CROWDING IN SCHOOL CHOICE

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Abstract. We consider the market design problem of matching students to schools in the presence of crowding effects. These effects are salient in parents' decision making and the empirical literature; however, they cause major difficulties in the design of satisfactory mechanisms and, as such, are not currently considered. We propose a new framework and an equilibrium notion that accommodates crowding, no-envy, and respect for priorities. The equilibrium has a student-optimal element that induces an incentive compatible mechanism and is implementable via a novel algorithm. Moreover, analogs of fundamental structural results of the matching literature—the Rural Hospitals Theorem, welfare lattice, etc.—survive.

Keywords: School choice with crowding; Rationing crowding equilibrium; Student optimality; Strategy-proofness

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1. Introduction

Parents believe that if a school is overcrowded, then the quality of their children's education will suffer. This is supported by empirical literature. Card and Krueger [1992], Krueger and Whitmore [2001], Chetty et al. [2011], and Jackson et al. [2016] all show that decreasing crowding, as measured by per capita expenditure or teacher-student ratios, etc., has positive effects on measures such as test scores, students' lifetime expected income, and career development. Policymakers also share parents' concerns. From 1999-2001, the United States' federal government allotted over $4 billion to the Class-Size Reduction Program with the goal of smaller class sizes in grades K-3.[1] Similar programs exist in other countries.[2]

Matching theory has been astonishingly successful in practice, reforming the way students are assigned to schools in many municipalities worldwide. However, despite the salience of crowding effects, the theoretical school choice literature has largely avoided the issue. This is because whenever externalities are introduced into the standard model, its appealing features—like the existence of fair (in terms of respect for priorities) or stable matchings, the ordered structure of the set of such matchings, and the existence of incentive compatible mechanisms—vanish (see Example 1 and Section 1.2). As such, while students' preferences take into account crowding, the design of school choice mechanisms currently fails to do so.

The contribution of this paper is a novel framework to analyse the *school choice problem with crowding*. Our framework generalizes the now-standard model of Abdulkadiroğlu and Sönmez [2003] and restores its core features. There, each student has a preference over the finite set of schools. In our model, each student has a preference over the *two dimensions* of school identity *and* the total amount of educational resources that they consume at each school. The more crowded a school is, the less resource each student enjoys, and so the value of this second dimension at each school will emerge endogenously. The feature that allows us to cut the proverbial Gordian knot is modelling the level of educational resources as a continuous variable. This is of course an abstraction; however, the return on this abstraction is high—recovering existence of an ordered set of *fair* matchings, and a class of *fair* and *strategy-proof* rules. Our techniques also connect the school choice literature with research

[1] U.S. Department of Education, Office of the Deputy Secretary, Policy and Program Studies Service, "A Descriptive Evaluation of the Federal Class-Size Reduction Program: Final Report," Washington, D.C., 2004.

[2] For example, the "Plus de Maîtres que de Classes" program in France in 2013 aimed to reduce class sizes in socioeconomically depressed areas (Réseaux d'Éducation Prioritare).

on Walrasian equilibria in the presence of price rigidities (see Section 1.2). Finally, our framework naturally opens up new avenues for policy analysts: Eliciting preferences over crowding would provide powerful data for welfare analysis, and the model very naturally accommodates novel comparative statics like shifting resources between schools.

Case Study: Crowding in the Wake County Public School System. Wake County is the most populous county in North Carolina and, in the last decade, was the third fastest-growing in the United States among those with over a million residents.[3] The Wake County Public School System (WCPSS) currently serves over 150,000 K-12 students. In 2016, WCPSS projected the arrival of 32,000 new students, which indeed materialized into student population growth that has outpaced that of physical infrastructure. For example, in Abbotts Creek Elementary School for the 2019-2020 year, 870 students enrolled, or 141.2% of the school's total building capacity.[4] This is not an uncommon phenomenon; Figure 1 lists crowding statistics for several other schools in the WCPSS, with some enrollments at almost twice the original building capacity. Student populations also fluctuate significantly from year to year. In Abbotts Creek, for the 2016-2017 year, there were 796 enrolled students, increasing to 870 for the 2019-2020 year, and dropping to 854 in the following year.

The National Center for Education Statistics (NCES) also provides a wide range of data relevant to crowding, including enrollments, financial expenditure, number of teachers, and basic demographics on all public schools in the United States. The recently initiated NCES School-Level Finance Survey program collects and releases data on faculty and staff salaries, spending on supplies, spending on technology related equipment, etc. For example, in 2016-2017 (latest available), Abbotts Creek had financial expenditures of $3.5 million (or $4,100 per student).[5]

This case study immediately points to several items outside of the purview of the standard model. First, parents have access to data relevant to crowding and can condition their preferences over schools in this dimension. The figures above are publicly available via the

[3] U.S. Census Bureau.

[4] Figures regarding Abbotts Creek accessed Nov 10, 2021 at https://www.wcpss.net/domain/100.

[5] https://nces.ed.gov/ccd/.

	2019-2020 Population	2020-21 Building Cap	Trailers	Crowding %	Crowding % w/ Trailers
Abbotts Creek	870	616	0	141.2%	141.2%
Alston Ridge	1,053	914	0	115.2%	115.2%
Beaverdam	790	616	4	128.2%	116.9%
Cedar Fork	848	508	8	166.9%	126.9%
Combs	767	410	13	187.1%	118%
Highcroft Drive	921	508	15	181.3%	114%
Holly Grove	940	814	3	115.5%	107.6%
Hortons Creek	911	716	0	127.2%	127.2%
Lead Mine	500	410	1	122%	122%
Mills Park	828	716	3	115.6%	106.7%
Oakview	901	716	6	125.8%	110.4%
Olive Chapel	978	612	14	159.8%	112.2%
Rogers Lane	730	616	4	118.5%	104.9%
Scotts Ridge	857	616	6	139.1%	116.4%
Sycamore Creek	1,082	914	0	118.4%	118.4%
Weatherstone	850	508	11	167.3%	116.8%
White Oak	747	616	0	121.3%	121.3%

Figure 1. Crowding statistics for elementary schools in WCPSS that enacted enrollment caps for the 2020-2021 school year.[6] These amounted to 17 out of 120 elementary schools.

WCPSS and NCES websites, as well as regularly reported on by the local media.[7] Second, the fact that student enrollments frequently and grossly exceed building capacity challenges the standard assumption that enrollment capacity is *ex ante* exogenously fixed. A school may almost always accommodate some extra students by adding seats in a classroom or a trailer. In the Feb. 4, 2020 WCPSS Board of Education meeting, the Assistant Superintendent for School Choice, Planning, and Assignment *proposed* enrollment caps for each of the schools in Figure 1 (among others)—after years of overcrowding.[8] Many of the schools

[6]Statistics from Feb. 4, 2020 WCPSS Board of Education Meeting Minutes section "Capping Recommendations." Accessed April 9, 2022 at https://www.wcpss.net/Page/3728.

[7]See, for example: Editorial Board, "More Growth Challenges Wake Schools," *News and Observer*, March 26, 2016, accessed November 7, 2021, https://www.newsobserver.com/opinion/editorials/article68451967.html.

[8]See Footnote 6. See also: T. Keung Hui, "Wake County Puts Enrollment Caps On 19 Schools For 2020. One More May Be Added," *News and Observer*, February 5, 2020, accessed November 7, 2021, https://www.newsobserver.com/news/local/education/article239808038.html.

in Figure 1 are therefore now considered *at* capacity. This implies, however, that many others are *below* capacity, with the potential for fluctuations year to year.[9]

Our model addresses all of these points. We accommodate inviolable caps while providing a framework for analyzing the still-salient crowding levels at all schools, and in particular those below capacity. Parents are able to explicitly express preferences over such levels. Moreover, the *strategy-proof* and *fair* rules we propose make crowding levels a publicly announced quantity in the manner of an auctioneer, eliminating the need for parents to infer future crowding from the lagged data available.

The remainder of this section provides a sketch of the model and an overview of the formal results. Each student consumes a bundle consisting of a school and a level of educational resources. We require that all the students at a given school consume the same level, an important benchmark representing the ideal that all students are equally entitled to the resources of their school.[10] An aggregate allocation is thus composed of a matching of students to schools and a vector of resource levels, one level for each school.

We propose a new equilibrium concept—the Rationing Crowding Equilibrium (RCE). The core of our innovation is in realizing that the vector of resource levels can function like a price. Consider a competitive solution applied in our context. We may imagine that an auctioneer announces a resource vector, which then determines a (finite) list of school and resource-level pairs. Each student will then demand (generically) one of these pairs, and we can ask the usual market clearing question: Does there exist a resource vector at which, for each school, the demand for educational resources is equal to its supply? We show that the answer is yes, if we allow for an error of *one* seat.[11,12] Since each student faced the same budget set, the resulting allocation satisfies *no-envy*, at least for schools that have not reached their enrollment cap.

[9]Apart from such proposals for enrollment caps, many states also have laws specifying a minimum teacher-student ratio and therefore implying such caps. For example, North Carolina requires at least one teacher per 18 students in grades K-3 (G.S. 115C-301 on "Allocation of teachers; class size"). Districts may also "manually" adjust capacities to accommodate unmatched students with specific requirements only available at certain schools e.g. disabilities, large number of siblings.

[10]This is a restriction, and yet we have found an appealing solution to the problem. It follows that finding an appealing solution for the *unrestricted* case should be easier.

[11]More precisely, if a school at an RCE admits k students, each student is guaranteed to consume more than $1/k+1$ fraction of the school's total resources (see Remark 1).

[12]Excluding several specialized schools, the smallest (Wake STEM Early College) and largest (Apex Friendship) schools in Wake County in 2020-2021 had 256 and 2,733 students, respectively. In practice, then, the presence or not of a single seat at either school is negligible.

For schools constrained by their caps, our equilibrium incorporates *fairness* via respect for priorities: If student i prefers school s at an RCE, then school s has exhausted its capacity and all students in that school have higher priority than student i. To continue the Walrasian analogy, a school exhausting capacity is tantamount to the auctioneer no longer being able to lower the announced resource level of a school, and thereby reduce demand—akin to a price ceiling.[13] If schools are below capacity, then assignments will be decided by our market-like mechanism without price rigidities.

Our main results are as follows. Fix a school choice problem with crowding. We show that an RCE exists for this problem so long as it satisfies a regularity condition that holds true generically (Theorem 1). We then establish a version of the Rural Hospitals Theorem for our environment, i.e. in each RCE, each school is matched with the same number of students (Theorem 2). We show that the set of RCEs constitute a closed upper semi-lattice under the Pareto dominance partial order (Theorem 3), and so there exists a student-optimal RCE (Proposition 3). These are analogs of fundamental results in the literature, but the presence of crowding necessitates substantial difference in technique.

A *maximal RCE mechanism* recommends, for each problem in our domain, a maximal RCE (all in this set are welfare-equivalent and student-optimal). We show that these mechanisms are *strategy-proof* (Theorem 4), and we further find an algorithm to calculate them on a natural subdomain (Theorem 5).[14] It is in general hard to propose an algorithm for equilibrium computation when students' preferences are not quasi-linear (see the discussion after Theorem 1). This algorithm uses tools from both the multi-item auction [Demange et al., 1986], with rationing constraints on students' demands, and the Deferred-Acceptance algorithm [Gale and Shapley, 1962]. The latter adjusts the matching for schools whose capacity constraint have been met, and the former adjusts the distribution for unconstrained schools.

1.1. Discrete Versus Continuous Crowding: Our Methodological Novelty.

We model crowding via the consumption of a continuous good that comes in fixed supply. We have touted the benefits of this approach, weighed against the negligible "within-one-seat" relaxation of the market clearing condition. Thus, one might wonder if we could have achieved the same without such abstraction, by simply saying that if n students arrive at a school,

[13] If the school admits further students, then crowding increases and the resource level each student enjoys would decrease. So an upper bound on student enrollment is equivalent to a lower bound on the resource level.

[14] The general domain is infinite dimensional. We show the algorithm works on a finite-dimensional subdomain.

then its aggregate resources are split n ways. We immediately face difficulties: two key solution concepts in the school choice literature, *no justified envy* and *non-wastefulness*, are incompatible. Recall, that if student i envies student j, who is attending school s, then this envy is *justified* if school s prioritizes i over j. An allocation is *wasteful* if a student would want to switch to a school at which there is currently unfilled capacity.

Example 1 (Incompatibility of *No Justified Envy* and *Non-Wastefulness*). Consider a simple problem with only two schools, s_1 and s_2, and three students, 1, 2, and 3. Assume that s_1 and s_2 each have capacity three; each school also prioritizes student 3 over the others. Suppose student 1 has such strong preferences for s_1 that they prefer to attend s_1 regardless of the number of students there. Assume 2 has similarly strong preferences for s_2. Then at any *non-wasteful* allocation we have 1 at s_1 and 2 at s_2. Finally, suppose student 3 would rather be alone than share either s_1 or s_2 with another student. If we put 3 at s_1, then since they have highest priority at s_2, they have justified envy of student 2. A symmetric statement holds if we put 3 at s_2.

In work independent of ours, Copland [2021] studies exactly this model. They propose a natural relaxation of the standard solution concept: the envy that 3 had for 2 above is not yet justified unless student 3 is also willing to attend school s_2, keeping all other students, including 2, at their place.[15] Given this definition, a solution always exists. They provide an algorithm to compute a solution, but it does not yield a *strategy-proof* mechanism. Unfortunately, other key properties are also lost: the Rural Hospitals Theorem and upper-lattice structure on the set of satisfactory matchings do not hold.

We review prior work in discrete models below in Section 1.2. The model just discussed is even more tightly structured than what had come before, allowing only crowding externalities as opposed to preferences over the identity of one's cohort or even the entire matching. Thus, it will not surprise the reader that those more general models also yield similarly negative results. Thus, so far, no discrete model yields structured solutions and *strategy-proof* rules as in the standard model without externalities.

1.2. **Related Literature.**

[15]The usual *no justified envy* embodies a "replacement" principle: if i has justified envy of j, then i can simply take the place of j. Copland's solution is subtly different—one student "joins" another school without replacing anyone else.

Matching with Externalities. Our problem is one of matching with externalities. In principle, an agent may have preferences over where all the other agents are matched. Sasaki and Toda [1996], Hafalir [2008], Bando [2012], and Bando [2014] propose and study various notions of stability in such an environment. For the case when there are general externalities *and* several ways for each pair of agents to match—the *matching with contracts* problem—Rostek and Yoder [2020] and Pycia and Yenmez [2021] identify general conditions on preferences under which stability is possible. In most applications, there is a natural structure on possible externalities. For example, agents may care only about who their peers are, not about those matched to other schools/firms [Dutta and Massó, 1997, Echenique and Yenmez, 2007, Leshno, 2021]. Narrowing down even further, a large literature considers matching with couples, siblings, or neighbors [Roth, 1984, Roth and Peranson, 1999, Klaus and Klijn, 2005, Kojima et al., 2013, Ashlagi et al., 2014, Nguyen and Vohra, 2018, Dur and Wiseman, 2019, Dur et al., 2022]. These studies find that externalities eliminate the structures that have been found in the classical literature. Preference over crowding introduces yet more structure to the problem, but that alone is not sufficient to yield all the standard results from the classical problem.

Tierney [2019] also studies crowding, but in an environmental resource allocation problem. In principle, we could treat school resources like environmental resources; however, adapting their equilibrium to our model is not appropriate for several reasons, chief among them being that the concept assumes agents can be allotted arbitrarily small quantities. This is unacceptable for school choice, as it implies that a single school can accommodate all the students in the entire system. Additionally, their equilibrium concept also satisfies *anonymity*. Here, respecting priorities necessitates retaining student identities.

Walrasian Approaches. Our RCE follows, in spirit, price-based equilibrium notions in matching with the possibility of monetary transfers. Shapley and Shubik [1971] first proved the existence and structural properties of the core and competitive equilibrium allocations for these models. These properties imply that, for each side of the market, there is a unique undominated utility vector it can achieve in the core. Mechanisms that realize one of these vectors are *efficient* for all and *strategy-proof* [Demange and Gale, 1985] for the side of the market whose utility is maximized. In matching models where prices are not fully flexible, e.g. restricted between price floors and ceilings, Drèze Equilibrium and Rationing Price Equilibrium were proposed as alternative notions [Drèze, 1975, Talman and Yang, 2008, Andersson and Svensson, 2014, Herings, 2018]. Under some mild domain restrictions, there are *constrained efficient* and *group strategy-proof* rules [Andersson and Svensson, 2014].

In several matching models without continuous transfers, the usual fairness properties are naturally expressed with an endogenously determined *cutoff* vector [Balinski and Sönmez, 1999, Sönmez and Ünver, 2010, Azevedo and Leshno, 2016, Dur and Morrill, 2018, Leshno and Lo, 2021]. In the case of school choice, the vector specifies for each school the lowest priority student able to attend. This is thus a competitive approach—cutoffs determine budget sets for agents, agents maximize therein, and markets clear. Note that the cutoffs in these models play a different role to that of our resource vector. The former control access to schools, while the latter modulates the character of what students eventually consume.

1.3. **Organization.** In Section 2, we define the school choice with crowding problem. In Section 3, we define the Rationing Crowding Equilibrium. Section 4 shows that RCE exist, and Section 5 investigates some properties of them, like the Rural Hospitals Theorem and the welfare upper lattice. In Section 6, we identify a *fair* and *strategy-proof* mechanism, and in Section 7, we find an algorithm that implements this mechanism on a restricted domain of preferences. In Section 8, we discuss the robustness of our results and some applications beyond school choice. Section 9 concludes with open questions.

