Optimal Execution in a Market with Small Investors*

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Abstract

We consider the dynamic trading strategies that minimize the expected cost of trading a large block of securities over a fixed finite number of periods and the endogenously determined price impact function that yields the execution prices for individual trades. This analysis is novel in that it introduces market participants other than institutional investors and constructing a general equilibrium model. We find that institutional investors are much more likely to speculate to exploit private informations and price impact function changes over time, which has been left unnoticed in the existing literature.

1 Introduction

We consider situations where an institutional investor has to execute large orders. For example, investment trust funds may have to liquidate parts of their positions when capital investors offer midterm cancellation. An investment bank may persistently buy stocks to take control of a listed company. Exchange Traded Funds have to adjust their allocation rates with stock name replacement or changeover of the proportion of stock price indices. Whenever such news is reported, stock prices swing wildly.

The market impact on prices has been analyzed both theoretically and empirically. Institutional investors usually transact portfolios of considerable size and thus incur permanent and temporary price impacts. The temporary impact represents the transitory cost of demanding liquidity and only affects an individual trade. On the contrary, the permanent component of the price impact not only influences the price of the first round of trade but also the prices of all subsequent rounds of trades of the institutional investor. Modelling this

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price dynamic explicitly enables us to derive cost-efficient execution strategies for multi-trade orders.

In an early string of theoretical studies, Bertsimas and Lo [4] considered situations where an institutional investor must execute a fixed share of securities in finite trade opportunities. They describe the price dynamic using a linear impact function, which has a permanent effect. An institutional investor minimizes the expected execution cost and evenly distributes shares between all trade opportunities if he is risk neutral.

Almgren and Chriss [1] analyze a model where the impact function has both permanent and temporary effects using a mean-variance objective function, and obtain a similar result. Almgren [2] analyzes a model with the nonlinear impact function. A discrete time setting is undesirable for such a problem. A natural way to address this issue would be take a continuous time limit of the discrete time formulation, but this leads to a degenerate situation in which the execution cost becomes strategy-independent. That is, the execution cost has a constant value under some circumstances no matter what the execution strategy may be. By introducing an additional cost penalizing speedy trades, Huberman and Stanzl [10] avoid this strange outcome in the continuous time limit. Obizhaeva and Wang [11] expand the model of Almgren and Chriss [1] and show that the optimal execution strategy involves both discrete and continuous trades when trading times are endogenously chosen.

The simple price impact functions in the previous work are exogenously specified and fail to capture the intertemporal nature of other market participants. It is interesting and important to take into consideration the fact that many investors behave strategically in actual markets. So we analyze a model with one risk-neutral institutional investor and risk-averse small investors. The impact function of our model is endogenously determined and turns out to be linear. We assume the small traders have constant absolute risk aversion and endogenously derive the linear impact and the magnitude of the impact. As a result, the equally divided selling order by the institutional investor does not work out.

Within the optimal liquidation literature, most research was directed to finding the optimal deterministic or statistic liquidation strategy. Some real-world investors, however, prefer agressive in-the-money or passive in-the-money strategies, which are provided by many sell side firms (see e.g., Kissell and Malamut [5] and Kissell and Malamut [6]). Only recently, academic research has started to investigate the optimization potential of agressive in-the-money strategies in a mean-variance setting (Almgren and Lorenz [3]). By introducing an endogenetic price impact, we can explain passive in-the-money strategies.

Prices have positive effect on future prices. We can identify it as "the permanent effect," which impounds the value of carrying small traders' inventory positions into the future. That is, when there is heavy selling, the present price is low and small traders are long inventory by a large margin. Then they will prefer to sell, and place lower price at the next period. Many previous researches investigate this issue. For example, Ho and Stoll [9] examine price setting in a model with competitive dealers. However, their model does not include orders determined by strategic traders. Our study formulates the inventory model under a general equilibrium setting.

2 Model

There are three types of traders in our model: one institutional investor (that we call II in what follows), a lot of small investors (SI), and a noise trader. They can trade at times $t = 0, 1, 2, \cdots, T$. II and the noise trader place market orders and the SIs place limit orders. We can interpret the SIs as market makers. They place the price after observing the amount of orders. This is the same setting as the model in Kyle [7], where the market makers are completely competitive and their expected profit must be zero in the equilibrium. We assume that the competition among the SIs is imperfect, so that the prices are more erastic compared to Kyle [7]'s model.

II has to sell W_1 units of security over this time period. W_1 follows a normal distribution that has a mean W_0 and a variance σ_0^2. None of the traders other than II knows the value W_1, but they all know the distribution of it. W_1 is realized at the beginning of $t = 1$, trades occur at $t = 1, 2, \cdots, T$, and those who have one unit of securities obtain dividends F_{T+1} at $t = T + 1$.

$$F_t = F_0 + \sum_{u=1}^{t} \varepsilon_u . \ (t = 1, 2, \cdots, T + 1) \,, \tag{1}$$

where F_0 is observed by all traders at $t = 0$. Each ε_t $(t = 1, 2, \cdots, T)$ follows a normal distribution that has a mean 0 and a variance $\sigma_{F_t}^2$ at the beginning of the period t independently of each other. All traders observe ε_t.

II places a market order S_t at every period $t = 1, 2, \cdots, T$. We require $\sum_{t=1}^{T} S_t = W_1$. We define $W_{t+1} = W_t - S_t$ $(t = 1, 2, \cdots, T)$.

There are infinitely many SIs. They are uniformly distributed over $[0, 1]$. The measure of each dewarf is 0. SIs have no position at $t = 0$. They can borrow some money or securities and place limit orders at $t = 1, 2, \cdots, T$. They face no liquidity constraint. The interest rate is 0 for simplicity. We denote the quantity possessed by a representative SI[1] at the end of t as B_t. His utility is

$$U(B_1, \cdots, B_T) = -\exp\left[-\rho \sum_{t=1}^{T} (P_{t+1} - P_t) B_t\right] , \tag{2}$$

where $P_{T+1} = F_{T+1}$. The trading volume of the representative SI is $MB_t = B_t - B_{t-1}$ (where $B_{-1} = 0$).

The noise trader randomly places a market order n_t at $t = 1, 2, \cdots, T$. n_t follows a normal distribution that has a mean 0 and a variance $\sigma_{n_t}^2$ independently of each other and of the ε_u $(u = 1, 2, \cdots, T)$.

[1] When all dwarfs place the identical orders, the aggregate dwarfs' order is the same as in the case where one trader with the risk aversion ρ places his order. So we can call this virtual tarder a "representative dwarf."

