# Anchoring Inflation Expectations*

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November 12, 2020

**Abstract**

We derive the restrictions on monetary policy which guarantee that inflation expectations are anchored in a simple monetary model. The restrictions are stronger than what is required for a globally unique rational expectations equilibrium. Loosely, the restrictions are captured by the idea that policy should 'lean against the wind, but not too aggressively'.

*We are grateful for conversations with Bernardino Adão, Marco Bassetto, Jeff Campbell, V.V. Chari, John Cochrane, Martin Eichenbaum, Joao Guerreiro, Patrick Kehoe, Ellen McGrattan, Emi Nakamura, Armen Nurbekyan, Matt Rognlie, Jón Steinsson, Marciano Siniscalchi and John Taylor. This draft replaces an earlier draft distributed under the title, 'Discouraging Deviant Behavior in Monetary Economics'.

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# 1 Introduction

Economists have long debated the question of how to anchor inflation expectations. Much progress has been made on this question since the introduction of rational expectations models in the 1970s. This literature has much to say about the conditions under which monetary policy guarantees a unique rational expectations equilibrium. But, it was soon understood that rational expectations leaves unanswered the key question of how, or even whether, non-cooperative agents would coordinate their beliefs on a rational expectations equilibrium (see, for example, Lucas (1978, section 6) and Guesnerie (1992)). We derive sufficient conditions for expectations to be anchored in a model that is globally tractable, and yet has the flavor of conventional monetary models used to analyze monetary policy.[1] We also derive necessary conditions for expectations to be anchored using local approximation methods that can be applied very generally. The analysis suggests a simple economic principle to guide policymakers seeking to anchor the public's expectations to a particular target.

The focus of our analysis is on the monetary policy advocated by John Taylor in a series of papers. We refer to this policy as the *Taylor strategy*. The best known part of that strategy is the Taylor rule. According to this rule, policymakers move the market interest rate more than one-for-one in response to an increase in inflation. Taylor (1996) argues that this policy can work well in normal times,[2] but points out that there are times when the rule might lead the economy awry. For example, that could occur if a rise in inflation leads to a rise in the interest rate which then creates, by a Fisher-type channel, expectations of even higher inflation. In this case, policymakers applying the Taylor rule could inadvertently become part of an inflation spiral. Taylor (1996) notes that the converse, a deflation spiral, could happen too. With these considerations in mind, Taylor (1996, page 37) suggests what we call the Taylor strategy: follow the Taylor rule under normal circumstances but if inflation gets out of hand, invoke an escape clause and switch to a money growth rule to get inflation under control.[3]

There exist studies that lend support to the Taylor strategy, by showing that it can produce a globally unique rational expectations equilibrium in which inflation corresponds to the policymaker's target.[4] But, as noted above, existence and uniqueness of an equilibrium is no guarantee that agents will coordinate on that equilibrium. That is, it does not ensure that expectations are anchored.

The literature also suggests that a monetary growth rule is not necessarily the right rule to switch to in the event that things go wrong in the Taylor rule regime.[5] For example, one could also consider switching to an Obstfeld and Rogoff (1983; 2017)-type strategy or a variant on the fiscal theory of

[1] The competitive equilibrium of the model is observationally equivalent to the models in Benhabib et al. (2001a), Woodford (2003, chapter 2) and Cochrane (2011).

[2] The model-based case that the Taylor rule works well can be made in the New Keynesian model with price-setting frictions and shocks. For additional discussion, see Section 2.2.2 below.

[3] In Taylor's words, "...I would argue that interest rate rules need to be supplemented by money supply rules in cases of either extended deflation or hyperinflation."

[4] See Benhabib et al. (2002) and Christiano and Rostagno (2001). These papers build on the work of Obstfeld and Rogoff (1983). For more recent work, see Atkeson et al. (2010).

