Macroeconomic Dynamics of Human Development and the Creation of a Market Economy

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This paper investigates macroeconomic dynamics of a poor society under Human Development using a framework of the well-known one-sector RBC model with factor-generated externalities by Benhabib and Farmer (1994). We introduce into it the "productive consumption hypothesis (PCH)" whose idea is essentially the same as the efficiency wage hypothesis. First, we find that this model can apply to poor economies because even in the absence of capital externality indeterminacy occurs when the PCH externality is strong enough. Second, in contrast to the standard growth models, the "intertemporal *complementarity* between present and future consumption" may hold when a steady-state equilibrium (SE) is saddle-point stable. Then per capita consumption is initially chosen at a high level and then begins to decrease along a transition path starting from low initial capital stock. Third, when an SE is perfectly stable, an equilibrium path typically converges to the SE with cyclical movements. It may either diverge from the SE with cyclical movements or exhibit an endogenous cycle when the supercritical Hoph bifurcation occurs. These results are consistent with real data of low-income developing economies for the past several decades. Forth, when the subcritical Hoph bifurcation occurs, no equilibrium paths exist if initial capital stock falls short of a critical level. This is the first theoretical explanation for how a poor society fails to establish a market economy system. "Big Push" may help escape from this new type of "underdevelopment trap". Finally, an introduction of human development aid can induce a cyclical path converging to a new SE with higher welfare. We cannot always expect that a poor economy will smoothly follow a monotonic growth process even if the aid improves SE welfare.

Keywords: Dynamic General Equilibrium Model, Hoph Bifurcation, Human Development, Indeterminacy of Equilibrium, Productive Consumption Hypothesis

JEL Classification Codes： O11, O15, E13

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1. Introduction

Since the beginning of this century the Millennium Development Goals (MDGs) have been playing a central role in the world-wide development strategies for poor developing countries. Looking at the eight goals in Table 1, we find that they put more emphasis on "poverty reduction" through an improvement of nutrition, health, sanitation and/or basic education. In contrast to the previous development efforts such as the World Bank's "structural adjustments" in the 1980s, these development strategies have been based on the concept of "human development (HD)", which primarily means an enhancement of human rights due to an improvement of these consumption (utility-increasing) activities.

The poverty reduction strategies toward the MDGs have, at least implicitly, presumed an economic growth mechanism that enables a poor economy to take off. This mechanism seems different from those in the standard growth models like Solow-type neoclassical growth model or Ramsey-Cass-Koopmans-type optimal growth model. This is because an engine of growth is *investment* in those models while it is the *consumption* that can raise productivity of people through an improvement of nutrition, health, sanitation or education in the recent poverty reduction strategies. Taking into account that an investment-driven growth will be accelerated by *decreasing* present consumption from a given amount of income, the growth induced by an *increase* in consumption must probably have qualitatively different aspects from that in the standard investment-driven growth models.

In addition to this theoretical consideration, we could find some empirical evidence implying that the consumption-driven growth process observed in poor areas of the modern world may differ from the equilibrium (saddle-point) path which converges smoothly to the steady-state equilibrium (SE) path in the standard growth models. In "Tables: Indeterminacy Cases" and "Tables: Bifurcation Cases", you will find per capita real GDP (blue line) in poor Latin American and African countries has not always converge smoothly to the long-run trend (red line) that could be interpreted as a SE path. Rather, the data of per capita real GDP exhibits volatile behaviors around the long-run trend: some data keep cyclical movements around the trend, others converge with ups and downs toward the trend. Despite these theoretical and empirical rationales, an economic mechanism of the consumption-driven growth has never been investigated explicitly in the discussions on effectiveness of poverty reduction strategies including foreign aid.

This paper investigates how the consumption-driven growth mechanism may differ from that in the standard (investment-driven) growth models, and explores its implications for macroeconomic equilibrium dynamics. In order to build a dynamic general equilibrium (DGE) model suitable to this purpose, we introduce the "productive consumption hypothesis (PCH)" by Steger (2000, 2002) into Benhabib and Famer's (1994:BF) dynamic (business cycle) model with factor-generated externalities and endogenous labor supply, because it provides a highly general framework of macroeconomic dynamic analysis. The idea that an increase in income and thus in consumption improves workers' productivity has been studied, since Leibenstein (1957), in the form of the "efficiency wage hypothesis" in the traditional literature of development economics and macroeconomics. In 1970s-80s, it was analyzed in static models, focusing on rural labor markets in developing economies (Stiglitz, 1976; Bliss and Stern, 1978; Gersovitz, 1983; Dasgupta and Ray, 1986). In 1990s, it was often used as one of the important theories explaining wage rigidity and involuntary unemployment in Keynesian macroeconomics. Then interesting results such as the existence of multiple equilibria were derived based on dynamic models. (Ray and Streufert, 1993; Banerji and Gupta, 1997; Jellala and Zenoub, 2000). These studies tended to focus on the issues within labor market.

Since the beginning of this century, Steger (2000, 2002) has presented the new formulations of the idea of the "efficiency wage hypothesis" that enable us to analyze a growth process of the entire economy, going beyond labor market issues. The first formulation was the one in which an increase in per capita consumption accelerates (disembodied) human capital accumulation. The second formulation is the one in which an increase in per capita consumption raises each worker's labor productivity at a point in time (as in the "efficiency wage hypothesis"). Gupta (2003) has proposed another formulation in which some part of per capita consumption raises only the level of human capital (labor productivity) and the other only utility. Unfortunately, these previous models have either no balanced growth path (BGP) as an SE (Steger (2000, 2002)) or only a perfectly *un*stable SE (Gupta (2003)). Because of this deficiency, economists have not been able to use the standard methods of growth analysis such as phase diagram. Nor have we been able to engage in comparative statics and dynamics of the SE.[1] In this paper, by contrast, we introduce the PCH by using Steger's second formulation, and will present a tractable analysis of the consumption-driven growth process of poor developing economies.

We have obtained five conclusions about equilibrium dynamics under "human development" of a poor developing economy where the PCH applies. First, we find that

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1 An exception is Daitoh (2010). He shows that it is possible to build a tractable growth model in which we can use these standard methods, based on Steger's first formulation of the PCH. He explains the population dynamics in modern developing countries by considering a regime shift from a zero-saving phase to a positive-saving phase.

this model can apply to poor economies because even in the absence of capital externality indeterminacy occurs when the PCH externality is strong enough. It means that the Benhabib and Farmer's (1994) model could provide a good framework of macroeconomic dynamic analysis not only for advanced countries but for poor developing economies. Second, in contrast to the standard growth models, the "intertemporal *complementarity* between present and future consumption" may hold when a steady-state equilibrium (SE) is saddle-point stable. In this case per capita consumption is initially chosen at a high level and then begins to decrease along a transition path starting from low initial capital stock. In the other cases, both consumption and capital increase along the transition path, as in the standard models like Ramsey-Cass-Koopmans optimal growth model. Third, when an SE is perfectly stable, an equilibrium path typically converges to the SE with cyclical movements. It may either diverge from the SE with cyclical movements or exhibit an endogenous cycle when the supercritical Hoph bifurcation occurs. These results are consistent with real data of low-income (Latin American and African) developing economies for the past several decades. Forth, when the subcritical Hoph bifurcation occurs, no equilibrium paths exist if initial capital stock falls short of a critical level. This is the first theoretical explanation for how a poor society fails to establish a market economy system. "Big Push" may help escape from this new type of "underdevelopment trap". Finally, an introduction of human development aid can induce a cyclical path converging to a new SE with higher welfare. We should be careful when evaluating the success or failure of HDA, because we cannot always expect that a poor economy will follow a smooth growth process even if the aid improves SE welfare.

2. The Model

We introduce the "productive consumption hypothesis (PCH)" and exogenous population growth into the one-sector real business cycle (RBC) model with factor-generated externalities by Benhabib and Farmer (1994).

The aggregate production function is $Y_t = F(K_t, N_t; \bar{X}_t)$, where Y_t is real GDP, K_t capital input and N_t labor input in efficiency units. Each worker's labor supply in efficiency units is $h(c_t)n_t$, where n_t is the number of working hours that he/she supplies, and $h(c_t)$ is a worker's labor productivity per working hour. Representing as L_t total population which is assumed to grow at a constant rate g at a point in time ($\dot{L}_t / L_t = g \geq 0$), we have the aggregate labor input $N_t = h(c_t)n_t L_t$. We assume that $Y_t = F(K_t, N_t; \bar{X}_t)$ exhibits constant returns to scale in K_t and N_t. Thus per capita GDP $y_t = (Y_t / L_t)$ is determined by $y = f(k, h(c)n; \bar{X})$, which exhibits *constant* returns to scale in per capita capital k ($= K_t / L_t$) and per capita efficiency labor $h(c)n$ ($= N_t / L_t$). To make the analysis clear, we assume

$$f(k, h(c)n; \bar{X}) = k^a [h(c)n]^{1-a} \bar{X} \qquad 0 < a < 1 \tag{1}$$

The term $\bar{X} = \bar{X}(\bar{k}, h(\bar{c})\bar{n})$, as in Benhabib and Farmer (1994), represents positive externalities of capital and labor to individual firms. The variables with an upper bar $\bar{k}$, $\bar{n}$ and $h(\bar{c})$ will be regarded as expected values. The per capita production function thus exhibits *increasing* returns to scale from a social point of view. To capture this property concisely, we assume

$$\bar{X} = \bar{k}^{\alpha - a}[h(\bar{c})\bar{n}]^{\beta-(1-a)} \qquad \alpha \geq a\,, \quad \beta \geq 1-a \tag{2}$$

Since we have $\alpha + \beta > 1$, the social production function $f(k, h(c)n) = k^{\alpha}[h(c)n]^{\beta}$ certainly exhibits *increasing* returns to scale. For discussions later, we should pay attention to the facts that $\alpha = a$ holds for no capital externality and that $\beta = 1 - a$ holds for no labor externality.

We introduce the PCH in the form of $h(c_t)$ function: an increase in per capita consumption c_t raises each worker's labor productivity $h(c_t)$ per working hour at a point in time t ($h'(c_t) > 0$). This corresponds to the second formulation for the PCH proposed by Steger (2002), which is the same as in the efficiency wage hypothesis. Furthermore, we suppose that the "PCH function" $h(c_t)$ is external to individuals because people in a poor society cannot fully control how much their productivity rises by an increase in consumption (even if they can recognize it): in a society with poor social or health infrastructure, people cannot help drinking unsafe water with some virus or toxic substances for survival, or have medical services (e.g., injection) under poor sanitation with high risks of diseases (e.g., HIV/AIDS or hepatitis). If the average consumption in such a society increases, poor social environment will improve due to better nutrition (physical strength or immunity), health (quality of medical services) and basic education (access to knowledge useful to protect people against diseases). This will lead to an increase in labor productivity of an individual worker.[2] In this sense, per capita consumption in the PCH function $h(c)$ could be regarded as the average consumption in the society and thus as the expected value $\bar{c}$ to an individual worker.

[2] It could be interpreted as the concept of "Marshallian externalities" being applied to a consumption side.

Therefore, we will treat all $h(c)$ terms as $h(\overline{c})$ in the following analysis for a decentralized market economy.

We specify the PCH function $h(c)$ into

$$h(c) = c^{\varepsilon}, \qquad \varepsilon > 0. \tag{3}$$

It is concave for $0 < \varepsilon < 1$ while it is convex for $\varepsilon > 1$. Taking $ch''(c)/h'(c) = \varepsilon - 1$ into consideration, a large value of ε means that the graph of $h(c)$ has a small curvature (is close to linear) for $0 < \varepsilon < 1$ while it has a large curvature (is strongly convex) for $\varepsilon > 1$. The reason why we assume this simple function (with no inflection points) instead of the S-shaped curve in the efficiency wage models is that it must be more interesting to find the possibility of complicated dynamics based on a simpler function.

2.1 Equilibrium of Decentralized Market Economy

The representative consumer derives his instantaneous utility $u = u(c_t, l_t)$ from consumption c_t and leisure l_t. In a decentralized market economy, given the expectations of time paths $\{w_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$ and $\{h(\overline{c}_t)\}_{t=0}^{\infty}$, he chooses time paths $\{c_t\}_{t=0}^{\infty}$ and $\{n_t\}_{t=0}^{\infty}$ to maximize the intertemporal utility

$$U = \int_0^{\infty} u(c_t, l_t) e^{-\rho t} dt \qquad \rho > 0 \tag{4}.$$

where $u(c_t, l_t)$ is assumed to be concave in c_t and l_t (i.e., we assume $u_{cc} < 0$ and $u_{cc}u_{ll} - u_{cl}u_{lc} > 0$). Consumers own all the capital and each worker has one unit of time which can be allocated between leisure l_t and working time n_t. Thus the time constraint is

$$n_t + l_t = 1. \tag{5}$$

Given $\overline{X}$, the rental rate r of capital and the wage rate w for one unit of efficiency labor are equal to the marginal product of capital and labor, respectively:

$$r = \frac{\partial f(k, h(\overline{c})n; \overline{X})}{\partial k} = ak^{\alpha-1}n^{\beta}c^{\varepsilon\beta}, \quad w = \frac{\partial f(k, h(\overline{c})n; \overline{X})}{\partial [h(\overline{c})n]} = (1-a)k^{\alpha}n^{\beta-1}c^{\varepsilon(\beta-1)} \tag{6}$$

The aggregate flow budget constraint is $r_t K_t + w_t N_t = C_t + \dot{K}_t - \delta K_t$, which leads to

$$\dot{k}_t = (r_t - \delta - g)k_t + w_t h(\overline{c}_t)n_t - c_t \tag{7}.$$

The representative consumer maximizes (4) subject to the time constraint (5) and the flow budget constraint (7). The current-value Hamiltonian function is

$$H(c, n, k, p) = u(c, 1-n) + p[(r - \delta - g)k + wh(\overline{c})n - c] \tag{8}$$

where p_t is a costate variable. To clarify the analysis, we specify

$$u(c, l) = \log c - \frac{[1-l]^{1+\chi}}{1+\chi}. \qquad \chi \geq 0, \tag{9}$$

Under $\overline{k} = k$, $\overline{n} = n$ and $\overline{c} = c$, the first-order necessary condition (FOC) leads to

$$\frac{1}{c} = p \tag{10}$$

$$n^{\chi} = p(1-a)k^{\alpha}n^{\beta-1}c^{\varepsilon\beta} \tag{11}$$

$$\dot{p} = p[\rho + \delta + g - ak^{\alpha-1}n^{\beta}c^{\varepsilon\beta}] \tag{12}$$

and the transversality condition (TVC) $\lim_{t\to\infty} k_t p_t e^{-\rho t} = 0$. From (10), c and p change in the opposite direction while from (6) and (12) c and $r = ak^{\alpha-1}n^{\beta}c^{\varepsilon\beta}$ in the same direction. Notice here that the second term on the right-hand side of (8) is linear in c

and n from the consumer's viewpoint, because he regards $h(\bar{c})$ as given. Since $u(c, 1-n)$ is concave, the Hamiltonian function is concave in *c*, *n* and *k*. Thus the paths which satisfy the FOCs are certainly equilibrium paths.