The price P_t is determined to conform the amount of all buy order to all sell order at every $t = 1, 2, \cdots, T$:

$$S_t = MB_t + n_t. \tag{3}$$

At the end of period t, all traders observe the price P_t and own trading volume $S_t - n_t$.[2]

We define some notations. $E_t^{SI}[\cdot]$ and $Var_t^d[\cdot]$ are an expectation and a variance conditional on events that the SIs can observe by the end of the period t (i.e., after observing ε_t and before the trade at t) respectively. $E_t^{II}[\cdot]$ is a conditional expectation on events that II can observe by the beginning of the period t (i.e., after the trade at t). Furthermore, we define

$$\sigma_{P_{t+1}}^2 = Var_t^d[P_{t+1}], \tag{4}$$

$$\sigma_t^2 = Var_t^d[W_{t+1}]. \tag{5}$$

3 Equilibrium

An equilibrium is a tuple of prices and orders placed by all market participants at every period, where for every $t = 1, 2, \cdots, T$,

1. II maximizes his expected revenue:

$$\max_{S_t} E_t^{II}\left[\sum_{t=1}^{T} P_t S_t\right] \text{ subject to } \sum_{t=1}^{T} S_t = W_1,$$

2. the SIs maximizes their expected utilities:

$$\max_{B_t(P_t)} E_t^{SI}\left[U\left(B_1(P_1), \cdots, B_T(P_T)\right)\right], \text{ and}$$

3. the market must clear:

$$S_t = MB_t + n_t.$$

There exists a unique equilibrium:

$$\begin{aligned} S_t &= \alpha_t F_t + \beta_t\left(\gamma_t F_{t-1} - P_{t-1}\right) + \delta_t W_t + \zeta_t E_{t-1}^{SI}[W_t], \text{ and} \\ P_t &= \gamma_t F_t + \eta_t\left(\gamma_t F_{t-1} - P_{t-1}\right) + \theta_t W_t + \iota_t E_{t-1}^{SI}[W_t] + o_t n_t, \\ &\text{for } t = 1, 2, \cdots, T, \end{aligned}$$

where the parameters satisfy backward equations in Appendix.

[2] We will see that there is a one-to-one correspondence between the price P_t and the trading volume $MB_t = S_t - n_t$. Therefore it is enough for the dwarfs to observe either one.

4 Result of Numerical Calculations

Now the number of equations equals the number of variables to be solved. However, it is difficult to study the equilibrium prices and volume analytically. We see the result of the numerical calculations in the following.

We now examine in more detail the behavior of the parameters. We focus mainly on the case where $T = 5$, $\rho = \sigma_0^2 = 1$, and $\sigma_{F_t}^2 = \sigma_{n_t}^2 = 0.5$ (for all t). While specific patterns may be vary with the parameter values chosen, the qualitative features of those patterns are robust.

	$t = 1$	2	3	4	5
α_t	0	0	0	0	0
β_t	0	-0.01457002	-0.0752485	-0.19118447	0
γ_t	1	1	1	1	1
δ_t	0.001586576	0.01326256	0.105008597	0.3305576	1
ζ_t	0.002650136	0.032676014	0.102255573	0.139709962	0
η_t	0	-0.1144348	-0.2125039	-0.324649	-0.4173298
θ_t	-2.2898848	-1.9810515	-1.5031667	-0.9363194	-0.5
ι_t	0.9770156	0.9473948	0.7297779	0.4428	0.2086649
o_t	35.539782	7.357132	2.49821	1.094692	0.5
K_t	0.003168102	0.026081066	0.178545453	0.288735878	0
σ_t^2	0.9968243	0.9702186	0.7608741	0.2923727	0
$\sigma_{P_t}^2$	637.2649704	31.4757968	5.812746	1.7662286	0.6980932

II's order is small and the order of noise trader almost fill the gross order earlier on. The SIs avoid the risk of fluctuation of fundamentals σ_F^2, the order of noise trader σ_n^2, and II's private information σ_{t-1}^2. They do not place their limit order so much, so that the market impact o_t and the price variance $\sigma_{P_t}^2$ become large. It is almost impossible to gather valuable data from the observation of gross market order. Therefore their expectation about the amount of II's execution volume $E_t^{SI}[W_{t+1}]$ and σ_t^2 change very little. In other words, K_t is vanishingly low.

Since α_t is 0 all of the time, the level of the fundamentals F_t does not affect II's execution strategy.

γ_t is 1 all of the time, so that we can regard β_t as a sensitivity to the dissociation of the price at $t-1$ and the fundamentals F_{t-1}. Since $\beta_t < 0$, the lower the last price is, the smaller the present sell order is. When the last price is low, the present price has a tendency to be low. So II hesitates to sell.

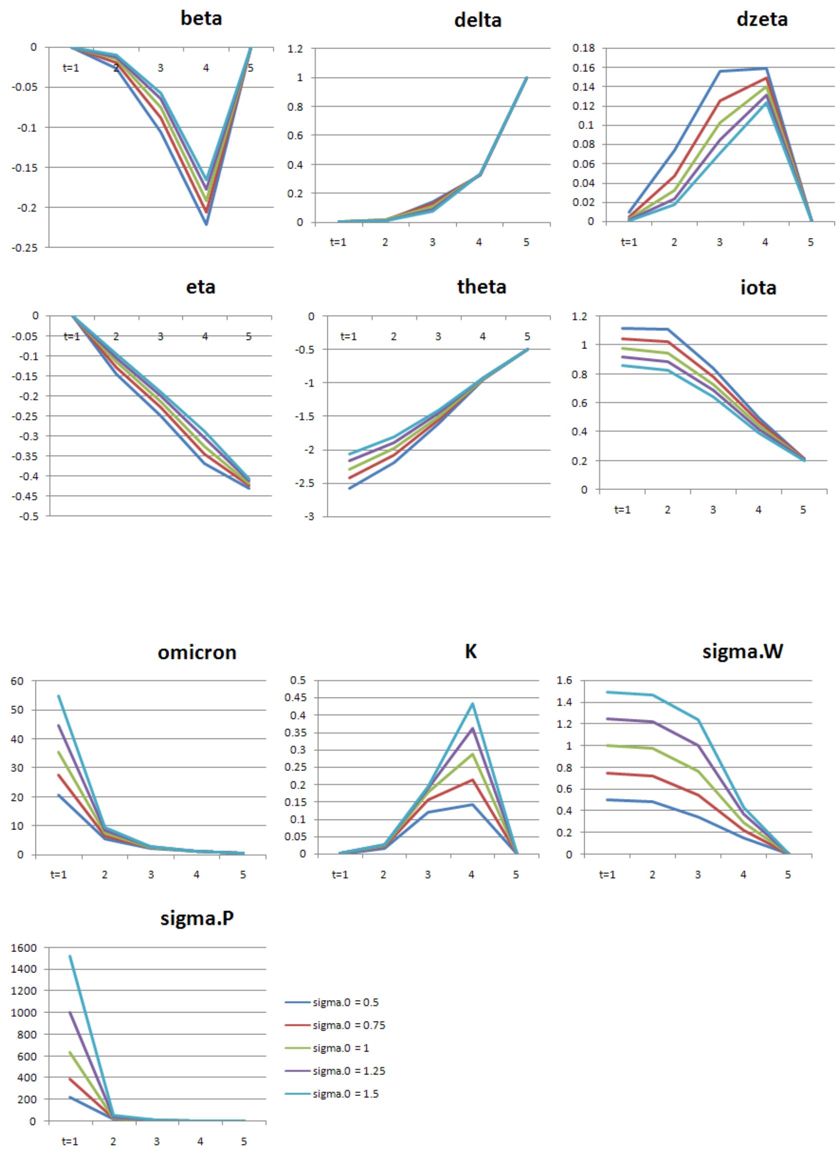
δ_t rises dramatically, which is consistent with the intuition that the larger the shares to execute is, the larger the execution earlier on.

