Eliminations of Dominated Strategies and Inessential Players: an Abstraction Process*†

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Abstract

We study the process of iterated elimination of (strictly) dominated strategies and inessential players from a finite strategic game (abbreviated as the IEDS process) A resulting (finite) sequence from this process is called a WIEDS, and the IEDS is the WIEDS where at each step of the process, all the dominated strategies and inessential players are eliminated. First, we show that any WIEDS preserves Nash equilibrium. The second result, an extension of the order-independence theorem, is that the IEDS sequence is the shortest and smallest WIEDS as well as that the resulting end game is the same. We have the third result about necessary and sufficient conditions for the possible sequences for the IEDS process. The conditions indicate a great variety of sequences possibly sustained as IEDS sequences for some games. We will interpret those results from the perspective of abstracting of social situations.

Key Words: Finite Strategic Form Games, Dominated Strategies, Inessential Players, Iterated Elimination, Order-Independence

1. Introduction

The notion of eliminations of dominated strategies is basic in game theory, and its relationships to other solution concepts such as rationalizability have been well discussed (cf., Osborne-Rubinstein [14], and Maschler, *et al.* [7]). Its nature, however, differs from other solution theories: It suggests negatively what would/should not be played, while

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other concepts suggest and predict what would/should be chosen in social/game situations. In this paper, we study eliminations of dominated strategies and of *inessential players* whose unilateral changes do not affect any players' payoffs including his owns. First, we describe this elimination process, and second consider its implications from the perspective of abstracting a social/game situation.

1.1. Elimination Process of Dominated Strategies and Inessential Players

We consider eliminations of (strictly) dominated strategies and inessential players in finite strategic form games, as well as their iterations. Eliminations of dominated strategies have a long history from Gale, *et al.* [3], but have studied extensively rather recently. We should mention two results in the literature, which are relevant to this paper.

One is the *preservation theorem* that Nash equilibrium is faithfully preserved in the elimination process, which is presented in Maschler, *et al.* [7], Theorem 4.35, p.109. The other result is more known in the literature that the elimination process results with the same endgame regardless of orders of eliminations of dominated strategies; this *order-independence theorem* was a kind of a folk theorem in the literature (some proofs are given in Gilboa *et al.* [4] and some others[1]), but now we find a comprehensive treatment of this theorem in Apt [1][2,3]. First, we will present generalizations of these two results in our framework allowing eliminations of inessential players, which will be given as Theorem 3.1 (preservation) and Theorem 5.1 (smallest and shortest).

Then, we will give another theorem on the IEDS process for the possible lengths and shapes in Theorem 6.1, which we call the *possible-shape theorem*.

Let us describe our framework and these three theorems. Eliminations of inessential players are newly introduced in this paper: A player is *inessential* iff his unilateral changes in strategies do not affect any players' payoffs including his own. We consider a possible (finite) sequence of games generated from the IEDS process. Such a sequence is called a *WIEDS* (*sequence*), where one game to the next game is connected by eliminations of dominated strategies and of inessential players; when we assume the eliminations of all dominated strategies and of all inessential players for each step, the WIEDS is called the *IEDS* (*sequence*). We are interested in the speed and shape of such a sequence, in addition to the resulting outcomes.

We have a few possible choices of orders of applications of those eliminations, but it will be shown in Lemma 2.3 that one order is more effective than the others. We take the order of eliminations of dominated strategies and then of inessential players.

[1] A full proof was given around 1990 by several people including T. Börgers and M. Stegman (see Apt [1] as well as Börgers [2]).

[2] Apt [1] treats iterated eliminations of various types of dominations, and can be read as a comprehensive survey of these subjects.

[3] It is well known that this order-independence theorem does not hold in the case of weak dominance, (cf., Myerson [12], p.60).

The preservation result (Theorem 3.1) is simply obtained for any WIEDS. Our smallest-shortest result (Theorem 5.1) states that from a given game, the IEDS is the shortest and smallest among all WIEDS's, and the resulting end games are the same. The last part is the order-independence theorem mentioned above. Since our process includes eliminations of inessential players, we need to develop various concepts to discuss the proof of this theorem.

The third result (Theorem 6.1) describes what shapes and lengths the IEDS sequences could have. First, we derive a certain set of necessary conditions for the IEDS sequence, which may appear insubstantial and far from sufficient conditions for the IEDS. However, when those conditions are given, we can construct a game so that the IEDS sequence from this game meet those conditions. For the 2-person case, these conditions give a specific information about the possible IEDS's, but for the case with more than 2 players, they do not give much restrictions. Hence, we have the implications that we have a large variety of IEDS (and WIEDS) sequences.

1.2. Choices of Relevant Actions/Players in an Abstraction Process

A social situation is a complex system containing a lot of seemingly relevant and/or irrelevant components. We, social scientists, focus on a target situation, by choosing relevant components and eliminating irrelevant ones. The standard economics textbooks start with this methodological view:

> "*... An economic model or theory is a simplified representation of how the economy, or parts of the economy, behave under particular conditions. In building a model, economists do not try to explain every detail of the real world. Rather, they focus on the most important influences on behavior because the real world is so complex*" (Thompson [15], p.10).

An analysis in game theory/economic theory, however, starts after such an abstraction is already done and given. Our study can be viewed as a trial of this abstraction process.

As described in Fig.1.1, our social world consists of many players, who are interdependent upon each other. Some interdependences are significant but many are insignificant. For example, the Battle of the Sexes is a 2-person game, while the corresponding social situation may include other boys and girls. When we have the Battle of the Sexes as an appropriate abstraction of the situation, we drop the other boys and girls. Let us consider a 3-person extension of the Battle of the Sexes.

Example 1.1 (Battle of the Sexes with the 2nd Boy). Consider the Battle of the Sexes situation including boy 1, girl 2, and another boy, 3. Let $\mathrm{BS}(1,2)$ be described as Table 1.1. Following Luce-Raiffa [13], boy 1 and girl 2 date at the boxing arena ($\mathbf{s}_{11} = \mathbf{s}_{21}$) or the cinema ($\mathbf{s}_{12} = \mathbf{s}_{22}$), but make decisions independently. Now, another boy, player 3, enters to this scene: Player 2 can play another BS game with player 3,

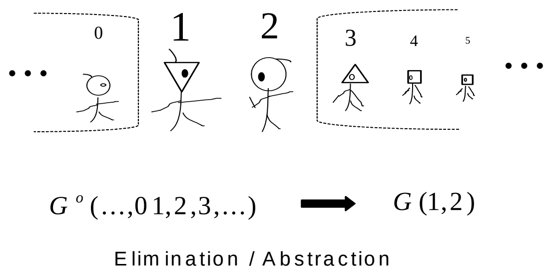

Figure 1.1: Relevant and Irrelevant People

dating in a different arena or cinema, which is described by Table 1.2. Here, player 2 has two more strategies. When 1 and 2 are considering their date, they would be happy even if they fail to meet, and player 3 does not enter to their minds at all. In this case, the choice by 3 does not affect the payoffs for 1 and 2. The numbers in the parentheses in Table 1.1 are 3's payoffs; we assume that his action has no effects on his payoffs. The dating situation for 3 and 2 is parallel to the situation for 1 and 2; only player 2 is much less happy than dating with player 1. Table 1.2; BS$(3,2)$ is also a Battle of the Sexes.

Now, player 2's two strategies $\mathbf{s}_{23}$ and $\mathbf{s}_{24}$ are dominated by $\mathbf{s}_{21}$ and $\mathbf{s}_{22}$. We eliminate those dominated strategies, and the resulting game is still a 3-person game. However, player 3 is inessential in the sense that 2 thinks only about dating with 1 and player 3's choice does not affect the players' payoffs at all. Thus, we can eliminate him as an inessential player, and we have the Battle of the Sexes BS$(1,2)$.

Table 1.1; BS$(1,2)$ game

1\2 (3)	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	15,10 (-10)	5,5 (-5)
$\mathbf{s}_{12}$	5,5 (-5)	10,15 (-10)

Table 1.2; BS$(3,2)$ game

3\2 (1)	$\mathbf{s}_{23}$	$\mathbf{s}_{24}$
$\mathbf{s}_{31}$	15,1 (-10)	0,0 (-5)
$\mathbf{s}_{32}$	0,0 (-5)	10,2 (-10)

Table 1.3; 3-person game

1, boy	$\overset{\text{BS(1,2)}}{\longleftrightarrow}$	2, girl	$\overset{\text{BS(3,2)}}{\longleftrightarrow}$	3, boy

Our study of the elimination process can be interpreted from the inside player's view as well as the outside scientist's view; the former interpretation requires a player's understanding of a situation, particularly, upon, experiential beliefs/knowledge on the situation, which can be understood from the viewpoint of inductive game theory (Kaneko-Matsui [6], Kaneko-Kline [5]). However, this requires a lot of restrictions for a player's experiences and understanding about the situation. In this paper, we will not consider such restrictions, but will take the outsider's perspective, i.e., will consider no restrictions on iterations of eliminations of dominated strategies and inessential players.

The *preservation* result (Theorem 3.1) has the implication that the eliminations of dominated strategies and inessential players do not affect the Nash equilibrium analysis after abstraction. The *smallest-shortest* result (Theorem 5.1) states that the iterated eliminations of the all the dominated strategies and then that of inessential players are the most efficient process. In Section 7, however, from the viewpoint of the number of preference comparisons, this is not necessarily true. The third result (Theorem 6.1) suggests that behind the abstracted game, there are vast possible social situations which go to the same abstracted game.

The paper is organized as follows: Section 2 gives basic definitions of dominance, an inessential player, and a few concepts of reductions of a game. Section 3 gives the preservation theorem. Section 4 defines the IEDS process, WIEDS and IEDS sequences. Section 5 gives and proves our version of the order-dependence theorem. Section 6 gives and proves the possible-shape theorem, Theorem 6.1. In Section 7, we will return to our original motivation stated above, and discuss the difficulties raised by our considerations from the viewpoint of the outside observer and that of an inside player.

2. Eliminations of Dominated Strategies and Inessential Players

We define three types of reductions of a game by eliminations of dominated strategies and inessential players, but we show that one type is more effective than the other two types. When we eliminate some (inessential) players, the player set shrinks; the domain of payoff functions for the remaining players change. Thus, we should be careful about the definition of a reduction.

Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a finite *strategic game*, where N is a finite set of *players*, S_i is a finite (nonempty) set of *strategies* and $h_i : \Pi_{j\in N} S_j \to \mathbb{R}$ is a *payoff function* for player $i \in N$. We allow N to be empty, in which case the game is the *empty game*, denoted as $G_\emptyset$. When we eliminate all the players in the IEDS process, the resulting game is the empty game $G_\emptyset$.

We use the following notation: Let I be a subset of N. Then, we may denote $s \in S_N := \Pi_{j\in N} S_j$ as $(s_I; s_{N-I})$, where $s_I = \{s_j\}_{j\in I}$ and $s_{N-I} = \{s_j\}_{j\in N-I}$. When $I = \{i\}$, we write S_{-i} for $S_{N-\{i\}}$ and $(s_i; s_{-i})$ for $(s_I; s_{N-I})$.

Let G be given, and $s_i, s_i' \in S_i$. We say that s_i' *dominates* s_i in G iff $h_i(s_i'; s_{-i}) > h_i(s_i; s_{-i})$ for all $s_{-i} \in S_{-i}$. When s_i is dominated by some s_i', we say simply that s_i is *dominated* in G.

We say that i *is* an *inessential player* iff for all $j \in N$,

$$h_j(s_i; s_{-i}) = h_j(s_i'; s_{-i}) \text{ for all } s_i, s_i' \in S_i \text{ and } s_{-i} \in S_{-i}. \tag{2.1}$$

A choice by i does not affect any players' payoffs including i's own, provided the others' strategies are arbitrarily fixed. We find a weaker version of this concept in Moulin [9]; he requires j to be i only. From the viewpoint of player i's own decision making, once i becomes inessential in this weak sense, he may stop thinking about his choice. However, his choice may still affect the others' payoffs; in this case, i's choice is still relevant to them. Some examples of inessential players will be discussed below[4].

As stated, we should be careful about the domains of the payoff functions when the player set changes. In fact, (2.1) can be extended to an arbitrary set of inessential players, stated in the following lemma, which guarantees to have the meaningful restrictions of payoff functions.

Lemma 2.1. Let I be a set of inessential players. Then, for all $j \in N$,

$$h_j(s_I; s_{N-I}) = h_j(s'_I; s_{N-I}) \text{ for all } s_I, s'_I \in S_I \text{ and } s_{N-I} \in S_{N-I}. \tag{2.2}$$

Proof. Let $I = \{i_1, ..., i_k\}$, and $I_t = \{i_1, ..., i_t\}$ for $t = 1, ..., k$. Also, let $s, s' \in S_N$ be arbitrarily fixed. We prove $h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t})$ by induction on $t = 1, ..., k$. Since $s, s' \in S_N$ are arbitrary, this for $t = k$ implies (2.2). The base case, i.e.., $h_j(s_{i_1}; s_{-i_1}) = h_j(s'_{i_1}; s_{-i_1})$, is obtained from (2.1). Suppose $h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t})$. Since $s = (s_{I_t}; s_{N-I_t}) = (s_{I_{t+1}}; s_{N-I_{t+1}})$, we have $h_j(s_{I_{t+1}}; s_{N-I_{t+1}}) = h_j(s_{I_t}; s_{N-I_t})$. Applying (2.1) to $h_j(s'_{I_t}; s_{N-I_t})$, we have $h_j(s'_{I_t}; s_{N-I_t}) = h_j(s'_{I_{t+1}}; s_{N-I_{t+1}})$. Now, we have, by the induction hypothesis, $h_j(s_{I_{t+1}}; s_{N-I_{t+1}}) = h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t})$ $= h_j(s'_{I_{t+1}}; s_{N-I_{t+1}})$. Thus, we have the assertion for $t + 1$. ■

Let I be a set of inessential players in G, $N' = N - I$, and let i be any player in N'. By Lemma 2.1, we can talk about the restriction of the payoff function $h_i : \Pi_{j \in N} S_j \to \mathbb{R}$ to the domain $\Pi_{j \in N'} S'_j$ with $\emptyset \neq S'_j \subseteq S_j$ for $j \in N'$. Then, the *restriction,* denoted by h'_i, of h_i to $\Pi_{j \in N'} S'_j$ is defined by

$$h'_i(s_{N'}) = h_i(s_{N'}; s_{N-N'}) \text{ for all } s_{N'} \in S'_{N'} \text{ and } s_{N-N'} \in S_{N-N'}. \tag{2.3}$$

The well-definedness of h'_i is guaranteed by Lemma 2.1.

We say that a game G' is a *D-reduction* of G iff the components of G' satisfy:

DR1: $N' \subseteq N$ and any $i \in N - N'$ is an inessential player in G;

DR2: for all $i \in N'$, $S'_i \subseteq S_i$ and any $s_i \in S_i - S'_i$ is a dominated strategy in G;

DR3: h'_i is the restriction of h_i to $\Pi_{j \in N'} S'_j$.

[4] Also, the concept of an inessential player may look related to the condition on the payoff functions, called the *transference of decision maker indifference*, due to Marx-Swinkels [8], p.5, which states that if two strategies s_i, s'_i for player i have unilaterally the same effects for s_{-i}, the others have the same effects. This is relative to two strategies s_i, s'_i, and s_{-i}, while an inessential player i has no effects on his and the others payoffs by his own unilateral changes.

A D-reduction allows simultaneous eliminations of dominated strategies and inessential players; both are relative to the initial game G. It would be clearer to separate between eliminations of dominated strategies and of inessential players.

First, we restrict a D-reduction as follows: Let G be a game, and G' a D-reduction of G. When $N' = N$ holds in DR1, G' is called a *ds-reduction* of G, denoted as $G \to_{ds} G'$, and furthermore, when $S_i - S_i' = \{s_i : s_i \text{ is a dominated strategy for } i \text{ in } G\}$ holds in DR2, it is the *strict* ds-reduction of G.

Returning to a D-reduction G' of G, when $S_i' = S_i$ for all $i \in N'$ in DR2, G' is called an *ip-reduction* of G, denoted by $G \to_{ip} G'$, and furthermore when $N - N' = \{i : i \text{ is an inessential player in } G\}$ holds in DR2, it is the *strict* ip-reduction of G.

In this paper, we choose the application order of a ds-reduction and then an ip-reduction. Hence, we have the following definition: We say that G' is a *DI-compound reduction* of G iff there is a game $\underline{G}$ such that $G \to_{ds} \underline{G}$ and $\underline{G} \to_{ip} G'$. We abbreviate a DI-compound reduction simply as a *DI-reduction*

$$\underbrace{G \to_{ds} \underline{G} \to_{ip} G'}_{DI\text{-reduction}} \qquad \underbrace{G \to_{ip} \underline{G} \to_{ds} G'}_{ID\text{-reduction}} \tag{2.4}$$

This still allows trivial cases, e.g., $G = \underline{G}$ or $\underline{G} = G'$. When $\underline{G}$ is the strict ds-reduction of G and G' is the strict ip-reduction of $\underline{G}$, we say that G' is the *strict* DI-reduction of G. In this case, we have $G \neq \underline{G}$ if G has some dominated strategies, and $\underline{G} \neq G'$ if $\underline{G}$ has some inessential players. The game $\underline{G}$ will be called an *interpolating game* of a DI-reduction.

We have another compound reduction: We say that G' is an *ID-reduction* of G iff $G \to_{ip} \underline{G} \to_{ds} G'$ for some $\underline{G}$. Lemma 2.3 will show the equivalence between D-reductions and ID-reductions.

The following lemma states that if a dominated strategy in G remains in a subgame G' of G, then it is still dominated in G'; and the parallel fact holds for an inessential player. The third assertion states that eliminations of only inessential players do not generate new dominated strategies. Example 2.2, however, shows that only eliminations of inessential players may generate new inessential players.

Lemma 2.2. Let $G' = (N', \{S_i'\}_{i \in N'}, \{h_i'\}_{i \in N'})$ be a D-reduction of G.

(1): If $s_i \in S_i'$ $(i \in N')$ is dominated in G, so is in G'.

(2): If $i \in N'$ is an inessential player in G, so is in G'.

(3): Suppose that $S_i' = S_i$ for all $i \in N'$. Let $s_i \in S_i$ and $i \in N'$. Then, a strategy s_i is dominated in G if and only if it is dominated in G'.

Proof. We prove only (1); and (2) is similarly proved. Suppose that s_i is dominated by s_i' in G. Then, $h_i(s_i'; s_{N-\{i\}}) > h_i(s_i; s_{N-\{i\}})$ for all $s_{N-\{i\}} \in S_{N-\{i\}}$. We can assume without loss of generality that s_i' is not a dominated strategy in G. Since G' is a subgame of G, we have, by (2.3), for all $s_{N-N'} \in S_{N-N'}$, $h_i'(s_i'; s_{N'-\{i\}}) = h_i(s_i'; s_{N'-\{i\}}; s_{N-N'})$ $> h_i(s_i; s_{N'-\{i\}}; s_{N-N'}) = h_i'(s_i; s_{N'-\{i\}})$ for all $s_{N'-\{i\}} \in S'_{N'-\{i\}}$. Thus, s_i is dominated by s_i' in G'.

(3): (1) is the *only-if* part. Now, consider the *if* part. Suppose that s_i is dominated by s_i' in G'. Then, $h_i'(s_i'; s'_{N'-\{i\}}) > h_i'(s_i; s'_{N'-\{i\}})$ for all $s'_{N'-\{i\}} \in S'_{N'-\{i\}}$. By the assumption, we have $S'_{N'-\{i\}} = S_{N'-\{i\}}$. Let $s'_{N'-\{i\}}$ be an arbitrary element in $S'_{N'-\{i\}} = S_{N'-\{i\}}$. Since G' is a subgame of G, we have, by (2.3), for all $s_{N-N'} \in S_{N-N'}$, $h_i(s_i'; s'_{N'-\{i\}}; s_{N-N'}) = h_i'(s_i'; s'_{N'-\{i\}}) > h_i'(s_i; s'_{N'-\{i\}}) = h_i(s_i; s'_{N'-\{i\}}; s_{N-N'})$ for all $s_{N'-\{i\}} \in S'_{N'-\{i\}}$. Thus, s_i is dominated by s_i' in G, too. ■

Although we decompose D-reductions into two parts and define compound reductions, the ID-reductions are equivalent to the D-reductions. However, the DI-reductions are more effective than the others. The converse of (2) does not hold.