Dividing (11) by (10), we obtain the labor market equilibrium condition:

$$c^{1-\varepsilon} n^{\chi} = (1-a) k^{\alpha} n^{\beta-1} c^{\varepsilon(\beta-1)} \tag{13}$$

Given *c*, the left-hand side represents the labor supply curve while the right-hand side the labor demand curve. Using (10) and (13), *n* can be represented as a function of *k* and *p*:

$$n = \left[(1-a) k^{\alpha} p^{1-\varepsilon\beta}\right]^{\frac{1}{1+\chi-\beta}} \tag{14}$$

The equilibrium system of a decentralized market economy is represented as a two-dimensional dynamical system of *k* and *p*:

$$\dot{k}_t = A k_t^{\Delta} p_t^{\Omega} - \left(\frac{1}{p_t}\right) - (g+\delta) k_t = K(k_t, p_t) \tag{15}$$

$$\dot{p}_t = p_t \left[\rho + \delta + g - aA k_t^{\Delta-1} p_t^{\Omega}\right] = P(k_t, p_t) \tag{16}$$

where

$$\Delta = \frac{\alpha(1+\chi)}{1+\chi-\beta} \qquad \Omega = \frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta} \qquad \text{with} \quad A = (1-a)^{\frac{\beta}{1+\chi-\beta}}.$$

2.2 Steady-state Equilibrium and its Stability

The steady-state equilibrium (SE) is defined by $\dot{k} = \dot{p} = 0$. The SE values (k^*, p^*) thus satisfy

$$A(k^*)^{\Delta} (p^*)^{\Omega} = \frac{1}{p^*} + (g+\delta) k^* \tag{17}$$

$$A(k^*)^{\Delta}(p^*)^{\Omega} = \left(\frac{\rho+g+\delta}{a}\right)k^* \tag{18}$$

We call the locus of (k, p) that satisfies (17) and (18) the *kk* curve and the *pp* curve, respectively. The slope of the *kk* curve and that of the *pp* curve in the neighborhood of an SE are, respectively,

$$\frac{dp}{dk} = -\frac{K_k^*}{K_p^*} = \frac{\delta+g-\Delta Ak^{\Delta-1}p^{\Omega}}{Ak^{\Delta}\Omega p^{\Omega-1}+(1/p^2)} \quad , \qquad \frac{dp}{dk} = -\frac{P_k^*}{P_p^*} = -\frac{(\Delta-1)p}{k\Omega} \tag{19}$$

However, their signs are ambiguous. In order to examine the stability of the SE, we derive the linearized system of (15) and (16) evaluated at the SE:

$$\begin{pmatrix} \dot{k} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} K_k^* & K_p^* \\ P_k^* & P_p^* \end{pmatrix} \begin{pmatrix} k-k^* \\ p-p^* \end{pmatrix} \tag{20}$$

Denoting the coefficient matrix as J^*, each term of J^* is

$$K_k^* = \Delta A(k^*)^{\Delta-1}(p^*)^{\Omega} - (g+\delta)$$

$$= \frac{\alpha(1+\chi)}{1+\chi-\beta}\left(\frac{\rho+\delta+g}{a}\right) - (g+\delta) = \frac{\alpha(1+\chi)\rho+[(\alpha-a)(1+\chi)+a\beta](g+\delta)}{a(1+\chi-\beta)}$$

$$K_p^* = \Omega A(k^*)^{\Delta}(p^*)^{\Omega-1} + \frac{1}{(p^*)^2} = \frac{1}{(p^*)^2}\left\{1+\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}\left[\frac{\rho+g+\delta}{\rho+(1-a)(g+\delta)}\right]\right\}$$

$$P_k^* = p^*\left[-aA(\Delta-1)(k^*)^{\Delta-2}(p^*)^{\Omega}\right] \qquad = p^*\left[1-\frac{\alpha(1+\chi)}{1+\chi-\beta}\right]\left(\frac{\rho+\delta+g}{k^*}\right)$$

$$= (p^*)^2\left[1-\frac{\alpha(1+\chi)}{1+\chi-\beta}\right]\frac{[\rho+(1-a)(g+\delta)](\rho+g+\delta)}{a}$$

$$P_p^* = p^*[-aA(k^*)^{\Delta-1}\Omega(p^*)^{\Omega-1}] = -(\rho+\delta+g)\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}$$

The SE is *saddle-point stable* if and only if Det. J^*<0 holds, while it is *perfectly stable* if and only if both Trace J^*<0 and Det. J^*>0 hold, where

$$Trace.J^* = K_k^* + P_p^*$$

$$= \left[\frac{(\alpha - a)(1+\chi)(\delta + g) + \rho[\alpha(1+\chi) - a\beta]}{a(1+\chi-\beta)} \right] + \frac{\varepsilon\beta(1+\chi)(\rho+\delta+g)}{1+\chi-\beta} \qquad (21)$$

$$Det.J^* = K_k^* P_p^* - K_p^* P_k^*$$

$$= -\frac{(\rho+\delta+g)(1+\chi)[\rho+(1-a)(\delta+g)]}{a} \frac{1-\alpha-\varepsilon\beta}{1+\chi-\beta} \qquad (22)$$

We have two important findings from the above expressions. First, the SE is unique if it exists. This is because the signs of these terms are unambiguously determined if a set of exogenous parameter values are given. If we assume there exist more than one SE, we should conclude that equilibria with different stability properties coexist. However, this leads to a contradiction.[3] Second, the labor externality needs to be as strong as in Benhabib and Farmer (1994) (i.e., $1+\chi<\beta$) for the SE to be perfectly stable (i.e., indeterminacy of equilibrium). To see this, suppose that $1+\chi>\beta$ holds. Then $Trace.J^*>0$ necessarily holds because $\alpha(1+\chi)-a\beta>0$ holds under $\alpha \geq a$.

In the present model, the SE can be either saddle-point stable or unstable for $1+\chi>\beta$, while it can be perfectly stable as well for $1+\chi<\beta$.[4]

Lemma 1: *In the present model under the PCH, (i) an SE is unique if it exists. (ii) The labor externality needs to be as strong as in Benhabib and Farmer (1994) (i.e., $1+\chi<\beta$) for the SE to be perfectly stable (i.e., indeterminacy of equilibrium).*

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3 The author appreciates Professor Takumi Naito for his comment about this point.
4 This property of the model is the same as in Benhabib and Farmer (1994).

3. Equilibrium Dynamics under Saddle-point Stability

In this section we explore the properties of equilibrium dynamics under the PCH for a saddle-point stable SE. In this case a transition path converging to the SE is uniquely determined and the properties of this saddle-point path will be qualitatively the same among most cases both for $1+\chi>\beta$ and for $1+\chi<\beta$: per capita consumption and per capita capital both increase along the equilibrium path starting from a low level of initial capital stock. However, under $1+\chi>\beta$, we will find an interesting case specific to the PCH, in which consumption is initially chosen at a high level and begins to decrease along a transition path with capital accumulation. To convey this message concisely, we will compare the case for small values of ε ($K_p^*>0$) and the case for large values of ε ($K_p^*<0$), by focusing on the case of $1>\varepsilon>1/(1+\lambda)$ under $1+\chi>\beta$ ($K_k^*>0$ always holds). In Appendix, we show the case of $1>\varepsilon(1+\lambda)$ under $1+\chi>\beta$ and the case of $1+\chi<\beta$.[5]

Before showing properties of equilibrium paths, let us consider in what situation an SE will be saddle-point stable. From the necessary and sufficient condition for it $Det.J^* = K_k^*P_p^*-K_p^*P_k^*<0$, we have $1-\alpha-\varepsilon\beta>0$ which is equivalent to $(1-\alpha)/\beta>\varepsilon$ (see (22)). Thus $1>\alpha$ is necessary. By $\alpha+\beta>1$, $1>\varepsilon$ is also necessary. That is, in order for an SE to be saddle-point stable, the capital externality α needs be smaller than unity and the PCH function need to be concave. Furthermore, we need to separate cases of $1>\varepsilon$ in more details because the slopes of the *kk* and *pp* curves depend on the sign of K_p^* (i.e., of $1-\varepsilon(1+\lambda)$). However, the qualitative

5 Under $1+\chi<\beta$, an SE can be saddle-point stable both for $1>\varepsilon(1+\lambda)$ and for $1<\varepsilon(1+\lambda)$.

properties of an equilibrium path have turned out the same among most cases of $1 > \varepsilon$. Thus, we will focus on the case of $1 > \varepsilon > 1/(1+\chi)$ as the representative case, and first show the equilibrium dynamics for small values of ε ($K_p^* > 0$) in Fig. 2-1.[6] Then we will proceed to the case for large values of ε ($K_p^* < 0$) in Fig. 2-2, which gives rise to a new and interesting finding of this paper.[7]

3.1 Equilibrium Dynamics for $K_p^* > 0$

When the value of ε is so small that $K_p^* > 0$ holds under $1 > \varepsilon > 1/(1+\chi)$, we find that the saddle-point path is unique which starts from the low level k_0 of initial per capita capital and converges to the SE (see Fig. 2-1). On this equilibrium path, per capita consumption $c = 1/p$ is initially chosen at a low level and increases with per capita capital k along the path toward the SE. To understand Fig. 2-1, one should note that, under $1 > \varepsilon > 1/(1+\chi)$, $P_p^* > 0$ always holds while the signs of K_p^* and P_k^* are ambiguous. Because $K_p^* > 0$ holds, $Det.J^* < 0$ leads to $0 > -(K_k^*/K_p^*) > -(P_k^*/P_p^*)$. Thus the decreasing *kk* curve has a flatter slope than the (decreasing) *pp* curve.[8]

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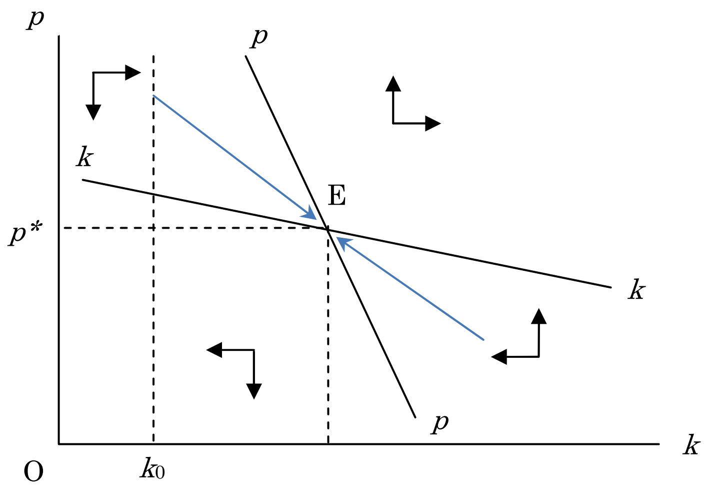
6 We show the dynamics for $1 > \varepsilon(1+\chi)$ in Fig. 1-1 and Fig.1-2 in Appendix, which are similar to this case.

7 The PCH parameter ε is included only in K_p^* and P_p^*. Suppose that the PCH effect is absent ($\varepsilon = 0$). Then $K_p^* > 0$ and $P_p^* < 0$ hold for $1+\chi > \beta$ while $P_p^* > 0$ (the sign of K_p^* is ambiguous) holds for $1+\chi < \beta$.

8 In Appendix we check whether each case of the figure is possible by the corresponding inequalities can hold simultaneously.