[5] In many models equilibrium is not unique under a money growth rule. In these models, switching to money growth rule would not anchor expectations.

the price level (see, e.g., Leeper (1991)). Still, the simplicity and familiarity of a money growth rule is convenient for illustrating principles which guarantee that inflation expectations are anchored.[6]

To understand how expectations come to be anchored, we obviously must drop the assumption that agents coordinate on the rational expectations beliefs. We instead assume that each agent is rational, understands the environment, and that these are common knowledge among agents (CK). Agents form their beliefs independent of each other by a private, logical reasoning process. To model the formation of beliefs by rational agents with CK, we exploit the mapping between a competitive equilibrium and a Nash equilibrium of a non-cooperative game.[7] This mapping puts us in a position to answer our anchoring question by making use of the seminal game-theoretic papers of Bernheim (1984) and Pearce (1984). They argue that in forming their beliefs, rational agents with CK only entertain beliefs that they deem 'reasonable'. By reasonable, Bernheim (1984) and Pearce (1984) mean that the belief is *rationalizable*. So, we ask what restrictions are required for the Taylor strategy to ensure that policymakers' desired rate of inflation is the unique rationalizable belief. When those restrictions are satisfied, we say that policymakers have successfully managed to anchor agents' expectations.

A key principle that determines unique rationalizability is that policy gives rise to a reduced form which features what we call 'leaning against the wind'. In our context, the word 'lean' and its connotations are crucial. The word literally means to 'to push back', and, accordingly, we find that having a coefficient on inflation bigger than unity in the Taylor rule is an important part of successful policy design. However, the word, 'lean', also carries the connotation that the push-back should be more like a gentle nudge and not be overly aggressive. Thus, we show that to anchor inflation expectations the coefficient on inflation in the Taylor rule should also not be too big. In addition, it is important that when policy switches to the money growth rule in case the escape clause is activated, the resulting recession must not be too deep. So, for the Taylor strategy to successfully anchor expectations, both the interest rate rule and money growth rule must be consistent with the principle of leaning against the wind, but not too aggressively.[8]

The outline of the paper is as follows. Section 2 provides a simple example that summarizes the results in our dynamic model. In addition, this section discusses the relationship between our paper and the literature. As part of that discussion, we derive necessary conditions for anchoring expectations in a simple New Keynesian model. Our methods for evaluating necessary conditions can be generalized to empirically relevant models. For space reasons, the technical details are relegated

[6] As is standard in the recent literature on monetary policy rules, we assume that monetary policy is active, fiscal policy is passive and the government never reneges on its policy. By 'active' and 'passive' we have in mind the ideas in Leeper (1991).

[7] For discussions of this mapping, see Bernheim (section 7 (b), 1984), Guesnerie (1992), Evans and Guesnerie (1993), Guesnerie (2002), Bassetto (2002; 2005), Evans and Guesnerie (2005), Atkeson et al. (2010), as well as the papers that they cite.

[8] The desirability of 'leaning against the wind' is well known in the macroeconomics literature and, indeed, it is stressed by Taylor (1996). It is also known, though by a different name, in the game-theoretic research that inspires the analysis in this paper. What we mean by 'leaning against the wind, but not too aggressively' corresponds to the idea behind Guesnerie (2002, page 456)'s conclusion, 'Coordination is favored whenever agents' actions are not too responsive to expectations'.

to Online Appendix E. Section 3 introduces the dynamic model and shows that the competitive equilibrium is globally unique under the Taylor strategy. This result is already known, but we include it here for completeness and set the stage for the later analysis. Section 4 reformulates the competitive equilibrium as the Nash equilibrium of a particular game. Section 5 provides our main results, by deriving sufficient conditions for expectations to be anchored. Section 6 introduces two shocks into the analysis, a money demand shock and trembles. Each allows us to show, in different ways, the superiority of the Taylor strategy to alternatives encountered in our analysis. A brief conclusion appears at the end.

# 2 Static Example and Literature Review

In this section we consider a drastically simplified, static version of our model which illustrates all the important aspects of our analysis and allows us to clarify the relationship between our work and the literature. We use the static model to show how (i) we can map the competitive equilibrium of a market economy into the Nash equilibrium of a particular game; (ii) the game can be used to formalize how agents form beliefs; and (iii) the game allows us to identify a principle ('leaning against the wind, but not too aggressively') which guarantees that the belief of each agent about inflation is anchored to the target chosen by the policy maker. To make the discussion as simple as possible, many of the economic relations are expressed in reduced form. In the main analysis, beginning in the next section, all features of the model are made explicit.