2. Model

Let S be the finite set of **schools**, and N be the finite set of **students**. Each school $s \in S$ has resources in the form of teachers, buildings, money, etc. We aggregate this to a single measure and for each school normalize this to one. When a student attends a school, the number of other students and the policy of the school together determine what fraction of those resources, that is, what **resource ratio**, they consume. Formally, each student's consumption space is $[0, 1] \times S$, where the first component is a resource ratio ρ_s and the second is the school s that they attend. Inclusion of an outside option in our model would not change the results and only add complication to the proofs.[16] This abstraction of the consumption space drives our theoretical innovations. For some applications, it may be essential that the resource ratio be restricted to *simple* fractions of the form $1/k$. This would be the case when crowding really depends *only* on the number of students at a school. Our model is still useful in these cases, since we shall recommend, for each school, a resource

[16] We show, however, in Section 3.1, that the richness of our preference space subsumes the outside option in the classical sense.

ratio that is within one student of being the simple fraction implied by the cohort they admit (see Remark 1).

Each student $i \in N$ has a complete and transitive **preference relation** R_i over $[0,1] \times S$. We assume that each is monotonic in the resource ratio: for each $\rho_s, \rho'_s \in [0,1]$ with $\rho'_s > \rho_s$, and each $s \in S$, $(\rho'_s, s)\ P_i\ (\rho_s, s)$. Let $\mathscr{R}$ be the set of monotonic preference relations, and $\boldsymbol{R} = (R_i)_{i \in N} \in \mathscr{R}^N$ denote a profile of preferences for students.

Each school $s \in S$ has a maximum natural number **capacity**. Let $b_s \in [0,1]$ be such that b_s^{-1} is equal to this capacity; we can then interpret b_s as a lower bound on the resource ratio it can provide. Let $\boldsymbol{b} = (b_s)_{s \in S}$ denote the profile of bounds for schools. We assume that $\sum_{s \in S} b_s^{-1} \geq |N|$, so all students can be admitted to some school. Each school $s \in S$ has a **priority order** $\succ_s$ over the set of students, where $i \succ_s j$ indicates that i has higher priority than j at s. Let $\succ = (\succ_s)_{s \in S}$ denote the profile of priorities for schools.

A **school choice problem with crowding** (hereafter just a *problem*) is a tuple $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$. The canonical school choice problem of Abdulkadiroğlu and Sönmez [2003] is a special case (see Section 3.1 for details). A **distribution** is a vector $\boldsymbol{\rho} \in [0,1]^S$ of resource ratios. A **matching** $\sigma : N \to S$ places each student at a school. Let $\mathcal{M}$ be the set of matchings, and for each $\sigma \in \mathcal{M}$ write $\sigma[s]$ as the set of students matched to s at σ.[17] An **allocation** is a pair $(\boldsymbol{\rho}, \sigma) \in [0,1]^S \times \mathcal{M}$ such that for each $s \in S$,

(1) (Distribution Feasibility) $\rho_s \cdot |\sigma[s]| \leq 1$ and
(2) (Respects Capacity) $\rho_s \geq b_s$.

Given a distribution $\boldsymbol{\rho} \in [0,1]^S$ and a school $s \in S$, we also write $(\boldsymbol{\rho}, s)$ to indicate bundle (ρ_s, s). Thus, for an allocation $(\boldsymbol{\rho}, \sigma)$, each $i \in N$ receives $(\boldsymbol{\rho}, \sigma(i))$. We refer to this as ***i*'s component** when the allocation at hand is clear.

A **mechanism** φ recommends, for each profile $\mathbf{R}$ in a domain $\mathscr{D} \subseteq \mathscr{R}^N$, an allocation. Denote by $\varphi_i(\mathbf{R})$ student i's component of the allocation recommended at problem $\mathbf{R}$. We formalize some senses in which these recommendations might either be good or achievable.

We consider fairness amongst the students. A student should be able to attend any school that is not yet at capacity. If a school is at capacity, then the student may be turned away if all others have higher priority. This embodies the classic *no-envy* condition of Foley [1966] as well as the standard notion of *no justified envy* in school choice. We combine these two ideas formally. Fix a problem $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$. The allocation $(\boldsymbol{\rho}, \sigma)$ is **fair** if

[17] Generally, for any function f and any element or set y, we let $f[y]$ denote the pre-image of y under f.

$(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$ implies that $\rho_s = b_s$ (s is at capacity) and, for each $j \in \sigma[s]$, $j \succ_s i$. We repeat this terminology for the associated property of mechanisms.

Fairness: For each problem, the allocation recommended by φ is *fair*.

There may be several *fair* allocations, and we distinguish those most preferred by the students. Allocation $(\boldsymbol{\rho}, \sigma)$ **Pareto-dominates** $(\boldsymbol{\rho}', \sigma')$ if for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\rho}', \sigma'(i))$, and for some $j \in N$, $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\boldsymbol{\rho}', \sigma'(i))$. A **student-optimal fair** allocation is *fair* and not Pareto-dominated by any other *fair* allocation.[18]

Student-optimal fairness: For each problem, the allocation recommended by φ is *student-optimal fair*.

Next, we consider the direct revelation incentive compatibility condition.

Strategy-proofness: For each problem $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$, each $i \in N$, and each preference relation $R'_i \in \mathcal{R}$ such that $(R'_i, \boldsymbol{R}_{-i}) \in \mathcal{D}$,

$$\varphi_i(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)\ R_i\ \varphi_i(S, N, (R'_i, \boldsymbol{R}_{-i}), \boldsymbol{b}, \succ).$$

One might worry about the potential for a single agent i to report R'_i and induce $(R'_i, \boldsymbol{R}_{-i}) \notin \mathcal{D}$. However, in all of the problems that we study, this event is negligible, either topologically or probabilistically. We provide a more detailed discussion in Section 4.

3. Solution Concept

Our solution concept is fundamentally Walrasian in spirit. There is a publicly announced vector (a distribution) that induces, for each student, a menu of options (a list of ratio-school pairs), and students select their most preferred. Two main points, though, distinguish our concept from price equilibrium. If too many students select one school, then rationing occurs as opposed to a price increase. Exogenously given priority information ($\succ$) determines which students are matched. Next, once a distribution is announced, they must consume the object at that distribution quantity (i.e. teacher-student ratio). They cannot "purchase" more or less of the object.

[18] There may be allocations that are *efficient* (Pareto-undominated by any allocation) but not *fair*. As we are in the context of school choice, we focus on respect for priorities.

Formally, a **Rationing Crowding Equilibrium (RCE)** is an allocation $(\boldsymbol{\rho}, \sigma)$ that satisfies three conditions:

(1) (Fairness) $(\boldsymbol{\rho}, \sigma)$ is *fair*.
(2) (Exhaustive Given $\boldsymbol{\rho}$) For each school s with $\sigma[s] \neq \emptyset$,
$$\lfloor \rho_s^{-1} \rfloor = |\sigma[s]|.$$
(3) (Inferior Empty Schools) For each school s with $\sigma[s] = \emptyset$, $\rho_s = 1$, and for each $i \in N$,
$$(\boldsymbol{\rho}, \sigma(i)) \; P_i \; (\boldsymbol{\rho}, s).$$

Note that our notion begins by offering each student the menu $\{(\boldsymbol{\rho}, s) : s \in S\}$. If $\rho_s > b_s$, then "demand" for s is just as one expects. Anticipating, however, that ρ_s cannot be reduced if it equals b_s, "demand" is first rationed via priorities in this case. Thus, *fairness* is the analog to consumer maximization. The second condition is the key to adapting our notion to the crowding environment and operationalizes the interpretation of ρ_s as a resource ratio. The amount of resources the school provides to each student is, up to rounding error, the total amount of resource (one) divided by the number of students matched to the school. Together with the definition of an allocation (feasibility), this is the analog of market clearing. The third condition states that each student finds any empty school strictly worse than their component of the allocation, and is therefore the analog of the requirement that, at equilibrium, unconsumed commodities should be available for free. Section 3.2 relates RCE to other notions in the literature encompassing competitive equilibria with price rigidities.

Example 2 (An RCE). Let $S = \{s_1, s_2\}$ and $N = \{1, 2, 3\}$. Agents' preferences are given by the following utility functions:

$$u_1(\boldsymbol{\rho}, s_1) = \frac{3}{22} + \rho_{s_1} \quad \text{and} \quad u_1(\boldsymbol{\rho}, s_2) = \rho_{s_2}$$
$$u_2(\boldsymbol{\rho}, s_1) = \frac{7}{12} + \rho_{s_1} \quad \text{and} \quad u_2(\boldsymbol{\rho}, s_2) = \rho_{s_2}$$
$$u_3(\boldsymbol{\rho}, s_1) = \frac{3}{11} + \rho_{s_1} \quad \text{and} \quad u_3(\boldsymbol{\rho}, s_2) = \rho_{s_2}$$

Each school s has minimum ratio $b_s = \frac{1}{2}$ (and thus capacity $b_s^{-1} = 2$). School s_1 has the priority order $1 \succ_{s_1} 2 \succ_{s_1} 3$. School s_2 has the priority order $3 \succ_{s_2} 1 \succ_{s_2} 2$.

Allocation $(\boldsymbol{\rho}, \sigma) = ((\rho_{s_1}, \rho_{s_2}), (\sigma(1), \sigma(2), \sigma(3))) = (1/2, 7/11, s_1, s_1, s_2)$ is an RCE. *Fairness*: Both students 1 and 2 find their component at least as good as others'.[19] Agent 3 prefers both 1 and 2's component to her own: $u_3(\boldsymbol{\rho}, \sigma(3)) = {}^{7}\!/_{11} < {}^{3}\!/_{11} + {}^{1}\!/_{2} = u_3(\boldsymbol{\rho}, s_1)$. Since $\rho_{s_1} = b_{s_1} = 1/2$ and $1 \succ_{s_1} 2 \succ_{s_1} 3$, however, *fairness* is still satisfied. *Exhaustiveness*: We have $\lfloor \rho_{s_1}^{-1} \rfloor = |\sigma[s_1]| = 2$ and $\lfloor \rho_{s_2}^{-1} \rfloor = |\sigma[s_2]| = 1$. *Inferior Empty Schools* is is trivially satisfied, as there is no empty school.

This example is also among the most extreme cases of mismatch between the number of students matched to a school and the reciprocal of the resource ratio. Since there is only one student at s_2, we should hope that $\rho_{s_2} = 1$. Let $\rho'_{s_2} = \rho_{s_2} + \epsilon$ and consider $\boldsymbol{\rho}' = (\rho_{s_1}, \rho'_{s_2})$. For student 1, $u_1(\boldsymbol{\rho}', \sigma(1)) = \frac{7}{11} < \frac{7}{11} + \epsilon = u_1(\boldsymbol{\rho}', s_2)$. Since $\rho'_{s_2} > b_{s_2} = 1/2$, 1 prefers (ρ', s_2) to their own component, in violation of *fairness*.

In this problem, the entire set of RCEs is

$$\{(\boldsymbol{\rho}, \sigma) : \rho_{s_1} = \frac{1}{2}, \frac{1}{2} < \rho_{s_2} \le \frac{7}{11}, \sigma(1) = s_1, \sigma(2) = s_1, \sigma(3) = s_2\}$$
$$\cup\{(\boldsymbol{\rho}', \sigma') : \rho'_{s_1} = \frac{1}{2}, \frac{7}{11} \le \rho'_{s_2} \le \frac{17}{22}, \sigma'(1) = s_2, \sigma'(2) = s_1, \sigma'(3) = s_1\}.$$

Note that we can have $\rho_{s_2} > 7/11$ when the matching is changed, but that in all cases, school 2 will have only 1 student and $\rho_{s_2} < 1$.

Remark 1 (Discrepancy between ρ_s and a simple ratio of the form $\frac{1}{k}$). The magnitude of the discrepancy at any RCE is

$$\frac{1}{|\sigma[s]|} - \rho_s = \frac{1}{|\sigma[s]|} - \frac{1}{\rho_s^{-1}} \le \frac{1}{|\sigma[s]|} - \frac{1}{\lfloor \rho_s^{-1} \rfloor + 1} = \frac{1}{|\sigma[s]|} - \frac{1}{|\sigma[s]| + 1}.$$

Thus, the difference between our modeled ratio and a strictly interpreted resource-to-student ratio is at most the addition of one more student. This is a negligible difference for the overwhelming majority of real world applications including school choice.

3.1. **Connection to the Standard School Choice Model.** Consider the canonical school choice model of Abdulkadiroğlu and Sönmez [2003]. We show how to embed this problem into school choice with crowding, then relate solution concepts across models.

Let schools S, students N, and priorities $\succ$ be defined as before. For each student $i \in N$, let P_i^* be a strict preference relation over schools and $\boldsymbol{P}^* = (P_i^*)_{i \in N}$ be the profile of such preferences. For each school $s \in S$, let $c_s^* \in \mathbb{N}$ be its capacity and $\boldsymbol{c}^* = (c_s^*)_{s \in S}$

[19] $u_1(\boldsymbol{\rho}, \sigma(1)) = \frac{3}{22} + \frac{1}{2} = \frac{7}{11} = u_1(\boldsymbol{\rho}, s_2)$, and $u_2(\boldsymbol{\rho}, \sigma(2)) = \frac{7}{12} + \frac{1}{2} > \frac{7}{11} = u_2(\boldsymbol{\rho}, s_2)$.

be the capacity profile for S. A **school choice problem** is a tuple $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \succ)$. An allocation is a matching $\sigma : N \to S$ such that for each $s \in S$, $|\sigma[s]| \leq c^*_s$. We recall two central properties in this model. A matching σ satisfies **no justified envy** if for each $i \in N$, there is no $j \in N \setminus i$ such that $\sigma(j)\ P^*_i\ \sigma(i)$ and $i \succ_{\sigma(j)} j$. A matching σ satisfies **non-wastefulness** if for each $i \in N$, and $s \in S$, $s\ P^*_i\ \sigma(i)$ implies $|\sigma[s]| = c^*_s$.

We now construct an associated school choice problem with crowding $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$. Let S, N, and $\succ$ be as in the school choice problem; only $\boldsymbol{R}$ and $\boldsymbol{b}$ need adjustment. For each $i \in N$, let R_i be such that for each $s, s' \in S$, $s\ P^*_i\ s'$ if and only if $(0, s)\ P_i\ (1, s')$. That is, at any distribution level, i prefers s to s' in R_i. For each $s \in S$, let $b_s = \frac{1}{c^*_s}$. Thus, the classical model embeds in ours as a preference restriction. The externality is still present: students are worse off when they have more classmates. However, on the restricted domain of classical preferences, there is no way for students to reveal this fact through their choices.

If we wish, we may include a special school, ϕ, in our model with $b_\phi < 1/|N|$. When the canonical school choice model is embedded in ours, this school may function as an outside option in the standard sense, since it can accept all students. This is despite the fact that in our model, we have elected not to consider the outside option.

Proposition 1. *Fix a school choice problem. The following statements are equivalent:*

(1) σ *satisfies* no justified envy *and* non-wastefulness*, and*
(2) *There is a distribution* $\boldsymbol{\rho}$ *such that* $(\boldsymbol{\rho}, \sigma)$ *is an RCE for the associated school choice problem with crowding.*

Proof. Fix a school choice problem $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \succ)$. Let $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$ be an associated school choice problem with crowding.

Let σ be a matching. Let $\boldsymbol{\rho} \in [0, 1]^S$ be such that for each empty school $s \in S$, $\rho_s = 1$ and for each non-empty school $s \in S$, $\rho_s^{-1} = |\sigma[s]|$. Thus $(\boldsymbol{\rho}, \sigma)$ is an allocation and exhaustive.

Matching σ is non-wasteful at empty schools if and only if, for each empty $s \in S$ and each $i \in N$, $\sigma(i)\ R^*_i\ s$. Since R^*_i is strict, this is true if and only if $\sigma(i)\ P^*_i\ s$. Then $(\boldsymbol{\rho}, \sigma(i))\ P_i\ (\boldsymbol{\rho}, s)$. Therefore, *non-wastefulness* for empty schools in $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \succ)$ is equivalent to *inferior empty schools* in $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$.

Now $(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\rho}, \sigma(i))$ if and only if $s\ P^*_i\ \sigma(i)$. So σ is non-wasteful for non-empty schools if and only if $|\sigma[s]| = c^*_s = b_s^{-1}$ if and only if $\rho_s = b_s$. Moreover, σ satisfies no justified envy for $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \succ)$ if and only if, for each $j \in \sigma[s]$, $j \succ_s i$. Conclude that *non-wastefulness* for non-empty schools and *no justified envy* in $(S, N, \boldsymbol{P}^*, \boldsymbol{c}^*, \succ)$ are equivalent to *fairness* in $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$. ■

3.2. **Competitive Foundations of RCE.** Our RCE concept is related to notions of competitive equilibria when prices exhibit rigidities. In the classical exchange problem, a price ceiling may cause demand to outstrip supply, resulting in the failure of market clearing and thus non-existence of equilibria. Drèze [1975] proposed and showed existence for a notion where prices are constrained by ceilings or floors and rationing occurs at such boundaries. In addition to prices, the notion includes a rationing scheme that specifies limits for the net trades of agents. Likewise, in matching models with price controls, Talman and Yang [2008], Andersson and Svensson [2014], and Herings [2018] introduce similar Drèze-style equilibrium concepts. None of these models consider consumption externalities. Our RCE can be seen as a conceptual parallel to their notions, but for the environment where agents have preferences over crowding.

To explore this relationship, we define several Drèze-style equilibrium notions for a school choice with crowding problem $(S, N, \boldsymbol{R}, \boldsymbol{b}, \succ)$, then compare them to those above. A rationing scheme provides each agent with a set of objects that is available for that agent to consume. For each agent $i \in N$, let $Q_i \subseteq S$ be the set of objects available to them. Let $\boldsymbol{Q} = (Q_i)_{i \in N}$ be the rationing scheme. A special case (defined below) is when rationing relies on priorities associated with the objects.