Lemma 2.3.(1): G' is a D-reduction of G if and only if G' is an ID-reduction of G.

(2): If G' is a D-reduction of G, then G' is a DI-reduction of G.

Proof. (1):(Only-If): Let G' is a D-reduction of G. It follows from Lemma 2.2.(1) that we can postpone and separate eliminations of dominated strategies from eliminations inessential players. Hence, G' can be an ID-reduction.
(If): Let G' be an ID-reduction of G, i.e., $G \rightarrow_{ip} \underline{G} \rightarrow_{ds} G'$ for some $\underline{G}$. Lemma 2.4.(3) states that $\underline{G}$ has the same set of dominated strategies as G. Hence, we can combine these two reduction processes to one, which yields the D-reduction G'.

(2): Let G' be an ID-reduction of G. Then, we have an interpolating $\underline{G}$ such that $G \rightarrow_{ip} \underline{G} \rightarrow_{ds} G'$. The game $\underline{G}$ may differ from G only with their player sets, i.e., $\underline{N} \subseteq N$. By Lemma 2.2.(3), the dominated strategies in G are the same as those in $\underline{G}$. Then, a set of dominated strategies $\underline{D}$ in $\underline{G}$ are eliminated and G' results. Since D is a set of dominated strategies in G, we can eliminate them from G, and we have $\underline{H}$, i.e., $G \rightarrow_{ds} \underline{H}$. By Lemma 2.2.(2), the inessential players in G remain inessential. Hence, we eliminate $N - \underline{N}$ from N in $\underline{H}$. This game is the same as G' and $\underline{H} \rightarrow_{ip} G'$. Hence, G' is an DI-reduction. ■

This is illustrated in the following example.

Example 2.1 (Large and Small Stores). Consider the leftmost game of Figure 2.1, which is interpreted as follows: 1 is a large supermarket, 2 is a small mart; and 1 ignores 2. Here, neither player is inessential, but $\mathbf{s}_{12}$ is dominated. By eliminating $\mathbf{s}_{12}$, we have the second left table, where player 1 is inessential, and by eliminating him. Now, we have the third table, where $\mathbf{s}_{22}$ is dominated, and goes to the fourth table. Finally, 2 is eliminated, and we have the empty game $G_\emptyset$. The third table is a DI-reduction of the first, but not a D-reduction. Also, the last empty game is a DI-reduction of the third. The fourth is a D-reduction of the second.

$$\left(\begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22}\\ \hline \mathbf{s}_{11} & 30,1 & 30,0 \\ \hline \mathbf{s}_{12} & 20,0 & 20,1\\ \hline\end{array} \underset{ds}{\longrightarrow} \begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22}\\ \hline \mathbf{s}_{11} & 30,1 & 30,0\\ \hline \end{array}\right) \underset{ip}{\longrightarrow} \left(\begin{array}{|c|c|c|}\hline 2 & \mathbf{s}_{21} & \mathbf{s}_{22}\\ \hline & 1 & 0\\ \hline\end{array} \underset{ds}{\longrightarrow} \begin{array}{|c|c|}\hline 2 & \mathbf{s}_{21}\\ \hline & 1\\ \hline\end{array}\right) \underset{ip}{\longrightarrow} G_{\emptyset}$$

Fig.2.1

The fourth game is a ID-reduction of the second, and also is a D-reduction. This equivalence will be shown in Lemma 2.3.(1).

In the above examples, eliminations of dominated strategies generate new inessential players. However, eliminations of inessential players may generate new inessential players, too.

Example 2.2 (Elimination of Inessential Players, only). The leftmost 2-person game has no dominated strategies, but player 1 is inessential. By eliminating 1, we have the second 1-person game, and by eliminating 2, we have the empty game.

$$\begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22}\\ \hline \mathbf{s}_{11} & 4,6 & 2,6 \\ \hline \mathbf{s}_{12} & 4,6 & 2,6\\ \hline\end{array} \underset{ip}{\longrightarrow} \begin{array}{|c|c|c|}\hline 2 & \mathbf{s}_{21} & \mathbf{s}_{22}\\ \hline & 6 & 6\\ \hline\end{array} \underset{ip}{\longrightarrow} G_{\emptyset}.$$

Fig.2.2

3. Preservation of Nash Equilibria

As discussed in Section 1, we study eliminations of dominated strategies and inessential players as negative criteria to eliminate irrelevant players as well as irrelevant actions for some players. From the abstraction point of view, it could be required that such eliminations should loose no essences in the target social situation. Here, we show that this is the case with respect Nash equilibrium.

Let G be a finite nonempty game. We say that $s \in S$ is a *Nash equilibrium* in G iff for all $i \in N$, $h_i(s) \geq h_i(s'_i; s_{-i})$ for all $s'_i \in S_i$. Let θ be the *null symbol,* i.e., for any $s \in S$, we stipulate $(\theta; s) = s$ and that the restriction of s to the empty game $G_{\emptyset}$ is the null symbol θ itself. Also, we stipulate that the *null symbol* θ is the Nash equilibrium in $G_{\emptyset}$.

Then, we have the following basic theorem, stating that eliminations of dominated strategies and inessential players do not affect Nash equilibrium. In the case of eliminations only of dominated strategies, the theorem is reduced to one given in Maschler, *et al.* [7], Theorem 4.35, p.109[5].

[5]Exactly speaking, their theorem states that Nash equilibrium remains in iterated eliminations of dominated strategies. But this follows from Theorem 2.1.

Theorem 3.1 (Preservation of Nash Equilibria). Let G' be a D-reduction of G. Then

(1): if s_N is an NE in G, then its restriction $s_{N'}$ to G' is an NE in G';

(2): if $s_{N'}$ is an NE in G', $(s_{N'}; s_{N-N'})$ is an NE in G for any $s_{N-N'}$ in $\Pi_{j \in N-N'} S_j$.

Proof. **(1):** Let s be an NE in G. For any $i \in N$, we have $h_i(s_i; s_{-i}) \geq h_i(s'_i; s_{-i})$ for any $s'_i \in S_i$. Let $i \in N'$. Then, s_i is not dominated in G, and thus, $s_i \in S'_i$. Let $s'_i \in S'_i$. Since G' is a D-reduction, we have $h'_i(s_i; s_{N'-\{i\}}) = h_i(s_i; s_{N-i}) \geq h_i(s'_i; s_{-i}) = h'_i(s'_i; s_{N'-\{i\}})$. Thus, $s_{N'}$ is an NE in G'.

(2): Let $s_{N'}$ be an NE in G'. We choose any $s_{N-N'} \in S_{N-N}$. We let $G^o = (N, \{S'_i\}_{i \in N}, \{h_i\}_{i \in N})$, where $S'_j = S_j$ for all $j \in N - N'$. First, we show that this $(s_{N'}; s_{N-N'})$ is an NE in G^o.

Let $i \in N'$. We have $h'_i(s'_{N'}) = h_i(s'_{N'}; s_{N-N'})$ for any $s'_{N'} \in S'_{N'}$ by Lemma 2.1, since the players in $N - N'$ are inessential in G. Since $s_{N'}$ is a NE in G', we have $h_i(s_i; s_{N'-\{i\}}; s_{N-N'}) = h'_i(s_i; s_{N'-\{i\}}) \geq h'_i(s'_i; s_{N'-\{i\}}) = h_i(s'_i; s_{N'-\{i\}}; s_{N-N'})$ for all $s'_i \in S'_i$. Let $i \in N - N'$. Then since i is inessential, we have $h^o_i(s_i; s_{N'-\{i\}}; s_{N-N'}) = h^o_i(s'_i; s_{N'-\{i\}}; s_{N-N'})$ for all $s'_i \in S^o_i$. Hence, $(s_{N'}; s_{N-N'})$ is an NE in G^o.

Suppose that $i \in N'$ has a strategy s''_i in G so that $h_i(s''_i; s_{N-\{i\}}) > h_i(s_i; s_{N-\{i\}})$. We can choose such an s''_i giving the maximum $h_i(s''_i; s_{N-\{i\}})$. Then, this s''_i is not dominated in G. Hence, s''_i remains in G', which contradicts that $s_{N'}$ is an NE in G'. ∎

Let $NE(G)$ and $NE(G')$ be the sets of Nash equilibria for G and G'. It follows from Theorem 2.1 that the set of Nash equilibria for G is given as

$$NE(G) = \Pi_{j \in N-N'} S_j \times NE(G'). \tag{3.1}$$

Hence, if $NE(G')$ satisfies the interchangeability in the sense of Nash [11] that $NE(G')$ is expressed as the product of Nash strategies, then so does $NE(G)$, and *vice versa.* Thus, the above theorem implies not only the preservation of Nash equilibria, but also preservation of the Nash noncooperative theory (in [11]). We can discuss preservations of other solution concepts such as rationalizability.

When G' is the empty game $G_\emptyset$, Theorem 3.1.(1) states that the resulting outcome is the null symbol θ, and (2) states that any strategy profile $s = (\theta; s)$ in G is a Nash equilibrium in G. This will be used to make a comparison with d-solvability due to Moulin [9], [10].

It follows from this theorem that in iterations of reductions such as *ds*- and *ip*-reductions that the Nash equilibrium is preserved in both directions. Hence, as far as Nash equilibrium is concerned as a positive decision criterion, eliminations of dominated strategies and inessential players would be a right procedure of abstraction.

4. The IEDS Process and Generated Sequences

Now, we define a sequence generated by the IEDS process in terms of DI-reductions. There are the other two alternative manners to define this process using D-reductions and ID-reductions. However, it suffices, by Lemma 2.3, to consider the IEDS process in terms of DI-reductions.

Let G be a given finite game. We say that $\langle G^0, G^1, \ldots, G^\ell \rangle$ is a *WIEDS sequence from* $G = G^0$ iff

$$G^{t+1} \text{ is a } DI\text{-reduction of } G^t \text{ and } G^{t+1} \neq G^t \text{ for each } t = 0, \ldots, \ell - 1; \tag{4.1}$$

$$G^\ell \text{ has no dominated strategies and no inessential players.} \tag{4.2}$$

We call ℓ the *length* of $\Gamma(G)$. We abbreviate a WIEDS sequence simply as a WIEDS. We denote the set of all WIEDS's from G by $\mathbb{W}(G)$.