## 2.1 Example

Each of a continuum of intermediate good firms, $i \in [0,1]$, sets its price, $p_i$, simultaneously and without communicating with the others. We assume that the optimal decision by the $i^{th}$ firm is to set $p_i = W$, where $W$ is the nominal marginal cost of production. The aggregate price index, $P$, is the average over each price setter's decision:

$$P = \int_0^1 p_i di. \tag{1}$$

Note that we can express $W$ as $P \times w$, where $w$ is real marginal cost. Resources are scarce, in the sense that $w$ is increasing in the level of aggregate economic activity. We assume that the government can influence the level of aggregate activity by changing the nominal rate of interest, $R$. High $R$ slows down economic activity (say, by reducing demand), which in turn reduces $w$. Monetary policy sets $R$ as an increasing function of inflation, so that (for a given level of the lagged aggregate price level), $R = f(P)$. Here, $f$ is an increasing function. In this way, we can express the nominal wage rate as follows:

$$W = Pw(f(P)). \tag{2}$$

In a competitive equilibrium each firm faces the same wage, so equation (1) implies $p_i = P$ for each $i$. A competitive equilibrium is a $P^*$ that satisfies $P^* = P^* w\left(f\left(P^*\right)\right)$, or,

$$w\left(f\left(P^*\right)\right) = 1. \tag{3}$$

We assume that $w\left(\cdot\right)$ and $f\left(\cdot\right)$ imply that the competitive equilibrium, $P^*$, is unique.

The question of how agents form their beliefs lies at the heart of the problem of how to design monetary policy, $f$, to ensure that each agents' belief is anchored at the policy maker's target. This question is sidestepped in equilibrium theory by the simple and unexplained assumption called 'clairvoyance' by Guesnerie (2002): agents believe prices take on their competitive equilibrium values. For many purposes, this assumption about beliefs may be a powerful and useful shortcut. But, for our purposes clairvoyance assumes away the problem of interest. By transforming the environment into a game we are able to exploit the formal tools developed in epistemic game theory for thinking about how agents form their beliefs.

The $i^{th}$ firm without clairvoyance finds itself in a conundrum. Note from equation (2) that the nominal wage is itself a function of $P$, which is the aggregate of the prices set by the other firms (see equation (1)). As a result, the $i^{th}$ firm cannot actually see $P$ (hence, $W$) until after firms have set their price. So, the $i^{th}$ firm must set its price based on a belief, $p_i^b$, about the prices being set simultaneously by the other firms.[9] We maintain the spirit of rational expectations in that we assume each firm understands the functions $w\left(\cdot\right)$ and $f\left(\cdot\right)$, knows that other firms face a symmetric problem, and understands that this knowledge is common knowledge (CK).

Conditional on a candidate belief, $p_i^b$, the $i^{th}$ firm computes the *continuation equilibrium* in which the nominal wage is $p_i^b w\left(f\left(p_i^b\right)\right)$.[10] So, the *best response* for the $i^{th}$ firm with belief, $p_i^b$, is

$$p_i = p_i^b w\left(f\left(p_i^b\right)\right) \equiv F\left(p_i^b; f\right), \tag{4}$$

where the notation is designed to highlight the dependence of $F$ on monetary policy, $f$. A Nash equilibrium is a $P^*$ such that $P^* = F\left(P^*; f\right)$. Since any such $P^*$ satisfies equation (3), we have in effect transformed the competitive equilibrium into the Nash equilibrium of a large game in which each player's best response function is given by $F$. For the purpose of the example, we assume that $F$ is linear.

In forming its belief, $p_i^b$, the $i^{th}$ firm is led to consider a range of possible values of $p_i^b$. If the slope of $f$ is sufficiently small then $F$ is increasing and crosses the 45° line from above, as in Figure 1a.[11]

[9] In this example, it is immaterial whether (i) $p_i^b$ corresponds to a belief about what each other firm will do; or (ii) $p_i^b$ is simply a belief about the average decision of the other firms. In our dynamic model (i) and (ii) are very different and we adopt assumption (i).

[10] The aggregate price index in the continuation equilibrium is $P = p_i^b$.

[11] This diagram bears a resemblance to the one associated with the seminal work of Barro and Gordon (1983). It is indeed very similar, but recall that we assume policymakers have the ability to commit to their policy strategy, while Barro and Gordon (1983) assume they do not have the ability to commit. Also, the object on the vertical axis of Figure 1a corresponds to the decision of an atomistic agent, while in Barro and Gordon (1983) the object on the vertical axis pertains to the aggregate price level. In addition, the graph in Figure 1a implicitly assumes that price

The anchoring question is whether each firm, $i \in [0,1]$, will independently and without coordination arrive at the same belief, $p_i^b = P^*$. If the answer is 'yes', we say that expectations are anchored.