The *rationed demand set* of agent i at distribution $\boldsymbol{\rho}$ and rationing scheme $\boldsymbol{Q}$ is

$$D_i(\boldsymbol{\rho}, Q_i) = \{s \in Q_i : (\boldsymbol{\rho}, s) \; R_i \; (\boldsymbol{\rho}, s'), \forall s' \in Q_i\}.$$

Consider a tuple $(\boldsymbol{\rho}, \sigma, \boldsymbol{Q})$ consisting of an allocation and a rationing scheme. Each equilibrium notion is defined by subsets of the conditions below:

(1) Each agent is matched to a school in their rationed demand set.
(2) For each school s with $\sigma[s] \neq \emptyset$, $\lfloor \rho_s^{-1} \rfloor = |\sigma[s]|$.
(3) For each pair of agent i and school s such that $s \notin Q_i$, $\rho_s = b_s$.
(4) For each agent $i, j \in N$, and school $s \in S$ such that $s \notin Q_i$ and $j \in \sigma[s]$, $j \succ_s i$.

A **Crowding Drèze Equilibrium (CDE)** is a tuple $(\boldsymbol{\rho}, \sigma, \boldsymbol{Q})$ that satisfies conditions (1)-(3). Note that $\boldsymbol{Q}$ is a general rationing scheme and does not depend on $\succ$. When Q is consistent with the priority profile $\succ$ (Condition 4), then we say that it is a **Priority-Compatible Drèze Equilibrium (PCDE)**. Our RCE is a further refinement of PCDE by additionally imposing the *inferior empty schools* condition.

Without crowding, these notions coincide with several of those in the literature. Condition 2 is the key difference—the number of students matched to a school is nearly the distribution associated with the school. Without this coupling, we revert back to the interpretation of the distribution as the price. Formally, without Condition 2, CDE coincides with the notion in Talman and Yang [2008], and if priorities are used, then PCDE, with additional requirement of *constrained efficiency*, coincides with that in Andersson and Svensson [2014]. Finally, notice that when agents' preferences satisfy the standard monotonicity and continuity assumptions, the existence for each of the aforementioned equilibrium notions is guaranteed. This is not true when there is crowding, as Example 3 demonstrates. With a mild domain restriction, however, we can recover existence (Theorem 1).

4. Existence of RCEs

There are profiles in $\mathcal{R}^N$ that do not admit an RCE. Such profiles are rare; profiles that *do* admit an RCE are generic. We first present an example and then introduce a domain restriction that excludes these cases and ensures existence.

Example 3 (The non-existence of RCE). Let $S = \{s_1, s_2, s_3\}$ and $N = \{1, 2, 3\}$. Agents have the following utility functions:

$$u_1(\boldsymbol{\rho}, s_1) = \rho_{s_1}, \quad u_1(\boldsymbol{\rho}, s_2) = \rho_{s_2}, \quad u_1(\boldsymbol{\rho}, s_3) = -\frac{1}{2} + \rho_{s_3}$$
$$u_2(\boldsymbol{\rho}, s_1) = \rho_{s_1}, \quad u_2(\boldsymbol{\rho}, s_2) = \rho_{s_2}, \quad u_2(\boldsymbol{\rho}, s_3) = -\frac{2}{3} + \rho_{s_3}$$
$$u_3(\boldsymbol{\rho}, s_1) = \rho_{s_1}, \quad u_3(\boldsymbol{\rho}, s_2) = \rho_{s_2}, \quad u_3(\boldsymbol{\rho}, s_3) = -\frac{3}{4} + \rho_{s_3}$$

Each school is allowed to have any priority order, and $b_{s_1} = 1/2$, $b_{s_2} = 1/2$, and $b_{s_3} = 0$.

We show that there is no RCE. By contradiction, suppose that there is. First we claim that no student is matched with school s_3. If there were, then by *exhaustiveness*, at least one of the other two schools, i.e., s_1 and s_2, should have a ratio greater than $1/2$. Thus the student matched with s_3 prefers the school with a ratio greater than $1/2$, contradicting *fairness*.

Since $b_{s_1} = b_{s_2} = 1/2$, in the case where s_1 takes two students, the ratio at s_1 is $1/2$ while s_2 only takes one student, and by *exhaustiveness*, has a ratio greater than $1/2$. Thus, any student matched with s_1 prefers s_2, contradicting *fairness*. The same reasoning works for the case where s_2 takes two students.

We now introduce our domain restriction. Given a preference profile $R \in \mathcal{R}^N$, two schools s_1 and s_{k+1} are **connected by indifference** if there is a distribution $\boldsymbol{\rho}$, a sequence of distinct students $\{i_1, \ldots, i_k\}$, and a sequence of schools $\{s_1, \ldots, s_{k+1}\}$ such that

(1) $\rho_{s_1} = \frac{1}{n}$ and $\rho_{s_{k+1}} = \frac{1}{m}$ for some $m, n \in \{1, \ldots, |N|\}$ and
(2) $(\boldsymbol{\rho}, s_i)\ I_i\ (\boldsymbol{\rho}, s_{i+1})$ for each student $1 \leq i \leq k$

Note that $\boldsymbol{\rho}$ in this case need not be part of an RCE, nor even be compatible with any feasible allocation; it is an arbitrary vector. A preference domain satisfies **no connection by indifference (NCBI)**, if it contains no profile that is connected by indifference. Denote by $\mathcal{D} \subseteq \mathcal{R}^N$ the subdomain of *all* profiles that are not connected by indifference.[20] Call this *the* NCBI domain. The preference profile presented in Example 3 is not in the NCBI domain as as all three students are indifferent between $(1/2, s_1)$ and $(1/2, s_2)$. On the other hand, the preference profile presented in Example 2 is not connected by indifference.

The NCBI domain $\mathcal{D}$ is not rectangular, but is open and dense in the full domain. Thus, we may invoke the Kuratowski-Ulam theorem to say that for a generic profile of the others, $\boldsymbol{R}_{-i}$, and a generic manipulation R_i' of i, the resulting profile $(R_i', \boldsymbol{R}_{-i}) \in \mathcal{D}$.[21] We are not permitted to say anything about the *probability* of landing outside $\mathcal{D}$; since $\mathcal{R}^N$ is infinite dimensional, there is no non-trivial, translation-invariant measure, and therefore no obvious extension of the concept *measure zero*. For our algorithm, however, we consider a natural finite-dimensional preference domain. Restricting to this domain, it is easy to see that the Lebesgue measure of the event that an agent forces the problem outside the domain is zero (see Remark 2 in Section 7).

Theorem 1. *Each preference profile in the NCBI domain admits an RCE.*

The proof of Theorem 1 is in Appendix A.4. We provide a sketch of the argument here. Start from a *fair* allocation. That is, we allow for the discrepancy between ρ_s^{-1} and the actual number of matched students to be arbitrarily large. The existence of *fair* allocations is trivial: set the distribution equal to vector $\boldsymbol{b}$ and run the Deferred Acceptance mechanism.

[20]NCBI is similar to the identically named condition in Andersson and Svensson [2014], although the two are applied to different environments. Our condition is stronger than theirs, and the latter is not sufficient to show existence of RCE in our model.

[21]A set is generic if it is the countable intersection of open-and-dense sets. When a particular generic set is fixed (e.g., a subdomain) its elements are called *generic*. Endow $\mathcal{R}$ with the topology of closed convergence [Hildenbrand, 2015]. Thus, $\mathcal{R}$ is metrizable and hence second-countable. Assume $\mathcal{D}$ is open and dense, which implies that it is both generic (also called comeager) in $\mathcal{R}^N$ and has the Baire property. Given $\boldsymbol{R}_{-i} \in \mathcal{R}^{N \setminus i}$, let $\mathcal{D}_{\boldsymbol{R}_{-i}} = \{R_i' \in \mathcal{R} : (R_i', \boldsymbol{R}_{-i}) \in \mathcal{D}\}$. Then the Kuratowski-Ulam theorem implies that the set $\{\boldsymbol{R}_{-i} \in \mathcal{R}^{N \setminus i} : \mathcal{D}_{\boldsymbol{R}_{-i}}$ is generic in $\mathcal{R}\}$ is generic in $\mathcal{R}^{N \setminus i}$.

The set of distributions that generate RCE, if non-empty, lie in the upper envelope of the set of distributions that generate a *fair* allocation. So, given that a *fair* allocation always exists, our argument begins with one of these, and seeks to increase the distribution vector while maintaining *fairness*. We do this with some graph theoretic tools.

With a distribution fixed, we study a graph with vertices S and such that an arc represents a student at a given school who finds another school at least as good. A *source set* is a set of vertices such that no arcs enter the set (there may be arcs among vertices in the set, so that no vertex is a source on its own). Lemma 1 tells us that if we can find a set of schools that is a source set in our graph, and for which all schools fail our exhaustiveness condition, then we can perturb upwards the ratios for these schools and arrive at another *fair* allocation.

Thus, our goal is to move students among schools such that we do not violate *fairness* and that we find a source set. This is the only part of the argument that requires NCBI, and is achieved in Lemma 2. In sum, beginning with a *fair* allocation, if it is not an RCE, then we can increase the ratio of some school and find another *fair* allocation. Along the way, we eliminate the problem of empty schools by simply putting students in them; NCBI ensures they will not hinder us in finding a source set.

Having established that we can perturb upwards any *fair* allocation that is not an RCE, it remains only to make a limit argument. This is done in Theorem 6 in the Appendix, which actually proves Theorem 1 and Proposition 3 below.

It is known that the exact auction of Demange et al. [1986] and its variants crucially depend on the quasi-linear assumption, and when agents have non-quasi-linear preferences, they are not appropriate methods to show the existence of equilibrium [Zhou and Serizawa, 2021]. Instead, the salary adjustment process of Crawford and Knoer [1981] and Kelso Jr and Crawford [1982], and its variants are frequently used to establish the existence of equilibrium [Herings, 2018, Fleiner et al., 2019]. In general, this method requires that agents always choose their favorite schools among those who have not rejected them yet, and an agent's welfare is independent of the number of tentatively matched agents. This is not true in our model, thus the method fails. Another well-known method is via Scarf's Lemma [Quinzii, 1984]: One first shows existence when agents have piece-wise quasi-linear utility functions and then takes the limit, approximating the original continuous utility functions. It is unclear how to establish the existence of RCEs even when agents have piece-wise quasi-linear utility functions, and the main challenges still come from showing *exhaustiveness*.

5. Structural Properties of RCEs

In the classical model, the set of no *justified envy* and *non-wasteful* allocations has a number of remarkable structural properties. Among them, the welfare lattice and the so-called *Rural Hospitals Theorem* stand out as particularly important [Roth and Sotomayor, 1990, Roth, 1986]. On the NCBI domain, we find analogous, sometimes identical properties. RCEs form a welfare upper lattice, which further implies the classical upper lattice result via the embedding. We also show that the number of students matched to any school remains constant across all RCE, which establishes a *Rural Hospitals Theorem* for our environment.

Along the way, we find a decomposition result that also has analogues in earlier literature, but was previously unknown. We require, first, a definition. A sequence of distinct students $(i_1, ..., i_k)$ constitutes a **trading cycle** from a matching σ to a matching τ if $\tau(i_l) = \sigma(i_{l+1})$ for each $1 \leq l \leq k-1$ and $\tau(i_k) = \sigma(i_1)$.

Proposition 2. *Consider two arbitrary RCEs, $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, for a given preference profile from the NCBI domain.*

(1) *When moving from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$, if student i experiences a strict welfare-improvement (welfare-reduction), then students in the same trading cycle as student i experience a non-decreasing welfare change (non-increasing welfare change).*
(2) *If $\tau(i) \neq \sigma(i)$, then i is involved in a trading cycle from σ to τ.*

The proof of Proposition 2 is in Appendix A.2. What follows is a sketch of the argument. Schools whose ratio either increases or remains the same cannot take on more students (*exhaustiveness*). Students attending a school whose ratio *increases* must be better off, as its increase causes it to rise above its lower bound and thereby be available for all. Then, using Walrasian-type reasoning, we find that these better-off students must go to a school whose ratio must not have decreased. This of course "closes off" the set of such schools and yields the proposition for them. The proposition for the other students comes from feasibility, since the schools whose ratios have decreased must take in the rest of the students, and from the properties of RCE—these schools must also have "high enough" ratios by *exhaustiveness*.

The full argument is complicated by several technical details, most difficult among them being: what if $(\boldsymbol{\gamma}, \tau(i))\ I_i\ (\boldsymbol{\rho}, \sigma(i))$, $\gamma_{\sigma(i)} = \rho_{\sigma(i)} > b_{\sigma(i)}$ and $\gamma_{\tau(i)} = \rho_{\tau(i)} = b_{\tau(i)}$? Student i may displace j at $\tau(i)$ who has $i \succ_{\tau(i)} j$, since i's indifference means that j's presence at $\tau(i)$, under allocation $(\boldsymbol{\sigma}, \rho)$, was not a priority violation. Student j's welfare may drop, and she may envy i at $(\boldsymbol{\gamma}, \tau)$, without priority violation. The problem is that seemingly nothing prevents i from being part of a trading path along which some previous student has increasing welfare and all other previous students have non-decreasing welfare. For this case we invoke NCBI, and the earlier Walrasian-type reasoning, as we find that such a student i must actually be the end of an indifference chain originating at a school $u \in S$ with $\rho_u = b_u$. Thus, it appears that NCBI is essential for this decomposition to hold.

Note the significance of claim 2 in the proposition. In general, if $\gamma_s^{-1} > \rho_s^{-1}$, then s might be able to take on more students under $(\boldsymbol{\gamma}, \tau)$, and so may be the endpoint of a trading *path* rather than a member of a trading cycle. In fact, this occurs among the *fair* allocations, and so our decomposition does not hold on that larger set of allocations.

Proposition 2 is reminiscent of the classical decomposition property of the marriage market [Roth and Sotomayor, 1990]: Moving from one stable outcome to another, there is a one to one correspondence between agents on the one side who have strictly increased welfare (resp. strictly reduced welfare) and those one the other side whose welfare is strictly reduced (resp. strictly increased). This property holds for competitive equilibrium/core outcomes in the two-sided matching models with transfers as well [Demange and Gale, 1985]. Nevertheless, such a decomposition does not hold in our model. Instead, we provide a generalized decomposition by associating welfare changes with trading cycles of components in an allocation. In our language, the existing decomposition results can be read as follows: if an agent experiences a strict welfare improvement, all the other agents in the same trading cycle will also experience strict welfare improvement. In contrast, claim 1 of Proposition 2 admits the possibility of unchanged welfare in a trading cycle.

We state our version of the *Rural Hospitals Theorem*—a clear corollary of Proposition 2:

Theorem 2. *Fix a preference profile from the NCBI domain and let $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ be two RCEs for this profile. Then for each school $s \in S$, the number of students matched to s under σ is equal to the number of students matched to s under τ.*

Consider the set of RCEs in Example 2. There are two RCE matchings. The first one is $\sigma = (\sigma(1), \sigma(2), \sigma(3)) = (s_1, s_1, s_2)$. The second one is $\sigma' = (\sigma'(1), \sigma'(2), \sigma'(3)) = (s_2, s_1, s_1)$. In either matching, school s_1 is always matched with two students and school s_2 is matched with one student. The standard Rural Hospitals Theorem also states that if a school has unfilled seats at a stable matching, then it matches to the same set of students in every stable matching. Example 2 also demonstrates that this fact is not true under crowding.

Given two distributions, $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$, let $\boldsymbol{\rho} \vee \boldsymbol{\gamma} \in [0,1]^S$ denote the vector whose s component, for each $s \in S$, is $\max\{\rho_s, \gamma_s\}$.

Theorem 3. *Assume that $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are RCEs for a preference profile from the NCBI domain. There is a matching μ such that $(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu)$ is an RCE, and for each $i \in N$,*

$$(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu(i)) \ R_i \ \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i)), (\boldsymbol{\gamma}, \tau(i)).\}$$

The theorem states that the set of RCE's has an upper-lattice structure in welfare space. It does *not* always have a lower-lattice.

The full proof of Theorem 3 is in Appendix A.2. Similar to extant results of similar character, a decomposition result, in our case Proposition 2, is the main tool. We simply begin with one of the two RCEs, say $(\boldsymbol{\rho}, \sigma)$, and to arrive at a candidate matching, μ, execute all the welfare-non-decreasing trading cycles from σ to τ. Then we show that $(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu)$ is an RCE. If $(\boldsymbol{\rho} \vee \boldsymbol{\gamma})_s = \gamma_s > \rho_s$, then any student $i \in \sigma[s]$ must have increased welfare, as otherwise they would prefer s at (γ, τ) and the previous inequality gives $\gamma_s > b_s$. Then Proposition 2 and some supporting results in the appendix allow us to conclude that i is part of a cycle among schools whose resource ratio is at least as high under $\boldsymbol{\gamma}$ as it is under $\boldsymbol{\rho}$. This further allows us to use the feasibility of $(\boldsymbol{\gamma}, \tau)$ to conclude the feasibility of $(\boldsymbol{\rho} \vee \boldsymbol{\gamma}, \mu)$. Since all students are partitioned by Proposition 2, the execution of these cycles will not interfere with each other. The foregoing argument studied the case when $\boldsymbol{\rho} \neq \boldsymbol{\gamma}$. However, the decomposition holds equally well when $\boldsymbol{\rho} = \boldsymbol{\gamma}$ and so demonstrates that RCE induce an upper-lattice in welfare space.

It is in general not true that the existence of an upper lattice in distributions implies the existence of an upper lattice in welfare (see Example 4 below). Here again the NCBI domain seems crucial.

Since we have an upper lattice in welfare space, a limit argument is sufficient to show the existence of a greatest RCE welfare vector. Several RCEs may induce this vector, all of which have the same distribution. Any RCE that induces this vector is called **maximal**. For Example 2, one such maximal RCE is given by distribution $(1/2, 17/22)$ paired with matching (s_2, s_1, s_1). We formalize the foregoing observations as follows.

Proposition 3. *Given a preference profile R from the NCBI domain,*

(1) *there is a greatest RCE distribution $\boldsymbol{\rho}^*(R)$,*
(2) *there is a maximal RCE, with distribution $\boldsymbol{\rho}^*(R)$, and all students are indifferent between all maximal RCEs, and*

(3) *among all RCEs, only the maximal RCEs satisfy* student-optimal fairness.