Since $\eta_t < 0$, the lower the last price is, the lower the present price is. One may think that the present price is likely to be higher on the surface because the present sell order become small. But it's just the contrary. The reasons include the fact that the SIs are risk-averse and hate buildup of inventory. In the case that there is heavy selling, the present price is low and small traders are long inventory by a large margin. Then they will prefer to sell, and place lower price at the next period.

θ_t is negative for every t. The larger the shares to execute is, the more the price is on a declining trend.

Now let us consider market impacts. For instance, what kinds of effects would one unit of sell order at period 1 have on behavior of each period? It drives down the price at the period 1 by $o_1 \simeq 35.5$, which can be interpreted as the temporary effect. It makes an impact on the price at period 2 through 2 routes. It drives down the price by $-o_1\eta_2 \simeq 4.07$ with the inventory effect, and by $K_1\iota_2 \simeq 0.0300$ through the change of the SIs' expectations. There is a total of price down $-o_1\eta_2 + K_1\iota_2 \simeq 4.07$. Hereinafter, we have $-(-o_1\eta_2 + K_1\iota_2)\,\eta_3 \simeq 0.865$ at the period 3, $(-o_1\eta_2 + K_1\iota_2)\,\eta_3\eta_4 \simeq 0.281$ at the period 4, and $(-o_1\eta_2 + K_1\iota_2)\,\eta_3\eta_4\eta_5 \simeq 0.117$ at the period 5, respectively. Thus, there is no simple-shaped "parmanent impact" assumed in the existing literature, and we find "long-lived temporary impact" that attenuates gradually in our model.

Now we try a variety of parameters. When the risk that the SIs face decreases (σ_F^2, σ_n^2, and σ_0^2 become smaller) or when SIs' risk aversion decreases, then the amount of the execution of an early period increases, and the market impact and the variance of prices becomes smaller. Trading is going on increasing period by period. The figures below plot the values of each parameters over time when σ_0^2 changes in the 0.5 to 1.5 range. The horizontal axes represent the time. The figure on the left of the first drawer is β_t, to the right of β_t is δ_t. Hereinafter, we put the figures of $\zeta_t, \eta_t, \theta_t, \iota_t, o_t, K_t, \sigma_t^2$, and $\sigma_{P_t}^2$.



When we explicate numerical calculations over a wider range of parameters, can we obtain the result where II sells the securities equally-divided, or liquidates longs at the early periods? The answer is partially yes. We show the case where

$T = 3,\ \rho = \sigma_0^2 = 1,\ \sigma_{F_2}^2 = \sigma_{F_3}^2 = 0$, and $\sigma_{F_4}^2 = \sigma_{n_t}^2 = 0.5$ (for all t).[3]

	$t = 1$	2	3
α_t	0	0	0
β_t	0	-0.3393326	0
γ_t	1	1	1
δ_t	0.7472907	0.3206204	1
ζ_t	0.07043507	0.23804079	0
η_t	0	-0.5525894	-0.4430819
θ_t	-1.453488	-0.970087	-0.5
ι_t	0.1207201	0.6562298	0.2215409
o_t	1.211787	1.123264	0.5
K_t	0.17841983	0.0130615	0
σ_t^2	0.03016787	0.01383835	0
$\sigma_{P_t}^2$	0.7979475	0.6592508	0.6284596

We obtain the result that II sells a large portion of his shares at the period 1, and divides equally between the period 2 and 3. (Of course, it depends on the SIs' expectations and the selling amounts of the noise trader.) We assume that there are no public information during the trading period, and the trade of the noise trader is not so active.[4] This result is consistent with Harris and Gurel [8]. That is, on the first trading day after an addition to S&P 500 list is announced, there is a large increase in volume, which is suggestive of a shift in demand. On that day, there is also an economically and statistically significant increase in price. Since the volume and price effects are not present in the first years of the sample, it is unlikely that the announcements, by themselves, cause the price changes observed in the latter years. Moreover, since the price increase is consistently reversed, it is unlikely that new information is the cause of the initial increase.

5 Concluding Comments

We have considered a multiperiod model of securities trading with small investors. We show that the institutional investor who has to sell a fixed share of securities within a fixed periods does not sell securities evenly even though he is risk neutral. We leave the discussion of the case where II is risk averse for future research.

[3]At heart, we would like to see the case where $T = 5$, but it is impossible to obtain the solution to the equations with the problem of numerical calculation, when the values of parameters are small. It has to be discussed continuously as a future issue.

[4]The latter is important. When we put $\sigma_{n_t}^2 = 1$ with the other parameters in status quo, the outcome is waiting strategy as is the above case of $T = 5$.

6 Appendix

We set up a strand of equation to obtain the equilibrium.

6.1 Decision Making of SIs

The conditional expectation and the conditional variance that the representative SI has at t $(t = 1, 2, \cdots, T)$ are

$$\begin{aligned} E_t^{SI}\left[\sum_{u=1}^{T}\left(P_{u+1}-P_u\right)B_u\right] &= \sum_{u=1}^{t-1}\left(P_{u+1}-P_u\right)B_u + \left(E_t^{SI}\left[P_{t+1}\right]-P_t\right)B_t \\ &\quad + E_t^{SI}\left[\sum_{u=t+1}^{T}\left(P_{u+1}-P_u\right)B_u\right], \text{ and} \qquad (6) \\ Var_t^d\left[\sum_{u=1}^{T}\left(P_{u+1}-P_u\right)B_u\right] &= \sigma_{P_{t+1}}^2 B_t^2 + Var_t^d\left[\sum_{u=t+1}^{T}\left(P_{u+1}-P_u\right)B_u\right]. \end{aligned}$$

We consider a linear equilibrium in the followings. Since a linear combination of variables that follow normal distributions also follows normal distributions, the price P_t follows normal distributions. In addition, all observations follow normal distributions, so that we can take it for granted that price P_t follows normal distributions conditional on the events by the beginning of t for all traders.