In forming its belief, $p_i^b$, will the $i^{th}$ agent choose the Nash equilibrium belief, or some other belief? One justification for the Nash equilibrium belief is that it is the unique belief, $p_i^b$, such that if everyone held that belief then no firm would ex post regret its action, $p_j = P^*$. The Nash equilibrium belief has the property that everyone has perfect foresight ('clairvoyance'). But, ex post considerations are irrelevant for justifying a Nash equilibrium because beliefs must be formed ex ante.[12] If a firm believed $p_i^b = 1.5$ ex ante then that firm would expect to feel regret if it did not choose $p_i = F(1.5; f) < P^*$. So, although each agent knows that $P^*$ is the unique Nash equilibrium, there is no *prima facie* reason to think that agents would form the belief, $p_i^b = P^*$. For the Nash belief (or, rational expectations equilibrium) to occur requires that $f$ be properly designed.

We follow the seminal work of Bernheim (1984) and Pearce (1984) in supposing that firms will only entertain beliefs that they deem to be *rationalizable*. To understand this approach to belief formation, consider the best response function, $F$, displayed in Figure 1a. Suppose firms only consider beliefs in the compact interval, $A$. In this example the exact boundaries of $A$ are irrelevant for the outcome of the analysis, except that $A$ must include the Nash equilibrium, $p_i^b = P^*$. Because $F$ is only a little flatter than the 45° line, monetary policy leans against the wind very slightly. The $i^{th}$ intermediate good firm would never choose $p_i \in A/B$ (see the set $B$ on the vertical axis in Figure 1a).[13] Understanding that the other agents are in the same situation, the $i^{th}$ firm deletes the beliefs, $p_i^b \in A/B$ from further consideration. No rational agent would believe that its opponent would take an action that is under no circumstances a best response. Using the 45° line to map $B$ onto the horizontal axis, we see that $B$ is strictly interior to $A$. The $i^{th}$ firm applies the same reasoning to the set of beliefs, $B$, and is driven to delete additional beliefs from further consideration. It is easily verified that, given the shape of $F$, this process leads to deleting all beliefs, except the Nash belief. So, in this case beliefs are uniquely anchored at $p_i^b = P^*$ for each $i \in [0,1]$.

The reason inflation expectations are anchored is: (a) monetary policy leans against inflation and (b) policy does so in a 'non-aggressive' way, in the sense that the range of best responses is smaller than any given range of beliefs. To explore (b), consider now the case where policy, $f$, leans against inflation more aggressively. Suppose the slope of $f$ is so large that $F$ is downward-sloping (see Figure 1b), but flatter than the 135° line (the dashed line).[14] If we ask, 'what beliefs are rationalizable' and we apply the same iterated deletion argument as before, we conclude once again that the only rationalizable belief is the Nash equilibrium. However, if monetary policy is even more aggressive, so that $F$ is steeper than the 135° line, then (b) is violated. In this case all beliefs are rationalizable