The proof of Proposition 3 is in Appendix A.4. It is not true that an RCE compatible with $\boldsymbol{\rho}^*(R)$ always maximizes students' welfare; maximal RCEs are a strict subset of the RCEs that are compatible with $\boldsymbol{\rho}^*(R)$.

We conclude this section with two remarks. First, in Section 3.1 we showed that the standard school choice model can be embedded in our model. Recalling that the embedded, standard model may have an outside option, even though our more general model does not, it follows that the standard Rural Hospitals Theorem of Roth [1986] is a corollary of Theorem 2. Theorem 3 and Proposition 3 mirror the standard welfare lattice results in Roth and Sotomayor [1990], but here we only consider the student side of the market and its corresponding direction in the lattice. Second, even when RCEs exist for profiles outside the NCBI domain, the structural results above may fail to hold.

Example 4 (Lack of structure on the general domain.). The reader will observe that the structures above fail for the same reason they do in the standard school choice model when students' preferences may have indifferences.

Let $S = \{s_1, s_2, s_3, s_4\}$ and $N = \{1, 2, 3\}$. Agents have the following preferences: For each $\boldsymbol{\rho} \in [0, 1]^S$,

$$(\boldsymbol{\rho}, s_1)\ I_1\ (\boldsymbol{\rho}, s_2)\ P_1\ (\boldsymbol{\rho}, s_3)\ P_1\ (\boldsymbol{\rho}, s_4)$$
$$(\boldsymbol{\rho}, s_1)\ P_2\ (\boldsymbol{\rho}, s_4)\ P_2\ (\boldsymbol{\rho}, s_3)\ I_2\ (\boldsymbol{\rho}, s_2)$$
$$(\boldsymbol{\rho}, s_2)\ P_3\ (\boldsymbol{\rho}, s_3)\ P_3\ (\boldsymbol{\rho}, s_1)\ I_3\ (\boldsymbol{\rho}, s_4)$$

Each school has unit capacity, i.e., $b_{s_1} = b_{s_2} = b_{s_3} = b_{s_4} = 1$. Schools have the following priority rankings: $1 \succ_{s_1} 2 \succ_{s_1} 3$; $1 \succ_{s_2} 3 \succ_{s_2} 2$; $3 \succ_{s_3} 2 \succ_{s_3} 1$; and $2 \succ_{s_4} 3 \succ_{s_4} 1$.

Let $\boldsymbol{\rho} = (1, 1, 1, 1)$. There are two RCEs compatible with $\boldsymbol{\rho}$, clearly making $\boldsymbol{\rho}^*(R) = \boldsymbol{\rho}$. The first one is $(\boldsymbol{\rho}, \sigma)$ such that $(\sigma(1), \sigma(2), \sigma(3)) = (s_1, s_4, s_2)$. The second one is $(\boldsymbol{\rho}, \tau)$ such that $(\tau(1), \tau(2), \tau(3)) = (s_2, s_1, s_3)$. It is not hard to see that there is no trading cycle from σ to τ, and so Proposition 2 fails to hold. Also at $(\boldsymbol{\rho}, \sigma)$ school s_3 is empty and at $(\boldsymbol{\rho}, \tau)$ school s_4 is empty school. Therefore, Theorem 2 fails to hold. Note that in both of the RCEs mentioned, two students get their favorite possible bundle, and one gets their second favorite. Thus, the only way to improve upon this is with matching $(\mu(1), \mu(2), \mu(3)) = (s_1, s_1, s_2)$ or matching $(\mu'(1), \mu'(2), \mu'(3)) = (s_2, s_1, s_2)$, both of which are infeasible.

However, student 2 prefers $(\boldsymbol{\rho}, \tau(2))$ to $(\boldsymbol{\rho}, \sigma(2))$. Thus Theorem 3 fails to hold. Since there are no maximal RCEs, Proposition 3 fails to hold as well.

Examples 3 and 4 highlight the role of our domain restriction, NCBI. However, it is worth noting that the structural properties hold under a weaker restriction. In particular, we may relax the first condition in the definition of connected by indifference, requiring only that $\rho_{s_1} = b_{s_1}$ and $\rho_{s_{k+1}} = b_{s_{k+1}}$. However, this domain restriction is *not* sufficient for our proof of the existence of RCE. See Appendix section A.1 for details.

6. Maximal RCE Mechanisms

Fixing an environment of schools, students, and lower-bounds, for each profile in the NCBI domain, there is a non-empty set of maximal RCE allocations, between which all students are indifferent (Proposition 3). A maximal RCE mechanism is a function that selects, for each profile in the NCBI domain, a maximal RCE allocation; we do not define these mechanisms on the full domain. Thus, all maximal RCE mechanisms are welfare equivalent.

Theorem 4. *Any maximal RCE mechanism on the NCBI domain is* strategy-proof.

The full proof of Theorem 4 is in Appendix B. For some intuition as to how it works, consider first the properties of $\boldsymbol{\rho}^*(\cdot)$, the greatest RCE distribution. Consider a student i who is not even weakly envied by a student at a different school. That is to say, all students not attending i's school strictly prefer their outcome to i's. Then it better not be feasible to raise the ratio at i's school, because if it were, then we could do so by a very small amount, make i and her classmates happier, and not induce any envy from other students. Thus, for each student i one of the following is true: 1) some student j from another school finds i's outcome at least as good as her own or 2) the ratio at i's school exactly corresponds with the number of students there. For case (2), this means that the ratio at i's school is of the form $1/k$, where k is the number of students at i's school. We say such a school is *completely exhausted*. The above reasoning then implies that, for each student for whom (1) is true, we must be able to find a chain of students $\{j, k, \ldots, l\}$ such that j finds i's outcome at least as good as her own, k finds j's outcome at least as good as her own, etc., and l's school is completely exhausted. This is the so-called *connectedness property*, seen in similar environments.

Now let i declare a R_i' such that her preference for her outcome is *strictly* stronger than before. That is, R_i' is a strict Maskin monotonic transform of R_i at her initial outcome. Ignoring complications, if she stays at her original school, then the connectedness property prevents the ratio at her school from rising. If she goes to another school, the connectedness property prevents the ratios of these schools from rising as well, so the only way her new outcome can be R_i better than the original is if she goes to a school that has hit its lower bound. Then we use our decomposition and the connectedness property to find that we cannot displace these students and make them happier, so i induces violation of *fairness* at this new school. In the appendix we show that, if the rule were manipulable, then it would be manipulable via a Maskin monotonic transform, and thus our argument is complete.

7. An Algorithm for Maximal RCE

As discussed after Theorem 1, it is non-trivial to find an algorithm that calculates an RCE in a finite number of steps, in particular, when students have general preferences as studied here. We can, however, on a restricted domain. Say a preference relation $R \in \mathscr{R}$ is *linear* if there is a vector $v \in \mathbb{R}^S_{++}$ such that the utility function $(r, s) \mapsto rv(s)$ represents R. For the remainder of this section, we fix a profile of linear preferences $\boldsymbol{R} \in \mathscr{D}$.

Remark 2 (NCBI in Linear Preferences). In the domain of linear preferences, a violation of NCBI is a sequence $\{1, \dots, J\}$ of students, a sequence $\{1, \dots, J+1\}$ of schools, and a distribution $\boldsymbol{\rho} \in \mathbb{R}^S$ such that, for each j on the sequence, $\rho_j v_j(j) = \rho_{j+1} v_j(j+1)$, and for the two end points, $\rho_1^{-1} \in \mathbb{N}$ and $\rho_{J+1}^{-1} \in \mathbb{N}$. By recursion on this sequence, we find there are p and $q \in \mathbb{N}$ such that

$$\frac{v_k(k+1)}{v_k(k)} = \frac{q}{p} \prod_{j \neq k} \frac{v_j(j+1)}{v_j(j)}.$$

Note that without loss of generality we may restrict attention to representations such that $v_i(1) = 1$, and therefore, given $\boldsymbol{R}_{-k}$, k has at-most-countably many reports that would induce a failure of NCBI. Thus, by Fubini's theorem, students almost surely report a profile that is within our domain.

The algorithm we introduce uses tools from the multi-item auction [Demange et al., 1986], and the Deferred-Acceptance algorithm [Gale and Shapley, 1962], presented in the framework of Drèze equilibrium (Section 3.2). In brief, starting from an initialized distribution, the algorithm will decrement it continuously, pausing occasionally to "reject" some students from some schools; the rejected students cannot thereafter demand these schools. In such cases, time is advanced discretely and, if possible, decrementing is re-initialized.

At each moment $z \in \mathbb{R}$ in time, each student i is permitted to demand a set Q_{iz} of objects, and so their rationed demand set is $D_i(\boldsymbol{\rho}_z, Q_{iz})$, shortened to D_{iz} for simplicity. Say that i *requires* S' if $D_{iz} \subseteq S'$. A set $S' \subseteq S$ is *in excess demand* if (1) the number of students i who require S' exceeds $\sum_{s \in S'} \lfloor \rho_s^{-1} \rfloor$, and (2) for each $S'' \subsetneq S'$, the number of students whose demand intersects S'', among those requiring S', is larger than $\sum_{s' \in S''} \lfloor \rho_{s'}^{-1} \rfloor$. There is a unique set E_z^* in the collection of sets in excess demand with maximal cardinality, which can be found in a computationally efficient way [Andersson et al., 2013].

The following decrementing process will be used in the algorithm. Let z be a time at which decrementing starts, or restarts. There is a set $\Omega_z \subseteq S$ whose ratios are continuously decreasing while $\boldsymbol{Q}_z = (Q_{iz})_{i \in N}$ remains constant. Each $s \notin \Omega_z$ maintains a constant ratio. Then, for $z' \geq z$ in the same decrementing phase as z, let

$$\rho_{sz'} = \begin{cases} \frac{\rho_{sz} z}{z'} & s \in \Omega_z \\ 0 & s \notin \Omega_z \end{cases}$$

The reader may verify that the resulting distribution path $\boldsymbol{\rho}_z$ has the following property, given linear preferences: if $D_{iz} \subseteq \Omega_z$, then $D_{i,z'}$ remains constant throughout the decrementing phase.

The Algorithm. Time begins at $z = 0$, the distribution is initialized at $\boldsymbol{\rho}_0 = (1, \ldots, 1)$, and each $Q_{iz} = S$. The algorithm consists of three subroutines, described below. In each case, assume for notational simplicity that the subroutine is entered at time z. The algorithm begins in the subroutine pause.

Pause: Calculate E_z^*.

If E_z^* is empty, the algorithm is terminated.

If all schools in E_z^* can be decremented, i.e., if each $s \in E_z^*$ has $\rho_{sz} > b_s$, then go to subroutine decrementing.

If E_z^* contains a school that cannot be decremented, go to subroutine rejection.

Regardless of the condition under which the algorithm exits this subroutine, the time at exit equals the time at entry.

Rejection: Let $s \in E_z^*$ have $\rho_{sz} = b_s$. Find the $i \in N$ who is $\succ_s$-lowest priority among those students j with $s \in D_{jz}$. Advance time 1 unit to $z+1$ and exit with $Q_{jz+1} = Q_{jz} \setminus \{s\}$ for all j with $i \succ_s j$ and for $j = i$. Go to subroutine pause. Note that s will reject at most

$|N| - b_s^{-1}$ times before it is no longer in excess demand, so this subroutine will be invoked at most $|S|\left(|N| - b_s^{-1}\right)$ times.

Decrementing: Advance time continuously and decrement $\boldsymbol{\rho}_z$ according to the path given above, with $\Omega_z = E_z^*$. Exit at time $z' > z$, when either

(1) there is $s \in E_z^*$ with $\rho_{sz'} = {}^1\!/\!{}_k$ for $k \in \mathbb{N}$ or
(2) there is i who requires E_z^* at z for whom $D_{iz'} \neq D_{iz}$.

Since the rationed demands of students who require E_z^* are locally constant, decrementing will run at least for an open set of time. Upon exit, go to subroutine pause.

When the algorithm terminates, say at time $\overline{z}$, Hall's theorem (Hall [1935]) guarantees the existence of an equilibrium assignment σ, which of course corresponds to the existence of a feasible allocation $(\boldsymbol{\rho}_{\overline{z}}, \sigma)$. In Appendix C we finish the proof of

Theorem 5. *For linear preference profiles from the NCBI domain, the algorithm terminates in finite time with finitely many elicitations of demand, and results in a maximal RCE.*

8. Discussion

8.1. **Dropping Inferior Empty Schools.** Let an allocation be a **weak RCE** if it satisfies all the conditions of RCE except for the *inferior empty schools* condition. There exists a weak RCE in the NCBI domain. A parallel result to Proposition 2 for weak RCEs would decompose changes into trading cycles of positive or negative welfare value, and indifference chains. The Rural Hospitals Theorem (Theorem 2) no longer holds for weak RCEs. Fortunately, all of the other remaining results can be established. The proofs of the above statements are analogous to those in the main text so we omit them.

8.2. **Applications beyond School Choice.** The first application is to the problem of **labor markets with financially constrained start-ups**. Consider the labor market where there is a finite set of start-ups F and a finite set of workers W. Each start-up $f \in F$ is subject to some (hard) financial constraints so that expenditure for labor employment $\kappa_f > 0$ is fixed for the modelled time period. Given this, the start-up selects employees following a priority order $\succ_f$. The labor market is protected by a minimum wage $\underline{w}$ such that $0 < \underline{w} \leq \kappa_f$ for each $f \in F$. Each worker $w \in W$ has a complete and transitive preference relation R_w over $\{(t, f) \in \mathbb{R} \times F :\ 0 \leq t \leq \kappa_f\}$ such that for each $f \in F$ and $t, t' \in [0, \kappa_f]$ with $t > t'$, $(t, f)\ P_w\ (t', f)$. The tuple $(F, W, \boldsymbol{R}, \boldsymbol{\kappa}, \succ, \underline{w})$ summarizes primitives of the problem.

We reformulate the problem to apply our results. Let F, W, and $\succ$ be as defined above. For each $f \in F$, let $b_f = \frac{\underline{w}}{\kappa_f}$. Each worker $w \in W$ has a complete and transitive preference relation R^*_w over $[0,1] \times F$ such that for each $f, f' \in F$ and each $t, t' \in [0,1]$, $(t, f)\ R^*_w\ (t', f')$ if and only if $(t\kappa_f, f)\ R_w\ (t'\kappa'_f, f')$. It is easy to see that all our results hold in the problem $(F, W, \boldsymbol{R}^*, \boldsymbol{b}, \succ)$ and therefore the insights can be carried over to the original problem $(F, W, \boldsymbol{R}, \boldsymbol{\kappa}, \succ, \underline{w})$.

The second application is to the problem of **allocating polluting firms**. Instead of students and schools, consider polluting firms and subnations. Each firm is willing to invest in at one most one subnation. Each subnation s is endowed with an amount of the same environmental resource, and the priority for selecting polluting firms is based on their industrial development policy. Let b_s^{-1} be the maximum number of polluting firms that the subnation is willing to admit. Each firm prefers to locate to a subnation where they are allowed a higher level of pollution. As this is simply a re-interpretation of the model, we are able to directly apply our results for this application.

9. Conclusion

We provide a new framework to model school choice with crowding and establish analogs of key results in the school choice and matching literature. Many issues of perennial interest—diversity considerations, efficiency improvements, sibling guarantees, etc.—can be reconsidered in this model. We also align with the empirical literature's explicit consideration of per capita crowding metrics.

The model also opens up completely new avenues of exploration not suited to the standard one. Most interesting is the question of redistributing resources *between* schools. We have assumed that each school has a fixed endowment of resources, but in practice the resources of all schools in a given district are usually controlled centrally by that district. For example, money or physical infrastructure (such as a trailer) can be shifted from one school to another; counselors routinely serve multiple schools. Thus, we can design new mechanisms encompassing such distribution between schools. We can also ask if there are ways to redistribute resources so that, *ex post*, our mis-match between supply and demand is completely eliminated. More generally, we can study comparative statics in this dimension and build a more dynamic model of educational resource allocation.

It is worth noting, in conclusion, that the model implicitly covers a wide variety of different crowding effects. We have modeled crowding in a hyperbolic way: n students share resources at (approximately) $1/n$ per-capita. However, our relatively unrestricted domain of preferences contains within it many subdomains that each embody different notions of crowding. For example, the reader may verify that a function of the form $(\rho_s, s) \mapsto v_s \frac{\rho_s^{1-\eta}}{1-\eta}$ exhibits decreasing marginal disutility for crowding if $\eta > 2$.[22]

Appendix A. Existence of RCE and maximal RCE

We proceed in four subsections. In Section A.1 we study the set of *fair* allocations, which contains the set of RCE, and which can be trivially shown to be non-empty on our domain. In Section A.2 we show that RCE, if they exist, induce an upper lattice in welfare space, which then leads to Theorem 3. Proposition 2 is necessary for this argument, so we give its proof here as well. In Section A.3, we uncover some Pareto dominance relations in the set of *fair* allocations ("the *fair* set" for short). These imply that RCE will lie in the welfare-upper-envelope of the *fair* set, and so will exist if the fair set induces a closed set in welfare space. Finally, we conclude in Section A.4 with the (simple) topological arguments that are required for this upper-envelope to exist.

A.1. Existence of *Fair* Allocations.

A preference domain $\mathcal{D}' \subseteq \mathcal{R}^N$ satisfies **no boundary indifference (NBI)**, if for each pair of schools s and t and each $R \in \mathcal{D}'$, there is no chain of indifference connecting (b_s, s) and (b_t, t). Note that if $\boldsymbol{R}$ satisfies NCBI, then it satisfies NBI.

Proposition 4. *A* fair *allocation exists on any NBI domain.*

Proof. Consider $\boldsymbol{\rho} \in [0,1]^S$ such that $\rho_s = b_s$. Then we have a standard school-choice problem where the capacity of school $s \in S$ is b_s^{-1}. NBI implies that student preferences are strict, so the set of stable matchings is non-empty [Roth and Sotomayor, 1990]. Let σ be a stable matching for this problem. Clearly, it satisfies no-blocking, in the sense of our model. Thus, $(\boldsymbol{\rho}, \sigma)$ is a *fair* allocation. ■

A.2. The upper-lattice property.

