff that bidder j's v^κ-type obtains by bidding a buy price B. The RHS of (6) is the expected payoff that bidder j's v^κ-type obtains by bidding the amount except a buy price B. Bidder j always wins against bidder 1 whose type is less than v^k. Furthermore, he wins against bidder 1 whose type is greater than v^{k+1}, but less than v^{k^*}.

From the same argument of Subsection 3.1, it suffices to consider (5) for $\kappa = k^*$ and (6) for $\kappa = k+1$. [7] Both inequalities are also necessary conditions for an equilibrium.

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[7] Let

$$\begin{aligned}I(x) := &\left\{\sum_{\nu=0}^{n-2}\binom{n-2}{\nu}\left(\frac{1-F_{k^*}}{n-\nu}+\frac{F_{k^*}}{n-1-\nu}\right)(1-F_k)^{n-2-\nu}(F_k)^\nu\right\}U(x-B)\\ &-\sum_{\mu=1}^{k}\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\}U(x-v^\mu) - (F_{k^*}-F_k)(F_k)^{n-2}U(x-b).\end{aligned}$$

Since we can show that $I(\cdot)$ is monotone increasing, it suffices to consider (6) for $\kappa = k+1$.

Proposition 6. *An asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is a Bayesian Nash equilibrium if and only if both* (5) *holds for $\kappa = k^*$ and* (6) *holds for $\kappa = k + 1$.*

In general, it is unclear whether there exists $b \in (v^k, B)$ such that the strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is an equilibrium. The following inequality is a necessary and sufficient condition for the existence of such b:

$$\begin{aligned} &\left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1 - F_{k^*}}{n - \nu} + \frac{F_{k^*}}{n - 1 - \nu} \right) (1 - F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \\ &\quad > \sum_{\mu=1}^{k} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} U(v^{k+1} - v^\mu) + (F_{k^*} - F_k)(F_k)^{n-2} U(v^{k+1} - B). \end{aligned} \tag{7}$$

Thus, if (7) holds, then we can find $b \in (v^k, B)$ such that (6) holds.

We have considered a specific asymmetric strategy profile. Indeed, other asymmetric strategy profiles might be equilibria. Even in those cases, we can always construct the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ that is an equilibrium.

Proposition 7. *Suppose that some asymmetric strategy profile $\sigma = (\sigma_1, \sigma^*_{-1})$ is an equilibrium. Then, there exists $b \in (v^k, B)$ such that the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is also an equilibrium.*

Proof. See Appendix.

By Proposition 6 and Proposition 7, (7) is also a necessary and sufficient condition for the existence of asymmetric equilibrium.

However, an asymmetric equilibrium does not always exist.

Proposition 8. *There is no asymmetric equilibrium if:*
(i) (5) *does not hold for $\kappa = k + 1$, and*
(ii) there are only two bidders.

Proof. See Appendix.

The first part of Proposition 8 states that there is no asymmetric equilibrium in the auction with a buy price $B \in (v^k, B^*_{k+1})$. Also, by the second part of Proposition 8, there is no asymmetric equilibrium in the two-bidder framework regardless of the number of bidders' types. It follows that there is no asymmetric equilibrium in the two-bidder, two-type framework of Budish and Takeyama [1].

4 Auction comparisons from a seller's point of view

In this section, we consider a seller's expected utility obtained from the auction with a buy price B. The seller's von-Neumann-Morgenstern utility function is $W : \mathbb{R}_+ \to \mathbb{R}_+$ with $W(0) = 0$. We assume that $W(\cdot)$ is strictly increasing and concave (possibly linear).

4.1 The comparison with the auction without a buy price B

First, we consider the auction with a buy price $B \in (v^{m-1}, v^m]$. There exists the symmetric equilibrium σ^* in the auction with a buy price $B \in (v^{m-1}, B^*_m]$ provided that (2) holds for $k = m - 1$. The seller's expected utility obtained from the outcome of the symmetric equilibrium σ^* is:

$$\begin{aligned} R^B = &\sum_{\mu=1}^{m-1} \left[nF_{m-1} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} - (n-1) \left\{ (F_\mu)^n - (F_{\mu-1})^n \right\} \right] W(v^\mu) \\ &+ \left\{ 1 - (F_{m-1})^n \right\} W(B). \end{aligned} \tag{8}$$

Since R^B is increasing with respect to B, we evaluate (8) at $B = B_m^*$ and denote by R_m^B.

