Equilibrium corporate finance*

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Abstract

We study a general equilibrium model with production where financial markets are incomplete. In this environment firms' corporate financing decisions are non trivial. At a competitive equilibrium firms take their production and financial decisions so as to maximize their value and we show that shareholders unanimously support such decisions. Furthermore, competitive equilibria are constrained Pareto efficient. Such results extend to the case where informational asymmetries are present and contribute to determine the firms' capital structure.

Preliminary

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1 Introduction

We study a general equilibrium economy with incomplete markets, production and non-trivial corporate financing decisions. Corporate financing decisions are non-trivial because constraints in financial markets, e.g., borrowing constraints on the part of the agents, incomplete financial markets, asymmetric information between corporate investors and managers or between bondholders and shareholders, guarantee that the Modigliani-Miller (MM) theorem does not hold and production and financing decisions of firms cannot be separated.

In this class of economies, indeed because production and financing decisions of firms cannot be separated, corporate finance quantities like the capital structure and inside ownership levels depend on aggregate shocks as well as on idiosyncratic shocks. Also, corporate finance quantities are determined jointly with production decisions and cash flows, therefore affecting asset prices.

Various foundational issues, in particular regarding the specification of a proper objective function of the firm when markets are incomplete have arguably hindered the study of the macroeconomic properties of these economies as well as the development of the integrated study of corporate finance with macroeconomics and asset pricing theory.

In this paper we hence concentrate on the foundational theoretical properties of these economies. To this end we restrict the analysis to a simple two-period economy along the lines of classical General Equilibrium models with Incomplete Markets (GEI).

We consider first the case where firms' equity cannot be sold short and show that shareholders unanimously support the firm's objective of maximizing the firm's value and that competitive equilibria are constrained efficient. We also show that, when firms cannot default on the debt issued, the capital structure of each firm is - typically - indeterminate, while the equilibrium capital structure of all firms in the economy is at least partly determinate. Thus the Modigliani Miller's irrelevance result only holds at the level of the individual firm, not at the aggregate level and we can then investigate how the capital structure varies with the aggregate state, or the business cycle.

The analysis is then extended to the case where firms can default on the debt they issue, which is then a risky asset as well as to the case where short sales are allowed. More specifically, we show that competitive equilibria are still constrained efficient even if we allow for short sales.

In the final sections of the paper we introduce informational asymmetries between the decision maker in the firm (e.g., the manager) and shareholders or equityholders, as in standard corporate finance models. We show that the unanimity property continues to hold with asymmetric information, both with moral hazard and adverse selection. Constrained efficiency also holds with moral hazard. In these economies, typically, Modigliani-Miller's theorem does not hold and incentive issues further contribute to determine the firms' capital structure.

2 The economy

The economy lasts two periods, $t = 0, 1$ and at each date a single consumption good is available. The uncertainty is described by the fact that at $t = 1$ one state out of the set $\mathcal{S} = \{1, ..., S\}$ realizes. We assume for simplicity that there is a single type of firm in the economy which produces the good at date 1 using as only input the amount k of the commodity invested in capital at time 0. The output only depends on k according to the function $f(k; s)$, where s is the state realized at $t = 1$. We assume that $f(k; s)$ is continuously differentiable, increasing and concave in k.[1]

In addition to firms, there are I types of consumers. Consumer $i = 1, .., I$ has an endowment of w_0^i units of the good at date 0 and $w^i(s)$ units at date 1 in each state $s \in \mathcal{S}$, thus the agent's endowment is also subject to the shock affecting the economy at $t = 1$. He is also endowed with θ_0^i units of stock of the representative firm. Consumer i has preferences over consumption in the two dates, represented by $\mathbb{E}u^i(c_0^i, c^i(s))$, where $u^i(\cdot)$ is also continuously differentiable, increasing and concave.

There is a continuum of firms, of unit mass, as well as a continuum of consumers of each type i, which for simplicity is also set to have unit mass.

2.1 Competitive equilibrium

We examine the case where firms take both production and financial decisions, and their equity and debt are the only assets in the economy. Let the outstanding amount of equity be normalized to 1 (the initial distribution of equity among consumers satisfies $\sum_i \theta_0^i = 1$) and assume this is kept constant. Hence the choice of a firm's capital structure is only given by the decision concerning the amount B of bonds

[1] The present analysis could be easily extended, only at the cost of increased notational complexity, to more general specifications of the technology: For instance we could allow the output to depend, in addition to k, on the choice of the loadings ϕ_h, $h = 1, .., H$, on H factors: $f(k, \phi; s) = \sum_h a_h(s)\phi_h g(k)$ where $a_h(s)$ is the productivity shock affecting factor h.

issued, which in turn identically corresponds to the firm's debt/equity ratio. The problem of the firm consists in the choice of its production plan k and its financial structure B. To begin with, we assume all firms' debt is riskless.[2]

Firms are perfectly competitive and hence take prices as given. The notion of price taking behavior has no ambiguity when referred to the bond price p. For equity, however, the situation is more complex. A firm's cash flow, and hence the return on equity, is $[f(k;s) - B]$; it varies with the firm's production and financing choices, k, B. Price taking therefore does not mean considering the price of equity as fixed, independent of the firm's decisions k, B. Rather, a competitive firm takes as given a price map $q(k, B)$ which specifies the market valuation of its cashflow for any possible value of k, B.[3]

Each firm chooses its production and financing plans k, B so as to maximize its value, as determined by such pricing map and the bond price.[4] The firm's problem is then:

$$V = \max_{k,B} -k + q(k, B) + p\ B \tag{1}$$

subject to the solvency constraint (ensuring that the bonds issued are riskfree):

$$f(k;s) \geq B \qquad \forall s \tag{2}$$

Let $\bar{k}, \bar{B}$ denote the solutions to this problem.

At $t = 0$, each consumer i chooses his portfolio of equity and bonds, θ^i and b^i respectively, so as to maximize his utility, taking as given the price of bonds, p and the price of equity q. In the present environment a consumer's long position in equity identifies a firm's shareholder, who may have a voice in the firm's decisions. It should then be treated as conceptually different from a short position in equity, which is not simply a negative holding of equity. To begin with, in line with part of the literature on incomplete markets economies with production, we rule out altogether the possibility of short sales and assume that agents cannot short-sell the firm equity nor its debt:[5]

$$b^i \geq 0, \quad \theta^i \geq 0, \ \forall i \tag{3}$$

[2] We shall allow for the possibility that firms' default on their debt in Section 4.1.

[3] These price maps are called *price perceptions* in Grossman-Hart (1979), Kihlstrom-Matthews (1990) and Magill-Quinzii (1999). See the next section for further discussion on their construction.

[4] We will later show that such decision is unanimously supported by the firm's shareholders.

[5] Such restriction is later removed, in Section 4.2, where the analysis is extended to the case where short sales are allowed.

The problem of agent i is then:

$$\max_{\theta^i, b^i, c^i} \mathbb{E} u^i(c_0^i, c^i(s)) \tag{4}$$

subject to (3) and

$$c_0^i \;= w_0^i \;+ [-k \;+ q \;+ p \;B\;] \,\theta_0^i - q \;\theta^i \;\; - p \;b^i \tag{5}$$

$$c^i(s) = w^i(s) + [f(k; s) - B\;] \,\theta^i \;\; + b^i, \qquad \forall s \in S, \tag{6}$$

Let $\bar{\theta}^i, \bar{b}^i, \bar{c}_0^i, (\bar{c}^i(s))_{s\in\mathcal{S}}$ denote the solutions of this problem.

In equilibrium, the following market clearing conditions[6] must hold, for the consumption good:

$$\begin{cases} \sum_i c_0^i \;+ k \;\; \leq \sum_i w^i \\ \sum_i c^i(s) \leq \sum_i w^i(s) + f(k; s) \text{ for all } s \in \mathcal{S} \end{cases}$$

or equivalently for the assets:

$$\begin{cases} \sum_i b^i \;\; \leq B \\ \sum_i \theta^i \;\; \leq 1 \end{cases} \tag{7}$$

In addition, the equity price map faced by the firm must satisfy the following consistency conditions:

i) $q = q(\bar{k}, \bar{B})$

ii) $q(k, B) = \max_i \mathbb{E}\left[MRS^i(s)(f(k; s) - B)\right]$ for all k, B. where $MRS^i(s)$ denotes the marginal rate of substitution between consumption at date 0 and at date 1 in state s for consumer i, evaluated at his optimal consumption choice $\bar{c}^i$.

Condition i) requires the price of equity faced by consumers to equal the value of the equity price map faced by firms corresponding to their optimal choice. Condition ii) then says that for any k, B the value of the equity price map $q(k, B)$ equals the highest marginal valuation - across all consumers in the economy - of the firm's cash flow associated to k, B. Note that the price taking assumption can be clearly seen in the specification of this condition: the equity price map takes into account the fact that a firm's choice of k, B modifies its cash flow, but the consumers' marginal rates

[6] We state here the conditions for the case of symmetric equilibria, where all firms take the same production and financing decision, so that only one type of equity is available for trade to consumers. They can however be easily extended to the case of asymmetric equilibria (considered for instance in the example of Section 2.4.1).

of sustitutions $MRS^i(s)$ used to determine the market valuation are independent of such choice of k, B.

To better understand the meaning of condition ii), note that the consumers with the highest marginal valuation for the firm's cash flow when the firm chooses k, B are those willing to pay the most for the firm's equity in that case and the only ones willing to buy equity when its price satisfies ii). Given i), as we show in (8) below such condition is clearly satisfied for the firms' equilibrium choice $\bar{k}, \bar{B}$. Condition ii) requires that the same is true for any other possible choice k, B, that is the value attributed to equity equals the maximum any consumer is willing to pay for it.

Summarizing,

Definition 1 *A competitive equilibrium of an economy with initial state s_0 is a collection $\left(\bar{k}, \bar{B}, \{\bar{c}^i, \bar{\theta}^i, \bar{b}^i\}_i, \bar{p}, \bar{q}, q(\cdot)\right)$ such that i) $\bar{k}, \bar{B}$ solve the firm problem (1) s.t. (2) given $\bar{p}, q(\cdot)$; ii) for all i, $\bar{c}^i, \bar{\theta}^i, \bar{b}^i$ solve consumer i's problem (4) s.t. (5) and (6) for given $\bar{p}, \bar{q}$; iii) markets clear, (7); iv) the equity price map is consistent, that is satisfies i) and ii).*

In equilibrium the price of equity and the bond satisfy:

$$\begin{aligned} \bar{q} &= \max_i \mathbb{E}\left[MRS^i(s)(f(\bar{k}; s) - \bar{B})\right] \\ \bar{p} &= \max_i \mathbb{E} MRS^i(s) \end{aligned} \tag{8}$$

2.2 Objective function of the firm

Starting with the initial contributions of Diamond (1967), Dreze (1974), Grossman-Hart (1979), and Duffie-Shafer (1987), a large literature has dealt with the question of what is the appropriate objective function of the firm when markets are incomplete.[7] The issue arises because firms' production decisions may affect the set of insurance possibilities available to consumers by trading in the asset markets, that is the asset span.

If agents are allowed infinite short sales of the equity of firms, at the same price at which equity can be purchased, as in the standard GEI model, a *small* firm will possibly have a *large* effect on the economy by choosing a production plan with

[7]See e.g., Bonnisseau-Lachiri (2002), Cres-Tvede (2001), DeMarzo (1993), Dierker-Dierker-Grodal (2002), Dreze-Lachiri-Minelli (2007), Kelsey-Milne (1996) and many others.

cash flows which, when traded as equity, change the asset span. It is clear that the price taking assumption appears hard to justify in this context, since changes in the firm's production plan have significant effects on allocations and hence equilibrium prices. In the environment considered here, as in part of the literature recalled above, consumers face a constraint preventing short sales, (3), which guarantees that each firm's production plan has an infinitesimal effect on the set of admissible trades and allocations available to consumers. As argued by Hart (1979) and Allen-Gale (1988), price taking behavior is then justified in this case, when the number of firms is large. Evidently, for price taking behavior to be justified a no short sale constraint is more restrictive than necessary and a bound on short-sales of equity would suffice. We will explore how to allow for short sales in Section 4.2.

When short sales are not allowed, a firm's decision has no effect on equilibrium allocations and market prices. However, of course, each firm's decision has a nonnegligible impact on its present and future cash flows. Price taking cannot therefore mean that the price of its equity is taken as given by a firm, independently of its decisions. The level of the equity price associated to out-of-equilibrium values of k, B is not observed in the market. It is rather *conjectured* by the firm. In a competitive environment we require such conjecture to be *competitive*, that is, determined by a given pricing kernel independently of the firm's decisions.

In addition, competitive price conjectures should satisfy some *consistency* conditions. The different equilibrium notions we find in the literature differ primarily in the specification of such consistency conditions. A minimal consistency condition is clearly given by i) in the previous section, which only requires the conjecture to be correct in correspondence to the firm's equilibrium choice. Duffie-Shafer (1987) indeed only impose such condition and find a rather large indeterminacy of the set of competitive equilibria. Additional consistency conditions impose some restrictions also on the value of the conjecture for other, out-of-equilibrium choices of the firms. Our condition ii) requires

$$q(k, B) = \max_i \mathbb{E} MRS^i(s) \left[f(k; s) - B \right] \text{ for all } k, B;$$

that is, $q(k, B)$ has to equal what would be the equilibrium price of equity if a negligible fraction of firms were to choose k, B. Thus we can view a price conjecture satisfying i) and ii) as one that ensures that all markets clear, not just the market for equity corresponding to the choice $\bar{k}, \bar{B}$ made by firms in equilibrium, for which the supply is the outstanding amount of equity, but also the market for equity corresponding to any other values of k, B, for which the supply is zero, and hence clears with zero trades.[8]

[8] An analogous specification of the price conjecture has been earlier considered, by Makowski

It is useful to compare our notion of equilibrium with that of Dreze (1974). In both cases the firm maximizes its value and evaluates alternative plans on the basis of a competitive price conjecture. The main distinction lies in the consistency condition imposed on such conjecture: in Dreze (1974), after translating his notation into ours, the consistency condition takes the following form:

$$q(k,B) = \mathbb{E}\sum_i \bar{\theta}^i MRS^i(s)\,[f(k;s) - B] \text{ for all } k, B\ . \tag{9}$$

Such condition requires the price conjecture for any plan k, B to equal the pro rata marginal valuation of the agents who at equilibrium are the firm's shareholders (that is, the agents who value the most the plan chosen by firms in equilibrium). It does not however require that the firm's shareholders are those who value the most any possible plan of the firm. Intuitively, the choice of a plan which maximizes the firm's value with $q(k, B)$ as in (9) corresponds to a situation in which the firm's shareholders choose the plan which is optimal for them[9] without contemplating the possibility of selling the firm in the market (so that it operates a plan preferred by the new equity buyers). Our consistency condition, on the contrary, by evaluating each plan according to the marginal valuation of the agent who values it the most, allows also to evaluate the benefits of selling the firm in the market.[10]

It is easy to see then that any allocation satisfying Definition 1 is also a Dreze equilibrium (with no short sales): all shareholders have in fact the same valuation for the firm's production plan and their marginal utility for any other possible production plan is lower, hence a fortiori the chosen plan maximizes the weighted average of the shareholders' valuations. But the reverse implication is not true, i.e., a Dreze equilibrium is not in general an equilibrium according to our definition.