SIs' utility at t is

$$U\left(B_1, \cdots, B_T\right) = -\exp\left[-\rho\sum_{t=1}^{T}\left(P_{t+1}-P_t\right)B_t\right]. \tag{7}$$

The first order condition is

$$\begin{aligned} \frac{\partial E_t^{SI}[U]}{\partial B_t} &= -\rho\left(E_t^{SI}\left[P_{t+1}\right]-P_t-\rho\sigma_{P_{t+1}}^2 B_t\right)E_t^{SI}[U] = 0 \qquad (8) \\ &\Rightarrow B_t\left(P_t\right) = \frac{E_t^{SI}\left[P_{t+1}\right]-P_t}{\rho\sigma_{P_{t+1}}^2}, \end{aligned}$$

which is the limit order that the SIs place.

6.2 Decision Making of II

We consider II's behavior by the inductive method.

6.2.1 Period T

By II's sellout constraint $S_T = W_T$ and by the market clearing condition (3),

$$P_T = F_T + \rho\sigma_{F_{T+1}}^2\left(n_T - W_T\right) - \frac{\sigma_{F_{T+1}}^2}{\sigma_{P_T}^2}\left(E_{T-1}^{SI}\left[P_T\right]-P_{T-1}\right). \tag{9}$$

The variance conditional on the events by $T-1$ is equal to

$$\sigma_{P_T}^2 = \sigma_{F_T}^2 + \left(\rho\sigma_{F_{T+1}}^2\right)^2 \left(\sigma_{n_T}^2 + \sigma_{T-1}^2\right). \tag{10}$$

Take the expectation of both sides conditional on the events by $T-1$, and obtain

$$\begin{aligned} E_{T-1}^{SI}[P_T] &= F_{T-1} - \rho\sigma_{F_{T+1}}^2 E_{T-1}^{SI}[W_T] - \frac{\sigma_{F_{T+1}}^2}{\sigma_{P_T}^2}\left(E_{T-1}^{SI}[P_T] - P_{T-1}\right) \\ &\iff E_{T-1}^{SI}[P_T] - P_{T-1} \\ &= \frac{\sigma_{P_T}^2\left(F_{T-1} - P_{T-1} - \rho\sigma_{F_{T+1}}^2 E_{T-1}^{SI}[W_T]\right)}{\sigma_{P_T}^2 + \sigma_{F_{T+1}}^2}. \end{aligned} \tag{11}$$

Substituting (11) to (9), we obtain

$$\begin{aligned} P_T &= F_T + \rho\sigma_{F_{T+1}}^2 (n_T - W_T) \\ &- \frac{\sigma_{F_{T+1}}^2}{\sigma_{P_T}^2 + \sigma_{F_{T+1}}^2}\left(F_{T-1} - P_{T-1} - \rho\sigma_{F_{T+1}}^2 E_{T-1}^{SI}[W_T]\right). \end{aligned} \tag{12}$$

Therefore the continuation selling amount is

$$\begin{aligned} V_T\left(F_T, F_{T-1}, P_{T-1}, W_T, E_{T-1}^{SI}[W_T]\right) &= F_T W_T + \rho\sigma_{F_{T+1}}^2 (n_T - W_T) W_T \\ &- \frac{\sigma_{F_{T+1}}^2}{\sigma_{P_T}^2 + \sigma_{F_{T+1}}^2}\left(F_{T-1} - P_{T-1} - \rho\sigma_{F_{T+1}}^2 E_{T-1}^{SI}[W_T]\right) W_T. \end{aligned} \tag{13}$$

6.2.2 Induction Hypothesis of Period $t+1$ ($t = T-1, T-2, \cdots, 2$)

We hypothesize the followings:

$$S_{t+1} = \alpha_{t+1}F_{t+1} + \beta_{t+1}(\gamma_{t+1}F_t - P_t) + \delta_{t+1}W_{t+1} + \zeta_{t+1}E_t^{SI}[W_{t+1}], \tag{14}$$

$$P_{t+1} = \gamma_{t+1}F_{t+1} + \eta_{t+1}(\gamma_{t+1}F_t - P_t) + \theta_{t+1}W_{t+1} + \iota_{t+1}E_t^{SI}[W_{t+1}] + o_{t+1}n_{t+1}, \text{ and} \tag{15}$$

$$\begin{aligned} &V_{t+1}\left(F_{t+1}, F_t, P_t, W_{t+1}, E_t^{SI}[W_{t+1}]\right) \\ &= \kappa_{t+1}F_{t+1}(\gamma_{t+1}F_t - P_t) + \lambda_{t+1}F_{t+1}W_{t+1} + \mu_{t+1}F_{t+1}E_t^{SI}[W_{t+1}] \\ &+ \nu_{t+1}(\gamma_{t+1}F_t - P_t)^2 + \xi_{t+1}(\gamma_{t+1}F_t - P_t)W_{t+1} + \pi_{t+1}(\gamma_{t+1}F_t - P_t)E_t^{SI}[W_{t+1}] \\ &+ \phi_{t+1}W_{t+1}^2 + \chi_{t+1}W_{t+1}E_t^{SI}[W_{t+1}] + \omega_{t+1}\left(E_t^{SI}[W_{t+1}]\right)^2 + (\text{expressions irrelevant to } S_t). \end{aligned} \tag{16}$$

6.2.3 Period t

By the market clearing condition (3),

$$\begin{aligned} P_t &= \gamma_{t+1}F_t + \frac{\theta_{t+1}}{1+\eta_{t+1}}W_{t+1} + \frac{\iota_{t+1}}{1+\eta_{t+1}}E_t^{SI}[W_{t+1}] \\ &- \frac{1}{1+\eta_{t+1}}\frac{\sigma_{P_{t+1}}^2}{\sigma_{P_t}^2}\left(E_{t-1}^{SI}[P_t] - P_{t-1}\right) + \frac{\rho\sigma_{P_{t+1}}^2}{1+\eta_{t+1}}(n_t - S_t). \end{aligned} \tag{17}$$

SIs expect the execution shares by the Kalman Filter:

$$E_t^{SI}[W_{t+1}] = K_t S_t + (\text{expressions irrelevant to } S_t). \tag{18}$$

On the equilibrium path, we have

$$\begin{aligned} E_t^{SI}[W_{t+1}] = {} & \delta_t (1 + K_t) W_t + (1 - \delta_t - \delta_t K_t) E_{t-1}^{SI}[W_t] \\ & - S_t - K_t n_t. \end{aligned} \tag{19}$$