Next, we consider the auction without a buy price B. The seller's expected utility is:

$$R^{NB} = \sum_{\mu=1}^{m} \Big[n\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\} - (n-1)\{(F_\mu)^n - (F_{\mu-1})^n\} \Big] W(v^\mu). \tag{9}$$

Since this auction is a second-price, sealed-bid auction, the winner pays the second highest bid v^μ to the seller.

Theorem 1. *Consider a n-bidder, m-type framework. Suppose that* (2) *holds for* $k = m-1$. *Then,*

$$R_m^B \geq R^{NB},$$

where the equality holds if and only if both the seller and the bidders are risk-neutral.

Proof. See Appendix.

Even if bidders are risk-loving, the symmetric equilibrium might exist. However, by the same argument of the proof of Theorem 1, the seller cannot improve her expected utility. [8]

In the two-type framework, by Proposition 4, (2) always holds for $k = 1$.

Corollary 1. *Consider a n-bidder, two-type framework. Then,*

$$R_m^B \geq R^{NB},$$

where the equality holds if and only if both the seller and the bidders are risk-neutral.

Corollary 1 generalizes the result of Budish and Takeyama [1] in the two-bidder, two-type framework with respect to the number of bidders and the seller's risk attitude.

4.2 The comparison with the auction with asymmetric equilibria

We consider whether a seller can obtain a higher expected utility than the expected utility R_m^B. As long as we limit attention to a symmetric equilibrium, the expected utility R_m^B is the highest of all auctions with a buy price $B \in (v^{m-1}, v^m]$. [9] We then extend the analysis to the case of asymmetric equilibria. The next subsection considers another possibility that a seller can obtain a higher expected utility from the auction with a buy price B between other valuations.

We continue to assume that (2) holds for $k = m-1$. Thus, by the definition of k^*, there is no asymmetric equilibrium in the auction with a buy price $B \in (v^{m-1}, B_m^*)$. From the same argument of Subsection 3.2, there exists an asymmetric equilibrium if and only if

$$\begin{aligned} &\sum_{\nu=0}^{n-2} \left\{ \frac{1}{n-1-\nu} \binom{n-2}{\nu} (f_m)^{n-2-\nu} (F_{m-1})^\nu \right\} U(v^m - B) \\ &\quad > \sum_{\mu=1}^{m-1} \{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\} U(v^m - v^\mu) + f_m (F_{m-1})^{n-2} U(v^m - B). \end{aligned} \tag{10}$$

Arranging (10), we have [10]

$$B < v^m - U^{-1}\left(\frac{(n-1) f_m \sum_{\mu=1}^{m-1} \{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\} U(v^m - v^\mu)}{1 - (F_{m-1})^{n-1} - (n-1)(f_m)^2 (F_{m-1})^{n-2}} \right). \tag{11}$$

[8] The seller obtains a strictly lower expected utility.

[9] Note that there is no symmetric equilibrium in the auction with a buy price $B \in (B_m^*, v^m]$.

[10] Note that

$$\sum_{\nu=0}^{n-2} \frac{1}{n-1-\nu} \binom{n-2}{\nu} (f_m)^{n-2-\nu} (F_{m-1})^\nu - f_m (F_{m-1})^{n-2} = \frac{1 - (F_{m-1})^{n-1} - (n-1)(f_m)^2 (F_{m-1})^{n-2}}{(n-1) f_m}.$$

Here let

$$\overline{B_m^*} := v^m - U^{-1}\left(\frac{(n-1)f_m \sum_{\mu=1}^{m-1}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^m - v^\mu)}{1-(F_{m-1})^{n-1} - (n-1)(f_m)^2(F_{m-1})^{n-2}}\right).$$

Note that there is no asymmetric equilibrium in the auction with a buy price $B \in [\overline{B_m^*}, v^m]$.