Grossman-Hart (1979) propose another consistency condition and hence a different equilibrium notion in a related environment. In their case (again, after a translation of the notation):

(1980, 1983), Makowski-Ostroy (1991) in a competitive equilibrium model with differentiated products, and by Allen-Gale (1989) and Pesendorfer (1995) in models of financial innovation. See also Magill-Quinzii (1999).

[9] It is in fact immediate to verify that the plan which maximizes the firm's value with $q(k, B)$ as in (9) is also the plan which maximizes the welfare of the given set of shareholders of the firm, when each of them evaluates any plan according to its own MRS.

[10] In our equilibrium notion, therefore, the firm evaluates different production plans using possibly different marginal valuations. This is not the case of Dreze (1974) where the marginal valuation is a fixed average of the valuations of the equilibrium shareholders. This is a fundamental distinguishing feature of our equilibrium notion with respect to the many others proposed in the GEI literature; see also the Grossman-Hart (1979) notion discussed next.

$$q(k,B) = \mathbb{E}\sum_i \theta_0^i MRS^i(s)\left[f(k;s) - B\right] \text{ for all } k, B;$$

We can interpret such notion as describing a situation where the firm's plan is chosen by the initial shareholders (i.e., those with some predetermined stock holdings at time 0) so as to maximize their welfare, again without contemplating the possibility of selling the equity to other consumers who value it more.

But *the proof is in the pudding*. Our equilibrium notion, besides being logically consistent as no *small* firm has *large* effects, also has some desirable properties: i) it produces equilibria which satisfy a constrained version of the First Welfare Theorem, ii) it delivers a Unanimity result and a local version of the Modigliani-Miller Theorem.

2.3 Unanimity

In our setup shareholders unanimously support the firm's choice of the production and financial decisions which maximize its value (or profits), as in (1). This follows from the fact that when the equity price map satisfies the consistency conditions i) and ii), the consumers' choice problem is equivalent to one where a continuum of types of equity is available for trade to consumers, corresponding to any possible choice of k, B the representative firm can make, at the price $q(k, B)$. Thus, as already mentioned, for any possible value of k, B a market is open where equity with a payoff $[f(k;s) - B]$ can be traded, and in equilibrium such market clears with a zero level of trades for the values of k, B not chosen by the firms.[11]

Unanimity then holds by the same argument as the one used to show it for Arrow Debreu economies. More formally, notice that we can always consider a situation where, in equilibrium, each consumer holds at most a negligible fraction of each firm. The effect of alternative choices by a firm can then be evaluated using the agent's marginal utility.For any possible choice k, B of a firm, the (marginal) utility of the agent if he holds the firm's equity is

$$\mathbb{E}MRS^i(s)\left[f(k;s) - B\right]$$

always less or at most equal to his utility if he sells the firm's equity at the market

[11] Note the crucial role of the no short sale condition for such property to hold (see also Hart (1979) for a unanimity result in a setup where no sort sales are allowed). In Section 4.2, we will show that the unanimity, as well as the constrained efficiency, results extend to the case where limited short sales are allowed, provided an appropriate specification of the markets for selling short assets is considered.

price, given by

$$\max_i \mathbb{E} MRS^i(s)\left[f(k;s) - B\right].$$

Hence the firm's choice which maximizes the latter also maximizes the shareholder's utility.

Proposition 1 *At a competitive equilibrium, shareholders unanimously support the production and financial decisions of firms $\bar{k}, \bar{B}$; that is, every agent i holding a positive initial amount θ_0^i of equity of the representative firm will be made - weakly - worse off by any other choice k', B' of a firm.*

2.4 Welfare properties

We show next that all competitive equilibria of the economy described exhibit desirable welfare properties. Evidently, since the hedging possibilities available to consumers are limited by the presence of the equity of firms and riskless bonds as the only assets, we cannot expect competitive equilibrium allocations to be fully Pareto efficient, but only to make the best possible use of the existing markets, that is to be constrained Pareto efficient in the sense of Diamond (1967).

To this end we say a consumption allocation $(c^i)_{i=1}^2$ is *admissible* if

1. it is *feasible*: there exists a production plan k of firms such that

$$\begin{aligned} \sum_i c_0^i + k &\leq \sum_i w_0^i \\ \sum_i c^i(s) &\leq \sum_i w^i(s) + f(k;s) \text{ for all } s \end{aligned} \tag{10}$$

2. it is *attainable with the existing asset structure*: there exists B and, for each consumer's type i, a pair θ^i, b^i such that:

$$c^i(s) = w^i(s) + \left[f(k;s) - B\right]\theta^i + b^i, \qquad \forall s \tag{11}$$

Next we present the notion of *efficiency* restricted by the *admissibility* constraints:

Definition 2 *A competitive equilibrium allocation is constrained Pareto efficient if we cannot find another admissible allocation which is Pareto improving.*

The validity of the First Welfare Theorem with respect to such notion can then be established by an argument essentially analogous to the one used to establish the Pareto efficiency of competitive equilibria in Arrow Debreu economies.[12]

Proposition 2 *Competitive equilibria are constrained Pareto efficient.*

Proof. Suppose $(\hat{c}^i)_{i=1}^I$ is admissible and Pareto dominates the competitive equilibrium allocation $(\bar{c}^i)_{i=1}^2$. This implies there exists $\hat{k}, \hat{B}$ and $\left(\hat{\theta}^i, \hat{b}^i\right)_i$ such that (10) and (11) are satisfied. This, together with the fact that $(\bar{c}^i)_{i=1}^I$ is the consumers' optimal choice at the equilibrium prices $\bar{q}, \bar{p}$ and the equity price map satisfies the consistency condition (ii), so that $\hat{q} = \max_i \mathbb{E}MRS^i(s)\left[f(\hat{k}; s) - \hat{B}\right]$, imply:

$$\hat{c}_0^i + \hat{q}\hat{\theta}^i \quad + \bar{p}\ \hat{b}^i - w_0^i \geq \bar{c}_0^i + \bar{q}\ \bar{\theta}^i + \bar{p}\ \bar{b}^i - w_0^i\ , \quad > \exists i$$

or equivalently,

$$\left[-\hat{k}\ \ + \hat{q}\ \ + \bar{p}\ \hat{B}\ \right] \theta_0^i + \tau^i \geq \left[-\bar{k}\ \ + \bar{q}\ \ + \bar{p}\ \bar{B}\ \right] \theta_0^i \text{ for each } i, \ \ > \exists i \tag{12}$$

for $\tau^i \equiv \hat{c}_0^i + \hat{q}\hat{\theta}^i \quad + p\ \hat{b}^i - \left[-\hat{k}\ \ + \hat{q}\ \ + \bar{p}\ \hat{B}\ \right] \theta_0^i - w_0^i$. Summing (12) over i yields:

$$\left[-\hat{k}\ \ + \hat{q}\ \ + \bar{p}\ \hat{B}\ \right] + \sum_i \tau^i > \left[-\bar{k}\ \ + \bar{q}\ \ + \bar{p}\ \bar{B}\ \right] \tag{13}$$

The fact that $\bar{k}, \bar{B}$ solves the firms' optimization problem (1) in turn implies that:

$$-\bar{k}\ \ + \bar{q}\ \ + \bar{p}\ \bar{B} \geq -\hat{k}\ \ + \hat{q}\ \ + \bar{p}\ \hat{B},$$

which, together with (13), yields:

$$\sum_i \tau^i > 0,$$

or equivalently:

$$\sum_i \hat{c}_0^i + \hat{k} > \sum_i w_0^i,$$

a contradiction to (10) at date 0. ■

[12]See also Allen-Gale (1989) for a constrained efficiency result in a related environment.

2.4.1 Efficiency and asymmetric equilibria

To better understand the result and contrast it with some different findings in the literature, it is useful to illustrate it by means of an example, where we consider essentially the same economy studied by Dierker-Dierker-Grodal (2002). For such economy they showed that all Drèze equilibria are constrained inefficient. We will show that, in contrast, a unique (asymmetric) competitive equilibrium exists according to our definition, and that it is constrained efficient. Given our focus here on efficiency we abstract from the firms' financial decisions and set $B = 0$.

Let $\mathcal{S} = \{s', s''\}$. There are two types of consumers, with type 2 having twice the mass of type 1, and preferences, respectively $u^1(c_0^1, c^1(s'), c^1(s'')) = c^1(s')/\left(1 - (c_0^1)^{9/10}\right)^{10/9}$, $u^2(c_0^2, c^2(s'), c^2(s'')) = c_0^2 + (c^2(s''))^{1/2}$, endowments $w_0^1 = .95$, $w_0^2 = 1$ and $w^1(s) = w^2(s) = 0$ for all $s \in \mathcal{S}$. The technology of each firm is described by $f(k; s) = \lambda k$ for $s = s'$ and $(1 - \lambda)k$ for $s = s''$, where λ can be freely chosen at any point in the interval $[2/3, 0.99]$. The firms' problem is then $\max_{\lambda,k} -k + q(k)$, where $q(k) = \max\left\{\frac{\partial u^1/\partial c^1(s')}{\partial u^1/\partial c_0^1}\lambda k; \frac{\partial u^2/\partial c^2(s'')}{\partial u^2/\partial c_0^2}(1-\lambda)k\right\}$.

We show first that in this case a symmetric equilibrium, where all firms choose the same value of k, λ, does not exist. Given the agents' endowments and preferences, both types of consumers buy equity in equilibrium. It is then easy to see that the firms' optimality condition with respect to λ can never hold for an interior value of λ nor for a corner solution[13].

Next, we show that an asymmetric equilibrium exists, where a fraction 1/3 of the firms choose $\lambda^1 = 0.99$ and $k^1 = 0.3513$ and the remaining fraction chooses $\lambda^2 = 2/3$ and $k^2 = 0.1667$, type 1 consumers hold only equity of the firms choosing λ^1, k^1 and type 2 consumers only equity of the other firms. At this allocation, we have $\frac{\partial u^1/\partial c^1(s')}{\partial u^1/\partial c_0^1} = 1.0101$, $\frac{\partial u^2/\partial c^2(s'')}{\partial u^2/\partial c_0^2} = 3$. Also, the marginal valuation of type 1 agents for the equity of firms choosing λ^2, k^2 is 0.1122, thus smaller than the market value of these firms' equity, equal to 0.1667, while the marginal valuation of type 2 agents for the equity of the firms choosing λ^1, k^1 is 0.0105, smaller than the market value of these firms' equity, equal to 0.3513. Therefore, at these values the firms' optimality conditions are satisfied and this constitutes a competitive equilibrium according to our definition. It can then be easily verified that the equilibrium allocation is constrained efficient.

[13]Consider for instance $\lambda = 0.99$. To have an equilibrium at this value the marginal valuation of equity for both consumers must be the same at $\lambda = 0.99$ and higher than at any other values of λ, but this second property clearly cannot hold for type 2 consumers.

In this economy, Dierker-Dierker-Grodal (2002) find a unique Drèze equilibrium where all firms choose a production plan with $\lambda \approx 0.7$. To understand the difference with respect to our findings, notice that the map associating the value of equity to alternative production plans is obtained here by taking as given the marginal rate of substitution of consumers at the candidate equilibrium allocation, which is justified when consumers hold a negligible fraction of each firm (in which case we can argue their price taking behavior for the equity they own is justified). On the other hand, in Dierker-Dierker-Grodal (2002), the expression of the firms' market value for alternative production plan is computed taking into account the effect of such choices on consumers' marginal rate of substitution when consumers hold non negligible shares of each firm.

3 Corporate Finance and Investment Decisions

In this section we study the properties of the firms' corporate finance and capital investment decisions at an equilibrium. To this end, it is convenient to introduce the notation I^e to denote the collection of all agents i such that

$$\bar{q} = \mathbb{E}MRS^i(s)\left[f(\bar{k};s) - \bar{B}\right]$$

that is, the collection of all agents that in equilibrium either hold equity or are indifferent between holding and not holding equity. We can similarly define the collection I^d of all agents i such that $\bar{p} = \mathbb{E}MRS^i(s)$, that is, the collection of all agents that in equilibrium either hold bonds or are indifferent between holding and not holding bonds. With some abuse of language we denote the agents in I^e as equityholders and those in I^d bondholders.

To derive the optimality conditions for the firm's choice of financing and production we should note that the equity price map $q(k,B) = \max_i \mathbb{E}MRS^i(s)\left[f(k;s) - B\right]$ may fail to be differentiable. $q(k,B)$ may also fail to be concave, so that in order to determine the optimality conditions with respect to B and k we also have to take into account the possibility of joint deviations. The first order conditions are then different according to whether the no default constraint (2) binds or not. Letting $\underline{s}$ denote the lowest output state, we obtain the following characterization of the firms' optimality conditions:[14]

Proposition 3 *The optimal production and financing decisions of a firm are obtained:*

[14]The proof can be found in the Appendix.

(i) either at an interior solution $f(k;\underline{s}) > B$ with:

$$\max_{i\in I^e} \mathbb{E}MRS^i(s) = \min_{i\in I^e} \mathbb{E}MRS^i(s) = p = \max_i \mathbb{E}MRS^i(s) \tag{14}$$

and

$$\max_{i\in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] = \min_{i\in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] = 1; \tag{15}$$

(ii) or at a corner solution $f(k;\underline{s}) = B$ with:

$$p \geq \max_{i\in I^e} \mathbb{E}MRS^i(s) = \min_{i\in I^e} \mathbb{E}MRS^i(s), \tag{16}$$

$$1 \geq \max_{i\in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] = \min_{i\in I^e} \left[\mathbb{E}MRS^i(s) f_k(s)\right], \tag{17}$$

and

$$f_k(\underline{s}) \left(p - \max_{i\in I^e} \mathbb{E}MRS^i(s)\right) = 1 - \max_{i\in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] \tag{18}$$

The economic meaning of such conditions can be described as follows: (14) says that *all equity holders are also bond holders* (while the reverse may not be true): $I^e \subseteq I^d$. That is, no one of the equity holder would like to short the bond. Condition (15) says that *all equityholders value equally the effect on equity of an infinitesimal increase in the investment level k* and they value it at 1, that is the same as the marginal cost of such investment. Conditions (14) and (15) imply that $\frac{\partial V}{\partial B}_+ = \frac{\partial V}{\partial B}_- = 0 = \frac{\partial V}{\partial k}_+ = \frac{\partial V}{\partial k}_-$.