Fig.2-1. Saddle-point stable SE for $1+\chi > \beta$, $1 > \varepsilon > 1/(1+\chi)$ and $K_p^* > 0$

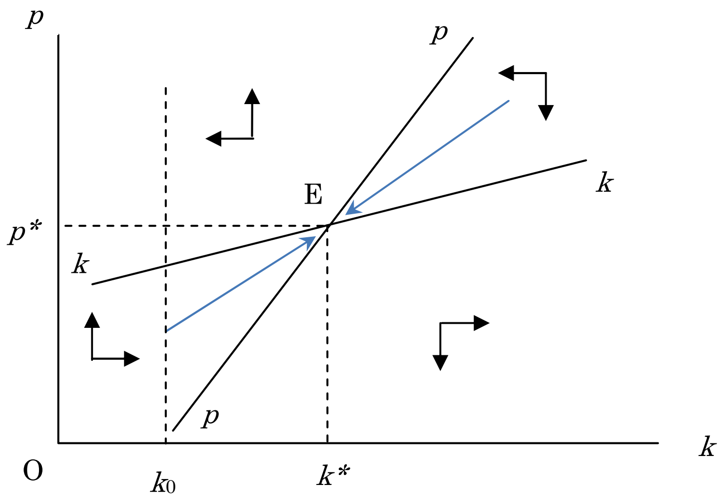


This property of equilibrium path is similar to that of Ramsey-Cass-Koopmans optimal growth model. However, an economic mechanism in our model has a different aspect because the PCH effect is present. Let us explain it intuitively. In the case of $1+\chi > \beta$, the labor demand curve is typically decreasing ($\beta - 1 < 0$). A low initial level of per capita consumption $c_0 = 1/p_0$ will make the labor demand curve $w = (1-a)k^{\alpha} n^{\beta-1} c^{\varepsilon(\beta-1)}$ lie in a high position but (because of $1-\varepsilon > 0$) the labor supply curve $w = c^{1-\varepsilon} n^{\chi}$ lie in a low position (see (13)). Both effects raise the equilibrium labor input n, and thus the labor input in efficiency units $h(c)n$ tends to be large. Therefore the rental rate r of capital will be high. By Euler equation (12), the costate variable p declines and thus per capita consumption c increases.

3.2 Equilibrium Dynamics for $K_p^* < 0$

Let us proceed to the case of $K_p^* < 0$ (ε is large) under $1 > \varepsilon > 1/(1+\chi)$. We find the equilibrium dynamics specific to the PCH in Fig.2-2: per capita consumption c is initially chosen at a high level and begins to decrease along the transition path starting from the low level k_0 of initial capital stock. To understand Fig. 2-2, one should note that $Det.J^* < 0$ leads to $0 < -(K_k^* / K_p^*) < -(P_k^* / P_p^*)$: the increasing *kk* curve is flatter than the *pp* curve.

Fig. 2-2. Saddle-point stable SE for $1+\chi > \beta$, $1 > \varepsilon > 1/(1+\chi)$ and $K_p^* < 0$



An intuitive explanation for this equilibrium path can be made by reversing the logic for the case of $K_p^* > 0$. A high level of initial per capita consumption c_0 will make the labor demand curve lie in a low position but (because of $1-\varepsilon > 0$) the labor supply curve lie in a high position (see (13)). Both effects decrease the equilibrium labor input n, and thus the labor input in efficiency units $h(c)n$ tends to be small. Therefore the rental rate r of capital will be low. By Euler equation (12), the costate variable p rises and thus per capita consumption c decreases. We summarize the results for a saddle-point stable SE including those in Appendix.

Proposition 1: (Equilibrium Dynamics for Saddle-point Stable SE)

Suppose that an SE is saddle point stable when the labor externality is so weak that $1+\chi > \beta$ *holds. Then both* $1 > \alpha$ *and* $1 > \varepsilon$ *necessarily hold.*

(i) If the value of ε *is so large that both* $1 > \varepsilon > 1/(1+\chi)$ *and* $K_p^* < 0$ *hold, per capita consumption is initially chosen at a high level and then begins to decrease with per capita capital accumulation along a transition path starting from a low level of initial capital stock.*

(ii) In the other cases, per capita consumption and per capita capital both increase along a transition path starting from a low level of initial capital stock (as in the standard growth model like Ramsey-Cass-Koopmans' optimal growth model).

Suppose that an SE is saddle-point stable when $1+\chi < \beta$ *holds, equilibrium dynamics has qualitatively the same properties as mentioned in (ii) above.*

Let us elucidate why this equilibrium dynamics can emerge in our PCH growth model and why it is interesting. First, recall the "intertemporal substitution (trade-off) between present and future consumption" in the standard (investment-driven) growth model like Ramsey-Cass-Koopmans optimal growth model. That is, when the representative consumer increases present consumption, the saving decreases from a given amount of income. It thereby decreases physical investment and thus capital accumulation will slow down. This gives rise to less future production and consumption.

By contrast, in the present model with the PCH effect, an increase in present consumption leads to an increase in present production at the same time, because labor productivity improves ($h'(c_t) > 0$). This weakens the corresponding decrease in saving and investment, and therefore mitigates the harsh intertemporal trade-off between present and future consumption. This insight has already been provided by Steger (2000, 2002). In our PCH (consumption-driven) growth analysis, we find that if the PCH effect on labor productivity is strong enough, an increase in present consumption will, not decrease but, *increase* investment in the present time. This gives rise to the "intertemporal *complementarity* between present and future consumption".[9] In such a situation, as shown in Fig. 2-2, consumers will choose high consumption expenditure in equilibrium and, by improving their labor productivity, get more savings at the initial point in time. Once capital begins to accumulate, they can afford to decrease consumption along the transition path, because an increase in capital k helps production grow until the economy reaches the SE. Note that we indeed have this special feature under $K_p^* < 0$, which says that an increase in c and thus a decline in $p = (1/c)$ increase the amount of saving K.

This finding must be theoretically interesting by itself. At the same time, it may provide a theoretical explanation for (at least, be consistent with) a seemingly paradoxical fact in the recent African growth miracle. According to Young (2012), sub-Saharan African countries have, for the past two decades, experienced a growth in living standards at about 3.4-3.7% per year, because of an increase in real consumption. More recently, African economies have shown a "consumption explosion" partly because of a rapid increase in international resource prices and thus in the resource revenues. However, taking into account the fact that African countries still suffer a shortage of physical capital including social infrastructure, the domestic rate of return of capital will be high and thus people should tend to invest, not consume, large part of

9 This does not always mean that consumers choose zero saving in our model, because they derive utility from a combination of consumption and leisure.

their income. This fact might be paradoxical because it suggests that African people choose high consumption expenditure when the level of per capita capital is still low (before capital accumulation sets in). The saddle-point path in Fig.2-2 which starts from a low level of initial capital stock must be consistent with this evidence.

4. Equilibrium Dynamics under Indeterminacy

Let us now investigate properties of equilibrium paths when an SE is perfectly stable. When an SE is perfectly stable, there exists a continuum of equilibrium paths that converge to the SE. We call this property indeterminacy of equilibrium.

As we have already pointed out, $1+\chi<\beta$ is necessary for indeterminacy to occur in our model as in Benhabib and Farmer (1994). This condition was criticized as unrealistic in 1990s because it implies the labor demand curve has a steeper slope than the (increasing) labor supply curve. Since the beginning of this century, however, it has already been well known that indeterminacy can easily occur even under small externalities in various realistic situations. For example, Benhabib and Nishimura（1998）and Mino（2001）found that in a two or three sector DGE model with sector-specific externalities, a small divergence between private and social factor intensity conditions generates indeterminacy rather easily even under constant-returns-to-scale technology.[10] Therefore, we could concentrate on exploring properties of macroeconomic dynamics

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[10] The reason why the parameter values for indeterminacy lie in unrealistic range in early studies was that they use too simple models, e.g., one-sector model or special specification of functions. It has been found that indeterminacy can easily occur under general formulations of the model from various sources such as non-separable utility function (Mino(1999), Bennett and Farmer (2000)), and imperfection of capital market.

under indeterminacy without worrying about the fact that $1+\chi<\beta$ is by itself unrealistic.

Before investigating properties of equilibrium paths, we should answer a theoretical question of whether an indeterminacy model can apply to poor developing economies with small or no capital externality. This is because both capital and labor externalities are needed for indeterminacy in the Benhabib and Farmer (1994) model (We can easily confirm it in the present model with $\varepsilon=0$ by finding that $Trace.J^{*}>0$ holds when either $\alpha=a$ or $\beta=1-a$ holds.) Poor developing economies are poor (at least partly) because they could not succeed in capital accumulation. The level of capital at each point in time could be regarded as a proxy for useful knowledge that has been accumulated through past production experiences, as shown in the formulation of learning-by-doing effects by Arrow (1962). In this sense, capital has a positive external effect. If poor economies have very small or no capital externality (this seems to be the case), indeterminacy cannot occur in those economies. However, in our model, indeterminacy can occur even without the capital externality if the PCH externality is strong enough. This is a new theoretical finding, implying that the Benhabib and Farmer (1994) model can apply not only to advanced countries but also to poor developing economies as a fundamental framework for macroeconomic dynamic analysis.

4.1 The PCH Externality as a Source of Indeterminacy

Let us first derive an implication of a necessary condition for indeterminacy under $1+\chi<\beta$. Solving $Trace.J^{*}<0$ for α, we get

$$\alpha>\frac{a(1+\chi)(\delta+g)+\beta[a\rho-\varepsilon(1+\chi)(\rho+\delta+g)]}{(1+\chi)(\rho+\delta+g)}=\underline{\alpha} \tag{23}$$

It means that the capital externality α needs to be strong enough even if the labor externality is so strong that $1+\chi<\beta$ holds. Benhabib and Farmer (1994) have already pointed out this property. A new finding of our paper is that the minimum value $\underline{\alpha}$ will be smaller when the value of the PCH parameter ε is larger.[11] That is, indeterminacy will occur more easily when the PCH externality is stronger.

Even if the capital externality is absent, indeterminacy can occur when the PCH externality is strong enough.[12] To see it, setting $\alpha=a$ and $\beta>1-a$, we get

$$Trace.J^* = \rho+\frac{\varepsilon\beta(1+\chi)(\rho+\delta+g)}{1+\chi-\beta} \tag{24}$$

Under $1+\chi<\beta$, $Trace.J^*<0$ is thus equivalent to

$$\varepsilon > \left(\frac{\rho}{\rho+\delta+g}\right)\left[\frac{1}{1+\chi}-\frac{1}{\beta}\right]>0 \tag{T}$$

So long as the labor externality is as strong as in Benhabib and Farmer (1994) (i.e., $1+\chi<\beta$ is satisfied), the necessary condition $Trace.J^*<0$ can be satisfied.[13] By $\alpha=a$, $Det.J^*>0$ is equivalent to $1-\alpha-\varepsilon\beta>0$, and thus

$$\varepsilon<\frac{1-a}{\beta} \tag{D}$$

In order for conditions (T) and (D) to be compatible, it is sufficient that

$$\frac{1-a}{\beta} > \left(\frac{\rho}{\rho+\delta+g}\right)\left[\frac{1}{1+\chi}-\frac{1}{\beta}\right]$$

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11 Another new finding is the role of population growth rate g on the value of $\underline{\alpha}$. See *Lemma 2*.

12 Benhabib and Farmer (1994) show that indeterminacy cannot occur in the absence of capital externality even when the labor externality is strong (p.29).

13 Conversely, $Trace.J^*>0$ necessarily holds when $\alpha>a$ and $\beta=1-a$ hold. Therefore the labor externality cannot be replaced with the PCH externality.

holds. Therefore, under $\alpha = a$ and $\beta > 1 - a$, indeterminacy will actually occur if the value of $\delta + g$ is so large that

$$\rho + \delta + g > \left(\frac{\rho}{1-a} \right) \left[\frac{\beta}{1+\chi} - 1 \right] \tag{K}$$

holds. For the case when $a = 0.33$, $\rho = 0.02$, $\delta = 0.1$, $\chi = 0$ and $g = 0.01$ hold, we have $\beta < 5.355$. Thus, other things being equal, indeterminacy will occur when the labor externality is relatively mild.

Proposition 2: (PCH Externality as a Source of Indeterminacy)

Suppose that the labor externality is so strong that $1 + \chi < \beta$ holds. In order for indeterminacy to occur, the capital externality α needs to be larger than $\underline{\alpha}$ defined by (23). Under these conditions, we find the following results.

(i) The range of α needed for indeterminacy is larger (the value of $\underline{\alpha}$ is lower) when *the value of ε is larger. That is, when the PCH externality gets stronger, the possibility of indeterminacy will expand in such a direction that weaker capital externality is needed.*

(ii) Even when the capital externality is absent ($\alpha = a$ and $\beta > 1 - a$), indeterminacy will actually occur if the value of $\delta + g$ is so large that (K) holds and if the PCH parameter ε satisfies both (T) and (D).

Let us make an intuitive explanation for why indeterminacy occurs due to the PCH externality by focusing on the labor market equilibrium. First of all, $1 + \chi < \beta$ means by (13) that the slope of the labor demand curve ($\beta - 1$) exceeds that of the labor supply curve (χ). In order to show that an SE is perfectly stable (indeterminate), we should say that when the value of p deviates from the initial SE (k^*, p^*), it will return to its initial value even if the SE value of k^* remains unchanged (i.e., we do not need to choose a new value of *k* from the saddle-point path). Let us now suppose that only the value of p increases from the SE (by some exogenous shock). Because consumption c decreases, labor productivity $h(c)$ declines due to the PCH externality. It gives rise to a downward shift of the (upward sloping) labor demand curve. If the corresponding rise in labor input n is large enough, labor input in efficiency units $h(c)n$ also increases. It raises the marginal productivity of capital, and thus the rental rate r of capital will increase. By Euler equation (12), $\dot{p} < 0$ holds. Therefore, the value of p declines toward its initial value. Conversely, if the PCH effect were absent in this process, the labor demand curve would not shift downward, and this mechanism would not work.

In addition, we could understand why indeterminacy can occur without the capital externality as follows. The mechanism above is independent of (a movement of) the capital externality, because we have made the above explanation on the presumption that the value of *k* remains unchanged at the SE value of k^*. In particular, even without the capital externality, this logic keeps working when the PCH externality is present under $1+\chi < \beta$.

4.2 Properties of Equilibrium Dynamics under Indeterminacy

We will proceed to explore equilibrium dynamics under Indeterminacy. First, let us examine in what situation they emerge by using the necessary and sufficient condition "$Trace.J^* < 0$ and $Det.J^* > 0$". Under $1+\chi < \beta$, we have $Det.J^* < 0$ whenever $\varepsilon(1+\chi) > 1$ holds, because $1-\alpha-\varepsilon\beta < 1-\alpha-\varepsilon(1+\chi) < [1-\varepsilon(1+\chi)] - \alpha < 0$ holds.