Given two allocations, $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, construct the labeled, directed *transfer graph* T on vertices S so that $s \xrightarrow{i} t \in T$ if $\sigma(i) = s$, $\tau(i) = t$,

[22] Calculate $\frac{d^2}{dn^2} u(1/n, s)$.

and $s \neq t$. Define the sets

$$S^+ = \{s \in S : \gamma_s > \rho_s\} \qquad N^+ = \{i \in N : (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i))\}$$
$$S^= = \{s \in S : \gamma_s = \rho_s > b_s\} \qquad N^= = \{i \in N : (\boldsymbol{\gamma}, \tau(i))\ I_i\ (\boldsymbol{\rho}, \sigma(i))\}$$
$$S^* = \{s \in S : \gamma_s = \rho_s = b_s\}.$$

We denote by $s \rightsquigarrow t \subseteq T(\sigma, \tau)$ a simple path in the transfer graph, which is to say, a path with no repeated arcs. Note that since our graph may contain several arcs, with the same orientation, between a given pair of vertices, there may be many distinct paths from s to t, even on the same ordered list of vertices. We distinguish between different paths either by decoration, so that $s \rightsquigarrow' t \neq s \rightsquigarrow t$, or superscript index, so that $s \rightsquigarrow^m t \neq s \rightsquigarrow^n t$ when $n \neq m$. A path is positive if it contains an N^+-labeled arc. A positive path is totally-positive if it contains only labels from $N^+ \cup N^=$.

The in-degree of a set of vertices $V \subseteq S$ is the number of edges $s \to t \in T$ with $s \notin V$ and $t \in V$. Symmetrically, the out-degree is the number of such edges where $s \in V$ and $t \notin V$.

Given the language just introduced, we rephrase Proposition 2.

Proposition 5. *Let $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ be two RCEs for a profile $\boldsymbol{R}$ satisfying NBI. Let T be the transfer graph from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$, and let the sets $S^+, S^=, S^*, N^+, N^=$ be defined as above. Then any path $s \rightsquigarrow t \subseteq T$ touching a S^+ school is positive. Any positive path is a totally-positive cycle, confined to $\overline{S} = S^+ \cup S^= \cup S^*$. Moreover, these cycles can be constructed so that they are all mutually disjoint.*

Proof. We first establish several claims.

Claim 1. *For each $s \in \overline{S}$ with $\sigma[s] \neq \emptyset$, the out-degree of s in T is at least as large as its in-degree.*

Proof of claim. For each $s \in \overline{S}$, with $\sigma^{-1}[s]$ non-empty,

$$\lfloor \gamma_s^{-1} \rfloor \leq \lfloor \rho_s^{-1} \rfloor = |\sigma^{-1}[s]|,$$

and so there cannot be more students at s under $(\boldsymbol{\gamma}, \tau)$ then under $(\boldsymbol{\rho}, \sigma)$, and so for any arc entering s, there must be at least one exiting. ◇

Claim 2. *If $s \xrightarrow{i} t \in T$ has $s \in \overline{S}$ and $i \in N^+ \cup N^=$, then $t \in \overline{S}$.*

Proof of claim. If $t \notin \overline{S}$, then $\gamma_t < \rho_t$, and furthermore, $\rho_t > b_t$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE and $\sigma(i) = s$, $(\boldsymbol{\rho}, s)\ R_i\ (\boldsymbol{\rho}, t)$. Since, $\gamma_t < \rho_t$, $(\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\gamma}, t)$, and since $\tau(i) = t$, $i \notin N^+ \cup N^=$. ◇

Claim 3. *$\sigma[S^+] \subseteq N^+$ and $\tau(N^+) \subseteq S^+ \cup S^*$.*

Proof of claim. For $s \in S^+$, $\gamma_s > b_s$. Therefore, for each $i \in N$, $(\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, s)$. In particular, for $i \in \sigma[s]$, preference monotonicity gives

$$(\boldsymbol{\gamma}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\rho}, s) = (\boldsymbol{\rho}, \sigma(i)) \,.$$

Thus, $i \in N^+$.

Let $i \in N^+$. If $\tau(i) \notin S^+$, then by preference monotonicity

$$(\boldsymbol{\rho}, \tau(i))\ R_i\ (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i)) \,,$$

and so $b_{\tau(i)} = \rho_{\tau(i)} \geq \gamma_{\tau(i)} \geq b_{\tau(i)}$. This yields $\tau(i) \in S^*$. ◇

Claim 4. *Consider a path in T of the following form:*

$$t \xrightarrow{i} u \xrightarrow{j} v$$

with $i \in N^+$ and $j \notin N^+$. Then $j \in N^=$, $u \in S^*$, $v \in S^+ \cup S^=$, and $\sigma[v] \neq \emptyset$.

Proof of claim. By Claim 3, $u \in S^+$ implies $j \in N^+$, and so $\gamma_u \leq \rho_u$. Since $i \in N^+$,

$$(\boldsymbol{\rho}, u)\ R_i\ (\boldsymbol{\gamma}, u) = (\boldsymbol{\gamma}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i)) \,.$$

It follows that $u \in S^*$ and, since $\sigma(j) = u$, $j \succ_u i$. Thus if j is made worse off going to $(\boldsymbol{\gamma}, \tau)$, we have

$$(\boldsymbol{\gamma}, u) = (\boldsymbol{\rho}, u) = (\boldsymbol{\rho}, \sigma(j))\ P_j\ (\boldsymbol{\gamma}, \tau(j)) \,,$$

implying, since $\tau(i) = u$, that $i \succ_u j$. In sum, we have $j \succ_u i$ and $i \succ_u j$, a contradiction. Therefore, $j \in N^=$. Since $u \in S^*$, if $\gamma_v = b_v$, then j is indifferent between (b_u, u) and (b_v, v), contradicting NBI. Moreover, if $\rho_v > \gamma_v$, then $\rho_v > b_v$ and

$$(\boldsymbol{\rho}, v)\ P_j\ (\boldsymbol{\gamma}, v) = (\boldsymbol{\gamma}, \tau(j))\ I_j\ (\boldsymbol{\rho}, \sigma(j)) \,,$$

contradicting that $(\boldsymbol{\rho}, \sigma)$ is an RCE. Conclude that $v \in S^+ \cup S^=$. Finally, if $\sigma[v] = \emptyset$, then

$$(\boldsymbol{\rho}, u) = (\boldsymbol{\rho}, \sigma(j))\ P_j\ (\boldsymbol{\rho}, v) = (1, v)\ R_j\ (\boldsymbol{\gamma}, v) = (\boldsymbol{\gamma}, \tau(j)) \,,$$

where the strict relation is by *inferior empty schools*, contradicting that $j \in N^=$. ◇

Let $s \xrightarrow{1} u \in T$ have $1 \in N^+$. We shall extend this to a path $s \rightsquigarrow t$. By Claim 3, $u \in \overline{S}$. If it were the case that $\sigma[u] = \emptyset$, then since $(\boldsymbol{\rho}, \sigma)$ is an RCE, *inferior empty schools* implies,

$$(\boldsymbol{\rho}, s)\ P_1\ (\boldsymbol{\rho}, u) = (1, u) = (\boldsymbol{\gamma}, u)\,,$$

contradicting that $1 \in N^+$. By Claim 1, there is $u \xrightarrow{2} v \in T$. If $2 \in N^+$, we could then start with this edge instead. Continuing inductively, let j be the first student on the path who is not in N^+ (if such an student does not exist, the argument yields a totally positive cycle, as desired). We have that $s \rightsquigarrow t$ decomposes to

$$s \xrightarrow{1} u \rightsquigarrow v \rightarrow \sigma(j) \xrightarrow{j} w,$$

where $u \rightsquigarrow v$, if it exists, is labeled by N^+ students. By Claim 4, $j \in N^=$, $\sigma(j) \in S^*$, $w \in S^+ \cup S^=$, and $\sigma[w] \neq \emptyset$. If $w \in S^+$, then the path is extended by an arc $w \xrightarrow{k} w'$ with $k \in N^+$ (Claims 1 and 3). Our argument has returned to its starting point; our goal is simply to show that a path initiated by a positive arc cannot have a terminal arc, remains within $\overline{S}$, and is labeled only by $N^+ \cup N^=$ students. Thus, we proceed constructively, and when we arrive at an arc with an N^+ label, call this the *escape condition* of our proof.

Assume, therefore, that $w \in S^=$. Claim 1 implies there is $w \xrightarrow{k} w' \in T$. Note that $k \in N^+ \cup N^=$, as otherwise,

$$(\boldsymbol{\gamma}, w) = (\boldsymbol{\rho}, w) = (\boldsymbol{\rho}, \sigma(k))\ P_k\ (\boldsymbol{\gamma}, \tau(k))\,,$$

violating that $(\boldsymbol{\gamma}, \tau)$ is an RCE. If $k \in N^+$, we have encountered the escape condition again, so assume $k \in N^=$. By Claim 2, $w' \in \overline{S}$. If $w' \in S^+$, then $\rho_{w'} < 1$ so $\sigma[w'] \neq \emptyset$, and there must be an outgoing N^+ arc from w' (Claims 1 and 3); again we have the escape condition. Thus, to continue the argument, assume $w' \in S^* \cup S^=$. Now if $w' \in S^*$, we have

$$\sigma(j) \xrightarrow{j} w \xrightarrow{k} w'$$

with $\sigma(j), w' \in S^*$, and $j, k \in N^=$. This is an indifference chain connecting two S^* schools, contradicting NBI. Conclude that $w' \in S^=$, so

$$(\boldsymbol{\rho}, w)\ I_k\ (\boldsymbol{\gamma}, w') = (\boldsymbol{\rho}, w')\,,$$

where the indifference is because $k \in N^=$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, by *inferior empty schools*, $\sigma[w'] \neq \emptyset$. We can then repeat the foregoing arguments and continue the path. In particular, w' must have an outgoing arc $w' \xrightarrow{k'} w''$, and $k' \in N^+ \cup N^=$. If $k' \in N^+$, we get the escape condition, and if $k' \in N^=$, we again conclude that $w'' \in S^=$ and $\sigma[w''] \neq \emptyset$.

Conclude that we can further decompose $s \rightsquigarrow t$ to

$$s \xrightarrow{1} u \rightsquigarrow v \to \sigma(j) \xrightarrow{j} w \rightsquigarrow x \to t$$

where

(1) $u \rightsquigarrow v$, is labeled by N^+ students, and is contained in $\overline{S}$ by Claim 3,
(2) $j \in N^=$ and $\sigma(j) \in S^*$,
(3) $w \rightsquigarrow x$, is within $S^=$ and labeled by $N^=$ students,
(4) $t \in S^+$.

These segments need not all exist. If the last segment exists, then we are back where we started and repeat the argument. In any case, we have shown that any N^+ labeled arc $s \to u$ induces a path that is always labeled by $N^+ \cup N^=$ students, is always within $\overline{S}$ (except possibly for the very first vertex, s), and can always be extended. It follows that we can find a cyclic sub-path, not necessarily including s. However, by deleting the cycle from T (viewing the graph as a set of labeled edges), we preserve the vertex degree inequality of Claim 1, and none of the other claims are affected. Thus, we may repeat the argument. Eventually, we must find a cycle involving s, implying $s \in \overline{S}$.

Note, finally, that if $s \to t \in T$ and $t \in S^+$, then since $\rho_t < \gamma_t \leq 1$, $\sigma[t] \neq \emptyset$, and by Claim 1, there is $t \xrightarrow{i} u \in T$, and $i \in N^+$. Invoking the argument above, we find that any path that touches a S^+ school is a totally positive cycle. ■

Now we apply Proposition 5 to the NCBI domain and complete the proofs of Proposition 2 and Theorem 3.

Proof of Proposition 2. Note that Proposition 5 can be applied in reverse, from $(\boldsymbol{\gamma}, \tau)$ to $(\boldsymbol{\rho}, \sigma)$ to get the first claim of this proposition. Thus, to complete the decomposition, it remains to show that there are no paths in T that are not cycles. We have already shown this for signed paths, so suppose $s \rightsquigarrow t \subseteq T$ is labeled only by $N^=$ students. Then each vertex on the path belongs to $S^* \cup S^=$. If $u \xrightarrow{i} v \in s \rightsquigarrow t$ has $\sigma[v] = \emptyset$, then by inferior empty schools, $(\boldsymbol{\rho}, u)\ P_i\ (1, v) = (\boldsymbol{\gamma}, v)$, contradicting that $i \in N^=$. If $\tau[u] = \emptyset$ then reverse the argument. Clearly only s or t could be empty in one of the two RCEs, and one of the two arguments just made applies to each, so no vertex touched by the path is empty at either RCE. Then, since $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$ are both exhaustive, for each vertex u touched by the path, $\lfloor \rho_u^{-1} \rfloor = |\sigma[u]|$ and $\lfloor \gamma_u^{-1} \rfloor = |\tau[u]|$. Since $u \in S^* \cup S^=$, $\rho_u = \gamma_u$ and so $|\sigma[u]| = |\tau[u]|$. Thus, $s \rightsquigarrow t$ can be extended to $u \to s \rightsquigarrow t \to v$. If $u \rightsquigarrow v$ is labeled only by

$N^=$ students, then we may repeat the argument. Since there are finitely many vertices, we can eventually extend to a cycle, either by exhaustion of the argument in this paragraph, or by extending to a signed path. ■

Proof of Theorem 3. Let T be the transfer graph from $(\boldsymbol{\rho}, \sigma)$ to $(\boldsymbol{\gamma}, \tau)$. Let μ be the matching that results from executing all the positive paths in T on σ. That is, if i labels an arc on a positive path, then $\mu(i) = \tau(i)$, and otherwise $\mu(i) = \sigma(i)$. Let $\boldsymbol{\zeta} = \boldsymbol{\rho} \vee \boldsymbol{\gamma}$. We show that $(\boldsymbol{\zeta}, \mu)$ is an RCE. Since positive paths are totally positive cycles, the number of students at each school is unchanged from σ to μ, so $(\boldsymbol{\zeta}, \mu)$ satisfies *exhaustiveness*. To check that $(\boldsymbol{\zeta}, \mu)$ is an allocation, it is sufficient to check the schools whose distribution has increased. That is, pick $s \in S^+$, where S^+ is defined as above. Since $|\tau^{-1}[s]| = |\sigma^{-1}[s]|$, and $\zeta_s = \gamma_s$, distribution feasibility at school s then follows from the distribution feasibility of $(\boldsymbol{\gamma}, \tau)$.

Let N' be the set of students on a totally positive cycle. By Proposition 5, totally positive cycles are confined to $\overline{S}$, so each $i \in N'$ gets $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))$. The total-positivity of these paths also yields

$$\forall i \in N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i)) \; R_i \; (\boldsymbol{\rho}, \sigma(i)) \,. \tag{1}$$

Each i not on a totally positive cycle gets $\mu(i) = \sigma(i)$. Let $\sigma(i) = s$. There are two cases, $s \in S^+$ or not. If $s \in S^+$, then Proposition 5 implies that i cannot label any arc in T, as any such arc is then part of a totally positive cycle. Thus, $\tau(i) = \sigma(i)$ and we have $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i))$. Then by preference monotonicity we have

$$\forall i \in \sigma[S^+] \setminus N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\gamma}, \tau(i)) \; P_i \; (\boldsymbol{\rho}, \sigma(i)) \tag{2}$$

If $s \notin S^+$, then $\zeta_s = \rho_s$ and so $(\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\rho}, \sigma(i))$. If $i \in N^+$, then since $\gamma_s \leq \rho_s$, it must be that $\tau(i) \neq \sigma(i) = s$. Then i would be on a totally positive cycle. Therefore, $i \notin N^+$ and again we conclude

$$\forall i \in \sigma[S \setminus S^+] \setminus N', (\boldsymbol{\zeta}, \mu(i)) = (\boldsymbol{\rho}, \sigma(i)) \; R_i \; (\boldsymbol{\gamma}, \tau(i)) \,. \tag{3}$$

In all cases, we have found that, at $(\boldsymbol{\zeta}, \mu)$, students are consuming either their bundle under $(\boldsymbol{\rho}, \sigma)$ or their bundle under $(\boldsymbol{\gamma}, \tau)$. Moreover, since

$$\left(\sigma[S^+] \setminus N'\right) \cup \left(\sigma[S \setminus S^+] \setminus N'\right) = \left(\sigma[S^+] \cup \sigma[S \setminus S^+]\right) \setminus N' = N \setminus N',$$

lines 1, 2, and 3 yield

$$\forall i \in N, (\boldsymbol{\zeta}, \mu(i)) \; R_i \; \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i)) \, , (\boldsymbol{\gamma}, \tau(i))\} \tag{4}$$

Suppose $(\boldsymbol{\zeta}, s)\ P_i\ (\boldsymbol{\zeta}, \mu(i))$, which by line 4 implies

$$(\boldsymbol{\zeta}, s)\ P_i \max_{R_i}\{(\boldsymbol{\rho}, \sigma(i)), (\boldsymbol{\gamma}, \tau(i))\}.$$

Assume there is $j \in \mu[s]$, so $(\boldsymbol{\zeta}, \mu(j)) = (\boldsymbol{\zeta}, s)$. Then since $(\boldsymbol{\zeta}, \mu(j)) \in \{(\boldsymbol{\rho}, \sigma(j)), (\boldsymbol{\gamma}, \tau(j))\}$, plugging the appropriate case into the previous line yields$j \succ_s i$.

Assume, therefore, that s is empty under μ. Then it is empty under σ, and so $\zeta_s = \rho_s = 1$. Thus, again invoking line 4,

$$(1, s) = (\boldsymbol{\rho}, s)\ P_i\ (\boldsymbol{\zeta}, \mu(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i)),$$

contradicting that $(\boldsymbol{\rho}, \sigma)$ is an RCE. Therefore, $(\boldsymbol{\zeta}, \mu)$ satisfies *fairness*.