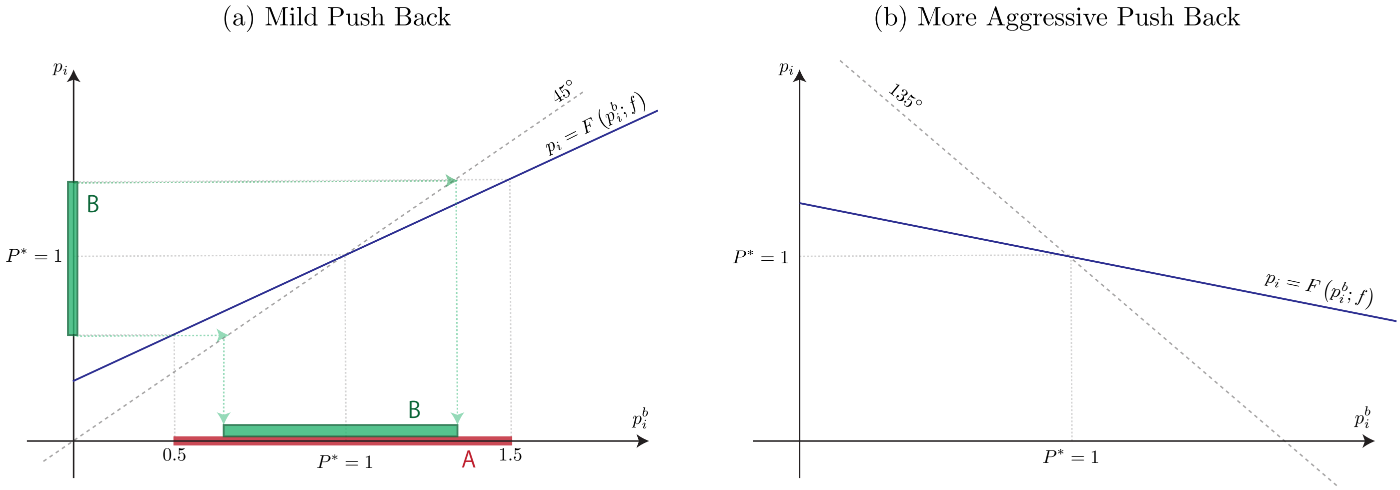
setting is characterized by strategic complementarity. Here, the nature of monetary policy plays a central role in determining whether price setting is characterized by complementarity or substitutability.

[12]This is a one-period model, so each firm is in a 'first play' situation. There is no history or 'norm' to coordinate on. The first play perspective is squarely in the spirit of rational expectations because those models are used to explore the consequences of policy strategies that differ from the ones currently in place.

[13]Here the set $A/B \equiv \{x : x \in A, x \neq B\}$.

[14]Here, we ignore any zero lower bound (ZLB) on the nominal rate of interest. In our dynamic example we do take into account the ZLB.

Figure 1: Anchored Expectations




and inflation expectations are not anchored.

Our conclusion is that for expectations to be anchored two conditions are sufficient: the best response function should cross the 45° line only once, so that there is a unique Nash equilibrium; and the best response function lies inside the butterfly-shaped diagram formed by the area between the 45° and 135° degree lines.[15] The second condition is our formalization of the idea that policy 'leans against the wind, but not too aggressively'.[16] For example, in the model described in the next section the coefficient, $\phi$, on inflation in the Taylor rule must be greater than unity to help guarantee a unique equilibrium. But, if $\phi$ is too large then policy is too aggressive and inflation expectations become unanchored despite the uniqueness of the competitive equilibrium.[17] In this case, there is no reason to expect that the coordination assumed by competitive equilibrium will occur.

An important input into the above analysis is that $F\left(p_i^b; f\right)$ is well-defined over a reasonable range of values of $p_i^b$. If the best response functions in the Nash equilibrium representation of a competitive equilibrium are well defined, then we say that the competitive equilibrium is a *strategy equilibrium*. As is evident from the example, for inflation expectations to be anchored at $p_i^b = P^*$ for each $i \in [0, 1]$ it is necessary, but not sufficient, that the equilibrium be a strategy equilibrium.

The main finding of the paper is that for inflation expectations to be anchored, it is sufficient that monetary policy is consistent with the 'leaning again the wind, but not too aggressively' prin-

[15]For a precise statement of the 'butterfly', see section 5.3. That section shows that being inside the butterfly diagram is sufficient for unique rationalizability. It also discusses necessity (see Proposition 10).

[16]Condition (iii) of Proposition 9 below provides a mathematical statement of the proposition that a sufficient condition for unique rationalizability requires that $F$ lie inside the butterfly diagram. Lying inside the butterfly diagram is what we mean by 'leaning against the wind, but not too aggressively'. The discussion immediately after Proposition 9 explains that unique rationalizability depends the graph of $F$ lying inside the butterfly, for some transformation of the domain and range of $F$. Although the upper and lower bounds that we placed on beliefs in our example played a role in the argument in the text, the precise value of those bounds does not affect the conclusion that $p_i^b = 1$ is the only rationalizable belief. The bounds do raise technical issues, which we address in Section 5.2. This type of bound issue is also considered in Kocherlakota (2018) and we relate our analysis to his in Footnote 76.

[17]A detailed discussion appears in section 5.5.

ciple. This principle guarantees that atomistic and non-cooperative agents arrive at the same beliefs independently, and that the desired rational expectations equilibrium is the unique outcome.