In particular, a WIEDS $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ is said to be the *IEDS sequence* iff

$$G^{t+1} \text{ is the strict } DI\text{-reduction of } G^t \text{ for all } t = 0, \ldots, \ell - 1. \tag{4.3}$$

A WIEDS may not be unique, but the IEDS is uniquely determined by a given G. The IEDS from G is denoted by $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$. We will show that this is the smallest and shortest WIEDS.

In Example 2.1, Fig.2.1 shows the unique WIEDS; *a fortiori,* it is the IEDS. It has the length 2 and is described as the sequence $\langle G^{*0}, G^{*1}, G^{*2} \rangle$, where G^{*1} is the third table and $G^{*2} = G_\emptyset$. The second table (the right one in the leftmost parentheses) is the interpolating game from G^{*0} to G^{*1}. The other interpolating game is the second game in the second parentheses.

In Example 2.2, the IEDS is given in Fig.2.2, which is the unique WIEDS. In this sequence, only an elimination of an inessential player occurs twice. Here, no non-trivial interpolating games occur.

Let us return to the 3-person $G = (\{1,2,3\}, \{S_i\}_{i=1}^3, \{h_i\}_{i=1}^3)$ of Example 1.1.

Example 1.1 (Continued). The IEDS is described as Fig.4.1. Player 2's strategies $\mathbf{s}_{23}$ and $\mathbf{s}_{24}$ are dominated by both $\mathbf{s}_{21}$ and $\mathbf{s}_{22}$, and by eliminating $\mathbf{s}_{23}$ and $\mathbf{s}_{24}$, we have the second interpolating 3-person game. Now, players 1 and 2 concentrate on their dating, ignoring player 3 as inessential. By eliminating him, we have the 2-person battle of the Sexes BS$(1,2)$.

$$\left(G^0 \underset{ds}{\rightarrow} \begin{array}{|c|c|c|}\hline 1\backslash 2\backslash 3 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline \mathbf{s}_{11} & 15,10,-10 & 5,5,-5 \\ \hline \mathbf{s}_{12} & 5,5,-5 & 10,15,-10 \\ \hline \end{array} \right) \underset{ip}{\rightarrow} \begin{array}{|c|c|c|}\hline 1\backslash 2 & \mathbf{s}_{21} & \mathbf{s}_{22} \\ \hline \mathbf{s}_{11} & 15,10 & 5,5 \\ \hline \mathbf{s}_{12} & 5,5 & 10,15 \\ \hline \end{array}$$

$\mathbf{s}_{31}$, or $\mathbf{s}_{32}$

Fig.4.1

A WIEDS sequence may be partitioned into two segments, $G^0, G^1, \ldots, G^{m_o}$ and $G^{m_o+1}, \ldots, G^{\ell}$ so that in the first segment, dominated strategies (and, maybe, inessential players) are eliminated, and in the second, only inessential players are eliminated, which is illustrated in (4.4).

$$\Gamma(G) = \langle \underbrace{G^0, G^1, \ldots, G^{m_o}}, \ \underbrace{G^{m_o+1}, ..., G^{\ell}} \rangle. \tag{4.4}$$

However, we need one small restriction for this observation. We say that $\Gamma(G) = \langle G^0, G^t, \ldots, G^{\ell} \rangle$ is *proper* iff if G^t has some dominated strategies, then $G^{t+1} \neq G'$; and if G' has some inessential players, then $G' \neq G''$. The IEDS $\Gamma^*(G)$ is proper.

We have the following theorem for any proper WIEDS.

Theorem 4.1 (Partition of a Proper WIEDS). Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^{\ell} \rangle$ be a proper WIEDS from $G^0 = G$. There is an m_o $(0 \leq m_o \leq \ell)$ such that

(i) for any $t \leq m_o$, at least one dominated strategy is eliminated in the step from G^{t-1} to G^t;

(ii) for any $t > m_o$, no dominated strategies are eliminated but at least one inessential player is eliminated in the step from G^{t-1} to G^t.

Proof. Suppose that G^t has no dominated strategies. Then, G^{t+1} is obtained from G^t by eliminating inessential players. It follows from Lemma 2.2.(3) that G^{t+1} has no dominated strategies. Hence, we choose the smallest m_o among such t's for m_o. ■

We call m_o given by Theorem 4.1 the *elimination divide*. Example 2.2, where $m_o = 0$, implies that the second segment may have a length greater than 1.

Let us apply Theorem 3.1 to a WIEDS $\Gamma(G) = \langle G^0, G^1, \ldots, G^{\ell} \rangle$.

Theorem 4.2 (Recovering Nash Equilibria from the Endgame). Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^{\ell} \rangle$ be a WIEDS from $G^0 = G$. Let $\underline{G}^t = (N^t, \{\underline{S}_i^t\}_{i \in N^t}, \{\underline{h}_i^t\}_{i \in N^t})$ be the interpolating game between G^t to G^{t+1} for $t = 0, ..., \ell - 1$. Then, $NE(G^0)$ is given as

$$NE(G^0) = \Pi_{j \in N^0 - N^1} \underline{S}_j^0 \times \cdots \times \Pi_{j \in N^{\ell-1} - N^{\ell}} \underline{S}_j^{\ell-1} \times NE(G^{\ell}). \tag{4.5}$$

Proof. It follows from (3.1) that for each $t = 0, ...\ell - 1$,

$$NE(G^t) = NE(\underline{G}^t) \ \text{ and } \ NE(\underline{G}^t) = \Pi_{j \in N^t - -N^{t+1}} \underline{S}_j^t \times NE(G^{t+1}).$$

Repeating this decomposition from $\ell - 1$, we have (4.5). ■

5. The Shortest and Smallest: the IEDS Sequence

It is the order-independence theorem that restricting the reduction steps to eliminations of dominated strategies, any WIEDS sequence has the same endgame (cf., Gilboa *et al.*

[4], and Apt [1]). Here, we extend this result, allowing eliminations of inessential players too, that for any given finite game G, the IEDS from G is the shortest and smallest among the WIEDS's with the same endgame. In Section 5.1, we formulate the theorem, and in Section 5, we prove it.

5.1. The IEDS Sequence

To make comparisons between two finite games, we introduce the concept of a subgame: We say that $G' = (N', \{S'_i\}_{i \in N'}, \{h'_i\}_{i \in N'})$ is a *subgame* of $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ iff *(i)* $N' \subseteq N$; *(ii)* $S'_i \subseteq S_i$ for all $i \in N'$; and *(iii)* for $i \in N'$, $h'_i : \Pi_{j \in N'} S'_j \to \mathbb{R}$ is given by (2.3). For the purpose of references, we state the following immediate result.

Lemma 5.1. If G' is a D-reduction of G, then G' is a subgame of G.

The D-reduction relation may not be transitive, while we drop the reasons for a reduction for a subgame. This allows the subgame relation to be partial ordering, which enables us to make meaningful comparisons between games.

Lemma 5.2 (Partial Ordering). The subgame relation is a partial ordering over the set of all finite games.

Proof. The subgame relation is reflexive, anti-symmetric, and transitive. Consider transitivity. Let G', G'' be subgames of G, G', respectively. It suffices to show that $h''_i(s_{N''}) = h_i(s_{N''}; s_{N-N''})$ for all $s_{N''} \in S''_{N'}$ and $s_{N-N''} \in S_{N-N''}$. Let s_N be an arbitrary strategy profile in S_N. By the supposition, we have $h''_i(s_{N''}) = h'_i(s_{N''}; s_{N'-N''}) = h'_i(s_{N'}) = h_i(s_{N'}; s_{N-N'}) = h_i(s_{N''}; s_{N-N''})$. The first and third equalities are due to (2.3) for G' and G'' and for G and G'. The second and fourth are simply changes of expressions. ■

By those lemmas, we have the following.

Lemma 5.3 (Monotonicity). Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ be a WIEDS sequence. For any $t, k = 0, ..., \ell$, if $t < k$, then G^k is a subgame of G^t.

The following theorem states that the IEDS sequence is the smallest and shortest in $\mathbb{W}(G)$ as well as its endgame is the same as that of any WIEDS sequence. The theorem will be proved in Section 5.2.

Theorem 5.1 (the IEDS; Shortest and Smallest). Let G be a finite game, and let $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$ be the IEDS from G. Then, for any WIEDS $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle \in \mathbb{W}(G)$,

(1): $\ell^* \leq \ell$;

(2): for each $t \leq \ell^*$, G^{*t} is a subgame of G^t;

(3): $G^{\ell^*} = G^\ell$.

Thus, the IEDS $\Gamma^*(G)$ is the shortest and smallest in $\mathbb{W}(G)$. In this sense, the IEDS sequence is the benchmark case. Furthermore, since a D-reduction and an ID-reduction are DI-reductions by Lemma 2.3, Theorem 5.1 covers all the sequences defined by D-reductions or ID-reductions. We should still be careful about a DI-reduction, which is a compound reduction consisting of a ds-reduction and a ip-reduction. Since these reductions are different, it would be natural to reach each DI-reduction as one step.

Theorem 5.1 would hold without eliminations of inessential players: The order-independence theorem is obtained by focussing on a WIEDS with an elimination of one strategy at a time and showing that any WIEDS has the same endgame with such a sequence. See Apt [1] for comprehensive discussions on this as well as order-independence theorems for various types of dominance relations. The "shortest" part follows this theorem.

It follows from Theorem 4.2 that if G^{ℓ^*} has a Nash equilibrium, then so does G. If G^{ℓ^*} is the empty game, which has the Nash equilibrium θ, then G has a Nash equilibrium. It follows Theorem 5.1 that these do not depend upon a choice of a WIEDS from G. Hence, it is a consequence that the decomposition given in (4.5) is independent upon the choice of a WIEDS.