Therefore $1 > \varepsilon(1+\chi)$ is necessary for $Det.J^* > 0$. Thus, by (22), we obtain $1-\alpha-\varepsilon\beta > 0$, which is equivalent to $(1-\alpha)/\beta > \varepsilon$. Therefore the SE is perfectly stable (indeterminate) when the capital externality α is smaller than unity and the PCH function is so strongly concave that $(1-\alpha)/\beta > \varepsilon$ holds (Note that $(1-\alpha)/\beta < 1/(1+\chi)$).

Let us now investigate two implications of equilibrium dynamics under Indeterminacy. The first implication has already been well known in the literature while the second one seems new and specific to our model under the PCH.

The first implication under indeterminacy is that there exists a continuum of (transition) equilibrium paths starting from a given initial capital stock. Each path will be selected by choosing the corresponding initial value of per capita consumption. (Fig.3-4 and Fig.3-5 illustrate one of the possible paths). Poor societies with the same economic characteristics may experience very different growth process on the path toward the SE even if they start from the same level of initial capital stock. Thus, if the consumption-driven economic growth is at work and “poverty reduction” is proceeding in low-income Asian, Latin American or African countries, they may exhibit various dynamic process, depending on the difference in non-economic (political, legal, social, cultural, or even religious) factors. In this situation, “coordination of expectations” will play a key role in choosing an equilibrium path. If a government can induce private agents to coordinate expectations on the growth path their society will follow, it may be an important development policy (see Krugman (1991), Matsuyama (1991), Fukao and Benabou (1993)).

Fig.3-4. Equilibrium dynamics under Indeterminacy for $1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and

$K_p^*<0$

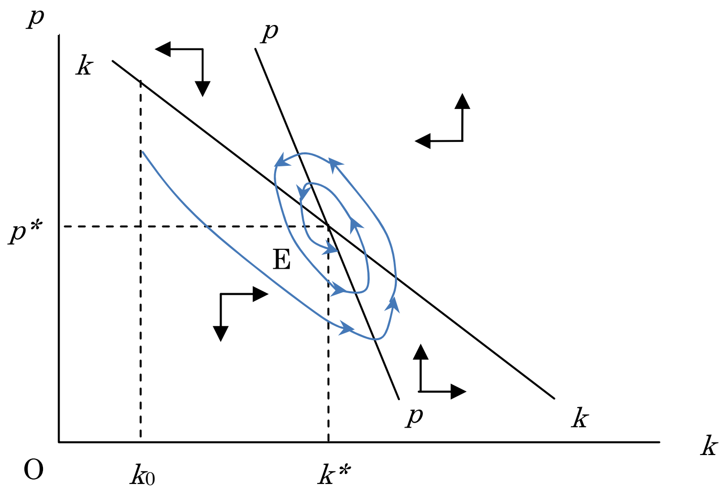
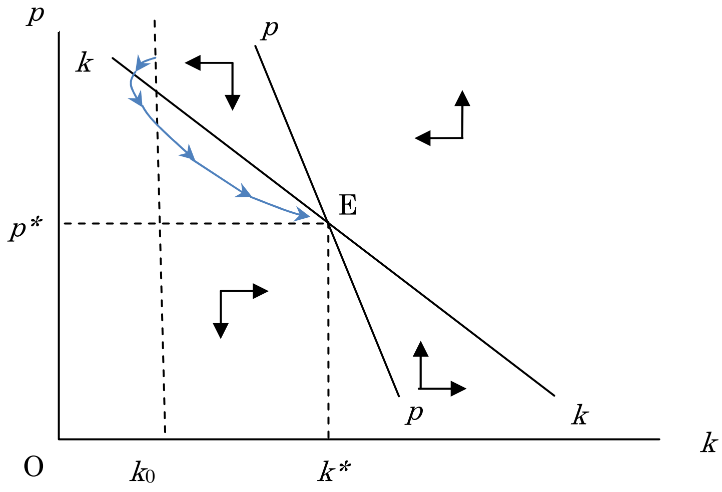


Fig.3-5. Equilibrium Dynamics under Indeterminacy for $1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and

$K_p^*<0$



The second implication will be much more important. Equilibrium dynamics under indeterminacy exhibit complicated behaviors, as shown in Fig.3-4 and 3-5. The dynamics along a saddle-point path are very different from a simple monotonic movement toward an SE. In Fig.3-4, the equilibrium path starting from a low level k_0 of initial capital stock will exhibit a damped oscillation, converging to the SE. That is, a poor economy where the PCH applies may experience “overshooting” (and “undershooting) of per capita capital beyond its SE level k^*, and follow cyclical movements until it reaches the SE. This implies that the macroeconomic dynamics of Human Development (i.e., a growth process under the PCH) may differ substantially from those in the standard (Solow-Swan type or Ramsey-Cass-Koopmans type) growth model. In this respect, the analysis of the PCH growth model in the present paper can contribute to a new understanding of the economic mechanism specific to poor economies. Fig.3-5 shows another possible equilibrium path. It exhibits no cyclical movements but a non-monotonic behavior.[14] In either case, we should not always expect a poor economy will follow a simple monotonic movement toward the SE even when it succeeds in (consumption-driven) economic growth or “poverty reduction” through HD.

Proposition 3: (Equilibrium Dynamics under Indeterminacy)

Suppose that the labor externality is so weak that $1+\chi > \beta$ *holds and that the SE is perfectly stable (indeterminate). Then we have two implications under indeterminacy.*

(i) Poor societies with the same economic characteristics which start to grow from the same level of initial capital stock may follow a very different equilibrium path of per capita consumption and per capita capital toward the SE, depending

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[14] An equilibrium path that converges to the SE monotonically can emerge if the representative consumer chooses the corresponding initial level of per capita consumption.

on non-economic factors. An equilibrium path will actually be chosen by coordination of expectations.

(ii) Transition paths of per capita consumption and per capita capital may either exhibit a cyclical behavior or simpler non-monotonic movements.

Let us examine whether these complicated behaviors of equilibrium paths could be consistent with reality. Since the independence of colonial areas in 1950s-60s, there have been low-income economies all over the world, especially in Asia, Latin America and Africa. Blue lines in "Table: Indeterminacy Cases" show the time series data (from *World Development Indicators*) of per capita real GDP in Latin American and African countries.[15] The red line in each table is the corresponding approximating curve. They represent the long-run trend, which could be interpreted as the SE path.[16] Then we find that per capita real GDP of Brazil, Guatemala, Algeria, Kenya and Swaziland first moved with ups and downs around the SE path, and their amplitudes have been damped gradually. These movements seem consistent with the oscillation in Fig.3-4. The data of per capita real GDP in Haiti during 1991-2011 may be consistent with the non-monotonic path in Fig. 3-5.

We might be able to add some more information for realistic relevance of using the Indeterminacy cases. It is often reported that the population growth rates in modern developing countries have been declining in recent years while they were very high in the era of the "population explosion". From (23), we can show the next lemma.[17]

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15 We use annual data of GDP because capital stock data are not available and consumption data lack in many years for low income and middle low income countries in WDI.

16 However, the upward or downward (and horizontal) slope of the SE path (red line in the Tables) cannot be explained in the present model. They may be determined by, e.g., the availability of infrastructure, frequency of political or social conflicts, etc..

17 Divide the numerator and denominator of (23) by ($\delta + g$) and differentiate partially the resulting

Lemma 2*: The range of the capital externality (α) needed for indeterminacy expands (i.e., the value of $\underline{\alpha}$ decreases) when the population growth rate (g) increases.*

Thus, Indeterminacy cases are more likely to happen during the period when the population growth rate is high.

5. Equilibrium Dynamics under Hoph Bifurcation

Finally we will investigate equilibrium dynamics when the Hoph bifurcation occurs. There may exist an invariant closed curve (non-stationary periodic solution or orbit) surrounding the SE point in our model. We show this fact based on the Hopf bifurcation theorem by considering a simpler model with no capital externality ($\alpha = a$ and $\beta > 1 - a$) in Appendix. Let us summarize this result in the next lemma.[18]

Lemma 3 *(Existence of Hoph Bifurcation): In the present model under the PCH, a periodic solution (an invariant closed curve) may emerge around the SE point as a stationary equilibrium when the PCH parameter ε changes. The bifurcation value is* ε_0 .

The implications of the existence of a closed orbit as an equilibrium path will be different, depending on in which range of the PCH parameter ($\varepsilon < \varepsilon_0$ or $\varepsilon_0 < \varepsilon$) the

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equation w.r.t. $x = \rho / (\delta + g)$.

[18] We do not always have to examine how many invariant closed curves exist, because it would not add any particular economic conclusions to the discussions below.

closed orbit will emerge. However, the condition for deciding in which range the closed orbit will emerge generally depends on the third partial derivatives of $K(k, p)$ and $P(k, p)$ (Guckenheimer and Holmes (2002), p.152). Taking into consideration that we cannot generally tell the economic meaning of the third derivative of a function, we will not be able to say which situation will happen more easily from an economic viewpoint. Thus we would like to show both cases here without explicitly showing the condition for separating these situations, and explore their economic implications.[19]

5.1 Stable Closed Orbit and Poverty Reduction by Human Development

If a close orbit emerges in the range of small values of ε ($\varepsilon < \varepsilon_0$), the SE point is unstable while the closed curve satisfies orbital stability (i.e., supercritical Hoph bifurcation). As shown in Fig.5-1, an equilibrium path will converge with cyclical movements to this closed orbit if it starts from the initial point either within this closed curve or outside of it. Because an economy stays on this closed curve, the value of $k(t)p(t)$ is finite at any point in time and thus the TVC $\lim_{t\to\infty} k(t)p(t)e^{-\rho t} = 0$ is satisfied by the effect of the discount rate $\rho > 0$. Therefore this closed orbit will certainly be a stationary equilibrium in the long-run. Per capita consumption and capital will not converge to the SE but instead endogenously follow a cyclical movement.

Proposition 4: (Equilibrium Dynamics under Supercritical Hopf Bifurcation)

When a closed orbit emerges in the range of small values of the PCH parameter ε *(*$\varepsilon < \varepsilon_0$*), the SE point is unstable while the closed curve satisfies orbital stability. An*

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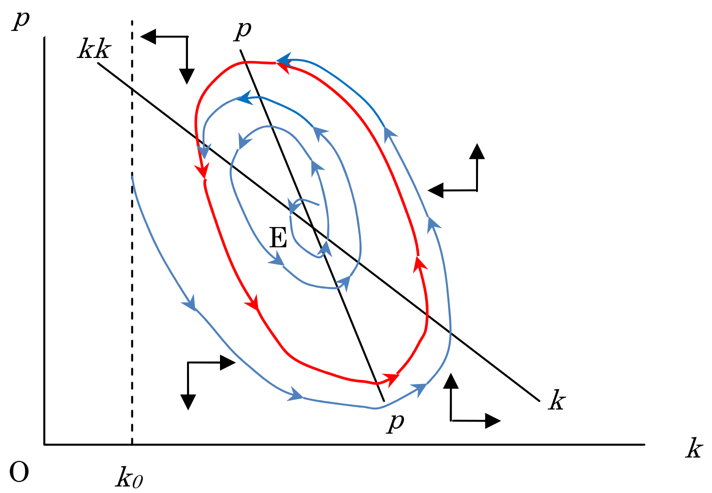
19 The similar way of presenting analytical results has been used in Benhabib and Miyao (1981) and Yoshida (2003).

equilibrium path does not converge to the SE and per capita consumption and capital follow an endogenous periodic cycle.

Fig.5-1 Endogenous Economic Cycle (Stable periodic orbit: supercritical Hopf bifurcation)



This possibility of an endogenous economic cycle has an important implication for understanding how the poverty reduction by HD (i.e., the equilibrium growth path under the PCH) will proceed. That is, when a poor developing economy where the PCH applies succeeds in consumption-driven economic growth, per capita capital will not grow smoothly to the SE path but move back and forth around it. Because of this new finding, we should *not* expect the growth process under the PCH will be qualitatively similar to that (a saddle-point path) in the standard (investment- driven) growth model like Ramsey-Cass-Koopmans optimal growth model.

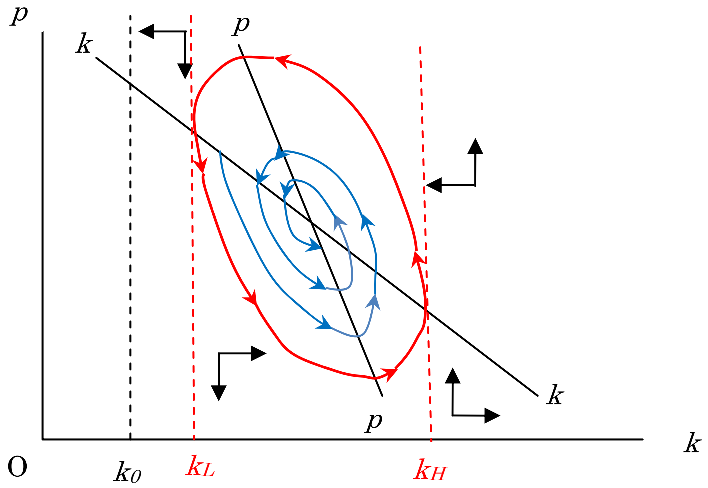
From a viewpoint of development policy, we should carefully suppose that economies who succeed in the consumption-driven growth that has been expected under poverty reduction strategies by HD may, not only exhibit a cyclical movement toward an SE (as shown in Fig.3-4), but also keep moving on an invariant closed curve without approaching to the SE point. We need to be careful in this respect when evaluating the success or failure of poverty reduction strategies including foreign aid.