From the above argument, there exists an asymmetric equilibrium in the auction with a buy price $B \in [B_m^*, \overline{B_m^*})$. We then consider the expected utility obtained from the auction with asymmetric equilibria. [11] An upper bound of the seller's expected utility with asymmetric equilibria is:

$$\begin{aligned} R^{\overline{B}} = &\sum_{\mu=1}^{m-1}\Big[nF_{m-1}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\} - (n-1)\{(F_\mu)^n - (F_{\mu-1})^n\}\Big]W(v^\mu) \\ &+ \sum_{\mu=1}^{m-1} f_m\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}W(v^\mu) + \{1-(F_{m-1})^{n-1}\}W(\overline{B_m^*}). \end{aligned} \tag{12}$$

Note that the bid of bidder 1's v^m-type is not actually paid to the seller.

Proposition 9. *Consider a n-bidder, m-type framework. Suppose that* (2) *holds for $k = m-1$ and that* (10) *holds. Then,*

$$R_m^B > R^{\overline{B}}.$$

Proof. See Appendix.

4.3 Discussions

We have only analyzed auctions with a buy price B between the highest valuation and the second highest valuation, $(v^{m-1}, v^m]$. In this subsection, we examine a more general case. That is, we take account of the auction with a buy price B between other valuations, $(v^k, v^{k+1}]$ $(k \neq m-1)$. We analyze the two-bidder, three-type framework as an example. We assume that a seller is risk-neutral and that bidders are risk-averse. Since we consider the two-bidder framework, by Proposition 8, we need not care about asymmetric equilibria.

First, we consider the auction with a buy price $B \in (v^2, v^3]$. The symmetric equilibrium exists if and only if (2) for $k = 2$ holds:

$$U(v^3 - v^2) \geq \frac{2f_1 U(v^3 - v^1)}{1 + f_1 - f_2}. \tag{13}$$

If (13) holds, by Theorem 1, then a seller can obtain a higher expected revenue from the auction with the buy price B_3^* than the auction without a buy price B.

Next, we consider the auction with a buy price $B \in (v^1, v^2]$. Since, by Proposition 4, (2) for $k = 1$ always holds, it suffices to compare seller's expected revenues in the same way of Subsection 4.1. The following condition ensures that a seller can obtain a higher expected revenue from the auction with the buy price B_2^* than the auction without a buy price B:

$$\begin{aligned} &U\left(\frac{-(f_3)^2 v^3 + \{1 - 2f_2 + 2f_1 f_2 - (f_1)^2 + (f_2)^2\}v^2 - 2f_1(1-f_1)v^1}{1-(f_1)^2}\right) \\ &\quad > \frac{2f_1}{1+f_1}U(v^2 - v^1). \end{aligned} \tag{14}$$

[11] In the auction with the buy price B_m^*, there exist both a symmetric equilibrium and an asymmetric equilibrium. Even in those cases, a seller can obtain a higher expected utility from the outcome of a symmetric equilibrium. This is because in any asymmetric equilibrium, one bidder whose type is greater than a buy price B pays less the buy price B_m^* to the seller with positive probability.

Example 2. *Let* $U(x) = 1 - e^{-\frac{1}{2}x}$, $f_1 = \frac{7}{10}$, $f_2 = \frac{1}{10}$, $v^1 = 0.25$, $v^2 = 0.50$, *and* $v^3 = 0.55$.

We have the followings:

$$\text{the LHS of (13)} \simeq 0.025 < 0.12 \simeq \text{the RHS of (13), and}$$
$$\text{the LHS of (14)} \simeq 0.0960 < 0.0968 \simeq \text{the RHS of (14).}$$

In Example 2, a seller cannot obtain a higher expected revenue from the auction with a buy price B even if bidders are risk-averse. Budish and Takeyama [1] showed that in the two-bidder, two-type framework, a seller can obtain a higher expected revenue from the auction with a buy price than the auction without a buy price if bidders are risk-averse. However, the same result is not obtained in the n-bidder, m-type framework.

Example 3. *Let* $U(x) = \sqrt{x}$, $f_1 = \frac{7}{10}$, $f_2 = \frac{1}{10}$, $v^1 = 0.25$, $v^2 = 0.50$, *and* $v^3 = 0.75$.

We have the followings:

$$\text{the LHS of (13)} = 0.50 < 0.62 \simeq \text{the RHS of (13), and}$$
$$\text{the LHS of (14)} \simeq 0.43 > 0.41 \simeq \text{the RHS of (14).}$$

In Example 3, there does not exist a symmetric equilibrium in the auction with a buy price $B \in (v^2, v^3]$. In the equilibrium that corresponds to the buy price B_2^*, bidder i's v^2-type submits the buy price B_2^*, and wins the auction with positive probability. That is, the item might be allocated to the bidder who does not have the highest valuation. This is exactly an inefficient allocation. Nevertheless, a seller can obtain a higher expected revenue from the auction with the buy price B_2^* than the auction without a buy price B.