The transition of the amount of securities that II must sell from $t+1$ on is

$$W_{t+1} = W_t - S_t. \tag{20}$$

The continuation revenue just after placing order at t is

$$\begin{aligned} & V_t(F_t, F_{t-1}, P_{t-1}, W_t) \\ & = \max_{S_t} [E_t^{II}[P_t] S_t \\ & + \kappa_{t+1} F_t \left(\gamma_{t+1} F_t - E_t^{II}[P_t]\right) + \lambda_{t+1} F_t W_{t+1} + \mu_{t+1} F_t E_t^{SI}[W_{t+1}] \\ & + \nu_{t+1} E_t^{II}\left[(\gamma_{t+1} F_t - P_t)^2\right] + \xi_{t+1}\left(\gamma_{t+1} F_t - E_t^{II}[P_t]\right) W_{t+1} \\ & + \pi_{t+1} E_t^{II}\left[(\gamma_{t+1} F_t - P_t) E_t^{SI}[W_{t+1}]\right] \\ & + \phi_{t+1} W_{t+1}^2 + \chi_{t+1} W_{t+1} E_t^{SI}[W_{t+1}] + \omega_{t+1}\left(E_t^{SI}[W_{t+1}]\right)^2 \\ & + (\text{expressions irrelevant to } S_t)]. \end{aligned} \tag{21}$$

Substituting (17), (18), and (20) to (21), we have the first order condition:

$$\begin{aligned} & \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho \sigma_{P_{t+1}}^2}{1+\eta_{t+1}}\right) S_t + E_t^{II}[P_t] \\ & - \kappa_{t+1} F_t \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho \sigma_{P_{t+1}}^2}{1+\eta_{t+1}}\right) \\ & - \lambda_{t+1} F_t \\ & + \mu_{t+1} F_t K_t \\ & + 2\nu_{t+1} \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho \sigma_{P_{t+1}}^2}{1+\eta_{t+1}}\right) \left(E_t^{II}[P_t] - \gamma_{t+1} F_t\right) \\ & + \xi_{t+1} \left\{-\left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho \sigma_{P_{t+1}}^2}{1+\eta_{t+1}}\right) W_{t+1} - \left(\gamma_{t+1} F_t - E_t^{II}[P_t]\right)\right\} \\ & + \pi_{t+1} \left\{-\left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho \sigma_{P_{t+1}}^2}{1+\eta_{t+1}}\right) E_t^{II}\left[E_t^{SI}[W_{t+1}]\right] + K_t \left(\gamma_{t+1} F_t - E_t^{II}[P_t]\right)\right\} \\ & - 2\phi_{t+1} W_{t+1} \\ & + \chi_{t+1} \left(-E_t^{II}\left[E_t^{SI}[W_{t+1}]\right] + K_t W_{t+1}\right) \\ & + 2\omega_{t+1} K_t E_t^{II}\left[E_t^{SI}[W_{t+1}]\right] \\ & = 0. \end{aligned} \tag{22}$$

Substituting (17), (19), and (20), we obtain

$$\begin{aligned} S_t = {} & C_{S_t,F} F_t + C_{S_t,EP} \frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}} \left(E^{SI}_{t-1}\left[P_t\right] - P_{t-1} \right) \\ & + C_{S_t,W} W_t + C_{S_t,EW} \left(1 - \delta_t - \delta_t K_t\right) E^{SI}_{t-1}\left[W_t\right], \end{aligned} \tag{23}$$

where

$$\begin{aligned} C_{S_t,S} = {} & \left(-\xi_{t+1} - \pi_{t+1} - 1 - 2\nu_{t+1} \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} - \frac{\iota_{t+1}}{1+\eta_{t+1}} - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) \right) \\ & \cdot \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) \\ & - 2\phi_{t+1} - \chi_{t+1} + \chi_{t+1} K_t + 2\omega_{t+1} K_t \\ & + \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} - \frac{\iota_{t+1}}{1+\eta_{t+1}} - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) \left(-\xi_{t+1} + \pi_{t+1} K_t - 1 \right), \end{aligned} \tag{24}$$

$$C_{S_t,F} = \left\{ -\kappa_{t+1} \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) - \lambda_{t+1} + \mu_{t+1} K_t + \gamma_{t+1} \right\} / C_{S_t,S}, \tag{25}$$

$$C_{S_t,EP} = \left\{ -2\nu_{t+1} \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) - \xi_{t+1} + \pi_{t+1} K_t - 1 \right\} \frac{1}{1+\eta_{t+1}} / C_{S_t,S}, \tag{26}$$

$$\begin{aligned} C_{S_t,W} = {} & \{ \left(-\xi_{t+1} - \pi_{t+1}\delta_t \left(1 + K_t\right) + 2\nu_{t+1} \left(\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} \delta_t \left(1 + K_t\right) \right) \right) \\ & \cdot \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) \\ & - 2\phi_{t+1} + \chi_{t+1} K_t + \delta_t \left(1 + K_t\right) \left(-\chi_{t+1} + 2\omega_{t+1} K_t \right), \text{ and} \\ & + \left(\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} \delta_t \left(1 + K_t\right) \right) \left(\xi_{t+1} - \pi_{t+1} K_t + 1 \right) \} / C_{S_t,S} \end{aligned} \tag{27}$$

$$\begin{aligned} C_{S_t,EW} = {} & \{ \left(-\pi_{t+1} + 2\nu_{t+1} \frac{\iota_{t+1}}{1+\eta_{t+1}} \right) \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} + \frac{\iota_{t+1}}{1+\eta_{t+1}} K_t - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}} \right) \\ & - \chi_{t+1} + 2\omega_{t+1} K_t + \frac{\iota_{t+1}}{1+\eta_{t+1}} \left(\xi_{t+1} - \pi_{t+1} K_t + 1 \right) \} / C_{S_t,S}. \end{aligned} \tag{28}$$

Substituting these equations to (17), we obtain

$$\begin{aligned} P_t = {} & C_{P_t,F} F_t + C_{P_t,EP} \frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}} \left(E^{SI}_{t-1}\left[P_t\right] - P_{t-1} \right) \\ & + C_{P_t,W} W_t + C_{P_t,EW} \left(1 - \delta_t - \delta_t K_t\right) E^{SI}_{t-1}\left[W_t\right] \\ & + C_{P_t,n} n_t, \end{aligned} \tag{29}$$

where

$$C_{P_t,F} = \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} - \frac{\iota_{t+1}}{1+\eta_{t+1}} - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}}\right) C_{S_t,F} + \gamma_{t+1}, \tag{30}$$

$$C_{P_t,EP} = \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} - \frac{\iota_{t+1}}{1+\eta_{t+1}} - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}}\right) C_{S_t,EP} - \frac{1}{1+\eta_{t+1}}, \tag{31}$$

$$C_{P_t,W} = \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} - \frac{\iota_{t+1}}{1+\eta_{t+1}} - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}}\right) C_{S_t,W} + \frac{\iota_{t+1}}{1+\eta_{t+1}}\delta_t\left(1+K_t\right) + \frac{\theta_{t+1}}{1+\eta_{t+1}}, \tag{32}$$

$$C_{P_t,EW} = \left(-\frac{\theta_{t+1}}{1+\eta_{t+1}} - \frac{\iota_{t+1}}{1+\eta_{t+1}} - \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}}\right) C_{S_t,EW} + \frac{\iota_{t+1}}{1+\eta_{t+1}}, \text{ and} \tag{33}$$

$$C_{P_t,n} = -\frac{\iota_{t+1}}{1+\eta_{t+1}}K_t + \frac{\rho\sigma^2_{P_{t+1}}}{1+\eta_{t+1}}. \tag{34}$$