## 2.2 Relation to the Literature

### 2.2.1 Strategy Equilibrium

Many previous authors have stressed that for policy to be successful, it is necessary that the equilibrium be what we call a strategy equilibrium. Diamond and Dybvig (1983) make this point in case of bank runs.[18] Bassetto (2005) stresses the importance of strategy equilibrium for implementation of Ramsey optimal allocations. Atkeson et al. (2010) and Cochrane (2011) consider the issue in an environment very similar to ours.[19] Notably, Cochrane (2011) presents a model in which the Taylor strategy produces a unique equilibrium, but it is not a strategy equilibrium. He correctly concludes that the Taylor strategy does not anchor inflation expectations in his model. However, we show that his conclusion is valid only because he assumes an endowment economy, in which the real interest rate must be the same on and off equilibrium paths. In our model the real interest rate can vary on out-of-equilibrium paths and this plays a key role to ensure that the unique equilibrium under the Taylor strategy in our model is a strategy equilibrium.

### 2.2.2 Rationalizability, Implementation and Leaning Against the Wind

The papers summarized in the previous subsection do not discuss rationalizability. But, a series of other papers has introduced rationalizability into macroeconomic analysis.[20] Those papers clearly state the insight that a relatively flat best response function helps to promote unique rationalizability.[21] However, the papers do not apply the insight to the policy design problem, as we do here.

Our analysis is closely related to the literature on policy implementation. Loosely, we say that a policy *uniquely implements* a desired equilibrium if the following three conditions are satisfied: (i) the unique (Nash) equilibrium is the desired equilibrium; (ii) the equilibrium is a strategy equilibrium; and (iii) the beliefs associated with the unique equilibrium are uniquely rationalizable.[22] If these conditions are satisfied then inflation expectations (as well as expectations about other variables) are

[18] Diamond and Dybvig (1983, section V) make a powerful case that the desired equilibrium must be a strategy equilibrium. They argue that although it is formally true that deposit insurance rules out the bank run equilibrium, it is important to study the off-equilibrium path in which a bank run occurs, to verify that the deposit insurance fund would have enough resources in that case. One supposes that if the no-run equilibrium were not a strategy equilibrium then agents would in fact not adopt the equilibrium belief that their money is safe in the bank.

[19] Bassetto (2005) refers to what we call a strategy equilibrium as a Schelling timing equilibrium, or a sequential equilibrium. Atkeson et al. (2010) call it a sophisticated equilibrium.

[20] See Guesnerie (1992), Evans and Guesnerie (1993), Guesnerie (2002), and Evans and Guesnerie (2005). For a review, see Desgranges (2014).

[21] Recall in particular the reference to Guesnerie (2002) in Footnote 8.

[22] A formal definition of our implementation concept is provided in Definition 10. In the body of the paper, we follow Atkeson et al. (2010) in working with one-shot sequential game representations of our dynamic equilibrium. We extend the one-shot concept of implementation to multiple shots in Definition 11 in the Online Appendix.

anchored. Our approach to implementation differs from existing approaches in the macroeconomics literature.

There exist at least two approaches to implementation in macroeconomics. One approach focuses on (i) only, by designing policy so that the rational expectations equilibrium is unique and has desirable characteristics.[23] The second approach requires (ii) in addition to (i). This is the approach advocated in Bassetto (2005) and Atkeson et al. (2010). But, the two approaches to implementation just described do not answer our anchoring question: 'what is required for policy to ensure that agents uniquely coordinate on the equilibrium beliefs?'. This question is answered by our approach to implementation. Our approach to implementation requires condition (iii), in addition to (i) and (ii).