This is related to Moulin's [9], [10] d-solvability: We say that a game G is *d-solvable* iff there is a sequence $\langle G^0, ..., G^\ell\rangle$ with $G^0 = G$, $G^{t-1} \to_{ds} G^t$ for $t = 1, ..., \ell - 1$, and for all $i \in N$,

$$h_i^\ell(s_i; s_{-i}) = h_i^\ell(s_i'; s_{-i}) \text{ for all } s_i, s_i' \in S_i \text{ and } s_{-i} \in S_{-i}. \tag{5.1}$$

As stated, this requires the constant payoffs for each player i with his unilateral deviation, while (2.1) requires for all the players' payoffs. Now, we have the following corollary.

Corollary 5.2. If a game G has a WIEDS $\Gamma(G) = \langle G^0, G^1, \dots, G^\ell\rangle$ with $G^\ell = G_\emptyset$, then G is d-solvable.

Proof. By Theorem 5.1, we can assume that it is the IEDS $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \dots, G^{*\ell^*}\rangle$ with $G^{\ell^*} = G_\emptyset$. Then, each G^{*t} is obtained from $G^{*(t-1)}$ so that $G^{*(t-1)} \to_{ds} \underline{G}^{(t-1)} \to_{ip} G^{*t}$ for some interpolating game $G'^{(t-1)}$. After the elimination divide m_0, only $\to_{ip}$ is applied to G^{*t}. Therefore, for the definition of a sequence for d-solvability, we consider $\langle G^{*0}, G^{*1}, \dots, G^{*\ell^*}\rangle$ up to m_0.

Then, we define a sequence $\langle H^0, H^1, \dots, H^{m_0}\rangle$ as follows: (1) $H^0 = G^{*0} = G$; and (2) for $t = 1, ..., m_0$, H^t is obtained from H^{t-1} by eliminating all the dominated strategies that are eliminated in $G^{*(t-1)} \to_{ds} \underline{G}^{(t-1)}$. Then, each H^t has the full set of players N, but $N - N^{*t}$ is a set of inessential players, and the payoff functions for the player in H^t are the same as those for G^{*t} with the domains of different dimension.

Since $G^{*\ell^*} = G_\emptyset$, G^{*m_0} has only inessential players if $m_0 < \ell$. Hence, in H^{m_0}, all the players in N are inessential, *a fortiori,* (5.1) holds. ■

The converse of Corollary 5.2 may not hold: Table 5.1, given in Moulin [10], is d-solvable, but this does not generate a WIEDS to the empty game.

Table 5.1

$1\backslash 2$	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	1,1	0,1
$\mathbf{s}_{12}$	1,0	0,0

5.2. Proof of Theorem 5.1

Now, let $\Gamma(G) = \langle G^0, \ldots, G^\ell \rangle$ be any WIEDS from G, and $\Gamma^*(G) = \langle G^{*0}, \ldots, G^{*\ell^*} \rangle$ the IEDS form G. We take two steps of the entire proof.

Lemma 5.4. G^ℓ is a subgame of $G^{*\ell^*}$.

Proof. For any $k \le \ell^* - 1$, we let $G^{*k} \to_{ds} \underline{G}^k \to_{ip} G^{*(k+1)}$. Let D^{*k} be the set of all dominated strategies in G^{*k}, and I^{*k} the set of all inessential players in $\underline{G}^k$. We prove by induction that for each $k = 0, ..., \ell^* - 1$, all strategies in D^{*k} and all inessential players in I^{*k} are eliminated before G^ℓ in $\Gamma(G) = \langle G^0, \ldots, G^\ell \rangle$. Equivalently, the remaining components of G^ℓ are still in $G^{*\ell^*}$, which implies that G^ℓ is a subgame of $G^{*\ell^*}$.

Let $k = 0$. All strategies in D^{*0} are dominated in $G^0 = G$. Therefore, it follows from Lemma 2.2.(1) that any of them remains dominated in G^t if it is a strategy in G^t. Hence, all strategies in D^{*0} are eventually eliminated in $\Gamma(G) = \langle G^0, \ldots, G^\ell \rangle$ since G^ℓ has no dominated strategies by (4.2). Also, all the players in I^{*0} are also eliminated eventually. We let G^{t_0} be the first game where D^{*0} and I^{*0} are all eliminated. Then, G^{t_0} is a subgame of G^{*1}

Now, we make the induction hypothesis that G^{t_k} is the first game where $D^{*0}, ..., D^{*k}$ and $I^{*0}, ..., I^{*k}$ are all eliminated. This G^{t_k} is a subgame of $G^{*(k+1)}$. If $D^{*(k+1)}$ and $I^{*(k+1)}$ are already all eliminated in G^{t_k}, then we let $t_{k+1} = t_k$. Suppose that some in $D^{*(k+1)}$ or $I^{*(k+1)}$ still remain in G^{t_k}. Hence, strategies in $D^{*(k+1)}$ are dominated in G^{t_k} if they remain in G^{t_k}; and players in $I^{*(k+1)}$ are inessential in G^{t_k} if they remain. Hence, we can find the first $G^{t_{k+1}}$ so that $D^{*(k+1)}$ and $I^{*(k+1)}$ are all eliminated. ■

The next step for the proof of Theorem 5.1 is to show that $\Gamma^*(G)$ is the shortest and smallest in the senses of (1) and (2) of the theorem. In Particular, transforming $\Gamma(G)$ by eliminating dominated strategies and inessential players, step by step, we construct a sequence of WIEDS's $\Theta^0(G) = \Gamma(G), ..., \Theta^{\ell^*}(G) = \Gamma^*(G)$ showing the relationships (1) and (2) between $\Gamma(G)$ and $\Gamma^*(G)$.

For such transformations, we need to consider the following lemma.

Lemma 5.5. Let $G \to_{ds} \underline{G} \to_{ip} G'$, where the strategies in D' are eliminated in $G \to_{ds} \underline{G}$ and the players in I' are eliminated in $\underline{G} \to_{ip} G$.

(1): Let D be any set of dominated strategies in G, and let H, $\underline{H}$, H' be obtained by eliminating D from from G, $\underline{G}$, G'. Then, $H \rightarrow_{ds} \underline{H} \rightarrow_{ip} H'$. This is illustrated in Fig. 5.1.

(2): Let I any set of inessential players in G, and let H, $\underline{H}$, H' be obtained by eliminating I from from G, $\underline{G}$, G'. Then, $H \rightarrow_{ds} \underline{H} \rightarrow_{ip} H'$. See Fig. 5.2.

G	$\rightarrow_{ds(D')}$	$\underline{G}$	$\rightarrow_{ip(I')}$	G'
$\downarrow_D$		$\downarrow_D$		$\downarrow_D$
H	$\rightarrow_{ds(D'-D)}$	$\underline{H}$	$\rightarrow_{ip(I')}$	H'

Fig.5.1

G	$\rightarrow_{ds(D')}$	$\underline{G}$	$\rightarrow_{ip(I')}$	G'
$\downarrow_I$		$\downarrow_I$		$\downarrow_I$
H	$\rightarrow_{ds(D')}$	$\underline{H}$	$\rightarrow_{ip(I'-I)}$	H'

Fig.5.2

Proof. (1): The games H and $\underline{H}$ are obtained by eliminating D from G and $\underline{G}$, respectively. The dominated strategies $D' - D$ remain in H. Hence, by eliminating these from H, we have $\underline{H}$. This means $H \rightarrow_{ds} \underline{H}$. Then, H' is obtained from G' by eliminating $D - D'$, and H' differs from $\underline{H}$ only with eliminations of the inessential players I'. Thus, $\underline{H} \rightarrow_{ip} H'$.

(2): Since I is a set of inessential players in G, the elimination of I from G and $\underline{G}$ do not affect the elimination of dominated strategies from G to $\underline{G}$ by Lemma 2.2.(3). Hence, $H \rightarrow_{ds} \underline{H}$. Let $\underline{G} \rightarrow_{ip} G'$ holds with eliminations of inessential players I'. Now, $I' - I$ remains inessential in $\underline{H}$ by Lemma 2.2.(2). We eliminate $I' - I$ from $\underline{H}$ to obtain H'. Thus, $\underline{H} \rightarrow_{ip} H''$. ■

Now, we prove the theorem.

Proof of Theorem 5.1. Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ be any WIEDS in $\mathbb{W}(G)$, and $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \ldots, G^{*\ell^*} \rangle$ the IEDS. By induction, we construct the following sequence of WIEDS's:

$$\Theta^k(G) = \langle H^{k0}, H^{k1}, \ldots, H^{k\ell_k} \rangle,\ k = -1, 0, ..., \ell^*.$$

We will show that the last WIEDS $\Theta^{\ell^*}(G) = \langle H^{\ell^*0}, H^{\ell^*1}, \ldots, H^{\ell^*\ell_{\ell^*}} \rangle$ coincides with $\Gamma^*(G) = \langle G^{*0}, G^{*1},\ \ldots, G^{*\ell^*} \rangle$, and will derive the properties (1), (2) and (3) of the theorem.

Now, let $\Theta^{-1}(G) = \Theta^0(G) = \Gamma(G)$. Let k be a natural number with $0 \leq k < \ell^*$. We suppose the induction hypothesis that $\Theta^k(G) = \langle H^{k0}, H^{k1}, \ldots, H^{k\ell_k} \rangle$ is a WIEDS from G satisfies

$$H^{kt} = G^{*t} \text{ for } t = 0, ..., k; \tag{5.2}$$

$$\ell_k \leq \ell_{k-1}; \tag{5.3}$$

$$H^{kt} \text{ is a subgame of } H^{(k-1)t} \text{ for } t = 0, ..., \ell_k. \tag{5.4}$$

For $k = 0$, these three conditions hold since $\Theta^{-1}(G) = \Theta^0(G) = \Gamma(G)$. Now, we define a WIEDS $\Theta^{k+1}(G) = \langle H^{(k+1)0}, H^{(k+1)1}, \dots, H^{(k+1)\ell_{k+1}}\rangle$ from $\Theta^k(G)$ and show that it satisfies (5.2), (5.3) and (5.4) for $k+1$.

We denote the set of dominated strategies in $H^{kk} = G^{*k}$ by D^{*k}, and the set of inessential players in the interpolating $\underline{G}^{*k}$ with $G^{*k} \to_{ds} \underline{G}^{*k} \to_{ip} G^{*(k+1)}$ by I^{*k}. Since $k < \ell^*$, we have $D^{*k} \neq \emptyset$ or $I^{*k} \neq \emptyset$.