The variance conditional on the events by $t-1$ is equal to

$$\sigma^2_{P_t} = C^2_{P_t,F}\sigma^2_{F_t} + C^2_{P_t,W}\sigma^2_{t-1} + C^2_{P_t,n}\sigma^2_{n_t}. \tag{35}$$

Take the expectation of both sides conditional on the events by $t-1$, and obtain

$$\begin{aligned} & E^{SI}_{t-1}\left[P_t\right] - P_{t-1} \\ & = \frac{1}{1 - C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}\{\left(C_{P_t,F}F_{t-1} - P_{t-1}\right) \\ & + \left(C_{P_t,W} + C_{P_t,EW}\left(1-\delta_t-\delta_tK_t\right)\right)E^{SI}_{t-1}\left[W_t\right]\}. \end{aligned} \tag{36}$$

Substituting (36) to(23) and (29), we obtain

$$S_t = \alpha_tF_t + \beta_t\left(\gamma_tF_{t-1} - P_{t-1}\right) + \delta_tW_t + \zeta_tE^{SI}_{t-1}\left[W_t\right], \text{ and} \tag{37}$$

$$P_t = \gamma_tF_t + \eta_t\left(\gamma_tF_{t-1} - P_{t-1}\right) + \theta_tW_t + \iota_tE^{SI}_{t-1}\left[W_t\right] + o_tn_t, \tag{38}$$

where

$$\alpha_t = C_{S_t,F}, \tag{39}$$

$$\beta_t = \frac{C_{S_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}{1 - C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}, \tag{40}$$

$$\gamma_t = C_{P_t,F}, \tag{41}$$

$$\delta_t = C_{S_t,W}, \tag{42}$$

$$\begin{aligned}\zeta_t &= C_{S_t,EW}\left(1 - \delta_t - \delta_t K_t\right) \\ &\quad + \frac{C_{S_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}{1 - C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}} \cdot \\ &\quad \left(C_{P_t,W} + C_{P_t,EW}\left(1 - \delta_t - \delta_t K_t\right)\right),\end{aligned} \tag{43}$$

$$\eta_t = \frac{C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}{1 - C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}, \tag{44}$$

$$\theta_t = C_{P_t,W}, \tag{45}$$

$$\begin{aligned}\iota_t &= C_{P_t,EW}\left(1 - \delta_t - \delta_t K_t\right) \\ &\quad + \frac{C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}}{1 - C_{P_t,EP}\frac{\sigma^2_{P_{t+1}}}{\sigma^2_{P_t}}} \\ &\quad \cdot \left(C_{P_t,W} + C_{P_t,EW}\left(1 - \delta_t - \delta_t K_t\right)\right), \text{ and}\end{aligned} \tag{46}$$

$$o_t = C_{P_t,n}. \tag{47}$$

Therefore the continuation revenue is

$$\begin{aligned}&V_t\left(F_t, F_{t-1}, P_{t-1}, W_t\right) \\ &= \kappa_t F_t\left(\gamma_t F_{t-1} - P_{t-1}\right) + \lambda_t F_t W_t + \mu_t F_t E^{SI}_{t-1}\left[W_t\right] \\ &+ \nu_t\left(\gamma_t F_{t-1} - P_{t-1}\right)^2 + \xi_t\left(\gamma_t F_{t-1} - P_{t-1}\right) W_t \\ &+ \pi_t\left(\gamma_t F_t - P_t\right) E^{SI}_{t-1}\left[W_t\right] \\ &+ \phi_t W_t^2 + \chi_t W_t E^{SI}_{t-1}\left[W_t\right] + \omega_t\left(E^{SI}_{t-1}\left[W_t\right]\right)^2 \\ &+ \left(\text{expressions irrelevant to } S_t\right),\end{aligned} \tag{48}$$