Finally, by Theorem 2, the set of empty schools remains the same in $(\boldsymbol{\rho}, \sigma)$ and $(\boldsymbol{\gamma}, \tau)$, and so also in $(\boldsymbol{\zeta}, \mu)$. Thus, by line 4, $(\boldsymbol{\zeta}, \mu)$ satisfies *inferior empty schools*. ■

A.3. **Domination lemmas.** Given an allocation $(\boldsymbol{\rho}, \sigma)$, let $s \xrightarrow{i} t \in \Gamma$ if $\sigma(i) = s \neq t$, and $(\boldsymbol{\rho}, t)\ R_i\ (\boldsymbol{\rho}, s)$. We say that Γ is the *weak envy* graph of $(\boldsymbol{\rho}, \sigma)$.

Recall that a *source set* in a directed graph is a set of vertices that no edge enters. Formally, it is a set $S' \subseteq S$ such that if $s \to t \in \Gamma$ and $s \notin S'$, then $t \notin S'$.

Say a school $s \in S$ is *totally exhausted* at $(\boldsymbol{\rho}, \sigma)$ if $|\sigma^{-1}[s]|\rho_s = 1$.

Lemma 1. *Let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation with weak-envy graph Γ. Suppose $S' \subseteq S \setminus S^*$, not empty, is a source set in Γ and that no school in S' is totally exhausted. Then there is an RCE $(\boldsymbol{\gamma}, \tau)$, Pareto-dominating $(\boldsymbol{\rho}, \sigma)$ and with $\gamma \gneq \rho$.*

Proof. Let $N' = \sigma^{-1}[S']$. For each $s \in S'$, let $n_s = |\sigma^{-1}[s]|$. We shall construct an assignment market isomorphic to the problem we currently face when restricted to N' and S'. To aid comparison of our current model with the assignment market we employ, we use the terms *stability* and *blocking*. For our model, clearly s and i block $(\boldsymbol{\rho}, \sigma)$ if $\sigma(i) \neq s$ and either $\sigma[s] = \emptyset$ or i has (justified) envy at s. An allocation is stable if there are no blocks. Let $\mathfrak{S}$ be a set of $\sum_{s \in S'} n_s$ elements. Let $f : \mathfrak{S} \to S'$ have $|f[s]| = n_s$. We view $\mathfrak{S}$ as the set of copies of the elements of S'.

Each $\mathfrak{s} \in \mathfrak{S}$ consumes a point $(l, i) \in \mathbb{R} \times N'$ and has simple preferences represented by utility function $W_{\mathfrak{s}}(l, i) = l$; copies of schools care only about resources. Each copy has an outside option denoted $\underline{w}_{\mathfrak{s}}$, so that $\mathfrak{s}$ will withdraw from the matching (now an option) before accepting a bundle giving utility less than $\underline{w}_{\mathfrak{s}}$. With abuse of notation, we retain the same notation for the students. Each $i \in N'$ consumes a point $(r, \mathfrak{s}) \in \mathbb{R} \times \mathfrak{S}$ and has preferences so that

$$(r, \mathfrak{s})\ R_i\ (r', \mathfrak{s}') \iff (r, f(\mathfrak{s}))\ R_i\ (r', f(\mathfrak{s}')).$$

Let U_i be a continuous utility function representation of R_i. Assume that the outside option utility for students is $-\infty$. When an student and a school match, one unit of divisible resource is produced, independent of their identities. We now have a one-to-one assignment market, matching the sets N' and $\mathfrak{S}$ together, and each matched pair having a unit of divisible resource to divide. Demange and Gale [1985] show that, in this model, there is a unique student-optimal stable utility profile $(\boldsymbol{u}, \boldsymbol{w})$, with at least one and possibly several matchings that yield these utilities. Moreover, there is at least one $\mathfrak{s} \in \mathfrak{S}$ with $w_{\mathfrak{s}} = \underline{w}_{\mathfrak{s}}$.

Let $\hat{\sigma} : N' \to \mathfrak{S}$ be a bijection such that each $\hat{\sigma}(i) \in f[\sigma(i)]$. The RCE $(\boldsymbol{\rho}, \sigma)$ induces the following allocation on the constructed assignment market: Each $i \in N'$ gets $(\rho_{f(\mathfrak{s})}, \mathfrak{s})$ and $\mathfrak{s} \in \mathfrak{S}$ gets $(1 - \rho_{f(\mathfrak{s})}, \hat{\sigma}^{-1}(\mathfrak{s}))$.[23]

We show in this paragraph that, so long as $\underline{w}_{\mathfrak{s}} \leq 1 - \rho_{f(\mathfrak{s})}$, the allocation $(\boldsymbol{\rho}, \hat{\sigma})$ in the constructed assignment market is stable. Clearly, no individual rationality constraints are violated. Since $(\rho_{f(\mathfrak{s})}, f(\mathfrak{s}))\ R_i\ (\rho_{f(\mathfrak{s}')}, f(\mathfrak{s}'))$ when $f(\mathfrak{s}) = \sigma(i)$, i and $\mathfrak{s}'$ could only form a blocking pair by giving i at least $\rho_{f(\mathfrak{s}')}$, leaving only $1 - \rho_{f(\mathfrak{s}')}$ for $\mathfrak{s}'$.

Fix $\varepsilon > 0$ and set each $\underline{w}_{\mathfrak{s}} = 1 - \rho_{f(\mathfrak{s})} - \varepsilon$. Let $(\boldsymbol{u}, \boldsymbol{w})$ be the student-optimal utility profile for this problem, and let $\hat{\mu}$ be a matching that supports it. Since $(\boldsymbol{\rho}, \hat{\sigma})$ is stable, Demange and Gale [1985] also show that, for each $\mathfrak{s} \in \mathfrak{S}$, $w_{\mathfrak{s}} \leq 1 - \rho_{f(\mathfrak{s})}$.

Suppose there are $\mathfrak{s}, \mathfrak{s}' \in f[s]$ with $w_{\mathfrak{s}} < w_{\mathfrak{s}'}$. Let $j = \hat{\mu}^{-1}(\mathfrak{s}')$. By feasibility, j is getting no more than $r_j = 1 - w_{\mathfrak{s}'}$ units of resource at $\mathfrak{s}'$. By monotonicity of j's preferences, and since she cannot distinguish $\mathfrak{s}$ and $\mathfrak{s}'$, $U_j(r_j, \mathfrak{s}) \geq u_j$. However,

$$W_{\mathfrak{s}}(1 - r_j, j) = 1 - r_j = w_{\mathfrak{s}'} > w_{\mathfrak{s}},$$

and so j and $\mathfrak{s}$ form a blocking pair. Conclude then that copies of the same school all get the same level of utility.

For each $s \in S'$, let $\gamma_s = 1 - w_{f[s]}$, where this latter is an abuse of notation but is well-defined by our previous observation. For each $s \notin S'$, let $\gamma_s = \rho_s$. For $s \in S'$ we have

$$\gamma_s = 1 - w_{f[s]} \geq 1 - (1 - \rho_s) = \rho_s,$$

[23]We use $f^{-1}(x)$ to denote the *unique* inverse of a bijection and $f[x]$ to denote the set-valued pre-image.

and since $w_{\mathfrak{s}} = \underline{w}_{\mathfrak{s}}$ for some copy of some school, the above inequality is strict for at least one $s \in S'$. Next, define μ so that, for each $s \in S'$, $\mu[s] = \{i \in N : \hat{\mu}(i) \in f[s]\}$, and for $s \notin S'$, $\mu[s] = \sigma[s]$.

Recall that for each $s \in S'$, $\rho_s > b_s$, so there is a block of $(\boldsymbol{\gamma}, \mu)$ involving $s \in S'$ if and only if there is $i \in N$ with $(\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\gamma}, \mu(i))$. Since S' is a source set in Γ, for ε small enough, it remains a source set in the weak-envy graph for $(\boldsymbol{\gamma}, \mu)$. Thus, there is no block with students outside N'. Since $(\boldsymbol{u}, \boldsymbol{w})$ is stable, for each $i \in N'$ and each $s \in S'$, $(\boldsymbol{\gamma}, \mu(i))\ R_i\ (\boldsymbol{\gamma}, s)$. Thus the only remaining possible block is between $i \in N'$ and $s \notin S'$. Suppose such a block exists. Then

$$(\boldsymbol{\rho}, s) = (\boldsymbol{\gamma}, s)\ P_i\ (\boldsymbol{\gamma}, \mu(i))\ R_i\ (\boldsymbol{\rho}, \sigma(i))\,,$$

where the last relation is because the stable match in the one-to-one problem Pareto dominates $(\boldsymbol{\rho}, \sigma)$ for the N' students. Thus, i would also like to block with s at $(\boldsymbol{\rho}, \sigma)$. However, since $\gamma_s = \rho_s$ and $\mu[s] = \sigma[s]$, if school s is party to the block at $(\boldsymbol{\gamma}, \mu)$ then it is at $(\boldsymbol{\rho}, \sigma)$ as well, contradicting that the latter is a *fair* allocation.

It remains to check that $(\boldsymbol{\gamma}, \mu)$ is feasible, which requires only checking feasibility for the S' schools. Since each $s \in S'$ has $n_s\rho_s < 1$, for ε small enough,

$$n_s\gamma_s = n_s(1 - w_{f[s]}) \leq n_s(1 - \underline{w}_{f[s]}) = n_s(\rho_s + \varepsilon) < 1,$$

as desired. ∎

Let Γ be the weak-envy graph of allocation $(\boldsymbol{\rho}, \sigma)$, and let $t \rightsquigarrow s \subseteq \Gamma$. Construct τ so that for each $i \in N$ with $u \overset{i}{\to} v \in t \rightsquigarrow s$, $\tau(i) = v$, and otherwise $\tau(i) = \sigma(i)$. We allow for $s = t$, so that the path may be a cycle. If $(\boldsymbol{\rho}, \tau)$ is an allocation, then we say that $t \rightsquigarrow s$ is *feasible*. We say that τ is the matching that results from *executing* the path on matching σ.

Given allocation $(\boldsymbol{\rho}, \sigma)$ with weak envy graph Γ, the set of *vertices upstream of* s is

$$U_s = \{t \in S : \exists t \rightsquigarrow s \subseteq \Gamma\}.$$

Lemma 2. *Given profile $\boldsymbol{R}$ from the NCBI domain, let $(\boldsymbol{\rho}, \sigma)$ be a* fair *allocation that is not an RCE. Then there is another* fair *allocation for $\boldsymbol{R}$ that Pareto dominates $(\boldsymbol{\rho}, \sigma)$.*

Proof. Observe that if the set of upstream vertices U_s is empty for some $s \in S$, and if ρ_s is less than 1, then we can set $\gamma_s = \rho_s + \epsilon$ and all else equal, and $(\boldsymbol{\gamma}, \sigma)$ is a *fair* allocation if ϵ is small enough. If $\sigma[s]$ is non-empty, we are done.

As usual, let $S^* = \{s \in S : \rho_s = b_s\}$. Execute the following procedure as many times as possible, starting with $\sigma_0 = \sigma$: choose $s \in S \setminus S^*$ with $|\sigma_m^{-1}[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$. Letting Γ_m be the weak-envy graph of $(\boldsymbol{\rho}, \sigma_m)$, find a (minimal) path $t \rightsquigarrow s \subseteq \Gamma_m$ with $t \in S^*$. That is, by taking sub-paths, $t \rightsquigarrow s$ touches S^* only at t. Execute the path to arrive at a new allocation $(\boldsymbol{\rho}, \sigma_{m+1})$. Observe that $(\boldsymbol{\rho}, \sigma_{m+1})$ is a *fair* allocation, as no student has entered a S^*-school, so no violations of *fairness* can be introduced. Of course, we now have a failure of exhaustiveness at t, if not before. Nonetheless, by the definition of the weak-envy graph, $(\boldsymbol{\rho}, \sigma_{m+1})$ Pareto weakly dominates $(\boldsymbol{\rho}, \sigma_m)$.[24] We have therefore proven the following claim:

Claim. *Let* Γ *be the weak-envy graph of a* fair *allocation* $(\boldsymbol{\rho}, \sigma)$. *Assume* $t \rightsquigarrow s \subseteq \Gamma$ *touches the set of constrained vertices,* S^**, only at* t. *Then the allocation* $(\boldsymbol{\rho}, \tau)$ *that results from executing* $t \rightsquigarrow s$ *on* $(\boldsymbol{\rho}, \sigma)$ *is a* fair *allocation that Pareto weakly dominates the original.*

If any $(\boldsymbol{\rho}, \sigma_{m+1})$ Pareto dominates $(\boldsymbol{\rho}, \sigma_m)$, we are done. Thus, we may assume $(\boldsymbol{\rho}, \sigma_{m+1})$ is welfare equivalent to $(\boldsymbol{\rho}, \sigma)$. This process can be repeated at most finitely many times. Let $(\boldsymbol{\rho}, \mu)$ be the result and Γ_μ the associated weak-envy graph.

Case 1: There is $s \in S \setminus S^*$ with $|\mu^{-1}[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$.

Our procedure above moves students *out* of S^* vertices along chains of weak-envy (actually, chains of indifference). Thus, since the procedure was executed to exhaustion, there are no S^* vertices in the set U_s of upstream vertices of s in graph Γ_μ.

By definition, U_s is a source in Γ_μ. If no school in U_s is totally exhausted under $(\boldsymbol{\rho}, \mu)$ then we may invoke Lemma 1 to arrive at our desired conclusion. Suppose, then, that there is $t \in U_s$ that is totally exhausted. Since $\{s\} \cup U_s \subseteq S \setminus S^*$, and since $(\boldsymbol{\rho}, \mu)$ is an RCE, all the arcs between these vertices in Γ_μ represent indifferences. Suppose there is $t' \in U_s$, $t' \neq t$, that is also totally exhausted. Then there are two chains of indifference, $t \rightsquigarrow s$ and $t' \rightsquigarrow' s$, in Γ_μ. The concatenation of these, $t \rightsquigarrow s \leftsquigarrow' t'$, represents a chain of indifference connecting t and t'. This violates NCBI as both of these vertices are totally exhausted and so $\rho_t^{-1}, \rho_{t'}^{-1} \in \mathbb{N}$. Therefore, t is the *only* member of U_s that is totally exhausted.

Execute $t \rightsquigarrow s$ on $(\boldsymbol{\rho}, \mu)$ to arrive at a *fair* allocation $(\boldsymbol{\rho}, \tau)$ with associated weak-envy graph Γ_τ. As above, if we have found a Pareto improvement, we are done, so we may assume it is welfare equivalent to $(\boldsymbol{\rho}, \mu)$. Our next task is to show that $\{s\} \cup U_s$ is a source in Γ_τ, recalling that U_s is the set of upstream vertices in Γ_μ.

[24]Allocation $(\boldsymbol{\rho}, \sigma)$ Pareto weakly dominates $(\boldsymbol{\rho}', \sigma')$ if for each $i \in N$, $(\boldsymbol{\rho}, \sigma(i))\ R_i\ (\boldsymbol{\rho}', \sigma'(i))$.

Let $u \xrightarrow{i} v \in \Gamma_\tau$ have $u \notin U_s$. If $\tau(i) = \mu(i)$ then clearly $u \xrightarrow{i} v \in \Gamma_\mu$, and since U_s is a source set in Γ_μ, $v \notin U_s$. If $\tau(i) \neq \mu(i)$, then i labels some arc on the path $t \rightsquigarrow s \subseteq \Gamma_\mu$ we just executed. Stated formally, there is $u' \xrightarrow{i} u \in t \rightsquigarrow s$, $\mu(i) = u'$, and $\tau(i) = u$. By construction, the only school not in U_s that is touched by this path is s, so in fact $u' \xrightarrow{i} u$ is the last arc of the path, and so $u = s$. Thus, we have shown that if $u \xrightarrow{i} v \in \Gamma_\tau$ has $u \notin U_s$ but $v \in U_s$, then $u = s$; the only arcs in Γ_τ (if there are any at all) that enter U_s are those coming from s.

Suppose there is a path $w \rightsquigarrow s \subseteq \Gamma_\tau$ that is not a path in Γ_μ. By taking sub-paths, assume we have the shortest such path, so that

$$w \rightsquigarrow s = w \xrightarrow{k} w' \rightsquigarrow s,$$

with $w' \rightsquigarrow s \subseteq \Gamma_\mu$. This latter inclusion, however, implies that $w' \in U_s$, along with all the other vertices touched by $w' \rightsquigarrow s$, and so by the previous paragraph, $w = s$. Conclude that the only paths to s in Γ_τ that are not in Γ_μ are of the form $s \to t \rightsquigarrow s$, where $t \rightsquigarrow s \subseteq \Gamma_\mu$. It follows that $\{s\} \cup U_s$ is a source set in Γ_τ.

Recall that our original path $t \rightsquigarrow s \subseteq \Gamma_\mu$ represented only indifferences. Since t is totally exhausted at $(\boldsymbol{\rho}, \mu)$, $\rho_t^{-1} \in \mathbb{N}$. By NCBI, it follows that ρ_s^{-1} is *not* an integer, implying that $\rho_s^{-1} > \lfloor \rho_s^{-1} \rfloor$. This further implies that s remains not totally exhausted if another student is added to it, and so is not totally exhausted at $(\boldsymbol{\rho}, \tau)$. The schools in the middle of the path have not changed the number of students they admit from μ to τ, so they remain not totally exhausted. Clearly,

$$|\tau^{-1}[t]| = |\mu^{-1}[t]| - 1 = \lfloor \rho_t^{-1} \rfloor - 1,$$

so t is not totally exhausted at $(\boldsymbol{\rho}, \tau)$. Since t was the *only* totally exhausted site in U_s under $(\boldsymbol{\rho}, \mu)$, we now have that $\{s\} \cup U_s$ is a source set in Γ_τ with no exhausted schools, and we therefore invoke Lemma 1.

Case 2: Each $s \in S$ with $|\mu^{-1}[s]| \leq \lfloor \rho_s^{-1} \rfloor - 1$ has $s \in S^*$, so $\rho_s^{-1} = b_s^{-1} \in \mathbb{N}$.