5 Conclusion

We have considered a second-price, sealed-bid auction with a buy price. To restrict an equilibrium, we have introduced the notion of partial truth-telling. We then have characterized all equilibria. We have shown that a seller can obtain a higher expected revenue from the auction with a buy price between the highest valuation and the second highest valuation if bidders are risk-averse. This result extends to the case in which a seller is risk-averse. On the other hand, we have shown an example that a seller cannot obtain a higher expected revenue from the auction with a buy price even if bidders are risk-averse. In case of three or more types, bidders' risk aversion is not a sufficient condition for revenue improvements. The result of Budish and Takeyama [1] does not extend to a general case.

We do not derive an optimal buy price. In the three or more type framework, we can consider not only the auction with a buy price between the highest valuation and the second highest valuation, but also the auction with a buy price between other valuations. To design an optimal auction, we need to compare seller's expected revenues further. Especially, in case of three or more bidders, we necessarily take account of a seller's expected revenue with an asymmetric equilibrium. It is left for future research.

Appendix

Proof of Proposition 1

We assume, by way of contradiction, that the strategy profile $\hat{\sigma}$ where some two bidders, say, bidder i and bidder j, do not play the strategy $\sigma_i^*(\cdot)$ is an equilibrium. For each bidder i and bidder j, there exist types $t_i \geq B$ and $t_j \geq B$ such that $\hat{\sigma}_i(t_i) \neq B$ and $\hat{\sigma}_j(t_j) \neq B$. Here, we denote the infimum of the support of $\hat{\sigma}_i(t_i)$ and $\hat{\sigma}_j(t_j)$ by $\underline{b}_i$ and $\underline{b}_j$, respectively. Without loss of generality, we assume that $\underline{b}_i \geq \underline{b}_j$.

In the equilibrium, by the definition of $\underline{b}_j$, it is optimal for the t_j-type to bid $\underline{b}_j$ or $\underline{b}_j + \varepsilon$ for sufficiently small ε. When the highest bid is not a buy price B, this auction is a second-price, sealed-bid auction. Thus, the winning probability of the t_j-type must be maximized among the bids except a buy price B.

Suppose that the t_j-type bids $b_j \in (\underline{b}_i, B)$. He can also win the auction when the second highest bid is in $(\underline{b}_j, b_j)$. By the definition of $\underline{b}_i$, this event occurs with positive probability. Thus, he can increase his winning probability by bidding b_j rather than $\underline{b}_j$ or $\underline{b}_j + \varepsilon$ for sufficiently small ε, which contradicts the assumption. This completes the proof.

Proof of Proposition 7

Suppose that some asymmetric strategy profile $\sigma = (\sigma_1(\cdot), \sigma^*_{-1})$ is an equilibrium. Since bidder 1 plays the equilibrium strategy,

$$\begin{aligned}&\sum_{\mu=1}^{k} \left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\} U(v^{k^*} - v^\mu) \\ &\quad \geq \left\{ \sum_{\nu=0}^{n-1} \frac{1}{n-\nu} \binom{n-1}{\nu} (1-F_k)^{n-1-\nu} (F_k)^\nu \right\} U(v^{k^*} - B).\end{aligned} \tag{15}$$

must hold.

Here let $\tilde{F}$ be the probability that bidder 1 whose type is greater than v^{k+1} does not bid a buy price B. Thus, $\tilde{F} \in (F_k, F_{k^*}]$. Since bidder j $(j = 2, \ldots, n)$ plays the strategy $\sigma^*_j(\cdot)$, for all j,

$$\left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-\tilde{F}}{n-\nu} + \frac{\tilde{F}}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \geq \pi_j(\varepsilon),$$

where $\pi_j(\varepsilon)$ is the expected payoff obtained from the winning by bidding the amount $B - \varepsilon$ for $\varepsilon > 0$. Since

$$\lim_{\varepsilon \to 0} \pi_j(\varepsilon) > \sum_{\mu=1}^{k} \left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\} U(v^{k+1} - v^\mu) + (\tilde{F} - F_k)(F_k)^{n-2} U(v^{k+1} - B),$$

we have

$$\begin{aligned}&\left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-\tilde{F}}{n-\nu} + \frac{\tilde{F}}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \\ &\quad > \sum_{\mu=1}^{k} \left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\} U(v^{k+1} - v^\mu) + (\tilde{F} - F_k)(F_k)^{n-2} U(v^{k+1} - B).\end{aligned}$$