where

$$
\begin{aligned}
\kappa_t &= \alpha_t\eta_t + \beta_t\gamma_t - \eta_t\kappa_{t+1} - \beta_t\lambda_{t+1} - \beta_t\mu_{t+1} - 2\left(\gamma_{t+1}-\gamma_t\right)\eta_t\nu_{t+1} \\
&\quad - \beta_t\left(\gamma_{t+1}-\gamma_t\right)\xi_{t+1} + \alpha_t\eta_t\xi_{t+1} - \beta_t\left(\gamma_{t+1}-\gamma_t\right)\pi_{t+1} + \alpha_t\eta_t\pi_{t+1} \\
&\quad + 2\alpha_t\beta_t\phi_{t+1} + \alpha_t\beta_t\chi_{t+1} + \alpha_t\beta_t\chi_{t+1} + 2\alpha_t\beta_t\omega_{t+1}, \qquad (49) \\
\lambda_t &= \alpha_t\theta_t + \gamma_t\delta_t - \theta_t\kappa_{t+1} + \left(1-\delta_t\right)\lambda_{t+1} + \delta_t\mu_{t+1}K_t - 2\left(\gamma_{t+1}-\gamma_t\right)\theta_t\nu_{t+1} \\
&\quad + \left(\gamma_{t+1}-\gamma_t\right)\left(1-\delta_t\right)\xi_{t+1} + \alpha_t\theta_t\xi_{t+1} + \delta_t\left(\gamma_{t+1}-\gamma_t\right)\pi_{t+1}K_t + \alpha_t\theta_t\pi_{t+1} \\
&\quad - 2\alpha_t\left(1-\delta_t\right)\phi_{t+1} - \alpha_t\delta_t\chi_{t+1}K_t - \alpha_t\left(1-\delta_t\right)\chi_{t+1} - 2\alpha_t\delta_t\omega_{t+1}K_t, \qquad (50) \\
\mu_t &= \alpha_t\iota_t + \gamma_t\zeta_t - \iota_t\kappa_{t+1} - \zeta_t\lambda_{t+1} + \left(1-\zeta_t-\delta_t-\delta_tK_t\right)\mu_{t+1} - 2\left(\gamma_{t+1}-\gamma_t\right)\iota_t\nu_{t+1} \\
&\quad - \left(\gamma_{t+1}-\gamma_t\right)\zeta_t\xi_{t+1} + \alpha_t\iota_t\xi_{t+1} + \left(\gamma_{t+1}-\gamma_t\right)\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\pi_{t+1} + \alpha_t\iota_t\pi_{t+1} \\
&\quad + 2\alpha_t\zeta_t\phi_{t+1} - \alpha_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\chi_{t+1} + \alpha_t\zeta_t\chi_{t+1} - 2\alpha_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\omega_{t+1}, \\
&\qquad (51) \\
\nu_t &= \beta_t\eta_t + \eta_t^2\nu_{t+1} + \beta_t\eta_t\xi_{t+1} + \beta_t\eta_t\pi_{t+1} + \beta_t^2\phi_{t+1} + \beta_t^2\chi_{t+1} + \beta_t^2\omega_{t+1}, \qquad (52) \\
\xi_t &= \beta_t\theta_t + \delta_t\eta_t + 2\eta_t\theta_t\nu_{t+1} - \left(1-\delta_t\right)\eta_t\xi_{t+1} + \beta_t\theta_t\xi_{t+1} - \delta_t\eta_t\pi_{t+1}K_t + \beta_t\theta_t\pi_{t+1} \\
&\quad - 2\beta_t\left(1-\delta_t\right)\phi_{t+1} - \beta_t\delta_t\chi_{t+1}K_t - \beta_t\left(1-\delta_t\right)\chi_{t+1} - 2\beta_t\delta_t\omega_{t+1}K_t, \qquad (53) \\
\pi_t &= \beta_t\iota_t + \zeta_t\eta_t + 2\eta_t\iota_t\nu_{t+1} + \zeta_t\eta_t\xi_{t+1} + \beta_t\iota_t\xi_{t+1} - \eta_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\pi_{t+1} + \beta_t\iota_t\pi_{t+1} \\
&\quad + 2\beta_t\zeta_t\phi_{t+1} - \beta_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\chi_{t+1} + \beta_t\zeta_t\chi_{t+1} - 2\beta_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\omega_{t+1}, \\
&\qquad (54) \\
\phi_t &= \delta_t\theta_t + \theta_t^2\nu_{t+1} - \left(1-\delta_t\right)\theta_t\xi_{t+1} - \delta_t\theta_t\pi_{t+1}K_t + \left(1-\delta_t\right)^2\phi_{t+1} + \delta_t\left(1-\delta_t\right)\chi_{t+1}K_t + \delta_t^2K_t^2\omega_{t+1}, \\
&\qquad (55) \\
\chi_t &= \delta_t\iota_t + \zeta_t\theta_t + 2\theta_t\iota_t\nu_{t+1} + \zeta_t\theta_t\xi_{t+1} - \left(1-\delta_t\right)\iota_t\xi_{t+1} - \theta_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\pi_{t+1} - \delta_t\iota_t\pi_{t+1}K_t \\
&\quad - 2\left(1-\delta_t\right)\zeta_t\phi_{t+1} + \left(1-\delta_t\right)\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\chi_{t+1} \\
&\quad - \delta_t\zeta_t\chi_{t+1}K_t + 2\delta_t\left(1-\zeta_t-\delta_t-\delta_tK_t\right)\omega_{t+1}K_t, \text{ and} \qquad (56) \\
\omega_t &= \zeta_t\iota_t + \iota_t^2\nu_{t+1} + \zeta_t\iota_t\xi_{t+1} - \left(1-\zeta_t-\delta_t-\delta_tK_t\right)\iota_t\pi_{t+1} \\
&\quad + \zeta_t^2\phi_{t+1} - \left(1-\zeta_t-\delta_t-\delta_tK_t\right)\zeta_t\chi_{t+1} + \left(1-\zeta_t-\delta_t-\delta_tK_t\right)^2\omega_{t+1}. \qquad (57)
\end{aligned}
$$

6.2.4 Period 1

By the market clearing condition (3),

$$
P_1 = \gamma_2F_1 + \frac{\theta_2}{1+\eta_2}W_2 + \frac{\iota_2}{1+\eta_2}E_1^{SI}\left[W_2\right] + \frac{\rho\sigma_{P_2}^2}{1+\eta_2}\left(n_1 - S_1\right). \qquad (58)
$$

The continuation revenue is

$$
\begin{aligned}
&V_1 (F_1, F_0, P_0, W_1) \\
&= \max_{S_1} E_1^{II} [P_1 S_1 + V_2 (F_2, F_1, P_1, W_2)] \\
&= \max_{S_1} E_1^{II} [\left(\gamma_2 F_1 + \frac{\theta_2}{1+\eta_2} W_2 + \frac{\iota_2}{1+\eta_2} E_1^{SI} [W_2] + \frac{\rho \sigma_{P_2}^2}{1+\eta_2} (n_1 - S_1)\right) S_1 \\
&+ \kappa_2 F_2 (\gamma_2 F_1 - P_1) + \lambda_2 F_2 W_2 + \mu_2 F_2 E_1^{SI} [W_2] \\
&\nu_2 (\gamma_2 F_1 - P_1)^2 + \xi_2 (\gamma_2 F_1 - P_1) W_2 + \pi_2 (\gamma_2 F_1 - P_1) E_1^{SI} [W_2] \\
&\phi_2 W_2^2 + \chi_2 W_2 E_1^{SI} [W_2] + \omega_2 \left(E_1^{SI} [W_2]\right)^2 + (\text{expressions irrelevant to } S_1)].
\end{aligned}
\tag{59}
$$

SIs expect the execution shares by the Kalman Filter:

$$E_1^{SI} [W_2] = K_1 S_1 + (\text{expressions irrelevant to } S_1) \tag{60}$$

On the equilibrium path, we have

$$E_1^{SI} [W_2] = \delta_1 W_1 + (1 - \delta_1) W_0 - S_1 + K_1 (\delta_1 W_1 - \delta_1 W_0 - n_1). \tag{61}$$