Turning then to corner solutions, (16) says that *all equity holders share the same valuation for the bond, possibly strictly less than its price p* and (17) that *they value equally the effect on equity of an infinitesimal increase in k, possibly strictly less than its cost* 1. Moreover, (18) says that whenever all equityholders value the bond less than p (that is, no equityholder is also a bondholder), so that they would all benefit from an increase in B, then it must be that they value also an increase in k strictly less than its cost and the "gap" in the two expressions is exactly equal.[15] We have so $\frac{\partial V}{\partial B}_+ = \frac{\partial V}{\partial B}_- \geq 0$, $\frac{\partial V}{\partial k}_+ = \frac{\partial V}{\partial k}_- \leq 0$ and $f_k(\underline{s})\frac{\partial V}{\partial B}_+ = f_k(\underline{s})\frac{\partial V}{\partial B}_- = -\frac{\partial V}{\partial k}_+ = -\frac{\partial V}{\partial k}_-$.

3.1 Modigliani Miller

What are the implications of the above characterization of the firm's optimality conditions for the firm's optimal financing choice, represented by B? Is such choice

[15] Also, the ratio at which bond supply and capital investment can be decreased while keeping the solvency constraint satisfied is given by $dB = f_k(\underline{s}_1)dk$.

indeterminate? Equivalently, does the Modigliani-Miller irrelevance result hold in our setup? The answer clearly depends on whether the solution of the firm's problem obtains at a point where the no default constraint is slack or binds. We consider each of these two cases in turn.

When $f(k; \underline{s}) > B$, we have shown in the previous section that the value of the firm V is locally invariant with respect to any change in B. Furthermore, this invariance result extends to any other admissible change in B: all equityholders are in fact indifferent with respect to any discrete change ΔB.[16] The other agents might not be indifferent, but the optimality of B, k implies their valuation of the firm is always lower. Thus V is invariant with respect to any admissible change in B, whether positive or negative.

When the optimum obtains at a corner, $f(k; \underline{s}) = B$, either the same property still holds (V is invariant with respect to any admissible change in B), or V is strictly increasing in B, which occurs when no equityholder is also a bondholder, in which case the firm's problem has a unique solution for B.

To sum up, except in this last case at a competitive equilibrium V is invariant with respect to any admissible change in B. It is important to note that, while the capital structure is indeterminate for any individual firm, it does not mean that the capital structure of the economy, that is of all firms in the economy is also indeterminate. In particular, as argued above B has to be such that all equityholders are also bondholders and this imposes a lower bound on the aggregate value of B (given by $\min_{i \in I^e} \bar{b}^i / \bar{\theta}^i$). We have thus established the following:

Proposition 4 *At a competitive equilibrium, the capital structure of each individual firm is indeterminate, except in the case where the firm's no default constraint binds and no equityholder is also a bondholder (when there is a unique optimal level of B, at $f(\bar{k}; \underline{s})$). On the other hand, the equilibrium capital structure of all firms in the economy is, at least partly, determinate: the equilibrium capital structure B of all firms in the economy is such that all equityholders are also bondholders, in which case any other admissible $B' > B$ as well as any $B" < B$ in an appropriately defined neighborhood (possibly empty) also configures an equilibrium*

Thus the Modigliani-Miller irrelevance result only holds at the level of an individual firm, not for all firms in equilibrium. The reason for the latter is the presence of

[16] An upper bound on admissible increases in B is obviously given by the no default constraint. Similarly, the lower bound on admissible decreases in B is given by B itself, that is changes that keep the total amount of bonds outstanding non negative are admissible. Any admissible variation in the debt level B of a firm can then be 'undone' by an appropriate adjustment in the shareholders' bondholdings.

borrowing constraints, which restrict the set of equilibrium values of the capital structures to an interval.[17]

3.2 Capital structure and business cycles

We illustrate the properties of the equilibrium and the firms' optimal investment and financial decisions by considering a simple example, with two types of consumers, $H = 2$; both consumers have initial stockholdings $\theta_0 = .5$ and preferences described by $u^i(c_0^i, c^i(s)) = u(c_0^i) + \beta u(c^i(s))$ for all i, with $u = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma = 2$ and $\beta = 0.95$. The production technology exhibits multiplicative shocks: $f(k; s) = a(s)k^\alpha$, with $\alpha = 0.75$. There are two possible states in each period, $\mathcal{S} = \{\underline{s}, \overline{s}\}$, with the following structure of endowment and productivity shocks:

	$\underline{s}$	$\overline{s}$
w^1	2	3
w^2	3	8
a	2.1429	4.7143

Hence: i) $\underline{s}$ describes a recession state, $\overline{s}$ a boom; ii) agent 2's endowments are higher than 1's both in boom and recession (166% higher in boom, 50% in recession); iii) the productivity shocks a are computed so that $\frac{a}{a+w^1+w^2} = .3$, a capital share of 30% if k is equal to 1.

We intend to investigate how the properties of the equilibria, and in particular of the firms' optimal choices, vary with respect to some parameters of the model like the persistence of the shocks and the initial state. We consider first the case where at date 0 the state is also recession, i.e. $w_0^i = w^i(\underline{s})$ for all i, and $\pi(\underline{s}) = .8$, $\pi(\bar{s}) = .2$, or the persistence of the shocks is fairly high.

We find that in this case there is a unique equilibrium allocation where firms' investment is $k = 0.39751$ while their capital structure is given by any level of B lying in the interval $[0.64149, 1.0728]$. The firms' capital structure is then partly indeterminate, as we showed in Proposition 4 is the case when the default constraint does not bind (and it can be readly verified that at the above values this never happens, except at $B = 1.0728$).

In order to better understand the determinants of the firms' equilibrium capital structure, it is useful to examine the situation where B is treated parametrically. Let B^{ex} denote the exogenously given level of the debt issued by each firm; for any value

[17] See Stiglitz (1969) for a first result along these lines.

of B^{ex} we find the investment level k which maximizes firms' value[18], the individual consumption and portfolio holdings $\{c^i, \theta^i, b^i\}_{i=1}^2$ solving (4) and the prices $\{q, p\}$ such that markets clear and the consistency conditions for q hold. In Figure (3.2) we plot, as B^{ex} is varied from 0 to 1.0728, the values obtained for the consumers' asset holdings, on the first line, and their willingness to pay for the assets, on the second line. We can then use this figure to determine when we have an equilibrium: this happens when the optimality condition for the firms' financing decisions found in Proposition 3 holds.

Obviously at $B^{ex} = 0$ the default constraint does not bind and so the firms' optimality condition stated in part i) of Proposition 3 applies, requiring that all stockholders have the same willingness to pay, at the margin, for the bond. At $B^{ex} = 0$, however, both consumers hold equity (see the top left panel) while consumer 1 has a higher willingness to pay for the bond than consumer 2 (lower right panel). Therefore, $B = 0$ is not an equilibrium value: when firms can choose both k and B, at $B = 0$ any firm can increase its value[19] by issuing debt.

As B^{ex} is progressively increased from 0 to 0.64149, it is still always true that consumer 1 has a higher willingness to pay for the bond, while for equity the two consumers have the same marginal valuation[20]. Since for all these values the default constraint does not bind, the same argument as above applies and implies that for all values of B^{ex} in this interval we do not have an equilibrium.

At $B^{ex} = 0.64149$, instead, the two consumers have the same willingness to pay for the bond (bottom right panel) and only consumer 2 holds equity. Thus, the condition in part i) of Proposition 3 is now satisfied and hence the prices and allocation obtained when $B^{ex} = 0.64149$ (with $k = 0.39751$) constitute an equilibrium of our model. As B^{ex} increases beyond 0.64149, up to its maximal level such that the no default condition is satisfied (1.0728), the same allocation and bond prices still constitute an equilibrium[21]. Values of $B^{ex} > 1.0728$ can only be sustained if the firm's investment is increased so as to satisfy the no default constraint: we find however that this is never an equilibrium.

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[18] As shown in Propositions 3, the firms' optimal choice of k must satisfy (15) at an interior solution and (17) at a corner solution.

[19] The firms' value is determined using the equity price map obtained, as stated in the consistency condition ii) of Definition 1, from the consumers' MRS at the consumption allocation under consideration (associated to $B^{ex} = 0$).

[20] Strictly speaking, for values of B^{ex} in the subinterval $(0.6, 0.63)$, agent 2's marginal valuation for equity becomes higher than 1's and hence holds all the equity. This is still not an equilibrium by the same argument.

[21] At $B^{ex} = 1.0728$, the default constraint binds and the optimality conditions in Part ii) of Proposition 3 are satisfied with equality.

To sum up, the equilibrium consumption and investment levels are uniquely determined while the capital structure of all the firms in the economy is partly indeterminate, given by any $B \in [0.64149, 1.0728]$. This finding is in accord with our Modigliani-Miller result, Proposition 4, which states that - partial - irrelevance holds when all equityholders are also bondholders, as in the current situation. The result also says that the capital structure of any individual firm is in this case completely indeterminate. This property is shown in Figure (2), where the value of an arbitrary firm $-k + q(k, B) + pB$ is plotted[22] for different levels of k and B: we see that its maximal level is attained at $k = 0.39751$ and all $B \in [0, 1.0728]$.

In the bottom left column of Table 3.2 below, we have reported the equilibrium values of the investment, asset prices and firms' leverage ratio corresponding to the lower bound of B in the equilibrium region. The other columns of the table report the corresponding equilibrium values for the other cases, where instead the persistence of the initial state is low (0.2) and/or the initial state is boom ($w_0^i = w^i(\bar{s})$).

	$\underline{s}$ (recession)	$\bar{s}$ (boom)
Low persistence		
k	0.21457	2.3497
B	0.28172	2.3748
q	0.21497	1.4054
p	0.25245	0.72749
$-k + pB + q$	0.071522	0.78325
$pB/(-k + pB + q)$	0.99436	2.2057
θ^1	1	0.49128
b^1	0	0
High persistence		
k	0.39751	1.6595
B	0.64149	2.3758
q	0.2351	1.1253
p	0.45973	0.4577
$-k + pB + q$	0.1325	0.55318
$pB/(-k + pB + q)$	2.2257	1.9657
θ^1	1	0.49106
b^1	0	0

[22] Again the value of the equity price map $q(k, B)$ is determined using the consumers' MRS at the candidate equilibrium allocation.

Comparing the different equilibrium values we can see in particular how firms' leverage varies along the cycle and with the persistence of the shocks. Obviously, given the simplified nature of the model, the implications are to be interpreted from a qualitative point of view.

There is a large body of literature about the cyclical properties of leverage. There seems to be consensus in the literature that leverage is countercyclical at the aggregate level but also that there is a high degree of heterogeneity across firms, with the leverage of smaller and more constrained firms being uncorrelated (or even slightly positively correlated) with the business cycle. For example[23], Korajczyk and Levy (2003) examine the determinants of time variation in firms' leverage ratios and security issue choices between 1984 and 1998. Their sample is divided on the basis of a measure of financial constraints faced by the firms. They find that the response of firms to cyclical fluctuations depends upon the stringency of financing constraints. Less constrained firms issue debt counter-cyclically and equity pro-cyclically. Consequently, these firms exhibit pronounced counter-cyclical variation in leverage ratios. In contrast, the financing mix of more constrained firms is insensitive to the business cycle[24].

In our example, we see that when the persistence of the shocks is low, firms' leverage increases from 0.99436 to 2.2057 going from recession to boom (in a boom, with low persistence, consumers expect to face hard times in the future and firms' productivity to be low; hence they demand debt relatively more than equity because, in this situation, debt represents a better hedge than equity against expected low idiosyncratic shocks). On the other hand, when the persistence of the shock is high, firms' leverage decreases from 2.2257 to 1.9657 from recession to boom.

While emphasizing once more that our model is extremely stylized, we want to stress the importance of our equilibrium approach for a thorough understanding of the

[23]See also Choe, Masulis, and Nanda (1993), Gertler and Gilchrist (1993), Kashyap, Stein, and Wilcox (1993), Gertler and Gilchrist (1994), Covas and Den Haan (2007), Hennessy and Levy (2007).

[24]Korajczyk and Levy (2003) find that these results are robust to both book and market measures of leverage, where the latter is defined as the book value of debt over the book value of debt plus the market value of equity. In our two-period environment there is no intermediate date between issuance of the securities and their maturity: therefore, book and market values coincide and we can define and refer to leverage simply as $\frac{pB}{-k+q+pB}$. Also in our stylized context, there is no heterogeneity across firms in size or productivity. Moreover, the only financial constraints are on the "demand-side" and are represented by the no-borrowing and short-sale constraint imposed on the agent. Since all firms are identical, these constraints apply equally to all firms. Our simulations for aggregate leverage therefore do not "hide" any heterogeneity across firms.

empirically observed facts in corporate finance.

The example just presented, for instance, far from having any quantitative relevance, is meant to show how cyclical variations in corporate leverage can be explained in terms of cyclical variations in demand for different securities by different agents, in turn determined by the structure of (aggregate) productivity and (idiosyncratic) endowment shocks, their persistence over time and the hedging opportunities provided by the existing markets. In particular, the case with high persistence of the shocks generates a counter-cyclical leverage as there is relatively more demand for debt in a recession than in a boom.

4 Extensions

We study two main extensions of the economy studied in the previous section, relaxing the main restrictions we have imposed on financial markets. In the first extension firms are allowed to use also risky debt as a financing instrument. In the second extension we allow for short-sales of equity.

4.1 Risky debt

Suppose the no default constraint (2) is no longer imposed on the firms' decision problem. Hence firms may default in some states of nature. In that case the debt issued by them is a risky asset and its yield varies, like equity's, with their production (k) and financial decisions (B). The market valuation for the debt issued by a firm is then also not fixed but varies with its decisions (k, B) and is given by a map $p(k, B)$. The firm's problem becomes so:

$$V = \max_{k,B} -k + q(k, B) + p(k, B)B \tag{19}$$

In the consumers' budget constraints the expressions of the yields of equity and bonds need to be accordingly modified:

$$c_0^i \ = w_0^i \ + [-k \ + q \ + pB\ ]\, \theta_0^i - q\ \theta^i \ - p\ b^i \tag{20}$$

$$c^i(s) = w^i(s) + \max\{0, f(k; s) - B\}\theta^i \ + \min\left\{1, \frac{f(k; s)}{B}\right\} b^i, \qquad \forall s. \tag{21}$$

The consumers' choice problem consists in solving problem (4) subject to the no short sale constraint (3) and the above budget constraints.

Finally, in equilibrium both the bond and equity price maps faced by firms must satisfy some suitable consistency conditions:

i′) $q = q(\bar{k}, \bar{B})$, $p = p(\bar{k}, \bar{B})$ at $\bar{k}, \bar{B}$ solving (19);

ii′) for all k, B: $q(k, B) = \max_i \mathbb{E}\left[MRS^i(s) \max\{0, f(k; s) - B\}\right]$ and
$p(k, B) = \max_i \mathbb{E}\left[MRS^i(s) \min\left\{1, \frac{f(k;s)}{B}\right\}\right]$.

In other respects the definition of a competitive equilibrium is the same as in Definition 1.