Assume $N' = \{j \in N : (\boldsymbol{\rho}, s)\ P_j\ (\boldsymbol{\rho}, \mu(j))\}$ is non-empty, and let $j = \min_{\succ_s} N'$. Define matching τ so that $\tau(j) = s$ and otherwise $\tau(i) = \mu(i)$. Then $(\boldsymbol{\rho}, \tau)$ is clearly a *fair* allocation that Pareto dominates $(\boldsymbol{\rho}, \mu)$, and therefore $(\boldsymbol{\rho}, \sigma)$. We proceed, therefore, under the assumption that each arc $t \xrightarrow{i} s \in \Gamma_\mu$ represents indifference.

If there is $t \in U_s$ with $\rho_t^{-1} \in \mathbb{N}$, then by taking sub-paths, assume $t \rightsquigarrow s$ is a minimal path starting from such a t. That is, for every $s' \in S$ touched by the path except t and s, $\rho_{s'} > b_{s'}$. Decompose $t \rightsquigarrow s$ as $t \to u \rightsquigarrow v \to s$. Then $u \rightsquigarrow v$ touches no S^* vertices and so, since $(\boldsymbol{\rho}, \mu)$

is a *fair* allocation, the edge $t \to u$ and all edges in $u \rightsquigarrow v$ represent indifference. We showed in the previous paragraph that $v \to s$ represents indifference. Thus, since $\rho_t^{-1} \in \mathbb{N}$, this path is a contradiction to NCBI. Conclude that U_s contains neither a totally exhausted vertex, nor a S^* vertex, and so we invoke Lemma 1. ■

A.4. **Topological argument to complete the proof.**

Theorem 6. *Given $\boldsymbol{R} \in \mathcal{R}^N$ satisfying NCBI, let $\mathcal{E}$ be the set of RCE for $\boldsymbol{R}$. Then*

(1) *$\mathcal{E}$ is not empty,*
(2) *$\mathcal{E}$ induces a closed upper-lattice in welfare space, and*
(3) *the set of distributions supporting the elements of $\mathcal{E}$ has a $\leq$-greatest element, $\boldsymbol{\rho}^*(\boldsymbol{R})$, which itself supports the welfare-greatest elements of $\mathcal{E}$.*

Proof. For each $i \in N$, let u_i be a continuous utility function representation for R_i. Fixing a matching σ, the function $\boldsymbol{\rho} \in [0,1]^S \overset{U}{\mapsto} (u_i(\boldsymbol{\rho}, \sigma))_{i \in N}$ is continuous. Closed subsets of $[0,1]^S$ are compact and so map to compact sets under this function. The set $\mathcal{D}^\sigma \subseteq [0,1]^S$ of distributions $\boldsymbol{\rho}$ such that $(\boldsymbol{\rho}, \sigma)$ is a *fair* allocation is closed: To see this, recall simply that a violation of *fairness* requires strict preference, and no new strict preference can be introduced in the limit of a sequence of distributions of *fair* allocations. Let $\mathcal{D} = \cup_\sigma \mathcal{D}^\sigma$. Since there are only finitely many possible matchings, $\mathcal{D}$ is compact.

Let $\mathcal{U} = U(\mathcal{D})$, which is compact. Let $\boldsymbol{u} \in \mathcal{U}$ be $\leq$-maximal. By Lemma 2, there is an RCE that induces $\boldsymbol{u}$. Thus, the $\leq$-upper envelope of $\mathcal{U}$ corresponds to RCE. By Theorem 3, the $\leq$-upper envelope of $\mathcal{U}$ is a lattice. Therefore, $\mathcal{U}$ has a $\leq$-greatest element. ■

Proof of Theorem 1. It follows directly from Part (1) of Theorem 6. ■

Proof of Proposition 3. By Lemma 2 and Theorem 6, it follows that the correspondence of welfare-greatest RCE, i.e., the maximal RCE, on the NCBI domain is non-empty, essentially single-valued, and satisfies *student-optimal fairness*. ■

Appendix B. Proof of Theorem 4: *Strategy-proofness*

First, we establish the following lemma, which is an immediate consequence of lemmas in Section A.3, but highlights a structural feature that will be important in the proof of *strategy-proofness* below.

Lemma 3. *Assume that $\boldsymbol{R} \in \mathcal{R}^N$ satisfies NCBI. Suppose $(\boldsymbol{\rho}, \sigma)$ is a* fair *allocation for $\boldsymbol{R}$ at which either $s \in S$ is not totally exhausted, or $s \in S^*$. Let U_s be the set of vertices* *upstream of s under the weak-envy graph of $(\boldsymbol{\rho}, \sigma)$. If U_s contains no totally exhausted schools, there is another* fair *allocation that Pareto dominates $(\boldsymbol{\rho}, \sigma)$.*

Proof. Observe that if U_s is empty, then we can set $\gamma_s = \rho_s + \epsilon$ and all else equal, and $(\boldsymbol{\gamma}, \sigma)$ is an RCE if ϵ is small enough. Thus, we may assume that U_s is non-empty for all $s \in S$.

By Case 2 of Lemma 2, we may assume each S^* school is totally exhausted. Thus if some $s \in S$ has U_s containing no totally exhausted schools, then it contains no S^* schools either. That is, $U_s \subseteq S \setminus S^*$, and is non-empty. We now invoke Lemma 1 to get the desired result. ■

Recall that preference relation R' is a Maskin Monotonic transform of preference relation R at bundle (x, m) if $(y, t)\ R'\ (x, m)$ implies that $(y, t)\ R\ (x, m)$. Let $\mathcal{T}(R, (x, m))$ be the set of Maskin monotonic transforms of R at (x, m). It is obvious that the correspondence of RCE is Maskin monotonic, which is to say that if $(\boldsymbol{\rho}, \sigma)$ is a RCE for $\boldsymbol{R}$, and $\boldsymbol{R}'$ has, for each $i \in N$, $R'_i \in \mathcal{T}(R_i, (\boldsymbol{\rho}, \sigma(i)))$, then $(\boldsymbol{\rho}, \sigma)$ is a RCE for $\boldsymbol{R}'$. We first uncover some structural properties of φ with respect to Maskin Monotonic Transforms.

An undirected graph is a *tree* if there is exactly one path in the graph between any pair of vertices. In particular, a tree is simple—there is at most one edge between any pair of vertices. With abuse of terminology, we shall call a directed graph a *tree* if its underlying undirected graph is a tree *and* there is a special vertex r, called the *root*, from which all paths emerge. That is, for all non-root vertices s, there is a path $r \rightsquigarrow s$ in the graph. Finally, a directed graph is a *forest* if it is comprised of disjoint directed trees, having no edge or vertex in common. Let Γ be the weak-envy graph for allocation $\varphi(\boldsymbol{R})$ with preferences $\boldsymbol{R}$. By Lemma 3, we can find a subgraph $\Gamma' \subseteq \Gamma$ that is a directed forest and such that each totally exhausted $s \in S$ with $\boldsymbol{\rho}^*_s(\boldsymbol{R}) > b_s$ is a root vertex. Call such Γ' a *minimal forest* for $\boldsymbol{R}$. The following observations imply that, for generic profiles, the minimal forest is unique. In any case, these structures lead to the following theorem:

Theorem 7 (The Locality Theorem)**.** *Let $\boldsymbol{R}' \in \mathcal{D}$ be a preference profile such that, for each $i \in N$, $(\boldsymbol{\rho}^*(\boldsymbol{R}), s)\ I_i\ \varphi_i(\boldsymbol{R})$ implies $(\boldsymbol{\rho}^*(\boldsymbol{R}), s)\ I'_i\ \varphi_i(\boldsymbol{R})$. Then $\boldsymbol{\rho}^*(\boldsymbol{R}') = \boldsymbol{\rho}^*(\boldsymbol{R})$.*

Proof. Let Γ' be a minimal forest for $\boldsymbol{R}$. First consider $\boldsymbol{R}'' \in \mathcal{D}$ such that, for each $i \in N$, $R''_i \in \mathcal{T}(R_i, \varphi_i(\boldsymbol{R}))$ and such that the weak-envy graph of $\boldsymbol{R}''$ at $\varphi(\boldsymbol{R})$ is *precisely* Γ'. By Maskin monotonicity, $\varphi(\boldsymbol{R})$ is an RCE for $\boldsymbol{R}''$, so $\boldsymbol{\rho}^*(\boldsymbol{R}'') \geq \boldsymbol{\rho}^*(\boldsymbol{R})$. By the lattice property, $\boldsymbol{R}''$ welfare can only increase from $\varphi(\boldsymbol{R})$ to $\varphi(\boldsymbol{R}'')$. By Theorem 2, the change in school-assignment between these two consists entirely of trading cycles. By Proposition 2, all trading cycles between these two allocations must be welfare non-negative, and therefore must be cycles in Γ'. However, Γ' has no cycles, and therefore the matching under $\varphi(\boldsymbol{R}')$, say σ, is the same as that under $\varphi(\boldsymbol{R})$. Now if $\boldsymbol{\rho}^*_s(\boldsymbol{R}') > \boldsymbol{\rho}^*_s(\boldsymbol{R})$, then clearly s is not totally exhausted at $\varphi(\boldsymbol{R})$. Thus there is a path $t \rightsquigarrow s \subseteq \Gamma'$. In particular, there is $u \xrightarrow{i} s \in \Gamma'$. However, we then have $\boldsymbol{\rho}^*_s(\boldsymbol{R}'') > b_s$ and

$$(\boldsymbol{\rho}^*{}_s(\boldsymbol{R}''), s)\ P''_i\ (\boldsymbol{\rho}^*{}_s(\boldsymbol{R}), s)\ R''_i\ (\boldsymbol{\rho}^*{}_s(\boldsymbol{R}), u)\,,$$

implying, since $\varphi(\boldsymbol{R}'')$ is an RCE, that $\boldsymbol{\rho}^*_u(\boldsymbol{R}'') > \boldsymbol{\rho}^*_u(\boldsymbol{R})$. It follows that u is not totally exhausted at $\varphi(\boldsymbol{R})$ and so we may repeat the argument. In fact, we may repeat the argument all the way up the path $t \rightsquigarrow s$ to vertex t, getting a contradiction to feasibility since $|\sigma[t]|\, \boldsymbol{\rho}^*_t(\boldsymbol{R}) = 1$. We conclude, therefore, that $\boldsymbol{\rho}^*(\boldsymbol{R}'') = \boldsymbol{\rho}^*(\boldsymbol{R})$.

Now let $\boldsymbol{R}' \in \mathcal{D}$ have, for each $i \in N$, $R'_i \in \mathcal{T}(R_i, \varphi(\boldsymbol{R}))$ and that Γ' is a subgraph of the weak-envy graph of $\boldsymbol{R}'$ at $\varphi(\boldsymbol{R})$. As above, $\boldsymbol{\rho}^*(\boldsymbol{R}') \geq \boldsymbol{\rho}^*(\boldsymbol{R})$. However, note that we may choose $\boldsymbol{R}''$ above so that, for each $i \in N$, $R''_i \in \mathcal{T}(R'_i, \varphi(\boldsymbol{R}))$. Thus, $\boldsymbol{\rho}^*(\boldsymbol{R}) = \boldsymbol{\rho}^*(\boldsymbol{R}'') \geq \boldsymbol{\rho}^*(\boldsymbol{R}')$ and so $\boldsymbol{\rho}^*(\boldsymbol{R}') = \boldsymbol{\rho}^*(\boldsymbol{R})$. ■

We are now prepared to prove the incentive compatibility of φ.

Proof of Theorem 4. Let $\boldsymbol{R}' = (R'_i, R_{-i}) \in \mathcal{D}$. Suppose $\varphi_i(\boldsymbol{R}')\ P_i\ \varphi_i(\boldsymbol{R})$. Let

$$R''_i \in \mathcal{T}(R'_i, \varphi_i(\boldsymbol{R}')) \cap \mathcal{T}(R_i, \varphi_i(\boldsymbol{R}))$$

have the following properties. For each $s \in S$, if $(\boldsymbol{\rho}^*(\boldsymbol{R}'), s) \neq \varphi_i(\boldsymbol{R}')$ then $\varphi_i(R')\ P''_i$ $(1, s)$. Also, let R''_i have the same indifference set through $\varphi_i(\boldsymbol{R})$ as R_i does. Note that this assumption implies $\varphi_i(\boldsymbol{R}')\ P''\ \varphi_i(\boldsymbol{R})$. Let $\boldsymbol{R}'' = (R''_i, \boldsymbol{R}_i)$. By the Locality Theorem, $\boldsymbol{\rho}^*(\boldsymbol{R}'') = \boldsymbol{\rho}^*(\boldsymbol{R})$. By Maskin monotonicity, $\varphi(\boldsymbol{R}')$ is an RCE for R''. Therefore $\varphi_i(R''_i, \boldsymbol{R}_{-i})\ R''_i\ \varphi_i(\boldsymbol{R}')\ P''_i\ \varphi_i(\boldsymbol{R})$. It follows that $\varphi_i(R''_i, \boldsymbol{R}_{-i})\ P_i\ \varphi_i(\boldsymbol{R})$. Therefore, if i can manipulate φ at $\boldsymbol{R}$, then i can manipulate via a preference such as R''_i. Without loss of generality, we assume henceforth that $\boldsymbol{R}' = \boldsymbol{R}''$.

We are considering two allocations, $\varphi(\boldsymbol{R})$ and $\varphi(\boldsymbol{R}')$ with the same distribution vector $\boldsymbol{\rho} = \boldsymbol{\rho}^*(\boldsymbol{R}) = \boldsymbol{\rho}^*(\boldsymbol{R}')$. We shall construct a classical school choice problem from these and derive a contradiction to the *strategy-proofness* of the student-optimal stable rule [Roth and Sotomayor, 1990] in this context.

The set of classical schools is denoted S. As usual, let $S^* = \{s \in S : \boldsymbol{\rho}_s = b_s\}$ and call these (crowded) schools *constrained*. We collapse all the unconstrained schools into one classical school, ȿ. School priorities in the classical model will be denoted $\triangleleft$. For $s \in S^*$, which maps to $\mathsf{s} \in \mathsf{S}$, set $\lhd_{\mathsf{s}} = \succ_s$. Set $k \lhd_{\grave{\mathsf{s}}} j$ if $k \in \tau[S \setminus S^*]$ and $j \in \tau[S^*]$. We shall not need to further specify $\lhd_{\grave{\mathsf{s}}}$.

Next we break ties in student preferences. We begin with an intermediate step, deciding that $\grave{\mathsf{s}}$ shall inherit the rank of the highest ranked unconstrained school. That is, let $s \in S \setminus S^*$ have, for each $t \in S \setminus S^*$, $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, t)$. Then, for $\mathsf{u} \in \mathsf{S}$, $\mathsf{u}\ \mathsf{R}_j\ \grave{\mathsf{s}}$ only if $(\boldsymbol{\rho}, u)\ R_j\ (\boldsymbol{\rho}, s)$. With this step, we have defined the weak preference R_j on S. It remains to break ties on this relation. Note that by NCBI, R_j is in fact strict when restricted to $\mathsf{S} \setminus \{\grave{\mathsf{s}}\}$, so there is at most one non-singleton indifference class, and it has the form $\{\mathsf{t}, \grave{\mathsf{s}}\}$. Before completing our tie-breaking specification, let us first make the following observation:

Claim 5. *Let $(\boldsymbol{\rho}, s)\ I_j\ (\boldsymbol{\rho}, \sigma(j))$ or $(\boldsymbol{\rho}, s)\ I'_j\ (\boldsymbol{\rho}, \tau(j))$. Then $s \in S \setminus S^*$.*

Proof of claim. Recall Lemma 3. First, if $\sigma(j)$ is exhausted at $(\boldsymbol{\rho}, \sigma)$, then the claim follows directly from NCBI. Otherwise, there is $t \rightsquigarrow \sigma(j) \subseteq \Gamma$, where Γ is the weak-envy graph of $\boldsymbol{R}$ at $(\boldsymbol{\rho}, \sigma)$, with t totally exhausted. By taking subpaths we may find assume this is a shortest (by length) path with this property. Thus, at most one school touched by the path is in S^*, and it must be t, as otherwise we could shorten the path further. Therefore, the path must consist entirely of indifferences, and so $t \rightsquigarrow \sigma(j) \to s$, with $s \in S^*$, contradicts NCBI.

Note that the symmetric proof holds for $(\boldsymbol{\rho}, s)\ I'_j\ (\boldsymbol{\rho}, \tau(j))$. ◇

We now break the tie in the indifference set $\{\mathsf{t}, \grave{\mathsf{s}}\}$. Here are the rules:

(1) If $\sigma(j)$ maps to $\grave{\mathsf{s}}$, then $\grave{\mathsf{s}}\ \mathsf{P}_j\ \mathsf{t}$.
(2) Otherwise $\mathsf{t}\ \mathsf{P}_j\ \grave{\mathsf{s}}$.

We now show that $\bar{\sigma}$ is stable for the classical school-choice problem with preferences $\mathbf{R}$. Suppose $\mathsf{s}\ \mathsf{P}_j\ \bar{\sigma}(j)$. There is $s \in S$ such that $(\boldsymbol{\rho}, s)\ R_j\ (\boldsymbol{\rho}, \sigma(j))$. If this relation is strict, then $s \in S^*$, since $(\boldsymbol{\rho}, \sigma)$ is an RCE. This further implies that $\lhd_{\mathsf{s}} = \succ_s$ and that, for each $k \in \sigma[s]$, $k \succ_s j$. If the relation is an indifference, then by the claim, $\mathsf{s} = \grave{\mathsf{s}}$. However, $\grave{\mathsf{s}}\ \mathsf{P}_j\ \bar{\sigma}(j)$ could only have happened via Rule (1), which could only happen if $\bar{\sigma}(j) = \grave{\mathsf{s}}$, a contradiction. In sum, all envy is justified by the priorities.