Here let

$$\begin{aligned}J(x) :=& \left\{ \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \left(\frac{1-x}{n-\nu} + \frac{x}{n-1-\nu} \right) (1-F_k)^{n-2-\nu} (F_k)^\nu \right\} U(v^{k+1} - B) \\ &- \sum_{\mu=1}^{k} \left\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\right\} U(v^{k+1} - v^\mu) - (x - F_k)(F_k)^{n-2} U(v^{k+1} - B).\end{aligned}$$

$J(\cdot)$ is linear with respect to x. By the assumption, $J(\tilde{F}) > 0$. In addition, $J(F_k) \leq 0$. This is because if $J(F_k) > 0$, then (1) holds for $\kappa = k + 1$, which contradicts the assumption.

Thus, $J(\cdot)$ is increasing. Hence, $J(F_{k^*}) > 0$. That is,

$$\begin{aligned}
&\left\{\sum_{\nu=0}^{n-2}\binom{n-2}{\nu}\left(\frac{1-F_{k^*}}{n-\nu}+\frac{F_{k^*}}{n-1-\nu}\right)(1-F_k)^{n-2-\nu}(F_k)^{\nu}\right\}U(v^{k+1}-B) \\
&\quad > \sum_{\mu=1}^{k}\left\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\right\}U(v^{k+1}-v^{\mu}) + (F_{k^*}-F_k)(F_k)^{n-2}U(v^{k+1}-B).
\end{aligned} \tag{16}$$

Since both (15) and (16) hold, by Proposition 6, there exists $b \in (v^k, B)$ such that the asymmetric strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot; b), \sigma^*_{-1})$ is an equilibrium. This completes the proof.

Proof of Proposition 8

We only prove the second part. Suppose, by way of contradiction, that an asymmetric equilibrium exists. Then, by (5) and (7),

$$\begin{aligned}
&\sum_{\mu=1}^{k}\left\{(F_\mu)-(F_{\mu-1})\right\}U(v^{k^*}-v^{\mu}) \\
&\quad \geq \frac{1}{2}(1-F_k)U(v^{k^*}-B) + (F_k)U(v^{k^*}-B),
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\frac{1}{2}(1-F_{k^*})U(v^{k^*}-B) + (F_{k^*})U(v^{k^*}-B) \\
&\quad > \sum_{\mu=1}^{k}\left\{(F_\mu)-(F_{\mu-1})\right\}U(v^{k^*}-v^{\mu}) + (F_{k^*}-F_k)U(v^{k^*}-B).
\end{aligned} \tag{18}$$

Hence,

$$\begin{aligned}
&\frac{1}{2}(1-F_{k^*})U(v^{k^*}-B) + (F_{k^*})U(v^{k^*}-B) \\
&\quad > \sum_{\mu=1}^{k}\left\{(F_\mu)-(F_{\mu-1})\right\}U(v^{k^*}-v^{\mu}) + (F_{k^*}-F_k)U(v^{k^*}-B) \\
&\quad \geq \frac{1}{2}(1-F_k)U(v^{k^*}-B) + (F_{k^*})U(v^{k^*}-B).
\end{aligned} \tag{19}$$

Rearranging (19), we have

$$F_k > F_{k^*}. \tag{20}$$

However, (20) does not hold because $k^* > k$ and $f_\mu > 0$ for all μ. Thus, both (17) and (18) do not hold simultaneously, which contradicts the assumption. This completes the proof.