Let us explore whether these dynamics could be regarded as consistent with reality. “Tables: Bifurcation Cases” show that per capita real GDP of Latin American countries such as Belize, Panama and Puerto Rico moved along the long-run trend during the first decade of 1960-2011, and then diverged from it with expanding cyclical movements during the recent 30 years. We observe similar evidence for African countries such as Mali and Tunisia. These data seem consistent with a cyclical equilibrium path starting from a point within the closed orbit and converging to this orbit in Fig.5-1. In contrast, the per capita real GDP data of other African countries such as Chad, Democratic Republic of the Congo, the Gambia and Zambia show cyclical movements (i.e., ups and downs) around the SE path during the whole period. These movements seem consistent with the endogenous economic cycle in Fig.5-1.

5.2 Unstable Closed Orbit and the Creation of a Market Economy by “Big Push”

If a close orbit emerges in the range of large values of ε ($\varepsilon_0 < \varepsilon$), we obtain the result that is quite different from above. In this case, the SE point is stable while the closed curve satisfies orbital *in*stability (i.e., subcritical Hoph bifurcation). As shown in Fig.5-2, a dynamic path will converge with cyclical movements to the SE if it starts from the initial point within this closed curve. Since the TVC is satisfied at the SE, this path is certainly an equilibrium path. However, if a dynamic path starts from the initial point outside of the unstable closed orbit, it will go farther from this orbit with cyclical movements. Thus it reaches either the *k* axis or the *p* axis in a finite time, violating the capital accumulation equation or Euler equation. Therefore this dynamic path is *not* an equilibrium path of the decentralized market economy.

Fig.5-2. Creation of a Market Economy by “Big Push” (Unstable periodic orbit: subcritical Hopf bifurcation)



These results represent an interesting and important insight. In Fig.5-2, if the initial capital stock k_0 lies above k_L and below k_H (which is not shown explicitly), the representative consumer chooses the initial level c_0 of per capita consumption either from an equilibrium path located within the closed orbit or from the closed orbit itself. In the former case the equilibrium path converges to the SE point while in the latter case the economy gets into an endogenous economic cycle.

By contrast, the initial level k_0 of per capita capital stock lies below k_L, there exist no corresponding equilibrium paths for a decentralized market economy. That is, even if a poor society which the PCH applies intends to take a decentralized market economy system, they cannot establish that system. This is a new theoretical finding of this paper and must be worth being emphasized, because none of the previous research on non-linear economic dynamics has recognized it. Rather, they put emphasis on the concept of "corridor stability" (e.g., Benhabib and Miyao (1981)), which means that when an exogenous shock changes an SE, an economy will converge to a new SE again if the shock is small while it may get into a cycle when the shock is large. The reason for this neglect must probably be that the previous research did not always examine explicitly whether a dynamic path satisfies all the equilibrium conditions of a market economy or not. By contrast, we have carefully examined whether each path satisfies equilibrium conditions. In other words, what is important and interesting from a viewpoint of economic theory will be research on non-linear *equilibrium* dynamics, not simply on non-linear dynamics.

From a viewpoint of development policy, the result mentioned above implies a new kind of "underdevelopment (poverty) trap". It implies that an initial shortage of capital accumulation may be a cause of the failure to establish a market economy system in poor developing countries. Such societies, by increasing the initial level k_0 of capital stock up to k_L (but below k_H), will be able to shift from this regime to a regime where it can establish a decentralized market economy system. In this sense, the "Big Push" may be a useful development policy to create and establish a market economy system in poor developing economy. This insight on the role of the "Big Push" for the creation of a market economy is also a new finding of this paper, which has never been derived theoretically.

Proposition 5: (Equilibrium Dynamics under Subcritical Hopf Bifurcation)

(i) *When a closed orbit emerges in the range of large values of the PCH parameter* ε *(*$\varepsilon_0 < \varepsilon$*), the SE point is stable while the closed curve satisfies orbital instability. An equilibrium path starting from the initial capital stock* k_0 *above* k_L *(and below* k_H*) will converge to the SE or stays on the closed orbit, that is, per capita consumption and per capita capital follow an endogenous periodic cycle.*

(ii) *If the initial level* k_0 *of per capita capital stock lies below* k_L*, there exist no corresponding equilibrium paths for a decentralized market economy. In this situation, a society will be able to shift to a regime where it can establish a decentralized market economy system by increasing the initial level* k_0 *of capital stock up to* k_L *(but below* k_H*)., that is, "Big Push".*

6. Dynamic Analysis of "Human Development Aid"

In this section we will analyze what kind of equilibrium path will be induced by foreign aid that enhances workers' labor productivity through an improvement of nutrition, health and basic education. We call this kind of aid "human development aid (HDA)", with a slight abuse of terminology. An interesting question here must probably be whether we can always expect that an introduction of HDA will succeed in inducing an equilibrium path that converges monotonically to an SE with higher welfare. If we can, we will be able to evaluate easily the success or failure of HDA. Otherwise, we should be more careful when evaluating it and suppose that a growth process induced by HDA may attain higher SE welfare even if it exhibits complicated (non-monotonic) movements.

To introduce the effect of HDA, we replace $h(c)$ with $\theta h(c) = \theta c^{\varepsilon}$ and assume an exogenous increase in parameter $\theta > 0$. It could be interpreted as this country receiving foreign aid because it incurs no cost for an increase in θ. In this formulation, each worker's labor productivity improves when the level of consumption remains unchanged. Examples of this kind of aid include cases when medical doctors from foreign countries teach people in developing countries an effective cure for dehydration or diseases, when specialists from advanced countries provide them with technological knowledge for obtaining clean water, or when an international development institution gives poor people suffering from malaria mosquito nets for free.

In this modified model, we can rewrite (14) into：

$$n = \left[\theta^{\beta}(1-a)k^{\alpha} p^{1-\varepsilon\beta}\right]^{\frac{1}{1+\chi-\beta}} \tag{14'}$$

The equilibrium dynamic system is:

$$\dot{k}(t) = \theta^{\frac{1+\chi}{1+\chi-\beta}} Ak(t)^{\frac{\alpha(1+\chi)}{1+\chi-\beta}} p(t)^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}} - \left(\frac{1}{p(t)}\right) - (\delta+g)k(t) = \tilde{K}(k, p:\theta) \tag{15'}$$

$$\dot{p}(t) = p(t)\left[\rho+\delta+g-\theta^{\frac{1+\chi}{1+\chi-\beta}} aAk(t)^{\frac{\alpha(1+\chi)}{1+\chi-\beta}-1} p(t)^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}}\right] = \tilde{P}(k, p:\theta) \tag{16'}$$

An SE is defined by $\tilde{K}(k, p:\theta) = 0$ and $\tilde{P}(k, p:\theta) = 0$. Totally differentiating them yields:

$$\begin{pmatrix} \tilde{K}_k^* & \tilde{K}_p^* \\ \tilde{P}_k^* & \tilde{P}_p^* \end{pmatrix}\begin{pmatrix} dk \\ dp \end{pmatrix} = \begin{pmatrix} -\tilde{K}_\theta^* \\ -P_\theta^* \end{pmatrix} d\theta$$

The coefficient matrix evaluated at the SE is the same as J^* in section 2 ($K_k{}^* = \tilde{K}_k{}^*$, $K_p{}^* = \tilde{K}_p{}^*$, $P_k^* = \tilde{P}_k^*$, $P_p^* = \tilde{P}_p^*$). Furthermore, the partial derivatives with respect to θ will be:

$$\tilde{K}_\theta = \left(\frac{1+\chi}{1+\chi-\beta}\right)\theta^{\frac{1+\chi}{1+\chi-\beta}-1} Ak^{\frac{\alpha(1+\chi)}{1+\chi-\beta}} p^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}}$$

$$\tilde{P}_\theta = p\left[-\left(\frac{1+\chi}{1+\chi-\beta}\right)\theta^{\frac{1+\chi}{1+\chi-\beta}-1} aAk^{\frac{\alpha(1+\chi)}{1+\chi-\beta}-1} p^{\frac{\beta[1-\varepsilon(1+\chi)]}{1+\chi-\beta}}\right]$$

The comparative-static results are：

$$\frac{dk^*}{d\theta} = \frac{\tilde{K}_p\tilde{P}_\theta - \tilde{P}_p\tilde{K}_\theta}{Det.J^*} \qquad \frac{dp^*}{d\theta} = \frac{\tilde{K}_\theta\tilde{P}_k - \tilde{P}_\theta\tilde{K}_k}{Det.J^*}$$

Let us look for situations where the welfare level in the SE

$$U^* = \frac{1}{\rho}\left[\log\frac{1}{p^*} - \frac{(n^*)^{1+\chi}}{1+\chi}\right]$$

improves by an introduction of HDA. The SE welfare U^* improves when p^* and n^* both decrease. To find that situation, we first see that

$$p^*k^* = \frac{a}{\rho + (1-a)(\delta+g)}$$

holds in an SE using (15') and (16'). Thus k^* and p^* move in the opposite directions. Next we derive a rate of change (represented by a hat notation) in n from (14'):

$$\hat{n}^* = \frac{1}{1+\chi-\beta}\left[\beta\hat{\theta} + \alpha\hat{k}^* + (1-\varepsilon\beta)\hat{p}^*\right] \qquad (14")$$

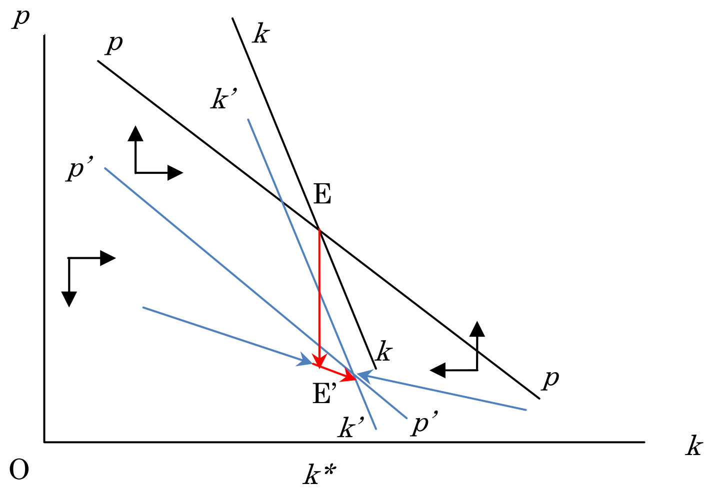
We find two situations when the SE welfare U^* necessarily improves, by investigating all the cases based on the criteria $1+\chi<\beta$ and $\varepsilon(1+\chi)>1$: under $1+\chi<\beta$, p^* and n^* both decrease if $\hat{k}^*>0$ and $1-\varepsilon\beta<0$ holds. In these two situations under $1+\chi<\beta$ and $1>\varepsilon(1+\chi)$, an increase in θ induces a downward shift both of the *kk* and of the *pp* curves. The new SE is located southeast of the initial SE (while k^* increase, p^* and n^* decrease. Thus the SE welfare is higher). Note that solving $1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $1-\varepsilon\beta<0$ simultaneously for ε yields $(1/\beta)<\varepsilon<1/(1+\chi)$. Thus the three inequalities hold simultaneously when the value of ε lies in this range.

Before showing the property of equilibrium dynamics, we should pay attention to the fact that labor supply n^* decreases by an introduction of HDA. Even when the SE welfare improves, people facing the HDA intend to decrease their labor supply. It means that HDA will weaken an incentive for work effort while it induces an increase in consumption and capital accumulation.

6.1 Welfare Improving Aid for Saddle-point Stable SE

The first situation of the SE welfare improving HDA happens when the SE is saddle-point stable, shown in Fig. 6-1 (corresponding to Fig.3-2). When HDA arrives from abroad, the economy will jump on to the new transition path by increasing consumption temporarily. Then capital stock *k* begins to accumulate. During this process consumption *c* increases along the path toward the new SE (E'). At point E', it enjoys higher welfare than in the initial SE. This seems a situation for welfare improving aid that we naturally expect.

Fig.6-1. Welfare Improving HDA for Saddle-point stable SE ($1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $K_p^*<0$)

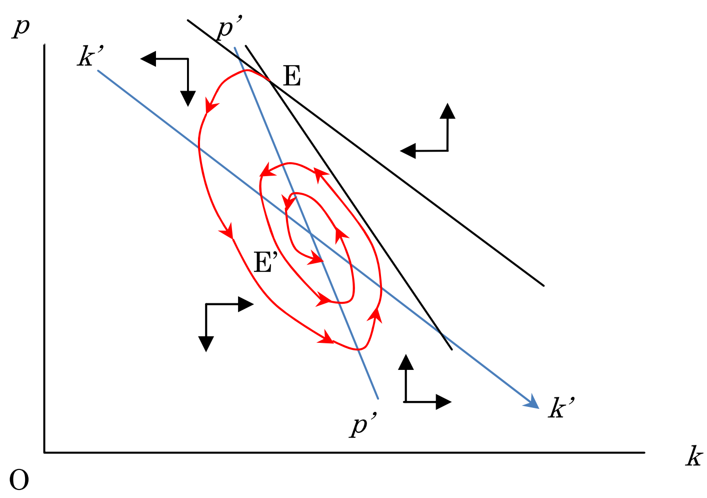


Furthermore, we should notice that welfare improves not only in the SE but also on the transition path in Fig. 6-1. To see it, recall $1+\chi<\beta$ and $1-\varepsilon\beta<0$ hold in (14'). A jump from point E means that p_t declines with k^* remaining constant. Thus labor supply n_t decreases. The instantaneous utility $u=\log(1/p(t))-n(t)^{1+\lambda}/(1+\chi)$ rises at that point in time. On the transition path toward E', the instantaneous utility increases because k_t rises and p_t declines (n_t rises). Therefore an introduction of HDA improves total welfare on the whole process from the initial SE to the new SE.