First, we eliminate D^{*k} from $H^{k(k+1)}, \dots, H^{k\ell_k}$, and obtain the sequence $H'^{k(k+1)}, \dots, H'^{k\ell_k}$, where each H'^{kt} is a subgame of H^{kt} for $t = k, ..., \ell_k$. By Lemma 5.5.(1), it holds that

$$H'^{kt} \to_{DI} H'^{k(t+1)} \text{ for all } t = k, ..., \ell_k - 1. \tag{5.5}$$

Thus, the sequence $\langle H'^{k(k+1)}, \dots, H'^{k\ell_k}\rangle$ may have already a repetition. We will take care of such repetitions later.

Then, we delete the players in I^{*k} from $\langle H'^{k(k+1)}, \dots, H'^{k\ell_k}\rangle$, and obtain the sequence $\langle H''^{k(k+1)}, \dots, H''^{k\ell_k}\rangle$, and by Lemma 5.5.(2), we have

$$H''^{kt} \to_{DI} H''^{k(t+1)} \text{ for all } t = k, ..., \ell_k - 1. \tag{5.6}$$

Now, the entire sequence $\langle H^{k0}, H^{k1}, \dots, H^{kk}, H''^{k(t+1)}, \dots, H''^{k\ell_k}\rangle$ may be a WIEDS, but it may contain some repetitions of the same game in $H''^{k(t+1)}, \dots, H''^{k\ell_k}$. In this case, we take one game for each repetition as a representative. The resulting sequence denoted by

$$\begin{aligned} \Theta^{k+1}(G) &= \langle H^{(k+1)0}, H^{(k+1)1}, \dots, H^{(k+1)\ell_{k+1}}\rangle \\ &= \langle H^{k0}, H^{k1}, \dots, H^{kk}, H^{(k+1)(k+1)}, \dots, H^{(k+1)\ell_{k+1}}\rangle \end{aligned}$$

is a WIEDS since it satisfies (5.6). Also, $\ell_{k+1} \leq \ell_k$, i.e., (5.3), since the length of the latter part may be the same or shorter than the $\ell_k - k$ because of the deletion of the repetitions. By definition, $H^{(k+1)(k+1)} = G^{*(k+1)}$, which together with (5.2) for k implies (5.2) for $k+1$.

We can see (5.4) for $k+1$ in the following manner. Let $t > k$. Recall that $H^{(k+1)t}$ is $H''^{kt'}$ for some $t' \geq t$ because of compressing each repetition to one game. Since $H''^{kt'}$ is a subgame of $H^{kt'}$ and $H^{kt'}$ is a subgame of H^{kt}, $H^{(k+1)t} = H''^{kt'}$ is a subgame of H^{kt} by Lemma 5.5.

Finally, we should show that the length ℓ_{ℓ^*} of $\Theta^{\ell^*}(G) = \langle H^{\ell^*0}, H^{\ell^*1}, \dots, H^{\ell^*\ell_{\ell^*}}\rangle$ coincides with ℓ^*. By the above induction proof, we have $\ell^* \leq \ell_{\ell^*}$ and $H^{\ell^*\ell_{\ell^*}} = G^{*\ell^*}$. This implies that if $\ell^* < \ell_{\ell^*}$, then $H^{\ell^*\ell_{\ell^*}} \to_{DI} H^{\ell^*(\ell_{\ell^*}+1)}$ implies $H^{\ell^*\ell_{\ell^*}} = H^{\ell^*(\ell_{\ell^*}+1)}$, which is impossible, since $\Theta^{\ell^*}(G)$ is a WIEDS. Thus, $\ell^* = \ell_{\ell^*}$ By (5.3), we have $\ell^* \leq \ell$.

By the construction of $\Theta^k(G), k = -1, 0, ..., \ell^*$, $H^{(k+1)\ell_{k+1}}$ is a subgame of $H^{k\ell_k}$ for all $k = 0, ..., \ell^* - 1$. By Lemma 5.5, $H^{\ell^*\ell_{\ell^*}} = G^{*\ell^*}$ is a subgame of G^ℓ. It follows from this and Lemma 5.4 that $G^{*\ell^*}$ coincides with G^ℓ. ■

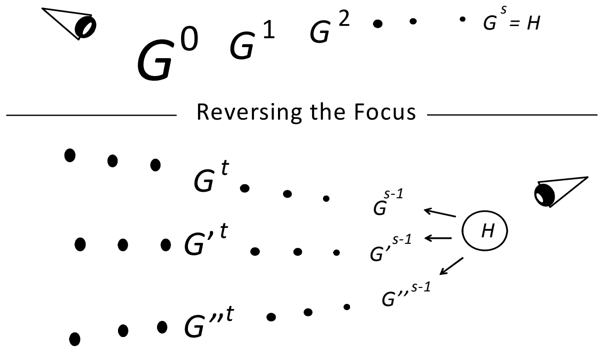

Figure 6.1: Start with the Final Game

6. Possible Shapes of IEDS Sequences

In Sections 2 to 5, we have studied the IDES and WIEDS sequences from a given game. In this section, we reverse the focus: We consider possible IEDS sequences to the given final game. Fig.6.1 describes both viewpoints: In the below, H is the given final game, and what IEDS sequences are possible to go to H. We give necessary and sufficient conditions with respect to the player set and the set of players having dominated strategies for such a sequence. Our results imply that sequences of quite arbitrary lengths and shapes can be sustained as the IEDS's from some games. Throughout this section, we consider only the IEDS'; thus we drop the asterisk $*$ for the IEDS, i.e., $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ is the IEDS from $G^0 = G$.

6.1. Possible Shapes of IEDS Sequences

Consider the IEDS sequence $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$, where $G^t = (N^t, \{S_i^t\}_{i \in N^t}, \{h_i^t\}_{i \in N^t})$ for $t = 0, ..., \ell$. Then, for each $t = 0, ..., \ell$, we let

$$T^t := \{i \in N^t : \text{player } i \text{ has a dominated strategy in } G^t\}. \tag{6.1}$$

We call T^t the *D-group* in G^t. Let us recall the elimination divide m_o given by Theorem 4.1, i.e., $T^t \neq \emptyset$ if $t < m_o$ and $T^t = \emptyset$ if $t \geq m_o$. Then, we call $\big[(N^0, T^0), \ldots, (N^\ell, T^\ell)\big]$ the *player-configuration* of $\Gamma(G)$.

The player configuration $\big[(N^0, T^0), \ldots, (N^\ell, T^\ell)\big]$ of $\Gamma(G)$ represents the structure of changes in players in $\Gamma(G)$. We focus only on the changes in the players, and drop the other information. However, this is enough for our consideration of possible lengths and shapes for IEDS sequences.

The following lemma gives simple observations, which turn out also to be sufficient conditions to sustain the IEDS sequence for some game G.

Lemma 6.1(Necessary Conditions). Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ be the IEDS-sequence with its elimination divide m_0 and player-configuration $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$. Then,

PC0: $T^t \subseteq N^t$ for $t = 0, ..., \ell$;

PC1: $N^0 \supseteq ... \supseteq N^{m_o} \supsetneq N^{m_o+1} \supsetneq ... \supsetneq N^\ell$ with $\left|N^\ell\right| \neq 1$;

PC2: for $t = 1, ..., m_o$, if $|T^{t-1}| = 1$, then $T^{t-1} \cap T^t = \emptyset$;

PC3: $T^{m_o} = T^{m_o+1} = \ldots = T^\ell = \emptyset$.

Proof. PC0 follows (6.1), and PC3 follows the definition of m_o.

PC1: Up to m_o, eliminations of inessential players may not occur; thus, we have weak inclusion relations up to m_o. After m_o, some inessential players are eliminated, and thus, we have strict inclusion relations after m_o.

PC2: Let $\left|T^{t-1}\right| = 1$, i.e., $T^{t-1} = \{i\}$. Then game G^t is the strict DI-reduction of G^{t-1}. If $i \notin N^t$, then $i \notin T^t$, *a fortiori*, $T^{t-1} \cap T^t = \emptyset$. Suppose $i \in N^t$. Then, let $G^{t-1} \to_{ds} \underline{G}^{t-1} \to_{ip} G^t$. Then, all the dominated strategies for player i in G^{t-1} are eliminated in $\underline{G}^{t-1}$. By Lemma 2.2.(3), player i has no dominated strategies in G^t. Hence, $T^{t-1} \cap T^t = \emptyset$. ■

Let us apply this lemma to a 2-person game G. Then, we have the following:

Corollary 6.2. Let G be a 2-person game, and $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ the IEDS with its elimination divide m_0 and player-configuration $\left[(N^0, T^0), \ldots, (N^\ell, T^\ell)\right]$. Then,

(1): $\ell - m_o \leq 2$;

(2): $N^0 = ... = N^{m_o-2} = \{1, 2\}$;

and there is some $k \leq m_o - 2$ such that

(3): $T^t = \{1, 2\}$ if $t \leq k$; and $\left|T^t\right| = 1$ if $t = k+1, ..., m_o - 1$;

(4): $T^t \cap T^{t+1} = \emptyset$ for $t = k+1, ..., m_o - 1$.

G^0	$\cdots$	G^k	G^{k+1}	G^{k+2}	$\cdots$	G^{m_o-1}	G^{m_o}	G^{m_o+1}
(N, N)	$\cdots$	(N, N)	$(N, \{i\})$	$(N, \{j\})$	$\cdots$	$(N, \{i\})$	$(\{i\}, \emptyset)$	$(\emptyset, \emptyset)$

Fig.6.2

Thus, up to G^k, the D-group remains the same as $T^t = N = \{1, 2\}$, but it starts alternating at G^{k+1} up to G^{m_o-1}. Then, the process continues for possibly two more steps, but may stop at G^{m_o}. Fig.6.2 describes one possiblity. This kind of monotonicity is observed only for the 2-person case. The following is a 3-person game, where T^0 is a singleton, but T^2 becomes the entire set.

Example 6.1 (Nonmonotonicity). Consider the following 3-person game G where each player has three strategies and the payoffs are described by the three tables. Each table has the PD game in the northwest corner for players 1 and 2.