Since the production and financial decisions of firms have now a wider impact on the asset returns, the conditions for an optimum of the firms' choices are more complex. The formal statement of these conditions as well as a calibrated example can be found in Appendix (8.2). We find that the condition for an interior optimum with regard to B implies now that not only all bondholders have the same valuation of the sum of the two components of the payoff of bonds but also that they have the same valuation for each of these two components separately. Secondly, it implies that all equityholders have the same valuation - and the same as bondholders - for the bonds' payoff (in the no default states). This however does not imply here that equityholders must all be bondholders, since there is a second component of bonds'payoffs (in the default states). The condition for an optimum with respect to k then says that all equityholders have the same valuation for the marginal productivity of capital in the no default states and all bondholders have the same valuation for it in the default states.

4.2 Intermediated short-sales

We allow here agents to sell short the firm's equity.[25] A short position on equity is, both conceptually and in the practice of financial markets, different from a simple negative holding of equity. A short sale is not a simple sale, it is a loan contract with a promise to repay an amount equal to the future value of equity. In order to model short sales, therefore, we introduce financial intermediaries, who can issue claims corresponding to both short and long positions (more generally, derivatives[26]) on the firm's equity.

[25] We could allow for short sales of the bond as well, at only notational cost.

[26] We could also allow for intermediation of different derivatives of the firm's equity, again at only notational cost.

As before, equity shares trade in the market at $t = 0$ in state s at a price q, the outstanding amount of equity is normalized to 1. Intermediaries bear no cost to issue claims, but face the possibility of default on the short positions they issue (i.e., on the loans granted via the sale of such positions).[27] To protect themselves against the risk of default on the short positions issued, intermediaries have to hold an appropriate portfolio of claims (which acts then as a form of collateral) and may charge a different price for long and short positions.

We consider here for simplicity the case in which the default rate on such positions is exogenously given and equal to δ in every state; this is primarily for simplicity and at the end of the section we discuss how the analysis can be extended to situations where the default rate varies with the type i and portfolio held by an agent.The best hedge against default risk on short positions on equity is clearly equity itself; again for simplicity we focus then our attention here on the case where only equity is held to hedge consumers' default risk and postpone till later the discussion of the more general case where any other security can also be held as a hedge.

The self-financing constraint of the intermediary intermediating m units of the derivative on the firm's equity is then:

$$m \leq m(1-\delta) + \gamma \tag{22}$$

where m is the number of long (and short) positions issued and γ the amount of equity of the firm retained as collateral by the intermediary.

Let q^+ (resp. q^-) be the price at which long (resp. short) positions in the derivative issued by the intermediary are traded. The intermediary chooses the amount of long and short positions in the derivative intermediated, $m \in R_+$, and the amount of equity held as collateral, $\gamma \in R_+$, so as to maximize its total revenue at date 0 :

$$\max_{m,\gamma}(q^+ - q^-)m - q\gamma \tag{23}$$

subject to the self-financing constraint (22).

The intermediation technology is characterized by constant returns to scale. A solution to the intermediary's choice problem exists provided

$$q \geq \frac{q^+ - q^-}{\delta}$$

and is characterized by $\gamma = \delta m$ and $m > 0$ only if $q = \frac{q^+ - q^-}{\delta}$.

[27]Any other cost of intermediation, as long as it is proportional to the amount intermediated, would give us the same results.

In this set-up derivatives are thus 'backed' by equity in two ways: (i) the yield of each derivative is 'pegged' to the yield of equity of the firm;[28] (ii) to issue any short position in the derivative, the intermediary has to hold - as a collateral against the risk of his customers' default - an appropriate amount of equity of the same firm to whose return the derivative is pegged.

Let $\lambda_+^i \in R_+$ denote agent i's holdings of long positions in the derivative, and $\lambda_-^i \in R_+$ his holdings of short positions. The consumer's budget constraints in this set-up[29] are then as follows:

$$c_0^i = w_0^i + [-k + q + p\,B\,]\,\theta_0^i - q\,\theta^i - p\,b^i - q^+\lambda_+^i - q^-\lambda_-^i \tag{24}$$

$$c^i(s) = w^i(s) + [f(k;s) - B\,]\,(\theta^i + \lambda_+^i - \lambda_-^i(1-\delta)) + b^i, \qquad \forall s. \tag{25}$$

The consumer's choice problem consists in maximizing his expected utility subject to the above constraints and $\left(\theta^i, b^i, \lambda_+^i, \lambda_-^i\right) \geq 0$.

The asset market clearing conditions are now, for equity

$$\gamma + \sum_{i\in I} \theta^i = 1,$$

and for the derivative security

$$\sum_{i\in I} \lambda_+^i = \sum_{i\in I} \lambda_-^i = m.$$

The firm's choice problem is the same as in in the previous section, (19). The equity price map $q(k, B)$ has however to be properly adjusted to reflect the fact that intermediaries as well as consumers may now demand equity:

$q(k, B)$ equals the maximal valuation, at the margin, among consumers and intermediaries of the equity's cashflow when the firm's decisions are given by k, B:

$$\begin{aligned} q(k,B) \;\; &= \;\; \max\left\{\max_i \mathbb{E}MRS^i(s)\,[f(k;s) - B]\,, \right. \\ &\qquad \left. \frac{\max_i \mathbb{E}MRS^i(s)\,[f(k;s) - B] - \min_i \mathbb{E}MRS^i(s)\,[f(k;s) - B]}{\delta}\right\}. \end{aligned} \tag{26}$$

[28] The role of equity as a benchmark to which the return on derivatives can be pegged can be justified on the basis of the fact that asset returns cannot be written as a direct function of future states of nature.

[29] In the expression of the ate 1 budget constraint we take into account of the fact that the consumer will default on a fraction δ of his short positions (equivalently, that he defaults with probability δ).

Note that in the above expression the intermediaries' marginal valuation can be interpreted as the *value of intermediation*. It is determined by consumers' marginal valuation for the corresponding derivative claim. The firm might capture the value of intermediation at the margin because it is the availability of its equity, retained as collateral by the intermediary, which makes intermediated short sales possible.

A competitive equilibrium of the economy with short sales can be defined along the lines of Definition 1. Two possible situations can arise then in equilibrium:

1. $q = (q^+ - q^-)/\delta > q^+$, which is in turn equivalent to $q^+ > q^-/(1-\delta)$. In this case equity sells at a premium over the long positions on the derivative claim issued by the intermediary (because of its additional value as input in the intermediation technology). Thus all the amount of equity outstanding is purchased by the intermediary, who can bear the additional cost of equity thanks to the presence of a sufficiently high spread $q^+ - q^-$ between the cost of long and short positions on the derivative.

2. $q = q^+$. In this case there is a single price at which equity and long positions in the derivative can be traded. Consumers are then indifferent between buying long positions in equity and the derivative and some if not all the outstanding amount of equity is held by consumers. When consumers hold all the outstanding amount of equity, intermediaries are non active at equilibrium and the bid ask spread $q^+ - q^-$ is sufficiently low (in particular, it is less or equal than δq).

Note that in this set-up consumers face no upper bound on their short positions on the derivative on equity, but the presence of a bid ask spread still limits their hedging possibilities. In contrast, a specification where there is no cost to issue the derivative claim, but consumers face a constraint on the level of short sales of the derivative, that is they can short at most, say, $\bar{K}$ units of the derivative on equity, is substantially different (see also Remark 2 below).

Remark 1 *The analysis and results for the above model extend to the more general environment where the default rate of any individual can vary with the agent's type and his portfolio choice, that is* $\delta(i, (\theta^i, b^i, \lambda^i_+, \lambda^i_-))$, *provided both type and portfolio choice are observable and the price of short positions in the derivative are allowed to depend on both (that is, to be type specific and be nonlinear,* $q^-(i, (\theta^i, b^i, \lambda^i_+, \lambda^i_-))$*). We can think of the map* $\delta(i, (\theta^i, b^i, \lambda^i_+, \lambda^i_-))$ *as being endogenously determined in equilibrium as the result of the default choice of individuals, when they face, for instance, some penalty for defaulting (as in Dubey et al. (2005)) and default is chosen at the initial date.*

By a similar argument as in Section 2.4 we can again show that the First Welfare Theorem holds:

Proposition 5 *Competitive equilibria of the economy with short-sales are constrained Pareto efficient.*

The idea of the proof is analogous to the one of Proposition 2, and again relies on the fact that the model described above can be viewed as equivalent to a model where all markets, that is not only the markets for equity and the bond attached to any possible choice k, B of firms, but also the markets for the corresponding derivative, are open. For all levels k', B' different from the one chosen by firms in equilibrium, in equilib rium we must have

$$\begin{aligned} \text{buying price (for long positions)} \quad &: \quad \frac{\max_i \mathbb{E}MRS^i(s)\,[f(k';s) - B']}{\delta} \\ \text{selling price (for short positions)} \quad &: \quad \frac{\min_i \mathbb{E}MRS^i(s)\,[f(k';s) - B']}{\delta} \end{aligned}$$

and at these prices the market must clear with a zero level of trade (both for long and short positions). This follows by construction from the previous characterization, hence the efficiency result.

Remark 2 *In Theorem 5, p. 1062, of Allen-Gale (1991) it is shown that the competitive equilibria of an economy with finite, exogenous bounds $\bar{K}$ on short sales are constrained inefficient.*[30] *In Allen-Gale's set-up, long and short positions trade at the same price, i.e. the bid ask spread is zero, and firms cannot internalize the effect of their choices, at the margin, on the value of intermediation. The inefficiency result in Allen-Gale (1991) then follows from the fact that in equilibrium the expression of market value which firms maximize ignores the effect of their decisions on the value of the intermediated short sale positions taken by agents. In other words, a firm is restricted not to exploit the gains from trade arising from the demand for short positions in the firm's equity.*
In our economy, instead, equity is an input in the intermediation process which allows short sales positions to be traded in the market. Hence the firm takes into account the value of its equity not only for the consumers but also for the intermediaries

[30] Though firms' decisions in Allen-Gale (1991) concern primarily which securities to issue, their analysis could be easily reformulated in a set-up where firms have to choose their level of output and take financial decisions, as in this paper.

when making its production and financial decisions. The gains from trading due to intermediation are so exploited by firms.[31]

Remark 3 *In Example 2, p. 96-7, of Pesendorfer (1995) it is shown that the competitive equilibria of an economy in which financial intermediaries may introduce complementary innovations in the market could get stuck at an equilibrium in which no intermediary innovates, even though in terms of efficiency all innovations should be traded. The result in the example is related to others in the theory of equilibrium with differentiated goods; notably, Hart (1980) and Makowski (1980). In fact the inefficiency arising in this economy is conceptually similar to that of Allen-Gale (1991) just discussed: each intermediary is implicitly restricted not to trade with other intermediaries; or, equivalently, equilibrium prices for non-traded innovations are restricted not to include at the margin the value of intermediation. If instead prices for non-traded innovations in equilibrium were defined as the maximum between the consumers' and the intermediaries' marginal valuation, as in our analysis, equation (26), efficiency would be restored at equilibrium.*

5 Asymmetric information

We have shown that production and financing decisions of firms cannot be separated, along the lines of the Modigliani-Miller result, when markets are incomplete and short sales are intermediated. Nonetheless, we have shown, unanimity and constrained efficiency characterize competitive equilibria in these economies. In this section we will study economies in which an additional link between production and financing decisions is due to asymmetric information, e.g., between debt-holders, shareholders, and the firm's management (i.e., the agents who manage the firm choose its production plans).

In corporate finance these class of economies have been studied for decades now, at least since the work of Jensen-Meckling (1976). Most of this work is however in the

[31] Another way to understand the difference between the set-up in this paper and Allen-Gale (1991)'s is in terms of the notion of market completeness. In our set-up, the markets for the derivative claim corresponding to any production plan k, B are open and clear at the equilibrium prices. If no firms chooses a particular production plan k, B, the market for the associated derivative is cleared using a different price for buying and selling positions: a possibly large spread clears the market at no trade. This is not the case in Allen-Gale (1991). To have an equilibrium in their set-up, where long and short positions are restricted to trade at the same price $\bar{K}$ must equal 0 for the claims corresponding to values of k, B which are different from the ones chosen by firms. Effectively, then, these markets are closed and an inefficiency might arise.

context of partial equilibrium models of agency and contracts. In this paper we are instead interested in the interaction between the contracts and properties of the firms' capital structure addressing these agency problems and macroeconomic variables, like the endogenous determination of aggregate risk in the economy and its implications for asset pricing. We pursue therefore the analysis in general equilibrium.

But while general equilibrium theory has been extended to the study of economies with asymmetric information, from the seminal work of Prescott-Townsend (1984) to e.g., the more recent work of Dubey-Geanakoplos-Shubik (2005) and Bisin-Gottardi (1999, 2006), most of this work concerns asymmetric information on the consumption side; exceptions include Acharya-Bisin (2008), Magill-Quinzii (2002), Dreze-Minelli-Tirelli (2008), Zame (2007), Prescott-Townsend (2006). In this paper therefore we shall concentrate on the conceptual issues concerning competitive equilibria in economies with asymmetric information in production, from the objective function of the firm to the effects of its financial decisions and efficiency. These issues appear to be of great relevance as foundations for macroeconomic models.

In the interest of clarity we will study several simple asymmetric information economies, workhorses of agency and contract theory in corporate finance.[32] We shall see that the moral hazard/adverse selection distinction is not important for unanimity, but it is for efficiency.

5.1 Unobservable risk composition - moral hazard

Consider the following economy, an extension of the one described in Section 2. As previously, it is a two-period economy with the stochastic shock governed by a Markov structure with transition probability matrix Π. There is a single type of firm and I types of consumers.

Production takes place according to the function $f(k, \phi; s)$, where ϕ represents a technological choice, affecting the stochastic structure of the firm's future output. The set of admissible values of ϕ is denoted by Φ, assumed for simplicity to be a finite set. The level of ϕ is chosen simultaneously with those of k and B, at time $t = 0$, *before financial markets open*, but, unlike the choice of B and k it is not observed by bond-holders nor by shareholders in financial market at time 0. In this economy, therefore, the characteristics of a firm's management, i.e., of who makes the firm's production and technological decisions, matter. Because of this the management, the

[32] This is attested by the fact that Jean Tirole's book on *The Theory of Corporate Finance*, MIT Press 2006, surveys the whole corporate finance literature by developing different versions of these same economies.

agent who makes such decisions, is endogenously chosen in equilibrium by the firm's shareholders.

To illustrate our general set-up, consider the following:

Example 1 *The firm's technology takes the following form:*

$$f(k,\phi;s) = [a(s) + \phi\epsilon(s)]\, k^{\alpha},$$

where $\epsilon(s)$ is an additional risk component and $\phi \in \{0,1\}$, a finite set, is the loading of the firm's cash-flow on such risk component. In other words, the choice of ϕ captures the choice of the risk composition of the firm's cash flow. Note that $\epsilon(s)$ is aggregate risk, which affects the whole production sector of the economy, that is, all the - identical - firms in the economy.