6.2 Welfare Improving Aid for Perfectly Stable SE

The second situation of the SE welfare improving HDA happens when the SE is perfectly stable, shown in Fig. 6-2 (corresponding to Fig.3-4). When HDA arrives from abroad, consumption and capital stock will begin to move toward the new SE by following a cyclical transition path. In this situation an economy can enjoy higher welfare in the SE. However, it will exhibit complicated dynamics for consumption and capital stock. That is, at first consumption c and capital stock k both begin to decrease, and then only consumption c begins to rise. After that, capital stock k also increases during the consumption-increasing process, and then consumption c begins to decrease while capital stock k increases. Further, both consumption and capital stock decrease. Following this cyclical movement, an economy will reach the new SE, where it enjoys higher welfare than in the initial SE. (It is difficult to evaluate welfare along the transition path in Fig.6-2.)

Fig.6-2. Welfare Improving HDA for Perfectly Stable SE ($1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $K_p^*<0$)



Based on this result, we need to be careful when evaluating the effects of HDA. A poor developing economy may begin to exhibit complicated dynamics when HDA is introduced. It may not show simple monotonic movements from which we can regard the aid as successful and effective for an increase in consumption and capital accumulation. The economy which will attain higher welfare ultimately (in the new SE) may exhibit a decrease in consumption or capital depreciation before it reaches the new SE. In such a situation, we are inclined to interpret the HDA as a failure. However, the welfare in the new SE is certainly higher than before the aid comes in (if time discount rate ρ is very low, the total welfare may increase).

Finally, let us explain three effects of an increase in θ. The first and the second effects are at work even if the PCH externality is absent while the third is specific to the PCH. First, an increase in θ, other things being equal, raises labor input in efficiency units $\theta h(c)n$ and thus the output. The saving and investment increase, given the initial level of consumption. Because of the corresponding capital accumulation, the rental rate r of capital declines along the marginal product of capital (MPK) curve. By Euler equation, this leads to $\dot{p} > 0$, and thus p^* tends to rise while consumption c^* to decrease. The second effect comes from the upward shift of the MPK curve due to an increase in $\theta h(c)n$. This upward shift leads to a rise in the rental rate $r = ak^{\alpha-1}[\theta h(c)n]^{\beta}$. Thus p^* tends to decline while consumption c^* to increase. These two effects, as you see, work in the opposite directions.

The third effect specific to the PCH will affect the labor market equilibrium through a change in the PCH function $h(c) = c^{\varepsilon}$. Note that given c, the labor demand curve $w = (1-a)k^{\alpha}[\theta h(c)n]^{\beta-1}$ is steeper than the increasing labor supply curve $w = c^{1-\varepsilon}n^{\chi}$ in Fig.6-1 and 6-2 (because of $\chi < \beta - 1$). Consider the situation where the first effect mentioned abov e dominates and consumption c decreases. Then a decrease in c makes the labor demand curve shift down because $h(c)$ decreases. Given an increasing labor supply curve $w = c^{1-\varepsilon}n^{\chi}$, labor input n tends to expand. However, in Fig.6-1 and 6-2 where $1 > \varepsilon(1+\chi)$ holds and thus the concavity of $h(c) = c^{\varepsilon}$ is relatively strong, one unit decrease in consumption induces a smaller decline in $h(c)$. Therefore labor input in efficiency units $\theta h(c)n$ tends to increase. Since the rental rate $r = ak^{\alpha-1}[\theta h(c)n]^{\beta}$ tends to rise, consumption c^* in the SE tends to increase. (This tendency will be offset partially if we take account of the downward shift of the labor supply curve $w = c^{1-\varepsilon}n^{\chi}$.) In this sense, the PCH external effect contributes to an increase in consumption and welfare improvement by an introduction of HDA.

Proposition 6 (Characterization of Equilibrium Dynamics due to Welfare Improving Human Development Aid): *An introduction of human development aid (HDA)can improve welfare in the SE under* $1+\chi<\beta$ *and* $1>\varepsilon(1+\chi)$.

(i)In the case of saddle-point stable SE in Fig.6-1, an economy will jump on to the new transition path by temporarily increasing consumption. Then capital stock k accumulates and consumption c increases along the equilibrium path toward the new SE (E'). Total welfare both in the SE and along the transition path improves.

(ii) In the case of perfectly stable SE in Fig.6-2, consumption and capital stock will begin to move toward the new SE by following a cyclical transition path. An economy enjoys higher welfare in the new SE. However, the welfare effect along the transition path is ambiguous.

(iii) Even if an introduction of HDA improves the SE welfare, it weakens an incentive for labor supply in the recipient country, decreasing their working hours.

7. Conclusions

This paper has investigated macroeconomic dynamics of a poor developing society in which an increase in per capita consumption improves worker's productivity. We have introduced the "productive consumption hypothesis (PCH)" by Steger (2000, 2002) into the one-sector RBC model with capital- and labor- generated externalities by Benhabib and Famer (1994), which seems to provide a highly general framework of macroeconomic dynamic analysis.

We have obtained five conclusions about equilibrium dynamics under "human development" of a poor developing economy where the PCH applies. First, we find that this model can apply to poor economies in that even in the absence of capital externality indeterminacy occurs when the PCH externality is strong enough. It means that the Benhabib and Farmer's (1994) model could provide a good framework of macroeconomic dynamic analysis not only for advanced countries but for poor developing economies. Second, in contrast to the standard growth models, the "intertemporal *complementarity* between present and future consumption" may hold when a steady-state equilibrium (SE) is saddle-point stable. In this case per capita consumption is initially chosen at a high level and then begins to decrease along a transition path starting from low initial capital stock. In the other cases, both consumption and capital increase along the transition path, as in the standard models like Ramsey-Cass-Koopmans optimal growth model. Third, when an SE is perfectly stable, an equilibrium path typically converges to the SE with cyclical movements. It may either diverge from the SE with cyclical movements or exhibit an endogenous cycle when the supercritical Hoph bifurcation occurs. These results are consistent with real date of low-income (Latin American and African) developing economies for the past several decades. Forth, when the subcritical Hoph bifurcation occurs, no equilibrium paths exist if initial capital stock falls short of a critical level. This is the first theoretical explanation for how a poor society fails to establish a market economy system. “Big Push” may help escape from this new type of “underdevelopment trap”. Finally, an introduction of human development aid (HAD) can induce a cyclical path converging to a new SE with higher welfare. We should be careful when evaluating the success or failure of HDA, because we cannot always expect that a poor economy will follow a smooth growth process even if the aid improves SE welfare.

Let us finally point out research agenda for the future. First, an extension to an open economy model must be important, because international trade and capital movements play an important role even in poor developing countries. Second, we could proceed to a two or three sector model, in which we will be able to derive more realistic conditions for indeterminacy. Third, it may be interesting to endogenize the population growth rate.

Appendix A. Equilibrium Dynamics for Saddle-point Stable SE

A.1 Cases of $1 > \varepsilon(1+\chi)$ *under* $1+\chi > \beta$ *:*

Under $1+\chi > \beta$, we have qualitatively the same equilibrium paths for $1 > \varepsilon(1+\chi)$ as those in Fig.2-1 in the text. In both cases shown in Fig. 1-1 and 1-2, $K_p^* > 0$ and thus $-(K_k^* / K_p^*) < 0$ holds (the *kk* curve is decreasing). By $Det.J^* < 0$, we get $-(K_k^* / K_p^*) <$ $-(P_k^* / P_p^*)$ (the slope of the *pp* curve is larger than that of the decreasing *kk* curve). Since $P_p^* < 0$ holds but the sign of P_k^* is ambiguous, the *pp* curve can be either increasing or decreasing. Fig. 1-1 shows the case of $P_k^* > 0$ while Fig. 1-2 that of $P_k^* < 0$. In either case, an equilibrium path will be uniquely determined if it starts from the low level k_0 of initial capital stock. Per capita consumption c and capital k both increase monotonically along the path, converging to the SE.

Fig.1-1.Saddle-point Stable SE under $1+\chi > \beta$, $1 > \varepsilon(1+\chi)$ and $P_k^* > 0$

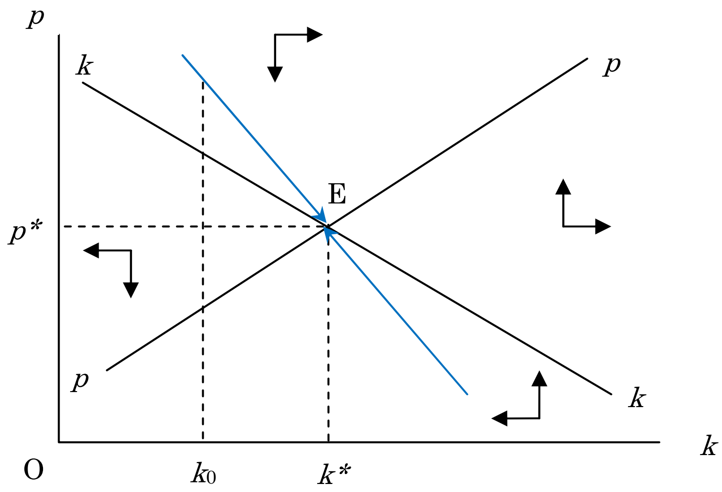
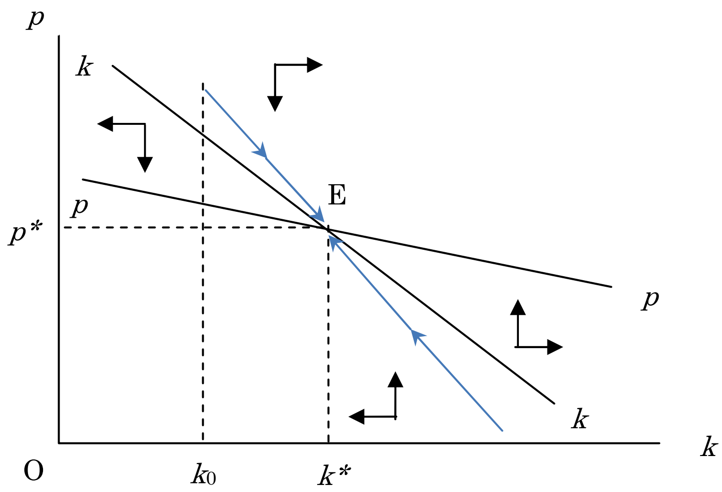


Fig. 1-2. Saddle-point Stable SE under $1+\chi>\beta$, $1>\varepsilon(1+\chi)$ and $P_k^*<0$



A.2 Cases under $1+\chi<\beta$ *:*

Under $1+\chi<\beta$, $Det.J^* = K_k^* P_p^* - K_p^* P_k^* < 0$ is equivalent to $1-\alpha-\varepsilon\beta<0$, i.e., $(1-\alpha)/\beta<\varepsilon$ (see (22)). That is, an SE will be saddle-point stable when the capital externality α is large and/or the PCH parameter ε is large (the PCH function is weakly concave, i.e., close to linear, or convex). We will separate three different cases, but the property of equilibrium paths is qualitatively the same among them. For the analysis below, note that we always have $K_k^*<0$ and $P_k^*>0$ under $1+\chi<\beta$.

A.2-1. Cases of $1>\varepsilon(1+\chi)$ *under* $1+\chi<\beta$ *:*

When $1>\varepsilon(1+\chi)$ holds under $1+\chi<\beta$, $P_p^*>0$ always holds because $(1-\alpha)/\beta<\varepsilon<1/(1+\chi)$ holds. The sign of K_p^* is ambiguous.[20] However, so long as the SE is saddle-point stable, the property of equilibrium paths is the same, regardless of whether $K_p^*>0$ or $K_p^*<0$ holds (See Appendix B for confirming these cases are

[20] These sign conditions are the same as in the case of $\varepsilon=0$. Thus the qualitative results for $1>\varepsilon(1+\chi)$ will remain in the model without the PCH.

possible). Fig.3-1 shows the case when $K_p^* > 0$ (and $Det.J^* < 0$ always) holds. Then, we get $-(K_k^* / K_p^*) > 0$ (the *kk* curve is increasing) and $-(P_k^* / P_p^*) < 0$ (the *pp* curve is decreasing). Thus an SE exists uniquely, and an equilibrium path is uniquely determined.

Fig.3-1. Saddle-point Stable SE under $1+\chi < \beta$, $1 > \varepsilon(1+\chi)$ and $K_p^* > 0$

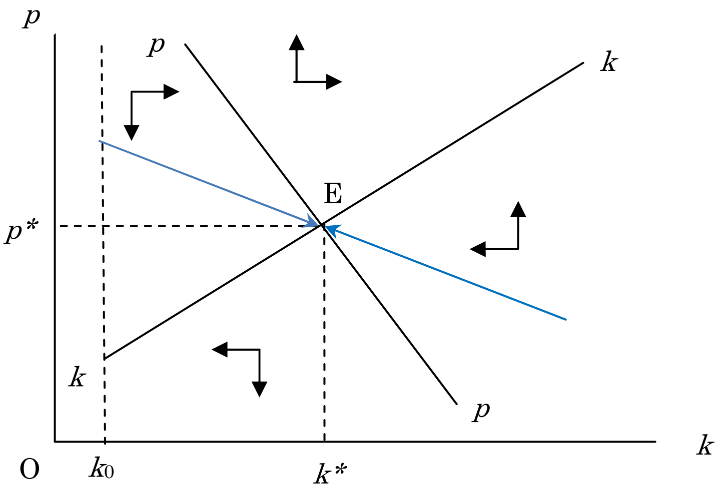
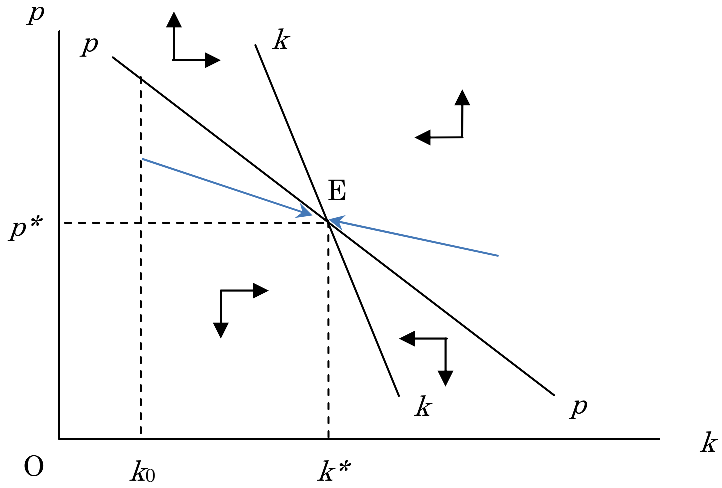


Fig.3-2 shows the case of $K_p^* < 0$. Then the *kk* and the *pp* curves are both decreasing. The sign of $Det.J^* = K_k^* P_p^* - K_p^* P_k^*$ can be either positive or negative. An equilibrium paths for $Det.J^* < 0$ is shown in the figure.