Table 6.1, $\mathbf{s}_{31}$

$1\backslash 2$	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	5,5,2	1,6,2	3,0,1
$\mathbf{s}_{12}$	6,1,2	3,3,2	1,0,1
$\mathbf{s}_{13}$	0,3,2	0,1,1	0,0,2

Table 6.2, $\mathbf{s}_{32}$

$1\backslash 2$	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	5,5,0	1,6,0	3,0,2
$\mathbf{s}_{12}$	6,1,0	3,3,0	1,0,1
$\mathbf{s}_{13}$	0,3,1	0,1,1	0,0,2

Table 6.3, $\mathbf{s}_{33}$

$1\backslash 2$	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	5,5,0	1,6,0	3,9,0
$\mathbf{s}_{12}$	6,1,0	3,3,0	1,9,0
$\mathbf{s}_{13}$	9,3,0	9,1,0	9,9,0

The IEDS from this game is as follows: In the game $G^0 = G$, player 3 has the dominated strategy $\mathbf{s}_{33}$, and in the resulting game G^1 from the elimination of $\mathbf{s}_{33}$, players 1 and 2 have dominated strategies $\mathbf{s}_{13}$ and $\mathbf{s}_{23}$. Here, the game G^2 obtained from G^1 by eliminating $\mathbf{s}_{13}$ and $\mathbf{s}_{23}$ has still three players, each of whom has 2 strategies. Game G^2 is expressed by the northwest corner of each table, where each has a dominated strategy. Here, the player-configuration is $\left[(N^0, T^0), \dots, (N^3, T^3)\right]$, where $N^0 = N^1 = N^2 = \{1, 2, 3\}$, $N^3 = \emptyset$, $T^0 = \{3\}, T^1 = \{1, 2\}, T^2 = \{1, 2, 3\}, T^3 = \emptyset$, and $m_o = 3$. This is described by Fig.6.3.

$(N^0, \{1\}) \rightarrow$	$(N^1, \{1, 2\}) \rightarrow$	$(N^2, \{1, 2, 3\}) \rightarrow$	$(\emptyset, \emptyset)$

Fig.6.3

Lemma 6.1 gives four conditions for the player-configuration of the IDES $\Gamma(G)$. Now, we reverse this statement. That is, for a given sequence $\left[(N^0, T^0), \dots, (N^\ell, T^\ell)\right]$ satisfying PC0-PC3, do we find a game G so that $[(N^0, T^0), \dots, (N^\ell, T^\ell)]$ is the player-configuration of $\Gamma(G)$? We give an affirmative answer to this queation. Thus, the possible shapes of the payer-configurations of the IEDS's are characterized by PC0-PC3.

Theorem 6.1 (Possible Shapes): Let $H = (N^H, \{S_i^H\}_{i \in N^H}, \{h_i^H\}_{i \in N^H})$ be a game with no dominated strategies and no inessential players. Let $[(N^0, T^0), \dots, (N^\ell, T^\ell)]$ be any sequence satisfying PC0-PC3 with $N^\ell = N^H$. Then, there exists a game G with the IEDS $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ from G such that

(a): $G^\ell = H$;

(b): $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ is the player-configuration of $\Gamma(G)$.

After all, PC0-PC3 form a sufficient condition to have a game G that its IEDS $\Gamma(G)$ has the suggested player-configuration. Since PC0-PC3 are not really restrictive, the assertion of the theorem is interpreted as meaning that the IEDS sequences have a great variety of lengths and shapes. Corollary 6.2 is now interpreted as implying that any player-configuration having (1)-(4) is sustained as the IEDS from some 2-person game G, and that we cannot restrict them any further.

6.2. Proof of Theorem 6.1

Consider a sequence $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ and $H = (N^H,\ \{S_i^H\}_{i \in N^H},\ \{h_i^H\}_{i \in N^H})$ given in the theorem. We construct a sequence $G^\ell, G^{\ell-1}, \ldots, G^0$ from $G^\ell = H$ along (N^ℓ, T^ℓ), ..., (N^0, T^0). It will be shown that for each $t = \ell, \ell - 1, ..., 0$, G^{t+1} must be the strict *DI*-reduction of G^t; thus, $\langle G^0, ..., G^\ell \rangle$ is the IEDS from G^0. Now, we construct a game G^t from G^{t+1} in the backward manner.

G^t	$\rightarrow_{ds}$	$\underline{G}^t$	$\rightarrow_{ip}$	G^{t+1}
(N^t, T^t)	$\Longleftarrow$(construction)		$\Longleftarrow$(construction)	(N^{t+1}, T^{t+1})
	Lemmas 6.3, 6.4		Lemma 6.2	

Fig.6.4

Lemma 6.2 is for the construction of the interpolating $\underline{G}^t$ from G^{t+1} in Fig.6.4.

Lemma 6.2. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game with $|S_i| \geq 2$ for all $i \in N$, and let I' be a nonempty set of new players. Then, there is a $G' = (N', \{S'_i\}_{i \in N'}, \{h'_i\}_{i \in N'})$ such that

(1): $N' = N \cup I'$;

(2): $|S'_i| \geq 2$ for all $i \in N'$;

(3): G is the strict *ip*-reduction of G'.

Proof. We choose the strategy sets $S_i, i \in N'$ so that $S'_i = S_i$ for all $i \in N$ and $S'_i = \{\alpha, \beta\}$ for all $i \in I'$. Then, we define the payoff functions $\{h'_i\}_{i \in N'}$ so that the players in I' are inessential in G' but no players in N are inessential in G'. Let I be the set of inessential players in G. For each $i \in I$, we choose a specific strategy $\mathbf{s}_{i1}$ from S_i. Then, we define $\{h'_i\}_{i \in N'}$ as follows:

(a): for any $j \in I'$, $h'_j(s_{N'}) = |\{i \in I : s_i = \mathbf{s}_{i1}\}|$ for $s_{N'} \in S_{N'}$;

(b): for any $j \in N$, $h'_j(s_{N'}) = h_j(s_N)$ for $s_{N'} \in S_{N'}$.

For any $j \in I'$, j's strategy s_j is nominal in (a) and (b) in the sense that s_j does not appear substantially in h'_i for any $i \in N \cup I$. Hence, the players in I' are all inessential in G'. On the other hand, each $i \in I$, as far as such a player exists in G, affects j's payoffs for $j \in I'$ because of (a) and $|S_i| \geq 2$. Hence, any $i \in I$ is not inessential in G'. Also, any $i \in N - I$ is not inessential in G' by (b). Hence, only the players in I' are inessential. Hence, G is the strict *is*-reduction of G'. ■

Now, we consider the step from $\underline{G}^t$ to G^t in Fig.6.4. For this construction, we need to show the following lemma: It is the backward argument that the dominated strategies in $\underline{G}^t$ are not dominated in the constructed G^t. Let G be a game, which is supposed to be $\underline{G}^t$. In the following, we write s_j dom_G s'_j iff s_j dominates s'_j in G. We want s'_j not to be dominated in the newly constructed game.

Lemma 6.3. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be an n-person game, and $j \in N$ a fixed player. There are real numbers $\{\alpha_j(s_j)\}_{s_j \in S_j}$ such that

$$\alpha_j(s_j) > 0 \text{ for all } s_j \in S_j; \tag{6.2}$$

$$\text{if } s_j \text{ dom}_G\ s'_j, \text{ then } \alpha_j(s_j) < \alpha_j(s'_j). \tag{6.3}$$

Proof: We write dom_G simply as dom. This relation dom is transitive and asymmetric. We call a sequence $\{s^1_j, ..., s^m_j\}$ a *descending chain* from s^1_j to s^m_j iff s^k_j dom s^{k+1}_j for $k = 1, ..., m-1$. For any given $s_j, s'_j \in S_j$, there may be no or multiple descending chains from s_j to s'_j.

We say that s_j is *maximal* in (S_j, dom) iff there is no $s'_j \in S_j$ such that s'_j dom s_j. Let $s^0_j, ..., s^k_j$ be the list of maximal elements in (S_j,dom). Then, we define the sets $A(s^0_j), ..., A(s^k_j)$ recursively by

$$A(s^0_j) = \{s^0_j\} \cup \{s_j \in S_j : s^0_j \text{ dom } s_j\}; \tag{6.4}$$

$$A(s^l_j) = \{s^l_j\} \cup \{s_j \in S_j - \cup_{t=0}^{l-1} A(s^t_j) : s^l_j \text{ dom } s_j\} \text{ for } l \leq k. \tag{6.5}$$

That is, we classify each $s_j \in S_j - \{s^0_j, ..., s^k_j\}$ to the first $A(s^t_j)$ with s^t_j dom s_j, which implies

$$\text{if } s^t_j \text{ dom } s_j \text{ and } s_j \in A(s^{t'}_j), \text{ then } t' \leq t. \tag{6.6}$$

Thus, these sets $A(s^0_j), ..., A(s^k_j)$ form a partition of S_j.

Now, we define $\{\alpha_j(s_j)\}_{s_j \in S_j}$ as follows: for $s_j \in A(s^t_j)$ and $t = 0, ..., k$,

$$\alpha_j(s_j) = -t\,|S_j| + l_{s_j}, \tag{6.7}$$

where l_{s_j} is the maximum length of a descending chain from s_j^t to $s_j \neq s_j^t$, and is 0 if $s_j = s_j^t$. When $k = 0$, l_{s_j} may be equal to $|S_j|$, but when $k > 0$, l_{s_j} is smaller than $|S_j|$.

Now, we show (6.3). Let $s_j, s_j' \in S_j$ and s_j dom s_j'. Also, let $s_j \in A(s_j^t)$ and $s_j' \in A(s_j^{t'})$. Since s_j^t dom s_j, we have s_j^t dom s_j', which implies $t' \leq t$ by (6.6). Now, we consider two cases: $t' = t$ and $t' < t$. Suppose that $t = t'$. Let $l_{s_j}, l_{s_j'}$ be, respectively, the maximal lengths of descending chains from s_j^t to s_j and s_j'. Since s_j dom s_j', we have $l_{s_j} < l_{s_j'}$. Thus, $\alpha_j(s_j) = -t|S_j| + l_{s_j} < \alpha_j(s_j') = -t|S_j| + l_{s_j'}$. Suppose $t' < t$. Since $|S_j| > l_{s_j}, l_{s_j'}$ as remarked above, we have $\alpha_j(s_j') - \alpha_j(s_j) = -t'|S_j| + l_{s_j'} - (-t|S_j| + l_{s_j}) = (t - t')|S_j| + (l_{s_j'} - l_{s_j}) > 0$.