An agent, if chosen as manager of a firm, will pick the level of ϕ of such firm so as to maximize his utility, since the choice of ϕ is not observable. The choice of ϕ affects the agent's utility both because the agent may hold a portfolio whose return is affected by ϕ but also because the agent may incur some disutility cost associated to different choices of ϕ (which can be interpreted as describing different 'inspection costs'). Let disutility costs be denoted $v^i(\phi)$. We will assume that the manager's portfolio is observable (in fact, without loss of generality, we assume that managers cannot trade their way out of the compensation package chosen by the shareholders).[33]

For simplicity, we continue to examine the case where the firm's equity and debt are the only assets in the economy. As in Section 4.1 we assume that the firm can issue risky debt. The problem of the (shareholders) of the firm is that of choosing the level of its physical capital k , its financial structure, represented by B (the amount outstanding of equity continues to be normalized to 1), as well as the type i of agent serving as its manager and his compensation package. The manager's compensation package consists of a net payment x_0, in units of the consumption good at date 0, together with a portfolio of θ^m units of equity and b^m units of bonds. Physical capital, financial structure, manager, and manager's compensation, are chosen by the firm (by its shareholders) so as to maximize the firm's market valuation, taking into account the manager's incentives (that is, the effect of the compensation on the manager's choice of ϕ).

The consumption side of the economy is as in Section 2: each consumer i is subject to endowment shocks w_0^i at date 0 and $w^i(s)$ at date 1 in state s and has an initial endowment of shares θ_0^i.

[33] See Acharya-Bisin (2008) and Bisin-Gottardi-Rampini (2008) for economies where much is made of the opposite assumption.

Each firm is still perfectly competitive and hence takes prices as given. The firm's cash flow, and hence the equity's dividend, in the presence of risky debt is $\max\{f(k,\phi;s)-B,0\}$. It depends on the firm's financing and production choices, that is, on B, k, and ϕ. As in the previous sections, price taking requires that the firm takes as given the market valuation of its future cash flow, that is the market value of equity, for any possible choice now of B, k and ϕ, described by a map $q(k,B,\phi)$. Similarly for the bonds, whose return is also risky in principle in this economy and given by $\min\{1, f(k,\phi;s)/B\}$ and the price map by $p(k,B,\phi)$. While ϕ is not observed by either bond-holders or shareholders in the financial markets at time 0, all agents in the economy can anticipate the manager's choice of ϕ given his/her incentives, that is, given his/her type i and his/her compensation package. In other words, agents anticipate that ϕ is determined by the appropriate incentive compatibility constraints.

Let $W^i(\phi,k,B;q,p)$ denote the total cost of the compensation package for a manager of type i, which induces him to choose ϕ when the firm's production and financial decisions are given by k, B and market prices are q, B. This cost is given by the sum of

- the payment made to this agent at date 0, x_0^i,
- plus the value of the portfolio $q(k,B,\phi)\left(\theta^{i,m}-\theta_0^{i,m}\right)+p(k,B,\phi)b^{i,m}$ attributed to him,
- minus the amount of the dividends due to this agent on account of his initial endowment $\theta_0^{i,m}$ of equity, $\theta_0^{i,m}\left[-k+p(k,B,\phi)B-W^i(\phi,k,B;q,p)\right]$.

After simplifying, the expression of $W^i(\phi,k,B;q,p)$ is equal to:

$$W^i(\phi,k,B;q,p)=\frac{\left\{x_0^i+q(k,B,\phi)\left(\theta^{i,m}-\theta_0^{i,m}\right)+p(k,B,\phi)b^{i,m}-\theta_0^{i,m}\left[p(k,B,\phi)B-k\right]\right\}}{1-\theta_0^{i,m}}$$

To analyse the firm's choice we proceed in two steps. We first state the firm's problem for a fixed manager's type i. It is then straightforward to formulate the firm's choice problem over the type of agent to be hired as manager.

The optimal choice problem of a firm who has a hired as manager a type i agent is then the following:

$$V^i=\max_{k,B,\phi,x_0^i,\theta^{i,m},b^{i,m}} q(k,B,\phi)+p(k,B,\phi)B-k-W^i(\phi,k,B;q,p) \qquad (27)$$

s.t.

$$\mathbb{E}u^i(w_0^i \ + x_0^i, w^i(s) + \max\{0, f(k,\phi;s) - B\}\theta^{i,m} \ + \min\left\{1, \tfrac{f(k,\phi;s)}{B}\right\} b^{i,m}) - v^i(\phi) \geq$$
$$\mathbb{E}u^i(w_0^i \ + x_0^i, w^i(s) + \max\{0, f(k,\phi';s) - B\}\theta^{i,m} \ + \min\left\{1, \tfrac{f(k,\phi';s)}{B}\right\} b^{i,m}) - v^i(\phi')$$
$$\text{for all } \phi' \in \Phi \tag{28}$$

$$\mathbb{E}u^i(w_0^i + x_0^i, w^i(s) + \max\{0, f(k,\phi;s) - B\}\theta^{i,m} + \min\left\{1, \frac{f(k,\phi;s)}{B}\right\} b^{i,m}) - v^i(\phi) \geq \bar{U}^i \tag{29}$$

The firm maximizes its value under constraints (28) and (29). The first is the incentive constraint of a type i manager, and the second is his/her participation constraint. The presence of the incentive constraint ensures that the firm internalizes the effect of its choices of k and B on ϕ. The reservation utility for a manager of type i, $\bar{U}^i$, is endogenously determined in equilibrium (see below).

The type $\bar{\imath} \in I$ of agent to be hired as manager is then chosen by selecting the type which maximizes the firm's value:

$$\max_{i \in I} V^i, \tag{30}$$

for V^i as determined in (27).

Each consumer of a given type j who is not hired as manager has then to choose, as in the previous sections, his portfolio of equity and bonds, θ^j and b^j, respectively, taking as given the price of bonds, p and the price of equity q, so as to maximize his utility.[34] The problem of agent i is then:

$$\max_{\theta^j, b^j, c^j} \mathbb{E}u^j(c_0^j, c^j(s)) \tag{31}$$

subject to

$$c_0^j \ = w_0^j \ + \left\{-k \ + q + pB - W^i(\phi, k, B; q, p)\right\} \theta_0^j - q \ \theta^j \ - p \ b^j \tag{32}$$

$$c^j(s) = w^j(s) + \max\{0, f(k,\phi;s) - B\}\theta^j \ + \min\left\{1, \frac{f(k,\phi;s)}{B}\right\} b^j, \qquad \forall s, \tag{33}$$

and

$$b^j \geq 0, \quad \theta^j \geq 0, \ \forall j \tag{34}$$

Let once again $\bar{\theta}^j, \bar{b}^j, \bar{c}^j$ denote the solutions of this problem. Let $\bar{U}^j$ the corresponding level of the agent's expected utility. It represents the endogenous reservation utility

[34] We maintain here the assumption that agents cannot short-sell the firm equity nor its debt. No conceptual difficulty is involved in allowing for intermediated short sales as in Section 4.2.

for a manager of type j. Let then $MRS^j(s)$ denote the marginal rate of substitution between consumption at date 0 and at date 1 in state s for consumer j evaluated at his optimal consumption choice $\bar{c}^j$.

In equilibrium, the bond and equity price maps faced by the firm must satisfy the following consistency conditions for all k, B, ϕ:

[p] $p(k, B, \phi) = \max_i \mathbb{E}MRS^i(s) \min\left\{1, \frac{f(k,\phi;s)}{B}\right\}$

[q] $q(k, B, \phi) = \max_i \mathbb{E}MRS^i(s) \max\{f(k, \phi; s) - B, 0\}$

These consistency conditions guarantee the following:

i) Investors correctly anticipate the payoff distribution of the risky bond and equity, given the observed levels of k and B and the manager's choice of the risk composition parameter ϕ given k, B and his compensation package. In particular, investors correctly anticipate that ϕ satisfies (28).

ii) The value of the bond and the equity price maps faced by each firm equal, for each k, B, ϕ, the highest marginal valuation across all consumers, evaluated at their equilibrium consumption choices, of the return on these assets.

Let $\bar{k}, \bar{B}, \bar{\phi}, \bar{\imath}, x_0^{\bar{\imath}}, \theta^{\bar{\imath},m}, b^{\bar{\imath},m}$ denote the solutions of the firms' problem (30) when $p(k, B, \phi)$ and $q(k, B, \phi)$ are as in [p], [q]. In equilibrium, the prices faced by consumers are $\bar{p} = p(\bar{k}, \bar{B}, \bar{\phi})$, $\bar{q} = q(\bar{k}, \bar{B}, \bar{\phi})$. In addition, the following market clearing conditions must hold[35]:

$$\sum_{i \neq \bar{\imath}} \bar{c}_0^i + \bar{k} + x_0^{\bar{\imath}} \leq \sum_{i \neq \bar{\imath}} w_0^i$$

$$\begin{aligned} \textstyle\sum_{i \neq \bar{\imath}} \bar{c}^i(s) + \max\{0, f(\bar{k}, \bar{\phi}; s) - \bar{B}\}\theta^{\bar{\imath},m} + \min\left\{1, \frac{f(\bar{k},\bar{\phi};s)}{\bar{B}}\right\} b^{\bar{\imath},m} \leq \\ \leq \textstyle\sum_{i \neq \bar{\imath}} w^i(s) + f(\bar{k}, \bar{\phi}; s) \text{ for all } s \end{aligned} \tag{35}$$

Summarizing,

[35] Recall that we have assumed for simplicity that the mass of agents of any given type i is equal to the mass of existing firms. This is obviously by no means essential.

Definition 3 *A competitive equilibrium of an economy with moral hazard is a collection*
$\left(\bar{k}, \bar{B}, \bar{\phi}, \bar{\imath}, x_0^{\bar{\imath}}, \theta^{\bar{\imath},m}, b^{\bar{\imath},m}, \{\bar{c}^i, \bar{\theta}^i, \bar{b}^i, \bar{U}^i\}_{i=1}^I, \bar{p}, \bar{q}, p(\cdot), q(\cdot)\right)$ *such that: i)* $\bar{k}, \bar{B}, \bar{\phi}, \bar{\imath}, x_0^{\bar{\imath}}, \theta^{\bar{\imath},m}, b^{\bar{\imath},m}$ *solve the firm problem (30) given* $p(\cdot), q(\cdot)$ *and* $\{\bar{U}^i\}_i$*; ii)* $p(\cdot), q(\cdot)$ *satisfy the consistency conditions [p] and [q], respectively, and* $\bar{p} = p(\bar{k}, \bar{B}, \bar{\phi})$, $\bar{q} = q(\bar{k}, \bar{B}, \bar{\phi})$*; iii) for all* i, $\bar{c}^i, \bar{\theta}^i, \bar{b}^i$ *solve consumer* i*'s problem (31) s.t. (32) and (33) for given* $\bar{p}, \bar{q}, \bar{k}, \bar{B}, \bar{\phi}$*, and* $\bar{U}^i = u(\bar{c}_0^i) + \beta \mathbb{E} u(\bar{c}^i(s))$*; iv) markets clear, (35).*

5.1.1 Unanimity and Welfare

In the economy with asymmetric information we described each firm chooses the production and financing plan which maximizes its value. The firm takes fully into account the effects that its production and financing plan as well as its choice of management and associated compensation package have on its value.

Consequently, by the same argument as the one developed in Section 2.2, shareholders' unanimity holds regarding the production and financing decisions of firms as well as the choice of management. That is, regarding the firms' observable decisions of k and B, as well as the choice of the manager and its compensation inducing the choice of ϕ.

Proposition 6 *At a competitive equilibrium of the economy with moral hazard, shareholders unanimously support the production and financial decisions of firms as well as the choice of management,* $\bar{k}, \bar{B}, \bar{\phi}, \bar{\imath}, x_0^{\bar{\imath}}, \theta^{\bar{\imath},m}, b^{\bar{\imath},m}$*; that is, every agent* i *holding a positive initial amount* θ_0^i *of equity of the representative firm will be made - weakly - worse off by any other admisible choice of a firm (that is, any* $k', B', \phi', i', x_0^{i'}, \theta^{i',m}, b^{i',m}$ *which satisfies (28) and (29)).*

We show next that all competitive equilibria of the economy described exhibit desirable welfare properties. Evidently, we cannot expect competitive equilibrium allocations to be fully Pareto efficient: first of all, the hedging possibilities available to consumers are limited by incomplete markets (equity and risky bonds are the only assets traded). Most importantly, the economy is characterized by the presence of moral hazard: the risk composition of the firms' cash-flow is chosen by the firms' managers, without being observed by the other agents (shareholders and bond-holders). Given these constraints, equilibrium allocatons are Pareto efficient, or constrained Pareto efficient in the sense of Diamond (1967) and Prescott-Townsend (1984).

More formally, a consumption allocation $(c^i)_{i=1}^I$ is *admissible* if:

1. It is *feasible*: there exists a production plan k and a risk composition choice ϕ of firms such that

$$\begin{aligned} \sum_i c_0^i \ + k \quad &\le \quad \sum_i w_0^i \\ \sum_i c^i(s) \quad &\le \quad \sum_i w^i(s) + f(k,\phi;s) \text{ for all } s \end{aligned} \tag{36}$$

2. It is *attainable with the existing asset structure*: that is there exists B and, for each consumer's type i, a pair θ^i, b^i such that

$$c^i(s) = w^i(s) + \max\{0, f(k,\phi;s) - B\}\theta^i \ + \min\left\{1, \frac{f(k,\phi;s)}{B}\right\} b^i, \qquad \forall s \tag{37}$$

3. It is *incentive compatible*: given the production plan k and the financing plan B, there exists $\bar{\imath}$ such that:

$$\begin{gathered} \mathbb{E}u^i(c_0^{\bar{\imath}}, w^{\bar{\imath}}(s) + \max\{0, f(k,\phi;s) - B\}\theta^{\bar{\imath}} \ + \min\left\{1, \tfrac{f(k,\phi;s)}{B}\right\} b^{\bar{\imath}}) - v^{\bar{\imath}}(\phi) \ge \\ \mathbb{E}u^i(c_0^{\bar{\imath}}, w^{\bar{\imath}}(s) + \max\{0, f(k,\phi';s) - B\}\theta^{\bar{\imath}} \ + \min\left\{1, \tfrac{f(k,\phi';s)}{B}\right\} b^{\bar{\imath}}) - v^{\bar{\imath}}(\phi') \quad \cdot \\ \text{for all } \phi' \in \Phi \end{gathered}$$

Constrained Pareto optimality is now straightforwardly defined as in Definition 2, with respect to the stronger notion of admissibility described above.

And the First Welfare theorem readily applies. It can be established by an argument essentially analogous to the one used to establish the Pareto efficiency of competitive equilibria in Arrow Debreu economies.