Proof of Theorem 1

At first, we recall that

$$B^*_m = v^m - U^{-1}\left(\frac{nf_m\sum_{\mu=1}^{m-1}\left\{(F_\mu)^{n-1}-(F_{\mu-1})^{n-1}\right\}U(v^m-v^{\mu})}{1-(F_{m-1})^n}\right).$$

If (8) > (9), then the seller can obtain a higher expected utility from the auction with the buy price B_m^*.

$$
\begin{aligned}
R_m^B - R^{NB} &= \sum_{\mu=1}^{m-1} \Big[nF_{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big] W(v^\mu) \\
&\quad + \{1 - (F_{m-1})^n\} W(B_m^*) \\
&\quad - \sum_{\mu=1}^{m} \Big[n\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big] W(v^\mu) \\
&= -nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} W(v^\mu) - \big\{1 - nf_m (F_{m-1})^{n-1} - (F_{m-1})^n\big\} W(v^m) \\
&\quad + \big\{1 - (F_{m-1})^n\big\} W(B_m^*) \\
&= \big\{1 - (F_{m-1})^n\big\} \Bigg[- \frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} W(v^\mu)}{1 - (F_{m-1})^n} \\
&\quad - \frac{\big\{1 - nf_m (F_{m-1})^{n-1} - (F_{m-1})^n\big\}}{1 - (F_{m-1})^n} W(v^m) + W(B_m^*)\Bigg] \\
&\geq \big\{1 - (F_{m-1})^n\big\} \Bigg[-W\Bigg(\frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} v^\mu}{1 - (F_{m-1})^n} \\
&\quad + \frac{\big\{1 - nf_m (F_{m-1})^{n-1} - (F_{m-1})^n\big\} v^m}{1 - (F_{m-1})^n} \Bigg) + W(B_m^*)\Bigg] \\
&\geq 0,
\end{aligned}
$$

where the first inequality follows because $W(\cdot)$ is concave and the second inequality follows because $W(\cdot)$ is monotone. Indeed,

$$
\begin{aligned}
& B_m^* - \frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} v^\mu}{1 - (F_{m-1})^n} - \frac{\big\{1 - nf_m (F_{m-1})^{n-1} - (F_{m-1})^n\big\} v^m}{1 - (F_{m-1})^n} \\
&= \frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} (v^m - v^\mu)}{1 - (F_{m-1})^n} \\
&\quad - U^{-1}\Bigg(\frac{nf_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^m - v^\mu)}{1 - (F_{m-1})^n} \Bigg) \\
&\geq 0,
\end{aligned}
$$

where the inequality follows because for all μ $(\mu = 1, \ldots, m-1), 0 < nf_m\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}/\big\{1 - (F_{m-1})^n\big\} < 1$, $nf_m \sum_{\mu}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}/\big\{1 - (F_{m-1})^n\big\} < 1$, and $U^{-1}(\cdot)$ is convex. Especially, the first equality holds if and only if a seller is risk-neutral. And the second equality holds if and only if bidders are risk-neutral. This completes the proof.

Proof of Proposition 8

At first, we recall that

$$
\overline{B_m^*} = v^m - U^{-1}\Bigg(\frac{(n-1) f_m \sum_{\mu=1}^{m-1} \big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} U(v^m - v^\mu)}{1 - (F_{m-1})^{n-1} - (n-1)(f_m)^2 (F_{m-1})^{n-2}} \Bigg).
$$

For notational simplicity, we rewrite B_m^* and $\overline{B_m^*}$ as:

$$B_m^* = v^m - U^{-1}\left(\frac{C}{C'}\right) \text{ and}$$
$$\overline{B_m^*} = v^m - U^{-1}\left(\frac{C}{C''}\right),$$

where $C = \sum_{\mu=1}^{m-1}\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\}U(v^m - v^\mu)$, $C' = \{1-(F_{m-1})^n\}/\{nf_m\}$, and $C'' = \{1-(F_{m-1})^{n-1} - (n-1)(f_m)^2(F_{m-1})^{n-2}\}/\{(n-1)f_m\}$.