This $\{\alpha_j(s_j)\}_{s_j \in S_j}$ does not satisfy (6.2), but adding some constant to $\alpha_j(s_j)$ uniformly, we have also (6.2). ■

Now, we go to the step from $\underline{G}^t$ to G^t in Fig.6.4. The following lemma is to show the construction of G^t from $\underline{G}^t$, and will be used in the main proof of Theorem 6.1.

Lemma 6.4. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game with N, and let $T \subseteq N$ satisfying the following condition:

$$\text{if } T = \{i\}, \text{ then there are no } s_i, s_i' \in S_i \text{ with } s_i \text{ dom}_G\ s_i'. \tag{6.8}$$

Then, there is a game $G' = (N, \{S_i'\}_{i \in N}, \{h_i\}_{i \in N})$ such that G is the strict ds-reduction of G' with the D-group T.

Proof. Without loss of generality, we can assume that G satisfies the condition:

$$h_i(s) > 0 \text{ for all } s \in N \text{ and } i \in N. \tag{6.9}$$

First, let β_j be a new strategy symbol for each $j \in T$. We define $\{S_j'\}_{j \in N}$ as follows:

$$S_j' = \begin{cases} S_j \cup \{\beta_j\} & \text{if } j \in T \\ \\ S_j & \text{if } j \in N - T \end{cases} \tag{6.10}$$

Then we extend h_j to $h_j' : \Pi_{i \in N} S_i' \to \mathbb{R}$ for $j \in N$ so that the restriction of h_j' to $\Pi_{i \in N} S_i$ is h_j itself and G is the strict ds-reduction of G'.

To be more precise, we take a few steps to define the payoff functions $\{h_j'\}_{j \in N}$. Let $j \in N$. First, h_j' is the same as h_j over $\Pi_{i \in N} S_i$, i.e.,

$$h_j'(s) = h_j(s) \text{ if } s \in \Pi_{i \in N} S_i. \tag{6.11}$$

For any $s \in S' - S$, if $j \in N - T$, then

$$h_j'(s) = \alpha_j(s_j);\ \text{ where } \alpha_j(s_j) \text{ is given in Lemma 6.3,} \tag{6.12}$$

and if $j \in T$, then

$$h'_j(s) = \begin{cases} \alpha_j(s_j) & \text{if } s_j \neq \beta_j \\ \\ 0 & \text{if } s_j = \beta_j. \end{cases} \tag{6.13}$$

First, let $j \in N - T$, and let $s_j, s'_j \in S_j = S'_j$. Suppose s_j dom_G s'_j. Then, there are $s, s' \in S' - S$ such that the j-th components of s and s' are s_j and s'_j. Hence, by (6.12), $h'_j(s) = \alpha_j(s_j) < \alpha_j(s'_j) = h'_j(s')$. Thus, Hence, s_j does not dominate s_j in G'. Hence, j has no dominated strategies in G'.

Second, let $j \in T$. We choose an $s^*_j \in S_j$ with $s^*_j \neq \beta_j$. By (6.13) and (6.2), we have, for any $s_{-j} \in S_{-j}$,

$$h'_j(\beta_j; s_{-j}) = 0 < \alpha_j(s^*_j) = h'_j(s^*_j; s_{-j}).$$

This does not depend upon s_{-j}; thus, s^*_j $\mathrm{dom}_{G'}$ β_j. By these two paragraphs, we showed that T is the D-group in G'.

It remains to show that s_j $\mathrm{dom}_{G'}$ s'_j does not hold for any $s_j, s'_j \in S_j = S'_j - \{\beta_j\}$ and $j \in T$. If s_j dom_G s'_j does not hold, then s_j $\mathrm{dom}_{G'}$ s'_j does not hold either. Now, we suppose s_j dom_G s'_j. By (6.8), we have $|T| > 1$. This guarantees that the existences of $s, s' \in S' - S$ such that their j-th components are s_j and s'_j. Then, by (6.13), we have $h'_j(s) = \alpha_j(s_j) < \alpha_j(s'_j) = h'_j(s')$. Hence, it is not the case that s_j $\mathrm{dom}_{G'}$ s'_j.

Thus, we have shown that G is the strict ds-reduction of G'. ■

Now, we can prove the Theorem.

Proof of Theorem 6.1: Let $G^\ell = H$. Since H has no dominated strategies and no inessential players, condition (4.2) holds.

Suppose that G^{t+1} is already defined with $\left|S_i^{t+1}\right| \geq 2$ for all $i \in N^{t+1}$. Condition PC2 guarantees condition (6.8). By Lemma 6.2, we find an interpolating game $\underline{G}^t$ so that G^{t+1} is the strict ip-reduction of $\underline{G}^t$ with its player set N^t and $\left|\underline{S}_i^t\right| \geq 2$ for all $i \in N^t$. By Lemma 6.4, we find another game G^t so that $\underline{G}^t$ is the strict ds-reduction of G^t with its D-group T^t and satisfying $\left|S_i^t\right| \geq 2$ for all $i \in N^t$.

Now, we have an IEDS sequence $\Gamma^*(G) = \left\langle G^{*0}, ..., G^{*\ell} \right\rangle$ such that $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ is the player-configulation of $\Gamma^*(G)$. ■

7. Conclusions

We have considered the process of iterated eliminations of (strictly) dominated strategies and inessential players (the IEDS process). Eliminations of inessential players are newly introduced in this paper, and get along well to eliminations of dominated strategies. Here, first, we give a summary of our technical contributions, and then we consider their implications for negative criteria for the abstraction process, i.e., from the perspective of modelling social situations.

The three main results in this paper are: Theorem 3.1 (Preservation), Theorem 4.1 (IEDS: Smallest/Shortest) and Theorem 6.1 (Possible Lengths and Shapes). The preservation theorem is a direct extension of one result given in Maschler, *et al.* [7], and leads to the recovering theorem (Theorem 4.2) about Nash equilibria. These results are important from the perspective of the abstraction process.

The second theorem is an extension of the order-independence theorem, which states that any sequences generated from the IEDS process ends up with the same game. In addition to this order-independence result, our theorem states that the IEDS sequence is the shortest and smallest among the WIEDS's from a given game G. To prove this theorem, we needed a lot of conceptual and technical developments. From them, we have learned a lot, for example, eliminations of dominated strategies and inessential players are quite parallel, but not completely symmetric, which is observed in Lemma 2.2 and from which we have the partition theorem (Theorem 4.1).

The third theorem is new. It gives necessary and sufficient conditions for possible shapes of the IEDS sequences. The theorem implies that the IEDS sequences have a vast variety of lengths and shapes. When we start with a given 2-person or 3-person game, it still suggests a variety of lengths and shapes of generated sequences.

What we have not touched upon this paper is to consider the required preference comparisons to calculate the IEDS sequence or a WIEDS sequence. The development given in this paper facilitates this consideration. In fact, we have some example of a game where the IEDS sequence is not the smallest from the viewpoint of preference comparions; some WIEDS sequence can be calculated by a smaller number of preference comparisons. This will be discussed in a separate paper.

Finally, we return to the perspective of abstracting social situations. The preservation theorem is relevant for this. That is, Nash equilibrium is a positive criterion for decision making by a player and/or prediction by an outside theorist. In inductive game theory (cf., Kaneko-Matsui [6], Kaneko-Kline [5]), an inside player takes this perspective. In either case, eliminations of dominated strategies and inessential players help to consider the process of choices of relevant components from a social situation.

From this perspective, we find an apparent restriction: The definition of an inessential player is too stringent to have no effects on the entire players. There are two directions to weaken this restriction. One is to consider one player's effects on some players, and the other is to introduce ε-effects. The presentation of payoffs in terms of real numbers leads to this problem. It may be possible to start with one's preference relation to be a partial ordering, which may be derived by taking only ε-effects. Those relaxations of basic concepts may lead to a better understanding of the abstraction process on social situations.

References

[1] Apt, K., R., (2004), Uniform Proofs of Order Independence for Various Strategy Elimination Procedures, *Contributions to Theoretical Economics* 4, Article 5.

[2] Börgers, T., (1993), Pure Strategy Dominance, *Econometrica* 61, 423-430.

[3] Gale, D., Kuhn, H., and Tucker, A., (1950), "Reductions of Game Matrices", *Contributions to the Theory of Games*, **I**, *Annals of Mathematics Studies*, **24**, Kuhn, H., and Tucker, A., ed., pp. 89-96, Princeton University Press.

[4] Gilboa, I., Kalai, E., and Zemel, E., (1990), On the Order of Eliminating Dominated Strategies, *Operations Research Letters* **9**, pp. 85-89.

[5] Kaneko, M., and J. J. Kline, Inductive Game Theory: A Basic Scenario, *Journal of Mathematical Economics* 44, (2008), 1332–1363.

[6] Kaneko, M., and A. Matsui, Inductive game theory: discrimination and prejudices, *Journal of Public Economic Theory* 1 (1999), 101-137.

[7] Maschler, M., E. Solan, and S. Zamir, (2013), *Game Theory*, Cambridge University Press, Cambridge.

[8] Marx, L. M., and J. M. Swinkels, (1997), Order Independence for Iterated Weak Dominance, Games and Economic Behaivor 18, 219-245.

[9] Moulin, H., (1979), "Dominance Solvable Voting Schemes", *Econometrica* 47, pp. 1137-1351.

[10] Moulin, H., (1986), *Game Theory for the Social Sciences*, 2nd and revised edition, New York University Press.

[11] Nash, J. F., (1951), Non-cooperative Games, *Annals of Mathematics* 54, 286-295.

[12] Myerson, R. B., (1991), *Game Theory*, Harvard University Press, Cambridge.

[13] Luce, R., and Raiffa, H., (1957), *Games and Decisions: Introduction and Critical Survey*, John Wiley & Sons, Inc.

[14] Osborne, M., and A. Rubinstein, (1994), *A course in Game Theory*, The MIT Press. Cambridge.

[15] Thompson, A. R., (1988), *Economics*, Addison-Wesley Publishing Company, New York.