Proposition 7 *Competitive equilibria of the economy with moral hazard are constrained Pareto efficient.*

5.1.2 Equilibrium capital structure and risk

In equilibrium the financing plans of the firm are determined by the demand of investors and by the incentives of managers. As in the economy of Section 3, the investors' demand for bonds and equity gives the firm the incentive to leverage its position and finance production with bonds. As we noted, this implies a lower bound on the quantity of corporate bonds issued by firms in equilibrium. When the firms' debt is risky, since the return on equity is a nonlinear function of B, there will typically also be an upper bound on the optimal level of B. But in the economy with moral hazard we have described the capital structure of the firm, together with the portfolio composition of its manager, also plays a role in determining the unobservable choice of ϕ and hence the returns of the firm's bonds and equity. This fact is used in turn to align the manager's incentives with those of the firm's shareholders. For instance, a manager of a leveraged firm with a large amount of the firm's equity in his portfolio has the incentive to load the firm heavily on the risk represented by $\epsilon(s)$, that is, to choose a high ϕ. This is because in this economy debt is risky and shareholders effectively face mostly upside risk. Bond-holders will therefore pay a premium for corporate bonds of less leveraged firms, whose managers also hold a larger proportion of debt than equity.

Thus both the capital structure and the portfolio composition of its manager can be used to enhance his incentives and hence to increase a firm's value. As a consequence, Modigliani-Miller's irrelevance result does not hold in this economy. As well known in corporate finance, the presence of incentive issues due to moral hazard in the firm's decision problem further contributes to determine its capital structure.

5.1.3 Other moral hazard economies

The analysis of the previous section can be easily extended to other moral hazard economies. We briefly sketch here only two examples.

Unobservable risk composition Consider the case in which the risk composition ϕ is not an unobservable choice of the manager of a firm, but rather is private information of the agent who is hired as manager of the firm at time $t = 0$, before the level of the firm's capital k and financial structure B are chosen. In this case, the manager of each firm is again chosen by the firm's initial shareholders, together with his/her compensation package and the recommendation of a production and financing plan contingent on ϕ: k_ϕ, B_ϕ. Let again

$$W^i_\phi(k_\phi, B_\phi; q, p) = \frac{1}{1-\theta^{i,m}_0}\left\{\begin{array}{c} x^i_{0\phi} + q(k_\phi, B_\phi; \phi)\left(\theta^{i,m}_\phi - \theta^{i,m}_0\right) + p(k_\phi, B_\phi; \phi) b^{i,m}_\phi - \\ -\theta^{i,m}_0\left[p(k_\phi, B_\phi; \phi) B_\phi - k_\phi\right] \end{array}\right\}$$

denote the cost of the compensation package for a manager of type i when the risk composition of the firm is ϕ.

Formally, the firm's problem, given the type i of the agent hired as manager, is then the following:

$$V^i = \max_{\left(k_\phi, B_\phi, x^i_{0\phi}, \theta^{i,m}_\phi, b^{i,m}_\phi\right)_{\phi\in\Phi}} \sum_{\phi\in\Phi} \Pr(\phi)\left[q(k_\phi, B_\phi;\phi) + p(k_\phi, B_\phi;\phi)B_\phi - k_\phi - W^i_\phi(k_\phi, B_\phi; q, p)\right] \tag{38}$$

s.t.

$$\begin{gathered}\mathbb{E}u^i(w^i_0 + x^i_{0\phi}, w^i(s) + \max\{0, f(k_\phi,\phi;s) - B_\phi\}\theta^{i,m}_\phi + \min\left\{1, \tfrac{f(k_\phi,\phi;s)}{B_\phi}\right\} b^{i,m}_\phi) \geq \\ \mathbb{E}u^i(w^i_0 + x^i_{0\phi}, w^i(s) + \max\{0, f(k_{\phi'},\phi;s) - B_{\phi'}\}\theta^{i,m}_{\phi'} + \min\left\{1, \tfrac{f(k_{\phi'},\phi;s)}{B_{\phi'}}\right\} b^{i,m}_{\phi'}) \\ \text{for all } \phi \text{ and all } \phi' \neq \phi\end{gathered} \tag{39}$$

$$\sum_{\phi\in\Phi} \Pr(\phi)\mathbb{E}u^i(w^i_0 + x^i_{0\phi}, w^i(s) + \max\{0, f(k_\phi,\phi;s) - B_\phi\}\theta^{i,m}_\phi + \min\left\{1, \frac{f(k_\phi,\phi;s)}{B_\phi}\right\} b^{i,m}_\phi) \geq \bar{U}^i \tag{40}$$

that is, to the incentive compatibility and participation constraints. The problem of optimally choosing the type of agent to serve as firm's manager is then again given by (30). It is tedious but not difficult to show that competitive equilibria for this economy also satisfy unanimity and constrained efficiency.

Unobservable leverage Another interesting moral hazard economy is one in which the debt-holders of each firm do not observe the total amount of debt issued by the firm in the market, i.e., its leverage. The production function is given by $y = f(k;s)$, as in Section 2, and shareholders at $t = 0$ choose the production plan k and the financial structure B (with no separate role for the firm's management). Shareholder choose k and B to maximize firm value. But B is not observable to bond-holders. They correctly anticipate it, given k, however. Formally, the firm's problem becomes:

$$V = \max_{k,B} -k + q(k,B) + p(k,B)B$$

s.t.

$$q(k,B) + p(k,B)B \geq q(k,B) + p(k,B)B' \text{ for all } B' \neq B$$

It is clear that in this situation the firm's shareholders have an incentive to expand the amount of debt issued to increase the revenue from its sale. This in turn reduces the equilibrium price of debt, as bondholders correctly anticipate the dilution of the payoff of the bonds issued. In particular, in the absence of any limit to the amount of debt which can be issued, or of any cost of default, the only value of B which satisfies the above incentive constraint is $B = \infty$ (and $p = 0$). Thus the only possible outcome in such case is the extreme one where a large amount of debt is issued, the firm defaults in every state on its debt which becomes then equivalent to equity.

Competitive equilibria for this economy continue to satisfy unanimity and constrained efficiency properties.

5.2 Unobservable manager's quality - adverse selection

Consider next an environment where the technology of an arbitrary firm is still described by the production function $f(k, \phi; s)$, but ϕ represents now its manager's quality, which affects the stochastic structure of the firm's future output. Thus $\phi \in \Phi$ (where Φ is still a finite set) is no longer an unobservable choice of the firm's manager, nor a characteristic of the firm which can only be learned by its manager; it is a privately observed characteristic of each agent in the economy which describes the productivity of the agent if hired as manager of a firm. We also assume managers receive benefits from control ς_{ϕ}, in units of the consumption good, diverted from the firm's output at time 1.

The problem of the (shareholders) of the firm is again that of choosing the production plan k and the financial structure B, as well as the type of agent serving as manager, were the type is now given by an observable component i and a second, unobservable component, ϕ, together with the associated compensation package. The manager's compensation package consists of an amount x_0 of the consumption good at date 0, θ^m units of equity and b^m of bonds. Since agents know their type at the beginning of date 0, before they may be hired as managers, this economy is one of adverse selection.

Let χ^i_{ϕ} denote the mass of agents of type i and quality ϕ. To ensure that firms are never rationed in equilibrium in their demand of managers we need to appropriately redefine the size of the mass of firms in the economy and set it here at a level smaller than χ^i_{ϕ} for all i, ϕ. Furthermore, we assume that the firms' technology is such that some production and financing levels and a compensation package can always be found so as to separate managers of different unobservable types. This is guaranteed by the following (stricter than necessary) *single crossing property* assumption:

Assumption 1 *The firms' technology is such that, for any tuple* $v = (x_0, b, \theta, B, k) \in$

$\mathbb{R}^5_+$ *the vectors*

$$D_v \mathbb{E} u^i(w_0^i + x_0, w^i(s) + \varsigma_\phi + \max\{0, f(k,\phi;s) - \varsigma_\phi - B\}\theta + \min\left\{1, \frac{f(k,\phi;s)-\varsigma_\phi}{B}\right\} b),$$

$\phi \in \Phi$

are linearly independent.

Let

$$W^i(\phi, k, B; q, p,) = \left\{ x_0^i + \frac{q(k,B,\phi)\left(\theta^{i,m} - \theta_0^{i,m}\right) + p(k,B,\phi)b^{i,m}}{\left(1-\theta_0^{i,m}\right)} - \frac{\theta_0^{i,m}}{1-\theta_0^{i,m}}\left[p(k,B,\phi)B - k - x_0^i\right] \right\}$$

denote the cost of the compensation package for a manager of type i and quality ϕ. It follows that the value maximization problem of a firm who is hiring as manager an agent of type i and unobservable quality ϕ takes the following form:

$$V^i(\phi) = \max_{k,B,x_0^i,\theta^{i,m},b^{i,m}} q(k,B,\phi) + p(k,B,\phi)B - k - W^i(\phi,k,B;q,p)$$

s.t.

$$\bar{U}^i \geq \mathbb{E}u^i(w_0^i + x_0^i, w^i(s) + \varsigma_{\phi'} + \max\{0, f(k,\phi';s) - \varsigma_{\phi'} - B\}\theta^{i,m} + \min\left\{1, \tfrac{f(k,\phi';s)-\varsigma_{\phi'}}{B}\right\} b^{i,m}) \quad \text{for all } \phi' \neq \phi \tag{41}$$

and

$$\bar{U}^i \leq \mathbb{E}u^i(w_0^i + x_0^i, w^i(s) + \varsigma_\phi + \max\{0, f(k,\phi;s) - \varsigma_\phi - B\}\theta^{i,m} + \min\left\{1, \frac{f(k,\phi;s)-\varsigma_\phi}{B}\right\} b^{i,m}) \tag{42}$$

Constraint (41) is the incentive compatibility constraint, which requires that a firm choosing a manager of type i and quality ϕ will set a compensation package which agents of the same type i but different quality $\phi' \neq \phi$ will not choose. This is because their reservation utility $\bar{U}^i$, describing as before the utility they can get by not being a manager and trading in the existing markets, is higher. Constraint (42) is then the participation constraint, which requires instead that an agent of type i and quality ϕ indeed prefers being hired as manager and receiving the proposed compensation package than receiving his reservation utility, $\bar{U}^i$.

The single crossing property guarantees that, for any i, there always exist a compensation scheme such that constraints (41-42) are satisfied non-trivially: only agents of quality ϕ become managers and all agents of quality $\phi' \neq \phi$ prefer not to.

However firms may also choose a production and a financing plan k, B and a compensation package such that a non singleton subset $\Phi' \subseteq \Phi$ of quality types prefer being hired as managers. The specification of the program yielding the maximal value $V^i(\Phi')$ of the firm in this case is analogous to the one above.

If at equilibrium the optimal choice of the firm is to hire a single quality type $\bar{\phi}$ as manager, we call the equilibrium separating, following Rothschild-Stiglitz (1979). On the other hand, if the optimal choice is to hire a nonsingleton set $\Phi' \subseteq \Phi$ of quality types, we say the equilibrium is (partially) pooling, where agents of different quality become managers.

By a similar argument as in Bisin and Gottardi (2006), we can show that competitive equilibrium are necessarily separating and moreover that, differently from the economy with moral hazard, equilibrium allocations are not in general constrained Pareto efficient, in the sense of Diamond (1967) and Prescott-Townsend (1984). On the other hand, unanimity still holds in this environment.

6 Conclusion

In the presence of financial frictions, such as incomplete markets and/or borrowing restrictions and informational asymmetries between managers and shareholders or bondholders, production decisions are not necessarily separated from financing decisions. Corporate financing decisions, in these economies, are therefore interesting and one can investigate their interaction with the properties of the equilibrium allocation and prices. The conceptual problems usually associated with modeling firm decisions when markets are incomplete or with asymmetric information can be overcome with *appropriate,* and *natural* modeling choices.

GEI with production is thus ready to be passed on to macroeconomists.

7 References

Acharya, V. and A. Bisin (2008): Managerial Hedging, Equity Ownership, and Firm Value, forthcoming, *Rand Journal of Economics.*

Allen, F. and D. Gale (1988): Optimal Security Design, *Review of Financial Studies*, 1 229-63.

Allen, F. and D. Gale (1991): Arbitrage, Short Sales, and Financial Innovation, *Econometrica*, 59(4), 1041-68

Bisin, A. and P. Gottardi (1999): General Competitive Analysis with Asymmetric Information, *Journal of Economic Theory*, 87(1), 1-48.

Bisin, A. and P. Gottardi (2006): Efficient Competitive Equilibria with Adverse Selection, *Journal of Political Economy*, 114, 3, 2006.

Bisin, A., P. Gottardi and A. Rampini (2008): Managerial Hedging and Portfolio Monitoring, *Journal of the European Economic Association,* 6(1), 158-209.

Bonnisseau, J.M. and O. Lachiri (2002): Dreze's Criterion in a Multi-Period Economy With Stock Markets, Cahier de la MSE, Cermsem.

Choe, H., R. Masulis, and V. Nanda (1993): On the Timing of Seasoned Common Stock Issues: Theory and Evidence, *Journal of Empirical Finance*, 1(2), 3-32.

Covas, F. and W.J. Den Haan (2007): The Role of Debt and Equity Finance over the Business Cycle, CEPR Discussion Paper No. 6145.

Cres, H. and M. Tvede (2001): Proxy Fights in Incomplete Markets: When Majority Voting and Sidepayments are Equivalent, HEC Groupe.

DeMarzo, P.M. (1993): Majority Voting and Corporate Control: The Rule of the Dominant Shareholder, *Review of Economic Studies,* 60, 713-734.

Diamond, P. (1967): The role of a stock market in a general equilibrium model with technological uncertainty, *American Economic Review*,

Dierker, E., H. Dierker, and B. Grodal (2002): Nonexistence of Constrained Efficient Equilibria when Markets are Incomplete, , *Econometrica*, 70, 1245-51.

Dreze, J.H. (1974): Investment Under Private Ownership: Optimality, Equilibrium and Stability, in Dreze J.H. (ed.), *Allocation Under Uncertainty: Equilibrium and Optimality*, Macmillan, New York.

Dreze, J.H., O. Lachiri, and E. Minelli (2007): Shareholder-efficient production plans in a multi-period economy, Core Discussion Paper 2007/82.

Dreze, J.H., E. Minelli and M. Tirelli (2008): Production and Financial Policies under Asymmetric Information, forthcoming, *Economic Theory*.

Dubey, P., J. Geanakoplos, and M. Shubik (2005): Default and punishment in general equilibrium, *Econometrica*, 73(1), 1-37.

Duffie, D. and W.J. Shafer (1987): Equilibrium and the role of the firm in incomplete markets, mimeo, EUI.

Gertler, M. and S. Gilchrist, 1993: The role of credit market imperfections in the monetary transmission mechanism: argument s and evidence, *Scandinavian Journal of Economics*, 95, 4363.

Gertler, M. and S. Gilchrist, 1994: Monetary policy, business cycles, and the behavior of small manufacturing firms, *Quarterly Journal of Economics*, 109, 309340.

Grossman, S. and O. Hart, 1979: A Theory of Competitive Equilibrium in Stock Market Economies. *Econometrica*, vol. 47(2), pages 293-329.