The transition of the amount of securities that II must sell from 2 on is

$$W_2 = W_1 - S_1. \tag{62}$$

Substituting (58), (60), and (62) to (59), we have the first order condition:

$$
\begin{aligned}
&\left(-\frac{\theta_2}{1+\eta_2}+\frac{\iota_2}{1+\eta_2}K_1-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)S_1\\
&+\gamma_2F_1+\frac{\theta_2}{1+\eta_2}\left(W_1-S_1\right)+\frac{\iota_2}{1+\eta_2}\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}S_1\\
&+\left(\frac{\theta_2}{1+\eta_2}-\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)\kappa_2F_1-\lambda_2F_1+\mu_2K_1F_1\\
&+2\nu_2\left(-\frac{\theta_2}{1+\eta_2}+\frac{\iota_2}{1+\eta_2}K_1-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)\\
&\cdot\left(\frac{\theta_2}{1+\eta_2}\left(W_1-S_1\right)+\frac{\iota_2}{1+\eta_2}\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}S_1\right)\\
&+\xi_2\left(\frac{\theta_2}{1+\eta_2}-\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)\left(W_1-S_1\right)\\
&-\xi_2\left(-\frac{\theta_2}{1+\eta_2}\left(W_1-S_1\right)-\frac{\iota_2}{1+\eta_2}\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}S_1\right)\\
&+\pi_2\left(\frac{\theta_2}{1+\eta_2}-\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)\\
&+\pi_2K_1\left(-\frac{\theta_2}{1+\eta_2}\left(W_1-S_1\right)-\frac{\iota_2}{1+\eta_2}\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}S_1\right)\\
&-2\phi_2\left(W_1-S_1\right)\\
&-\chi_2\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)+\chi_2K_1\left(W_1-S_1\right)\\
&+2\omega_2K_1\left(\delta_1W_1+\left(1-\delta_1\right)W_0-S_1+K_1\left(\delta_1W_1-\delta_1W_0\right)\right)\\
&=0.
\end{aligned}
\tag{63}
$$

Substituting (58), (61), and (62), we obtain

$$S_1=\alpha_1F_1+\delta_1W_1+\zeta_1W_0, \tag{64}$$

where

$$
\begin{aligned}
C_{S_1,S} &= (-1-\xi_2-\pi_2)\left(-\frac{\theta_2}{1+\eta_2}+\frac{\iota_2}{1+\eta_2}K_1-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right) \\
&+\left(1+2\nu_2\left(-\frac{\theta_2}{1+\eta_2}+\frac{\iota_2}{1+\eta_2}K_1-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)+\xi_2-\pi_2K_1\right)\left(\frac{\theta_2+\iota_2}{1+\eta_2}+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right) \\
&-2\phi_2-\chi_2+\chi_2K_1+2\omega_2K_1, \qquad (65)
\end{aligned}
$$

$$
\alpha_1 = \left(\gamma_2+\left(\frac{\theta_2}{1+\eta_2}-\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)\kappa_2-\lambda_2+\mu_2K_1\right)/C_{S_1,S}, \qquad (66)
$$

$$
\begin{aligned}
\delta_1 &= \{\left(1+2\nu_2\left(-\frac{\theta_2}{1+\eta_2}+\frac{\iota_2}{1+\eta_2}K_1-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)+\xi_2-\pi_2K_1\right)\frac{\theta_2+\iota_2\delta_1(1+K_1)}{1+\eta_2} \\
&+(\xi_2+\pi_2\delta_1(1+K_1))\left(\frac{\theta_2}{1+\eta_2}-\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)-2\phi_2 \\
&+(-\chi_2+2\omega_2K_1)\,\delta_1(1+K_1)+\chi_2K_1\}/C_{S_1,S}, \text{ and} \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_1 &= \{\left(1+2\nu_2\left(-\frac{\theta_2}{1+\eta_2}+\frac{\iota_2}{1+\eta_2}K_1-\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right)+\xi_2-\pi_2K_1\right)\frac{\iota_2}{1+\eta_2}(1-\delta_1-K_1\delta_1) \\
&+\pi_2(1-\delta_1-K_1\delta_1)\left(\frac{\theta_2}{1+\eta_2}-\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}\right) \\
&+(-\chi_2+2\omega_2K_1)(1-\delta_1-K_1\delta_1)\}/C_{S_1,S}. \qquad (68)
\end{aligned}
$$

The price is

$$P_1 = \gamma_1F_1+\theta_1W_1+\iota_1W_0+o_1n_1, \qquad (69)$$

where

$$\gamma_1 = \gamma_2-\alpha_1\frac{\theta_2+\iota_2+\rho\sigma_{P_2}^2}{1+\eta_2}, \qquad (70)$$

$$\theta_1 = \frac{\theta_2}{1+\eta_2}+\delta_1\frac{\iota_2}{1+\eta_2}(1+K_1)-\delta_1\frac{\theta_2+\iota_2+\rho\sigma_{P_2}^2}{1+\eta_2}, \qquad (71)$$

$$\iota_1 = \frac{\iota_2}{1+\eta_2}(1-\delta_1-\delta_1K_1)-\zeta_1\frac{\theta_2+\iota_2+\rho\sigma_{P_2}^2}{1+\eta_2}, \text{ and} \qquad (72)$$

$$o_1 = -\frac{\iota_2}{1+\eta_2}K_1+\frac{\rho\sigma_{P_2}^2}{1+\eta_2}. \qquad (73)$$

6.3 Expectation of SIs

What SIs can observe from the outcome at t are

$$y_1 \equiv \delta_tW_t-n_t, \text{ where} \qquad (74)$$

$$-n_t \sim N\left(0,\sigma_{n_t}^2\right). \qquad (75)$$

The transition of the state variable is

$$W_{t+1}+\alpha_tF_t+\beta_t(\gamma_tF_{t-1}-P_{t-1})+\zeta_tE_{t-1}^{SI}[W_t] = (1-\delta_t)W_t. \qquad (76)$$

By the Kalman filter,

$$
\begin{aligned}
&E_t^{SI}\left[W_{t+1}\right]=\left(1-\delta_t\right)E_{t-1}^{SI}\left[W_t\right]+\alpha_t F_t+\beta_t\left(\gamma_t F_{t-1}+P_{t-1}\right)+\zeta_t E_{t-1}^{SI}\left[W_t\right]\\
&+K_t\left(\delta_t\left(W_t-E_{t-1}^{SI}\left[W_t\right]\right)-n_t\right)\\
&=\delta_t W_t+\left(1-\delta_t\right)E_{t-1}^{SI}\left[W_t\right]-S_t+K_t\left(\delta_t W_t-\delta_t E_{t-1}^{SI}\left[W_t\right]-n_t\right), \qquad (77)\\
&Var_t^d\left[W_{t+1}+\alpha_t F_t+\beta_t\left(\gamma_t F_{t-1}-P_{t-1}\right)+\zeta_t E_{t-1}^{SI}\left[W_t\right]\right]\\
&=\sigma_t^2\\
&=\left(1-\delta_t\right)\sigma_{t-1}^2\left(1-\delta_t-K_t\delta_t\right),\ \text{and} \qquad (78)
\end{aligned}
$$

$$
K_t=\frac{\left(1-\delta_t\right)\delta_t\sigma_{t-1}^2}{\delta_t^2\sigma_{t-1}^2+\sigma_{n_t}^2}. \qquad (79)
$$

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