We now show that τ is stable for $\mathbf{R}'$. Observe that, by construction, P'_i top ranks $\bar{\tau}(i)$, so we may restrict attention to $j \neq i$. In this case, $\mathsf{R}'_j = \mathsf{R}_j$. Suppose $\mathsf{s}\ \mathsf{P}_j\ \bar{\tau}(j)$. Given the argument of the previous paragraph, it is clear we can skip to the case that $(\boldsymbol{\rho}, s)\ I_j\ (\boldsymbol{\rho}, \tau(j))$. The claim then implies that $\mathsf{s} = \grave{\mathsf{s}}$ and so $\bar{\tau}(j) \neq \grave{\mathsf{s}}$ and $j \in \tau[S^*]$. Thus for each $k \in \bar{\tau}[\grave{\mathsf{s}}]$, $k \in \tau[S \setminus S^*]$, and so $k \lhd_{\grave{\mathsf{s}}} j$. Again, all envy is justified by the priorities.

Now we claim that $\bar{\sigma}$ is the student optimal stable match for R. Suppose that $\bar{\mu}$ is a stable match that weakly dominates $\bar{\sigma}$. By our tie-breaking construction, if $(\boldsymbol{\rho}, \sigma(j))\ R_j\ (\boldsymbol{\rho}, t)$, then $\bar{\sigma}(j)\ \mathsf{P}_j\ \mathsf{t}$. In particular, since $(\boldsymbol{\rho}, \sigma)$ is an RCE for $\boldsymbol{R}$, either $\bar{\sigma}(j) = \grave{\mathsf{s}}$ or $\bar{\sigma}(j)\ \mathsf{P}_j\ \grave{\mathsf{s}}$. Thus, going from $\bar{\sigma}$ to $\bar{\mu}$ cannot involve moving students into $\grave{\mathsf{s}}$ who are not already there. Then by feasibility, no students can move out of $\grave{\mathsf{s}}$. Thus, $\bar{\mu}[\grave{\mathsf{s}}] = \bar{\sigma}[\grave{\mathsf{s}}]$. In other words, $\bar{\mu}$ is a reassignment of the students at constrained schools. Let μ be a matching in the crowded school model that coincides with $\bar{\mu}$ on S^* and with σ otherwise. Suppose there is $j \in N$ with $\bar{\mu}(j)\ \mathsf{P}_j\ \bar{\sigma}(j)$. Then $j \in \sigma[S^*]$ and $\mu(j) \in S^*$. Since $(\boldsymbol{\rho}, \sigma)$ is maximal for $\boldsymbol{R}$, there is $k \in N$ with $k \succ_{\mu(j)} j$ and $(\boldsymbol{\rho}, \mu(j))\ P_k\ (\boldsymbol{\rho}, \mu(k))$. By construction, $k \lhd_{\bar{\mu}(j)} j$. If $\mu(k) \in S^*$, then $\bar{\mu}(k) \neq \grave{\mathsf{s}}$ and $\bar{\mu}$ is blocked in the classical model, as preferences over S^* map directly to preferences over $\mathsf{S} \setminus \grave{\mathsf{s}}$. Thus, $\mu(k) = \sigma(k) \in S \setminus S^*$. Since $(\boldsymbol{\rho}, \sigma)$ is an RCE, for each $t \in S \setminus S^*$, $(\boldsymbol{\rho}, \sigma(k))\ R_k\ (\boldsymbol{\rho}, t)$, so recalling that $\grave{\mathsf{s}}$ inherits the rank of the highest unconstrained school, we have $\bar{\mu}(j)\ \mathsf{P}_k\ \bar{\mu}(k) = \grave{\mathsf{s}}$. Again we conclude that $\bar{\mu}$ is blocked in the classical model.

Now observe that $\bar{\tau}(i)$ is the top-ranked school for R'_i, so $\bar{\mu}(i) = \bar{\tau}(i)$ for any stable $\bar{\mu}$ that dominates $\bar{\tau}$ for R'. By assumption, $(\boldsymbol{\rho}, \tau(i))\ P_i\ (\boldsymbol{\rho}, \sigma(i))$, so by construction $\bar{\tau}(i)\ \mathsf{P}_i\ \bar{\sigma}(i)$, contradicting that i is not able to manipulate the student optimal stable rule. ■

Appendix C. Proof of Theorem 5: The Algorithm

We first introduce *threshold equilibrium*, which is a way of generalizing price equilibrium so that more abstract objects can play the role of the price. Then by providing an equivalent modification of the original algorithm via threshold equilibrium, the convergence result of the modified algorithm establishes Theorem 5.

Augment N with a null element ϕ having the property that, for each $s \in S$ and each $i \in N$, $i \succ_s \phi$. For each $s \in S$, let

$$\mathcal{T}_s = \{(r, i) \in [b_s, 1] \times N : i \neq \phi \implies r = b_s\}.$$

Define linear order $\sqsupset_s$ on $\mathcal{T}_s$ so that $(r, i) \sqsupset_s (q, j)$ if either $r < q$ or $(r = q = b_s$ and $i \succ_s j)$. Let $\sqsupseteq_s$ denote the reflexive enlargement of $\sqsupset_s$. With abuse of notation, write $(\boldsymbol{\rho}, \boldsymbol{a})$ to indicate the list $((\rho_s, a_s))_{s \in S}$. Let $(\boldsymbol{\rho}, \boldsymbol{a}) \sqsupseteq (\boldsymbol{\gamma}, \boldsymbol{b})$ if, for each $s \in S$, $(\rho_s, a_s) \sqsupseteq_s (\gamma_s, b_s)$.

Given a list of thresholds, student i's *constrained demand* is their favorite school from among those whose threshold they can cross:

$$C_i(\boldsymbol{\rho}, \boldsymbol{a}) = \{s \in S : i \succ_s a_s \text{ and } (i \succ_t a_t \implies (\boldsymbol{\rho}, s)\ R_i\ (\boldsymbol{\rho}, t))\}.$$

A threshold equilibrium $(\boldsymbol{\rho}, \boldsymbol{a}, \sigma)$ is a list of thresholds $(\boldsymbol{\rho}, \boldsymbol{a})$ and a matching σ such that $(\boldsymbol{\rho}, \sigma)$ is an allocation and, for each $i \in N$, $\sigma(i) \in C_i(\boldsymbol{\rho}, \boldsymbol{a})$.

Clearly, there is an equivalence between *fair* allocations and threshold equilibrium allocations. A maximal RCE is therefore also a student-optimal threshold equilibrium allocation. Whereas the distribution that supports maximal RCE is unique, this is not true of the threshold list. However, setting a_s equal to the highest-$\succ_s$-ranked student who has $(\boldsymbol{\rho}^*(\boldsymbol{R}), s)\ P_j\ (\boldsymbol{\rho}^*(\boldsymbol{R}), \sigma(j))$ will $\sqsupset_s$-minimize this threshold. Since the welfare at maximal RCE is well-defined, this is independent of any particular maximal RCE matching, and thus, the *minimal* threshold $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R}))$ is well-defined.

Given these concepts, we make a slight modification of the algorithm in the main text. Rather than decrementing the distribution, we shall *increment* the threshold list. Initialize the modified algorithm at $(\boldsymbol{\rho}_0, \boldsymbol{a}_0) = ((1, \ldots, 1), (\phi, \ldots, \phi))$. The decrementing subroutine remains the same. Rejection, however, is modified so that, with each iteration, a_{sz} is incremented one step. Thus, $a_{s,z+1} = \min_{\succ_s}\{i \in N : i \succ_s a_{sz}\}$. To see the equivalence of the two algorithms, let i be the student identified in the rejection subroutine at time z. Then excluding s from Q_{jz+1} for $j = i$ and all lower ranked students is the same as repeating the modified rejection subroutine until $a_{s,z+k} = i$. Moreover, nothing in the state of the economy changes until $a_{s,z+k} = i$ as, by definition, none of the j with $i \succ_s j$ have $s \in D_{jz}$.

Let $Z \subseteq \mathbb{R}$ be the set of times at which the algorithm is defined. It is the disjoint union of closed intervals (closures of decrementing subroutines) and isolated points (rejection followed by rejection). Clearly, $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R})) \sqsupseteq (\boldsymbol{\rho}_0, \boldsymbol{a}_0)$. Let $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R})) \sqsupseteq (\boldsymbol{\rho}_z, \boldsymbol{a}_z)$ and, for some $s \in S$, $(\rho_{sz}, a_{sz}) = (\rho^*_s, a^*_s)$. We show that (ρ_{sz}, a_{sz}) will remain unchanged.

By contradiction, suppose $s \in E^*_z$. Let $\Omega'_z \subseteq E^*_z$ contain s and be such that any two schools in Ω'_z are connected by chains of indifference. Let N' be the set of students who require Ω'_z at $(\boldsymbol{\rho}_z, \boldsymbol{a}_z)$. Let $S' \subseteq \Omega'_z$ be the set of schools such that $(\rho_{tz}, a_{tz}) = (\rho^*_t(\boldsymbol{R}), a^*_t(\boldsymbol{R}))$. By construction, an student's demand intersects Ω'_z only if they require Ω'_z. Thus, $|N'| > \sum_{s \in \Omega'_z} \lfloor \rho^{-1}_{tz} \rfloor$, so by Hall's Theorem (Hall [1935]) there is no equilibrium matching of N' to Ω'_z. Let μ instead be a maximal feasible matching such that each student gets something in their conditional demand. By NCBI, there is at most one $t \in \Omega'_z$ with

$\rho_{tz} = b_t$. Suppose such a t exists and $t \notin S'$. Then the algorithm will increment a_{tz} before decrementing ρ_{sz}. Thus, it is without loss of generality to assume that $t \in S'$. Then, for each $u \in \Omega'_z \setminus S'$, $\rho^*_u(\boldsymbol{R}) < \rho_{uz}$. It follows that *any* $j \in N'$ with $C_j(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \cap S' \neq \emptyset$ requires S' at $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R}))$. In particular, this is true for each $j \in \mu[S']$. Recalling condition (2) in the definition of excess demand, Hall's theorem then implies there is a student $i \in N' \setminus \mu[S']$ with $C_i(\boldsymbol{\rho}_z, \boldsymbol{a}_z) \cap S' \neq \emptyset$. Then the students who require S' at $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R}))$ are at least the set $\{i\} \cup \mu[S']$. However,

$$|\{i\} \cup \mu[S']| = 1 + |\mu[S']| = 1 + \sum_{s \in S'} \lfloor \rho_{sz}^{-1} \rfloor = 1 + \sum_{s \in S'} \lfloor \rho_s^{*-1} \rfloor,$$

where the second equality is because μ is a maximal matching and so covers S'. This equation implies that S' is overdemanded at $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R}))$, a contradiction.

By construction, the path $z \mapsto \boldsymbol{\rho}_z$ is continuous. Since our rejection procedure increments $\boldsymbol{a}$ one component, one step at a time, no s can cross its minimal threshold without first equaling its minimum equilibrium threshold. The foregoing argument shows that when a school first reaches its minimum equilibrium threshold, it is not incremented again. Thus, for all $z \in Z$, $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R})) \sqsupseteq (\boldsymbol{\rho}_z, \boldsymbol{a}_z)$. The modified algorithm must terminate: as mentioned, the number of rejection subroutines is bounded and the distribution is decremented at constant speed. However, the modified algorithm only terminates at a threshold equilibrium. In conclusion, the algorithm terminates at $(\boldsymbol{\rho}^*(\boldsymbol{R}), \boldsymbol{a}^*(\boldsymbol{R}))$. Thus, by letting σ be any matching satisfying demands, we have a maximal RCE $(\boldsymbol{\rho}^*(\boldsymbol{R}), \sigma)$.

References

Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

Tommy Andersson and Lars-Gunnar Svensson. Non-manipulable house allocation with rent control. *Econometrica*, 82(2):507–539, 2014.

Tommy Andersson, Christer Andersson, and Dolf Talman. Sets in excess demand in simple ascending auctions with unit-demand bidders. *Annals of Operations Research*, 211:27–36, 2013.

Itai Ashlagi, Mark Braverman, and Avinatan Hassidim. Stability in large matching markets with complementarities. *Operations Research*, 62(4):713–732, 2014.

Eduardo M Azevedo and Jacob D Leshno. A supply and demand framework for two-sided matching markets. *Journal of Political Economy*, 124(5):1235–1268, 2016.

Michel Balinski and Tayfun Sönmez. A tale of two mechanisms: student placement. *Journal of Economic Theory*, 84(1):73–94, 1999.

Keisuke Bando. Many-to-one matching markets with externalities among firms. *Journal of Mathematical Economics*, 48(1):14–20, 2012.

Keisuke Bando. A modified deferred acceptance algorithm for many-to-one matching markets with externalities among firms. *Journal of Mathematical Economics*, 52:173–181, 2014.

David Card and Alan B Krueger. Does school quality matter? returns to education and the characteristics of public schools in the united states. *Journal of Political Economy*, 100 (1):1–40, 1992.

Raj Chetty, John N Friedman, Nathaniel Hilger, Emmanuel Saez, Diane Whitmore Schanzenbach, and Danny Yagan. How does your kindergarten classroom affect your earnings? evidence from project star. *The Quarterly Journal of Economics*, 126(4): 1593–1660, 2011.

Andrew Copland. School choice and class size externalities. *Working Paper*, 2021.

Vincent P Crawford and Elsie Marie Knoer. Job matching with heterogeneous firms and workers. *Econometrica*, 49:437–450, 1981.

Gabrielle Demange and David Gale. The strategy structure of two-sided matching markets. *Econometrica*, 53:873–888, 1985.

Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94(4):863–872, 1986.

Jacques H Drèze. Existence of an exchange equilibrium under price rigidities. *International Economic Review*, 16:301–320, 1975.

Umut Dur and Thayer Morrill. Competitive equilibria in school assignment. *Games and Economic Behavior*, 108:269–274, 2018.

Umut Dur and Thomas Wiseman. School choice with neighbors. *Journal of Mathematical Economics*, 83:101–109, 2019.

Umut Dur, Thayer Morrill, and William Phan. Family ties: School assignment with siblings. *Theoretical Economics*, 17(1):89–120, 2022.

Bhaskar Dutta and Jordi Massó. Stability of matchings when individuals have preferences over colleagues. *Journal of Economic Theory*, 75(2):464–475, 1997.

Federico Echenique and M Bumin Yenmez. A solution to matching with preferences over colleagues. *Games and Economic Behavior*, 59(1):46–71, 2007.

Tamás Fleiner, Ravi Jagadeesan, Zsuzsanna Jankó, and Alexander Teytelboym. Trading networks with frictions. *Econometrica*, 87(5):1633–1661, 2019.

Duncan Karl Foley. *Resource allocation and the public sector*. Yale University, 1966.

David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

Isa E Hafalir. Stability of marriage with externalities. *International Journal of Game Theory*, 37(3):353–369, 2008.

Philip Hall. On representatives of subsets. *Journal of the London Mathematical Society*, s1-10:26–30, 1935.

P Jean-Jacques Herings. Equilibrium and matching under price controls. *Journal of Economic Theory*, 177:222–244, 2018.

Werner Hildenbrand. *Core and Equilibria of a Large Economy. (PSME-5):*. Princeton University Press, 2015. ISBN 9781400869473.

C Kirabo Jackson, Rucker C Johnson, and Claudia Persico. The effects of school spending on educational and economic outcomes: Evidence from school finance reforms. *The Quarterly Journal of Economics*, 131(1):157–218, 2016.

Alexander S Kelso Jr and Vincent P Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50:1483–1504, 1982.

Bettina Klaus and Flip Klijn. Stable matchings and preferences of couples. *Journal of Economic Theory*, 121(1):75–106, 2005.

Fuhito Kojima, Parag A Pathak, and Alvin E Roth. Matching with couples: Stability and incentives in large markets. *The Quarterly Journal of Economics*, 128(4):1585–1632, 2013.

Alan B Krueger and Diane M Whitmore. The effect of attending a small class in the early grades on college-test taking and middle school test results: Evidence from project star. *The Economic Journal*, 111(468):1–28, 2001.

Jacob Leshno. Stable matching with peer-dependent preferences in large markets: Existence and cutoff characterization. *Working Paper*, 2021.

Jacob D Leshno and Irene Lo. The cutoff structure of top trading cycles in school choice. *The Review of Economic Studies*, 88(4):1582–1623, 2021.

Thanh Nguyen and Rakesh Vohra. Near-feasible stable matchings with couples. *American Economic Review*, 108(11):3154–69, 2018.

Marek Pycia and M Bumin Yenmez. Matching with externalities. *University of Zurich, Department of Economics, Working Paper*, (392), 2021.

Martine Quinzii. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory*, 13(1):41–60, 1984.

Marzena Rostek and Nathan Yoder. Matching with complementary contracts. *Econometrica*, 88(5):1793–1827, 2020.

Alvin E Roth. The evolution of the labor market for medical interns and residents: a case study in game theory. *Journal of Political Economy*, 92(6):991–1016, 1984.

Alvin E Roth. On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica*, 54:425–427, 1986.

Alvin E Roth and Elliott Peranson. The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American Economic Review*, 89(4):748–780, 1999.

Alvin E Roth and Marilda Sotomayor. a study in game-theoretic modeling and analysis. *Econometric Society Monographs*, 18, 1990.

Hiroo Sasaki and Manabu Toda. Two-sided matching problems with externalities. *Journal of Economic Theory*, 70(1):93–108, 1996.

Lloyd S Shapley and Martin Shubik. The assignment game i: The core. *International Journal of Game Theory*, 1(1):111–130, 1971.

Tayfun Sönmez and M Utku Ünver. Course bidding at business schools. *International Economic Review*, 51(1):99–123, 2010.

Dolf Talman and Zaifu Yang. A dynamic auction for differentiated items under price rigidities. *Economics Letters*, 99(2):278–281, 2008.

Ryan Tierney. The problem of multiple commons: A market design approach. *Games and Economic Behavior*, 114:1–27, 2019.

Yu Zhou and Shigehiro Serizawa. Multi-object auction design beyond quasi-linearity: Leading examples. *Working Paper*, 2021.