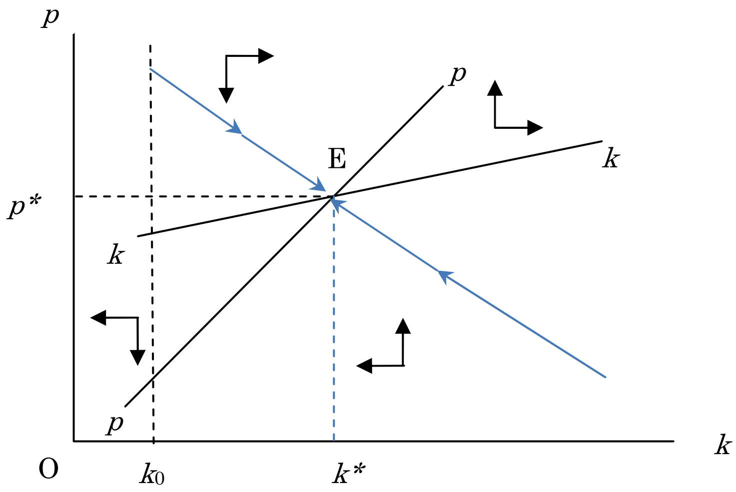
Fig.3-2. Saddle-point Stable SE under $1+\chi<\beta$, $1>\varepsilon(1+\chi)$ and $K_p^*<0$



A.2-2. Case of $\varepsilon(1+\chi)>1$ *under* $1+\chi<\beta$

Fig.3-3 shows the case of $\varepsilon(1+\chi)>1$ under $1+\chi<\beta$. Then $1-\alpha-\varepsilon\beta<1-\alpha-\varepsilon(1+\chi)=[1-\varepsilon(1+\chi)]-\alpha<0$, and by (22) $Det.J^*<0$ holds. Taking account of $K_p^*>0$, we obtain the law of motion there. An equilibrium path is uniquely determined.

Fig.3-3. Saddle-point Stable SE under $1+\chi<\beta$ and $\varepsilon(1+\chi)>1$



Let us explain intuitively why the equilibrium dynamics above occurs under $1+\chi<\beta$. Recall that the labor demand curve is steeper than the increasing labor supply curve ($\chi<\beta-1$). When the initial level c_0 of consumption is low, the labor demand curve lie in a low position (because of a large value of β, given ε) and thus the equilibrium labor input n tends to be large. Thus the labor input in efficiency units $h(c)n$ tends to be large, too. This leads to a nigh value of the rental rate r of capital. By Euler equation (12), p declines and therefore consumption c increases along the transition path.

Appendix B: Compatibility of Conditions for Saddle-point Stable Cases

B-1. Possibility of Fig.2-1 and Fig.2-2 :

We will examine whether $K_p^*>0$ ($K_p^*<0$) actually holds under $1+\chi>\beta$ and $\varepsilon(1+\chi)>1$. Condition $K_p^*>0$ ($K_p^*<0$) is equivalent to

$$\varepsilon(1+\chi)<(>)1+\left(\frac{1+\chi-\beta}{\beta}\right)\left[\frac{\rho+(1-a)(\delta+g)}{\rho+\delta+g}\right]$$

Since the right-hand side is larger than unity, this inequality can be compatible with $\varepsilon(1+\chi)>1$. Therefore $K_p^*>0$ ($K_p^*<0$) is possible.

B.2 Possibility of Fig.3-1 and Fig.3-2 :

We will examine whether $K_p^*>0$ ($K_p^*<0$) actually holds under $1+\chi<\beta$ and $1>\varepsilon(1+\chi)$. Condition $K_p^*>0$ ($K_p^*<0$) is equivalent to

$$\varepsilon(1+\chi)<(>)1+\left(\frac{1+\chi-\beta}{\beta}\right)\left[\frac{\rho+(1-a)(\delta+g)}{\rho+\delta+g}\right]$$

Since the right-hand side is smaller than unity, this inequality can be compatible with $1>\varepsilon(1+\chi)$. Therefore $K_p^*>0$ ($K_p^*<0$) is possible.

Appendix C: Proof of the Hopf Bifurcation

C.1 The Hoph Bifurcation Theorem:

Guckenheimer and Holmes (1983, pp.151-152) provide a sufficient condition for the Hopf bifurcation in Theorem 3.4.2. Asada (1997) applies it to economic dynamics. Here we use the version of the Hoph bifurcation theorem provided in Yoshida (2003), which use trace and determinant conditions. I will summarize the theorem for a 2- dimensional dynamical system, based on proposition 4.1 in Yoshida (2003, p.78).

Hopf Bifurcation Theorem (Yoshida (2003)):

Suppose that the system $\dot{x}=f(x;\varepsilon)$, $x\in\mathbb{R}^n$ with a bifurcation parameter $\varepsilon\in\mathbb{R}$ has an equilibrium (x^*,ε_0) at which the following properties (H-1), (H-2) and (H-3) are satisfied.

(H-1) Equilibrium value $x^*(\varepsilon)$ of this system is a smooth function of ε.

Denote the Jacobian matrix of $f(x;\varepsilon)$ evaluated at the equilibrium by $J^*(\varepsilon)$. When $n=2$ holds,[21]

[21] The characteristic equation for $n=2$ is $\lambda^2-(Trace.J^*(\varepsilon))\lambda+Det.J^*(\varepsilon)=0$.

(H-2) $Trace.J^{*}(\varepsilon_0)=0$ *and* $Det.J^{*}(\varepsilon_0)>0$

(H-3) $d[Trace.J^{*}(\varepsilon_0)]/d\varepsilon \neq 0$

Then there exists a periodic solution which bifurcates from $x^{*}(\varepsilon_0)$ *at* $\varepsilon=\varepsilon_0$, *and its amplitude is given by* $2\pi/\operatorname{Im}\lambda(\varepsilon_0)$ *approximately.* □

C.2 Proof of Existence of Periodic Solution:

To show that the Hoph bifurcation happens in the present model under the PCH, we focus on the model with no capital externality ($\alpha=a$ and $\beta>1-a$).

(H-1): An SE is defined by $K(k^{*},p^{*};\varepsilon)=0$ and $P(k^{*},p^{*};\varepsilon)=0$. Since $Det.J^{*}=K_k^{*}P_p^{*}-K_p^{*}P_k^{*}\neq 0$ holds, by the implicit function theorem, there exist smooth functions $k^{*}(\varepsilon)$ and $p^{*}(\varepsilon)$ that satisfy $K(k^{*}(\varepsilon),p^{*}(\varepsilon);\varepsilon)=0$ and $P(k^{*}(\varepsilon),p^{*}(\varepsilon);\varepsilon)=0$. Therefore the SE values $(k^{*}(\varepsilon),p^{*}(\varepsilon))$ are smooth functions of ε.

(H-2): Under $1+\chi<\beta$, the value of ε that satisfies

$$Trace.J^{*}=\rho+\frac{\varepsilon\beta(1+\chi)(\rho+\delta+g)}{1+\chi-\beta}=0$$

is

$$\varepsilon_0=\left(\frac{\rho}{\rho+\delta+g}\right)\left[\frac{1}{1+\chi}-\frac{1}{\beta}\right]>0$$

As was already explained in the text, if the value of $\delta+g$ is so large that

$$\rho+\delta+g>\left(\frac{\rho}{1-a}\right)\left[\frac{\beta}{1+\chi}-1\right] \tag{K}$$

holds, $Det.J^{*}>0$, which is equivalent to $\varepsilon<\dfrac{1-a}{\beta}$, is satisfied at $\varepsilon=\varepsilon_0$.

(H-3): Under $1+\chi<\beta$, we get

$$\frac{d[Trace.J^*(\varepsilon_0)]}{d\varepsilon} = \frac{\beta(1+\chi)(\rho+\delta+g)}{1+\chi-\beta} < 0$$

This term is not zero. Therefore, a periodic solution (an invariant closed curve) exists around the SE.

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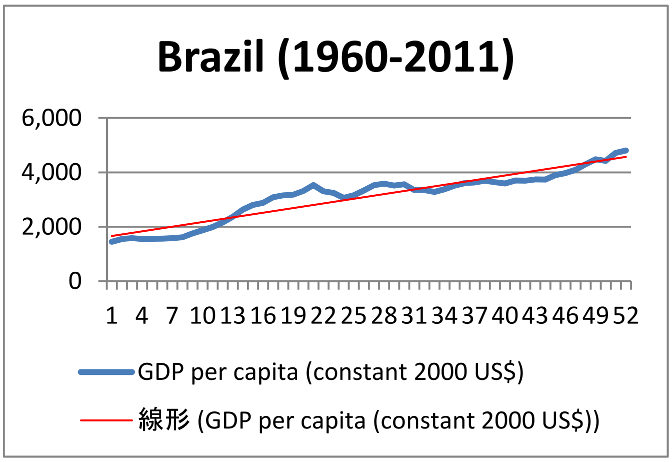
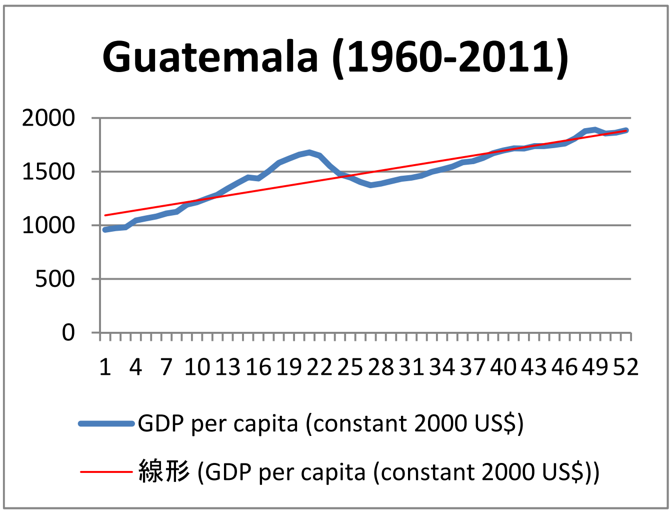
Table 1: The Millennium Development Goals

1. Eradicate extreme poverty and hunger
2. Achieve universal primary education
3. Promote gender equality and empower women
4. Reduce child mortality
5. Improve maternal health
6. Combat HIV/AIDS, malaria, and other diseases
7. Ensure environmental sustainability
8. Develop a global partnership for development

Tables: Indeterminacy Cases (Latin America)



Horizontal axis: the number of years (e.g., The 10th point on the horizontal axis is for 1969.)