If (8) > (12), then the seller can obtain a higher expected utility from the auction with the buy price B_m^*.

$$\begin{aligned}
R_m^B - R^{\overline{B}} &= \sum_{\mu=1}^{m-1}\Big[nF_{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big]W(v^\mu) \\
&\quad + \big\{1-(F_{m-1})^n\big\}W(B_m^*) \\
&\quad - \sum_{\mu=1}^{m-1}\Big[nF_{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\} - (n-1)\big\{(F_\mu)^n - (F_{\mu-1})^n\big\}\Big]W(v^\mu) \\
&\quad - f_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}W(v^\mu) - \big\{1-(F_{m-1})^{n-1}\big\}W(\overline{B_m^*}) \\
&= \big\{1-(F_{m-1})^n\big\}W(B_m^*) - f_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}W(v^\mu) \\
&\quad - \big\{1-(F_{m-1})^{n-1}\big\}W(\overline{B_m^*}) \\
&= W(B_m^*) - \Big[(F_{m-1})^nW(B_m^*) + f_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}W(v^\mu) \\
&\quad + \big\{1-(F_{m-1})^{n-1}\big\}W(\overline{B_m^*})\Big] \\
&\geq W(B_m^*) - W\Big((F_{m-1})^nB_m^* + f_m\sum_{\mu=1}^{m-1}\big\{(F_\mu)^{n-1} - (F_{\mu-1})^{n-1}\big\}v^\mu \\
&\quad + \big\{1-(F_{m-1})^{n-1}\big\}\overline{B_m^*}\Big) \\
&> 0,
\end{aligned}$$

where the first inequality follows because $W(\cdot)$ is concave and the second inequality follows because $W(\cdot)$ is monotone. Indeed,

$$
\begin{aligned}
& B_m^* - \left[(F_{m-1})^n B_m^* + f_m \sum_{\mu=1}^{m-1} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} v^\mu + \left\{ 1 - (F_{m-1})^{n-1} \right\} \overline{B_m^*} \right] \\
&= -\left\{ 1 - (F_{m-1})^n \right\} U^{-1}\left(\frac{C}{C'} \right) + f_m \sum_{\mu=1}^{m-1} \left\{ (F_\mu)^{n-1} - (F_{\mu-1})^{n-1} \right\} (v^m - v^\mu) \\
&\quad + \left\{ 1 - (F_{m-1})^{n-1} \right\} U^{-1}\left(\frac{C}{C''} \right) \\
&\geq -\left\{ 1 - (F_{m-1})^n \right\} U^{-1}\left(\frac{C}{C'} \right) + f_m (F_{m-1})^{n-1} U^{-1}\left(\frac{C}{(F_{m-1})^{n-1}} \right) \\
&\quad + \left\{ 1 - (F_{m-1})^{n-1} \right\} U^{-1}\left(\frac{C}{C''} \right) \\
&= \left\{ 1 - (F_{m-1})^n \right\} \left[-U^{-1}\left(\frac{C}{C'} \right) + \frac{f_m (F_{m-1})^{n-1}}{\left\{ 1 - (F_{m-1})^n \right\}} U^{-1}\left(\frac{C}{(F_{m-1})^{n-1}} \right) \right. \\
&\quad \left. + \frac{\left\{ 1 - (F_{m-1})^{n-1} \right\}}{\left\{ 1 - (F_{m-1})^n \right\}} U^{-1}\left(\frac{C}{C''} \right) \right] \\
&\geq \left\{ 1 - (F_{m-1})^n \right\} \left[-U^{-1}\left(\frac{C}{C'} \right) \right. \\
&\quad \left. + U^{-1}\left(\frac{f_m C}{\left\{ 1 - (F_{m-1})^n \right\}} + \frac{\left\{ 1 - (F_{m-1})^{n-1} \right\} C}{\left\{ 1 - (F_{m-1})^n \right\} C''} \right) \right] \\
&> 0,
\end{aligned}
$$

where the first two inequalities follow because $U^{-1}(\cdot)$ is convex and the last inequality follows because $U^{-1}(\cdot)$ is monotone. Indeed,

$$
\begin{aligned}
& \frac{f_m C}{\left\{ 1 - (F_{m-1})^n \right\}} + \frac{\left\{ 1 - (F_{m-1})^{n-1} \right\} C}{\left\{ 1 - (F_{m-1})^n \right\} C''} - \frac{C}{C'} \\
&= \frac{\left\{ 1 - (F_{m-1})^{n-1} - (n-1) f_m C'' \right\} C}{\left\{ 1 - (F_{m-1})^n \right\} C''} \\
&= \frac{(n-1)(f_m)^2 (F_{m-1})^{n-2} C}{\left\{ 1 - (F_{m-1})^n \right\} C''} \\
&> 0.
\end{aligned}
$$

This completes the proof.

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