Hart, O., 1979: On Shareholder Unanimity in Large Stock Market Economies. *Econometrica*, vol. 47(5), pages 1057-83.

Hart, O., 1980:

Jensen, M. and W. Meckling, 1976: Theory of the firm: managerial behavior, agency costs, and capital structure, *Journal of Financial Economics* 3, 305360.

Kashyap, A., Stein, J. and D. Wilcox, 1993: Monetary policy and credit conditions: evidence from the composition of external finance. *American Economic Review* 83 (1), 7897.

Kelsey, D. and F. Milne, 1996: The existence of equilibrium in incomplete markets and the objective function of the firm, *Journal of Mathematical Economics,* 25, 229-245.

Kihlstrom, R.E. and S.A. Matthews, 1990: Managerial incentives in an entrepreneurial stock market model, *Journal of Financial Intermediation,* 1, pp. 57–79.

Korajczyka, R.A. and Levy A., 2003: , Capital structure choice: macro economic conditions and financial constraints, *Journal of Financial Economics* 68, 75109.

Levy, A. and C. Hennessy, 2007: Why does capital structure choice vary with macroeconomic conditions?, *Journal of Monetary Economics* 54 (2007) 15451564.

Magill, M. and M. Quinzii (1999): Incentives and risk-sharing in a stock market equilibrium, in A. Alkan, D.D. Aliprantis and N.C. Yannelis (Eds), *Current Trends in Economics: Theory and Applications*, Springer Verlag, Berlin.

Magill, M. and M. Quinzii (2002): Capital market equilibrium with moral hazard, *Journal of Mathematical Economics,* 38, pp. 149–190.

Makowski, L., 1980:

Makowski, L., 1983: Competition and Unanimity Revisited, *American Economic Review*, Vol. 73, No. 3, pp. 329-339.

Makowski, and J. Ostroy, 1983:

Pesendorfer, W., 1995: Financial Innovation in a General Equilibrium Model. *Journal of Economic Theory*, vol. 65(1), pages 79-116.

Prescott, E. and R. Townsend, 1984: Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard, *Econometrica*, Vol. 52, No. 1, pp. 21-45.

Prescott, E.S. and R.M. Townsend (2006): Firms as clubs in a Walrasian market with private information, Journal of Political Economy, 114(4), 644-71.

Rothschild, M. and J. Stiglitz (1979):

Stiglitz, J.E., 1969: A re-examination of the Modigliani-Miller Theorem, *American Economic Review*, 59(5), 784-93.

Zame, W., 2007: Incentives, contracts, and markets: A general equilibrium theory of firms, *Econometrica*, 75(5), 1453-500.

8 Appendix

8.1 Proof of Proposition 3.

Note first that

$$q(k, B + dB) = \max_i \mathbb{E} MRS^i(s) \left[f(k; s) - B - dB \right].$$

Since for all $i \notin I^e$, $\mathbb{E}MRS^i(s)\,[f(k;s) - B] < q(k, B)$, the max in the above expression is attained for some $i \in I^e$ and hence

$$q(k, B + dB) = q(k, B) + \max_i \mathbb{E}MRS^i(s)\,[-dB]\,.$$

The right and left derivative of $q(k, B)$ with respect to B are then given by:

$$\frac{\partial q}{\partial B}_+ = -\min_{i \in I^e} \mathbb{E}MRS^i(s); \quad \frac{\partial q}{\partial B}_- = -\max_{i \in I^e} \mathbb{E}MRS^i(s). \tag{43}$$

and may differ. Similarly the derivatives with respect to k are:

$$\begin{aligned} \frac{\partial q}{\partial k}_+ &= \max_{i \in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] \\ \frac{\partial q}{\partial k}_- &= \min_{i \in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right], \end{aligned} \tag{44}$$

where f_k denotes the derivative of f with respect to k.

The first order conditions are then different according to whether the no default constraint (2) binds or not. Recalling that $\underline{s}$ denotes the lowest output state they are given by:

i. $f(k; \underline{s}) > B$ and

$$\begin{aligned} \frac{\partial V}{\partial B}_+ &= \frac{\partial q}{\partial B}_+ + p \leq 0, \;\; \frac{\partial V}{\partial k}_+ = \frac{\partial q}{\partial k}_+ - 1 \leq 0, \\ \frac{\partial V}{\partial B}_- &= \frac{\partial q}{\partial B}_- + p \geq 0, \;\; \frac{\partial V}{\partial k}_- = \frac{\partial q}{\partial k}_- - 1 \geq 0; \end{aligned} \tag{45}$$

Since from (43) we immediately see that $\frac{\partial q}{\partial B}_+ \geq \frac{\partial q}{\partial B}_-$, the above conditions are equivalent to:

$$\frac{\partial V}{\partial B}_+ = \frac{\partial q}{\partial B}_+ + p = \frac{\partial V}{\partial B}_- = \frac{\partial q}{\partial B}_- + p = 0,$$

that is:

$$\max_{i \in I^e} \mathbb{E}MRS^i(s) = \min_{i \in I^e} \mathbb{E}MRS^i(s) = p = \max_i \mathbb{E}MRS^i(s)$$

or (14) holds. Similarly, from (43) we immediately see that $\frac{\partial q}{\partial k}_+ \geq \frac{\partial q}{\partial k}_-$, so that the first order conditions with respect to k, (**??**) are equivalent to:

$$\frac{\partial q}{\partial k}_+ - 1 = \frac{\partial q}{\partial k}_- - 1 = 0,$$

that is,

$$\max_{i \in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] = \min_{i \in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right] = 1$$

or (15) holds.

ii. $f(k;\underline{s}) = B$ and
$$\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p \geq 0, \ \frac{\partial V}{\partial k_+} = \frac{\partial q}{\partial k_+} - 1 \leq 0. \tag{46}$$
This condition can be equivalently written as
$$p = \max_i \mathbb{E}MRS^i(s) \geq \max_{i \in I^e} \mathbb{E}MRS^i(s). \tag{47}$$
and
$$1 \geq \max_{i \in I^e} \mathbb{E}\left[MRS^i(s) f_k(s)\right]. \tag{48}$$
Note that (47) is always satisfied. In particular, it holds as equality when at least one equity holder is also a bond holder, or $I^e \cap I^d \neq \varnothing$, and as a strict inequality when no equity holder is also a bond holder, or all equityholders would like to short the riskless asset.
To verify whether a solution indeed obtains at $f(k;\underline{s}) = B$ when (48) holds, as argued above, we need to consider also the optimality with respect to joint changes in k and B,[36] or[37]:
$$\begin{aligned}\frac{\partial V}{\partial B_+}dB + \frac{\partial V}{\partial k_+}dk &= \left(\frac{\partial q}{\partial B_+} + p\right)dB + \left(\frac{\partial q}{\partial k_+} - 1\right)dk \leq 0 \text{ for } dB = f_k(\underline{s})dk > 0, \text{ and} \\ \frac{\partial V}{\partial B_-}dB + \frac{\partial V}{\partial k_-}dk &= \left(\frac{\partial q}{\partial B_-} + p\right)dB + \left(\frac{\partial q}{\partial k_-} - 1\right)dk \geq 0 \text{ for } dB = f_k(\underline{s})dk < 0.\end{aligned}$$

Using again (43),(44) to substitute for the derivatives of q wrt k and B into the above expression yields:
$$\left[\begin{array}{c} f_k(\underline{s})\left(-\min_{i \in I^e}\mathbb{E}MRS^i(s) + \max_i \mathbb{E}MRS^i(s)\right) + \\ \max_{i \in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right) - 1 \end{array}\right] \leq 0,$$
or
$$\begin{aligned} & f_k(\underline{s})\left(-\min_{i \in I^e}\mathbb{E}MRS^i(s) + \max_i \mathbb{E}MRS^i(s)\right) \\ \leq \ & 1 - \max_{i \in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right), \end{aligned} \tag{49}$$

where the term on the rhs is always nonnegative by (48) and the one on the lhs is obviously always nonnegative. Proceeding similarly with the second condition above, we get:
$$\left[\begin{array}{c} f_k(\underline{s})\left(-\max_{i \in I^e}\mathbb{E}MRS^i(s_1) + \max_i \mathbb{E}MRS^i(s)\right) \\ + \min_{i \in I^e} E_{s_0}\left(MRS^i(s_1)f_k(s)\right) - 1 \end{array}\right] \geq 0,$$

[36]This is obviously not necessary when the first order conditions wrt B are satisfied at an interior solution, that is when (14) and (15) hold.
[37]Without loss of generality, we can limit our attention to changes in B and k such that the no default constraint still binds, or $f_k(\underline{s})dk \geq dB$ holds as equality.

or

$$\begin{aligned} & 1-\min_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right) \qquad (50)\\ \leq\ & f_k(\underline{s})\left(-\max_{i\in I^e}\mathbb{E}MRS^i(s)+\max_i\mathbb{E}MRS^i(s)\right), \end{aligned}$$

and again both terms are non negative. Putting (49) and (50) yields

$$\begin{aligned} 1-\max_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right) &\geq \\ f_k(\underline{s})\begin{pmatrix} -\min_{i\in I^e}\mathbb{E}MRS^i(s)+ \\ \max_i\mathbb{E}MRS^i(s)\end{pmatrix} &\geq \\ f_k(\underline{s})\begin{pmatrix} -\max_{i\in I^e}\mathbb{E}MRS^i(s)+ \\ \max_i\mathbb{E}MRS^i(s)\end{pmatrix} &\geq \\ \geq 1-\min_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right), \end{aligned}$$

where the second inequality in the first line follows from the fact that

$$-\min_{i\in I^e}\mathbb{E}MRS^i(s)\geq-\max_{i\in I^e}\mathbb{E}MRS^i(s)$$

Since, by the same argument,

$$-\min_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right)\geq-\max_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right),$$

the condition can only hold as equality:

$$\begin{aligned} 1-\max_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right) &= \qquad (51)\\ f_k(\underline{s})\left(-\min_{i\in I^e}\mathbb{E}MRS^i(s)+\max_i\mathbb{E}MRS^i(s)\right) &= \\ f_k(\underline{s})\left(-\max_{i\in I^e}\mathbb{E}MRS^i(s)+\max_i\mathbb{E}MRS^i(s)\right) &= \\ = 1-\min_{i\in I^e}\mathbb{E}\left(MRS^i(s)f_k(s)\right) \end{aligned}$$

This implies that (16), (17), (18) hold, thus completing the proof.■

8.2 Risky Debt: Further Details

We study here in detail the economy of Section 4.1, where firms can default on their debt obligations, hence corporate debt is risky. Before stating the conditions for an optimum of the firms' decision problem in the presence of risky debt, it is useful to introduce some further notation. Given a face value of debt equal to B, let S^{nd} denote the collection of states in $t = 1$ for which $f(k; s) \geq B$ and by $\underline{s}^{nd}$ the lowest state in S^{nd}, that is the state with the lowest realization of the technology shock for which the firm does not default. Conversely, denote S^d the collection of states in $t = 1$ for which $f(k; s) < B$, i.e. the firm (partially) defaults on its debt.

Proposition A. 1 *The optimal investment and capital structure decision of a firm obtains either at an interior solution, where $f(k; \underline{s}^{nd}) > B$, with:*

$$
\begin{aligned}
& p = \min_{i \in I^d} \mathbb{E}(MRS^i(s) \left[\frac{f(k;s)}{B} \right] \Big| \, s \in S^d) \Pr\{s \in S^d\} + \\
& \left. \mathbb{E}(MRS^i(s) \left[\frac{f(k;s)}{B} \right] \Big| \, s \in S^d) \Pr\{s \in S^d\} \right\} = \\
& = \max_{i \in I^d} \mathbb{E}(MRS^i(s) \left[\frac{f(k;s)}{B} \right] \Big| \, s \in S^d) \Pr\{s \in S^d\} + \\
& \max_{i \in I^e} \mathbb{E} \left\{ MRS^i(s) \big| \, s \in S^{nd} \right\} \Pr\{s \in S^{nd}\}.
\end{aligned} \tag{52}
$$

and

$$
\begin{aligned}
& 1 = \max_{i \in I^e} \mathbb{E} \left\{ MRS^i(s) f_k(k, s) \big| \, s \in S^{nd} \right\} \Pr\{s \in S^{nd}\} + \\
& \max_{i \in I^d} \mathbb{E}(MRS^i(s) f_k(k; s) \big| \, s \in S^d) \Pr\{s \in S^d\} \\
= \; & \min_{i \in I^e} \mathbb{E} \left\{ MRS^i(s_1) f_k(k, s) \big| \, s \in S^{nd} \right\} \Pr\{s \in S^{nd}\} + \\
& \min_{i \in I^d} \mathbb{E}(MRS^i(s) f_k(k; s) \big| \, s \in S^d) \Pr\{s \in S^d\}
\end{aligned} \tag{53}
$$

or at a corner solution, $f(k; \underline{s}^{nd}) = B$.

The proof of this Proposition and the following Claim can be found in Appendix B, available at http://www.eui.eu/Personal/Gottardi/.

We now proceed to characterize the conditions for corner solutions.

Claim 1 *The conditions for an optimum at a corner,* $f(k;\underline{s}_1^{nd}) = B$, *are:*

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)\middle|\, s_1 \in S^{nd\prime}\right\} \Pr\{s_1 \in S^{nd\prime}\}+ \tag{54}$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(s_1)\left[\frac{f(k;s_1)}{B}\right]\Bigg|\, s_1 \in S^{d\prime}) \Pr\{s_1 \in S^{d\prime}\} \geq p \geq$$
$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}+$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(s_1)\left[\frac{f(k;s_1)}{B}\right]\Bigg|\, s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}$$

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)f_k(k,s_1)\middle|\, s_1 \in S^{nd\prime}\right\} \Pr\{s_1 \in S^{nd\prime}\}+ \tag{55}$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(s_1)\frac{f_k(k;s_1)}{B}\Bigg|\, s_1 \in S^{d\prime}) \Pr\{s_1 \in S^{d\prime}\} \geq 1 \geq$$
$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)f_k(k,s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}+$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(s_1)\frac{f_k(k;s_1)}{B}\Bigg|\, s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}$$

$$1-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)f_k(k,s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}- \tag{56}$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(s_1)\frac{f_k(k;s_1)}{B}\Bigg|\, s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\} =$$
$$\Bigg[-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}$$
$$-\min_{i\in I^d} E_{s_0}(MRS^i(s_1)\left[\frac{f(k;s_1)}{B}\right]\Bigg|\, s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\} + p\Bigg] f_k(\underline{s}_1^{nd}) =$$
$$\Bigg[-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}$$
$$-\max_{i\in I^d} E_{s_0}(MRS^i(s_1)\left[\frac{f(k;s_1)}{B}\right]\Bigg|\, s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\} + p\Bigg] f_k(\underline{s}_1^{nd}) =$$
$$1-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1)f_k(k,s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}$$
$$-\min_{i\in I^d} E_{s_0}(MRS^i(s_1)\frac{f_k(k;s_1)}{B}\Bigg|\, s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}$$

8.2.1 An example

We further illustrate the consequences of allowing for default and risky debt by means of an example. Consider an economy similar to the one of the example of Section

??, again with two types of consumers ($H = 2$) with the same preferences, initial stockholdings and technology as in that example. Only the stochastic structure of the endowment and productivity shocks are different:

	s_1	s_2	s_3
w^1	1	2	1
w^2	1	1	2
a	0.1053	1.2857	2

Thus there are now three possible states in each period, $\mathcal{S} = \{s_1, s_2, s_3\}$. We assume that at date 0 the state is also s_1, $w_0^i = w^i(s_1)$ for all i, and $\pi(s_1) = .02$, $\pi(s_2) = .28$ and $\pi(s_3) = .7$.[38]

We report in the following table the equilibrium values respectively for the case in which firms can only issue riskless debt and the one where they are allowed to default and issue risky debt:

	Riskless Debt	Risky Debt
k	0.084647	0.094879
B	0.016519	0.061534
θ^1	1	1
b^1	0	0
q	0.10551	0.10242
p	0.4453	0.39134
$-k+q+pB$	0.028216	0.031626
$pB/(-k+q+pB)$	0.26071	0.76141
U^1	-1.7539	-1.7565
U^2	-1.6034	-1.6002

Table

With only riskless debt the competitive equilibrium obtains at a corner solution, where $f(k; s_1) = B$.[39] At the values in the left column of the table, the conditions stated in Proposition ((16), (17) and (18)) are satisfied. In addition, the value of the firm, computed using the MRS's of the consumers at the candidate equilibrium allocation, indeed attains a maximum at $k = 0.084647$, $B = 0.016519$. All equity is held by type 1 consumers and debt by type 2.