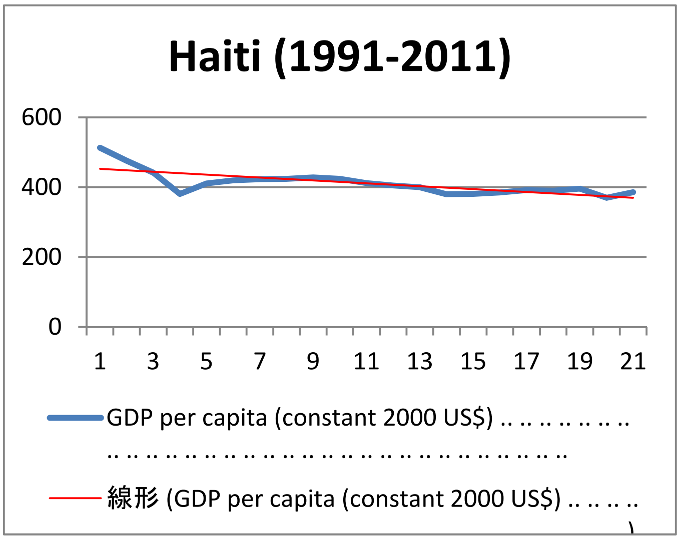
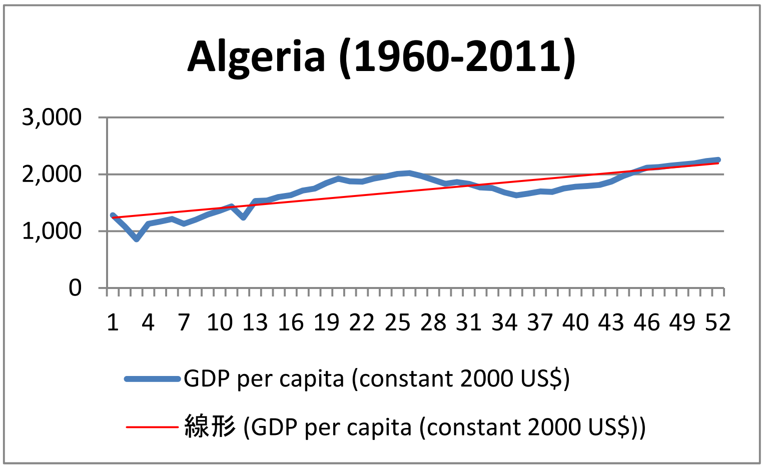
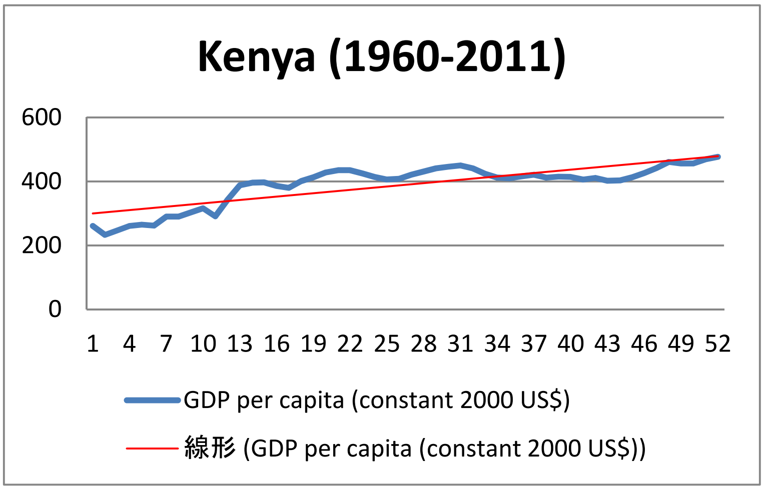
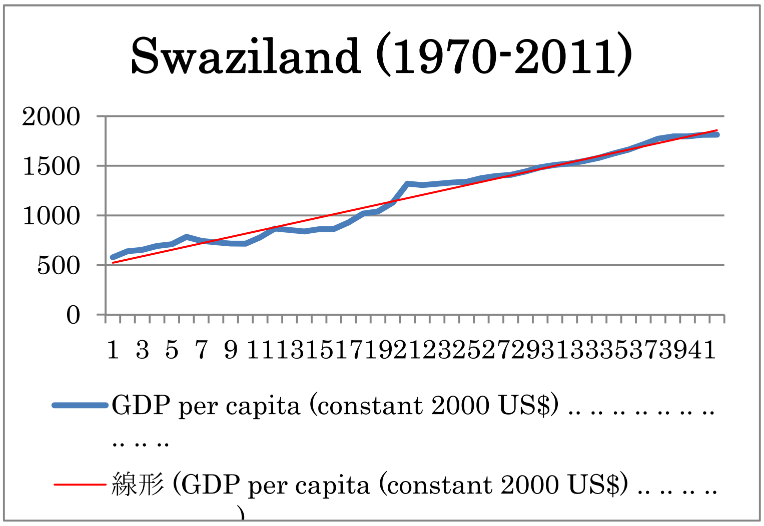


Table: Indeterminacy Cases (Africa)

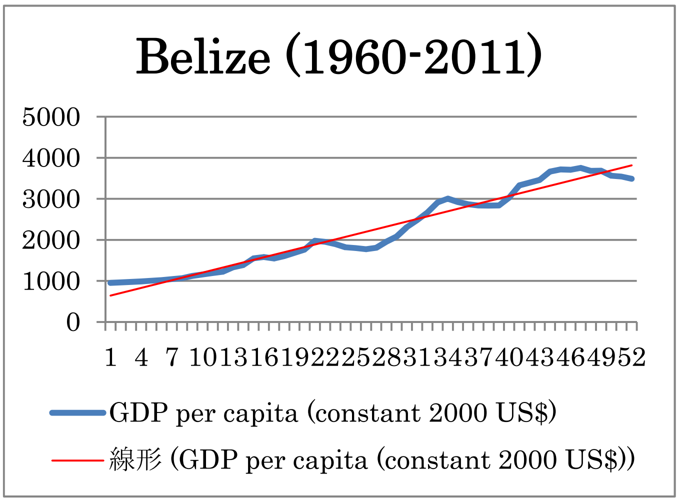




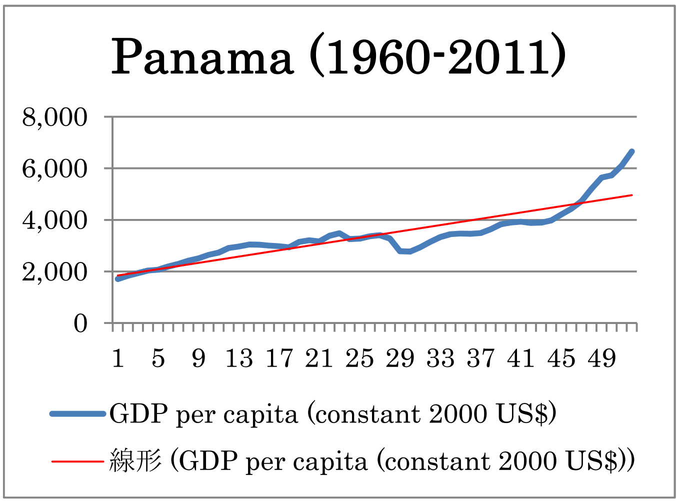


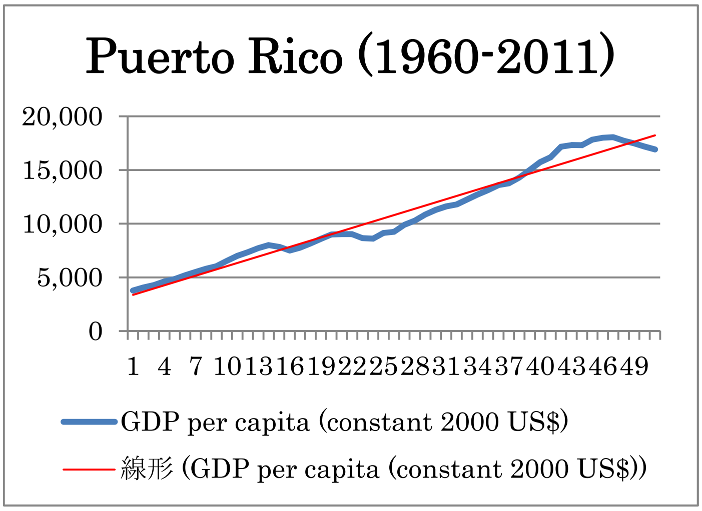
(Source: World Development Indicators)

Tables: Bifurcation Cases (Latin America)

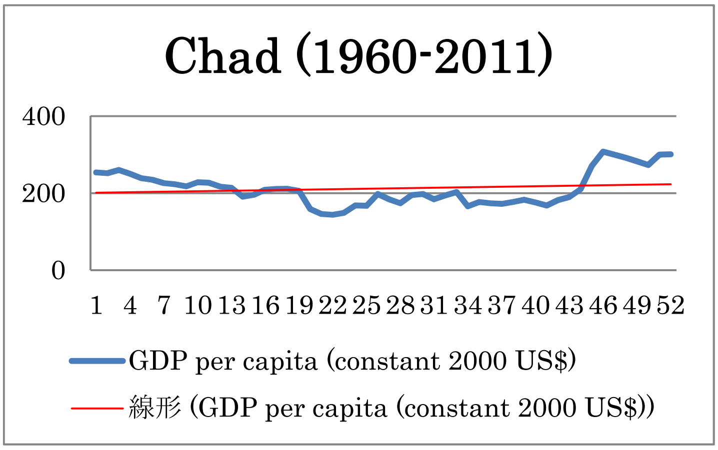


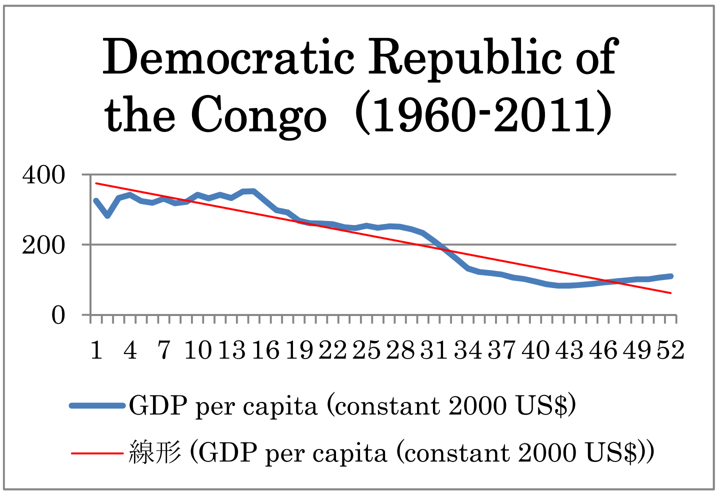
Horizontal axis: the number of years (e.g., The 10th point on the horizontal axis is for 1969.)

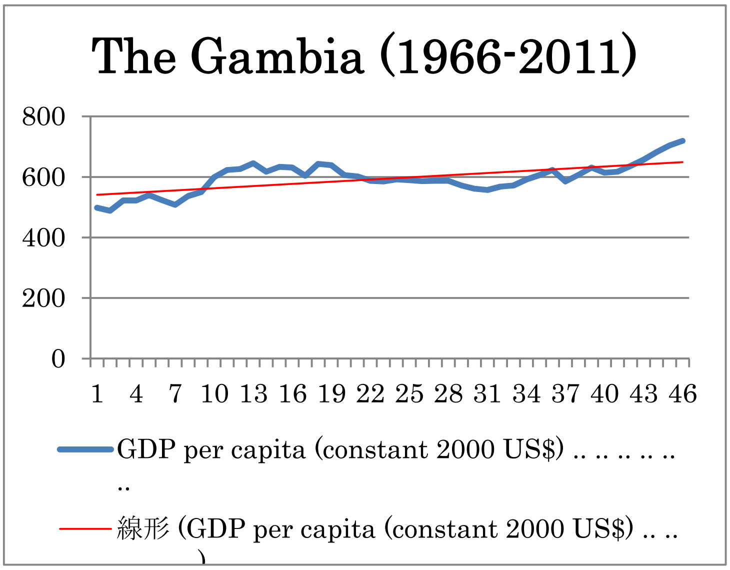


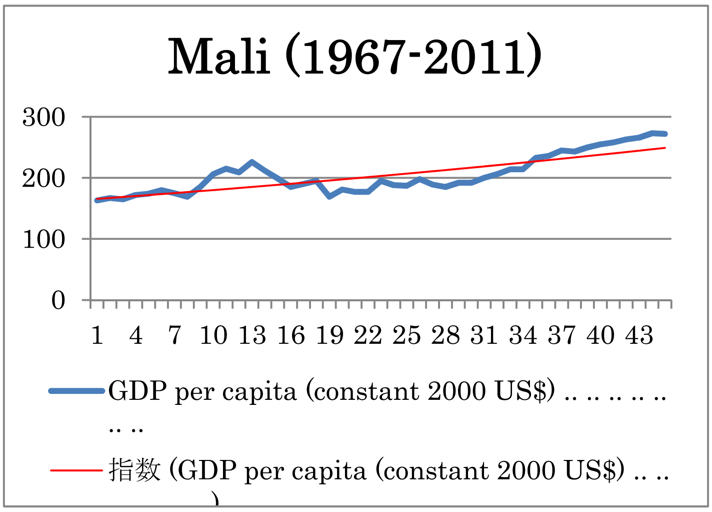


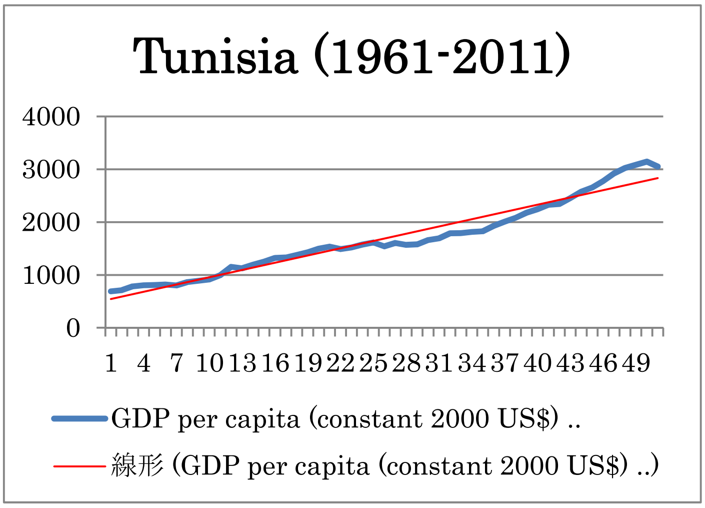
Tables: Bifurcation Cases (Africa)

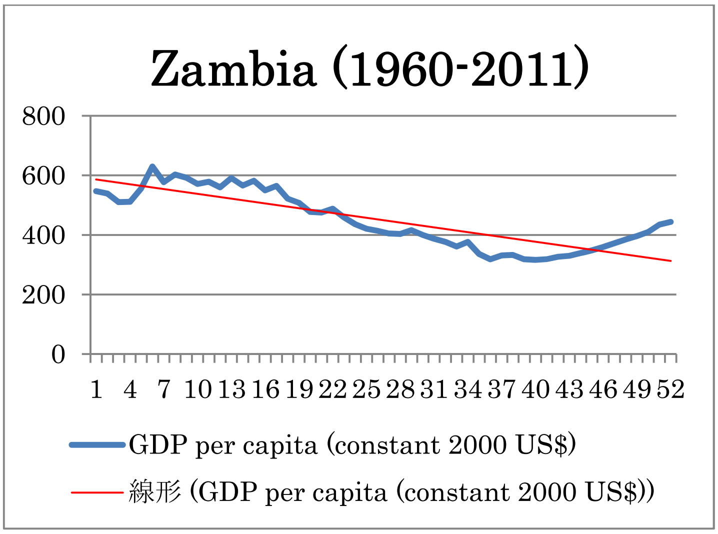












(Source: World Development Indicators)