[38] The correlations of the consumers' endowments with the firms' productivity shocks are in this case -0.7472 for 1 and 0.9140 for 2.

[39] In fact $(0.1053)\, k^{.75} - 0.016519 = 0$

When firms can issue risky debt, we find that in equilibrium firms choose a higher investment level ($k = 0.094879$) and a much higher debt level ($B = 0.061534$). Default occurs in state s_1 only[40]. Hence debt is a risky asset. The equilibrium obtains now at an interior solution, where $f(k; \underline{s}^{nd}) = 1.2857k^{\alpha} > B$ and conditions (52), (53) of the above Proposition 1 are satisfied. Agents' portfolio holdings are unchanged with respect to the case where there is only riskless debt, with type 2 consumers again holding all the debt issued by firms. The value of the firm increases ($0.028216 < 0.031626$) and so does the market value of the debt issued at $t = 0$ ($0.0074 < 0.0241$). As we see from the last two rows of Table 8.2.1 reporting the consumers' utility levels at the two equilibria, type 2 consumers gain ($-1.6034 < -1.6002$) while type 1 consumers loose ($-1.7539 > -1.7565$) when firms are allowed to default and issue risky debt.

For completeness, in Figure (3) we report the graph of the market value of a single firm for the different possible choices of k, B, computed using the equity and bond price maps constructed from the consumers' MRSs at the candidate equilibrium. Three different regions can be clearly distinguished in the figure, corresponding to different levels of debt and implied sets of default states: i) no default at all (for lower values of B); ii) default in state s_1 only (for intermediate values of B); iii) default in state s_1 and s_2 (for higher values of B). The value of the firm is maximized in the intermediate region where there is default in state s_1 only, with $k = 0.094879$ and $B \in [a(s_1)k^{\alpha} = 0.0180, a(s_2)k^{\alpha} = 0.2198]$[41]. Thus, in the situation under consideration the capital structure of all firms in the economy is exactly determined at $B = 0.061534$ whereas that of a single firm is only partially determined, with $B \in [0.0180, 0.2198]$.

[40] In fact $(0.1053)\,k^{.75} - 0.061534 = -0.0435$ while $(1.2857)\,k^{.75} - 0.061534 = 0.1583$.

[41] Recall that the solution is at the interior and thus conditions (52), (53) are satisfied. In particular, condition (52) imposes that equityholders (agent 1 here) and bondholders (agent 2 here) have the same valuation of the payoff of the bond in the no-default states. This implies that the value of the firm can be written as

$$-k + q + pB =$$

$$= -k + E_{s_0} MRS^1_{s \in S_{nd}} [f(s,k) - B] + E_{s_0} MRS^2_{s \in S_d} [f(s,k)/B] B + E_{s_0} MRS^2_{s \in S_{nd}} B =$$

$$= k + E_{s_0} MRS^1_{s \in S_{nd}} [f(s,k) - B] + E_{s_0} MRS^2_{s \in S_d} [f(s,k)/B] B + E_{s_0} MRS^1_{s \in S_{nd}} B =$$

$$= k + E_{s_0} MRS^1_{s \in S_{nd}} f(s,k) + E_{s_0} MRS^2_{s \in S_d} f(s,k) + E_{s_0} MRS^1_{s \in S_{nd}}$$

which is independent of B as long as the default regions (the set of states S_{nd} and S_d) are unchanged.

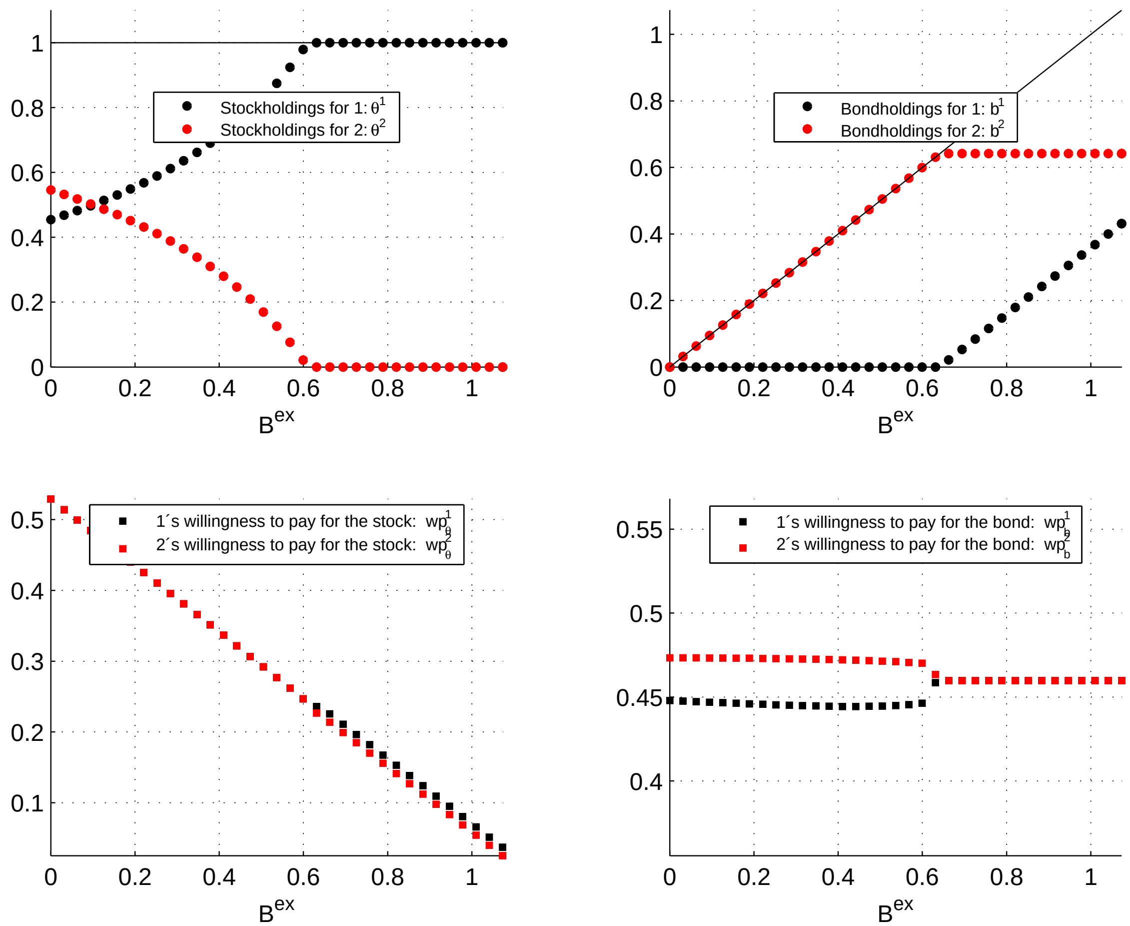

Figure 1: Parametric example with initial state $s_0 = \underline{s}_1$, "recession"and high persistence of the shock (0.8). $B^{ex} = [0, 1.0728]$. i) First row: asset holdings. ii) Second row: consumers' willingness to pay for the assets: for a type i consumer, this is given by $wp^i_\theta = E_{s_0} MRS^i(s_1) \left[a(s_1)k^\alpha - B^{ex}\right]$ for equity and $wp^i_b = E_{s_0} MRS^i(s_1)$ for bonds.

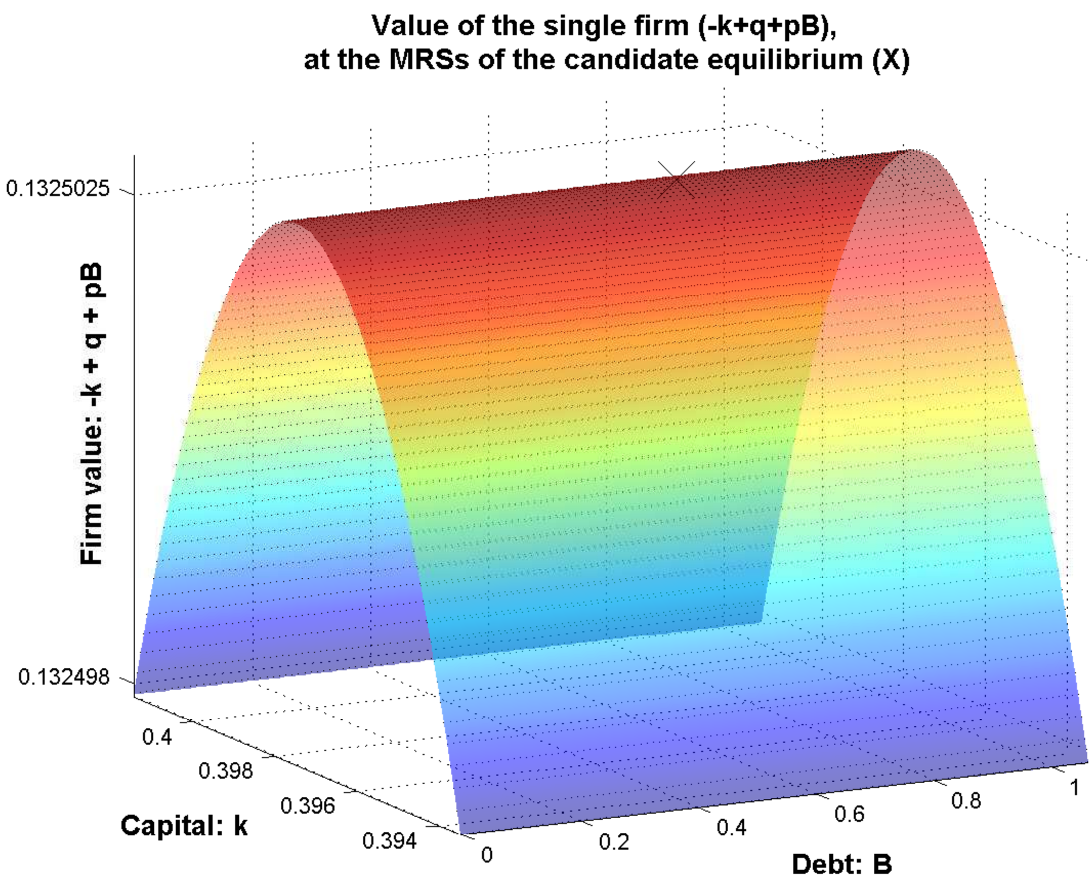

Figure 2: Value of the single firm at the MRSs of the candidate equilibrium (riskless debt).

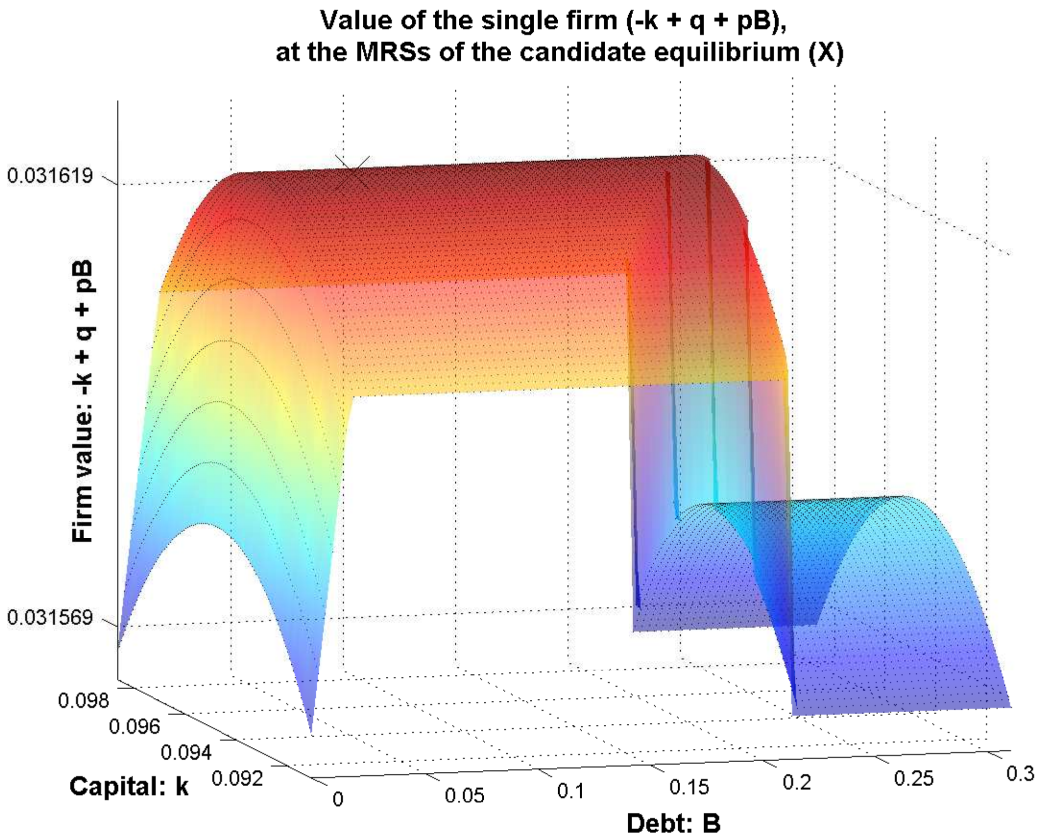

Figure 3: Value of the single firm at the MRSs of the candidate equilibrium (risky debt).