In addition to anchoring expectations, condition (iii) has another feature that is of interest. In particular, condition (iii) provides a formal foundation for ruling out policies that seem intuitively peculiar, even though they satisfy (i) and (ii). We provide two examples. In the first example, high expected inflation is ruled out by a commitment to make actual inflation even higher (see section 5.5). In terms of Figure 1a this corresponds to a scenario in which the best response function is steeper than the $45^o$ line. Intuitively, one might expect such an odd dynamic to manifest itself in the form of some kind of expectational instability, but (i) and (ii) offer no way to contemplate such a possibility. In contrast, in this example the unique equilibrium fails to be uniquely rationalizable (i.e., (iii) fails) precisely because an expectational instability prevents iterated deletion from eliminating all but the competitive equilibrium expectations. Our second example is the opposite of the first because that example involves a best response function with slope steeper than minus unity.[24] The example is a simple version of the New Keynesian model with Calvo-style price-setting frictions and two shocks. A known property of that model is that for a sufficiently large value of the coefficient on inflation, $\phi$, in the Taylor rule, the locally unique equilibrium allocations approach the first-best allocations.[25] Intuitively, it may seem peculiar that a central bank commitment to respond to inflation with arbitrarily large force could move the economy arbitrarily close to first-best. One might expect that some form of instability would occur instead. But, there is no room to consider any kind of instability in the implementation concepts, (i) and (ii). Under condition (iii), a form of instability indeed does occur when $\phi$ is too large.[26] A policymaker contemplating high values of $\phi$ would expect good results under (i) and (ii) alone. But, a policymaker that incorporates (iii) when designing policy would realize that when $\phi$ is large enough, expectations become unanchored and control over the economy is lost.

[23] For examples of this approach see what Atkeson et al. (2010, p. 49) refer to as 'unsophisticated implementation'.

[24] In terms of Figure 1b the best response cuts the $135^o$ line from above.

[25] The fact that, in equilibrium, a sufficiently high value of $\phi$ can close the gap between actual and first best output reflects in part that the model incorporates a tax subsidy that neutralizes a monopoly distortion in non-stochastic steady state (for completeness, all results for the New Keynesian model are proved in Online Appendix E). Our analysis of this model assumes that exists an escape policy which ensures that the locally unique equilibrium is also globally unique.

[26] For details, see Online Appendix E.

### 2.2.3 Coordination

From a broader perspective, our analysis is part of an effort that began well before the rational expectations revolution. Walras and Marshall in the 19th century also struggled to understand how markets coordinate on equilibrium prices. The work in recent decades on how agents manage to coordinate their beliefs is loosely divided by Binmore (1987) into two approaches: the eductive approach and evolutive approach. The former imagines that agents undergo a private, mental reasoning process in a pre-period to arrive at a set of beliefs. The eductive approach was illustrated in the example above. The second, evolutive, approach assumes that agents form their beliefs based on observing data as they accumulate over time. No doubt, the correct approach is a combination of the two.[27] More recently, Evans et al. (2018) make a convincing case that realistic models cannot completely avoid the evolutive approach.

As noted above, the standard rational expectations approach in macroeconomics sidesteps the coordination question altogether by simply assuming that coordination occurs (e.g., Guesnerie (2002)'s 'clairvoyance').[28] As we explain below, tractability drives us to use a combination of the eductive approach and clairvoyance. We take a step away from full rational expectations towards eduction (rather than the evolutive approach) because it is closest in spirit to the rational expectations framework used in the analysis of monetary policy rules.

### 2.2.4 Escape Clauses

Finally, our analysis is related to the general topic of escape clauses. Escape clauses allow governments to deviate from normal policy (which might work well in normal times) to an alternative policy in case things go wrong. An escape clause may work well by anchoring the normal equilibrium, even if the policy under the escape clause is not very efficient. For example, in the last section of the paper we introduce money demand shocks, so that a money growth rule is not efficient for the classic reasons spelled out in Poole (1970). Still the commitment to deviate to that money growth rule in the event that an inflation monitoring range is violated (as long as it is designed properly), may ensure that the conditions that trigger the escape clause never happen in the first place, as in the Diamond and Dybvig (1983) analysis of deposit insurance. Another example is the 'suspension clause' in the United States Constitution (Article I, Section 9, Clause 2), which gives the United States Congress the right to suspend habeas corpus '...in cases of rebellion or invasion [when] the public safety may require it.' An example that applies to monetary policy is the 'unusual and exigent circumstances' clause, Section 13.3, in the Federal Reserve Act, used to rationalize the use of unconventional monetary policy in the wake of the 2008 Financial Crisis and during the current COVID-19 crisis. The policy to which the government deviates under the escape clause might not be optimal in normal times, but

[27]For example, Binmore (1987, p. 185) states, "It is not denied that the middle ground between these extremes is more interesting than either extreme."

[28]This approach is also dominant in practical applications of game theory. It is common to assume that agents simply know (without further explanation) that everyone else will play the Nash strategy.

the mere existence of the escape clause may reduce chance that undesired equilibria form in the first place.

# 3 Competitive Equilibrium

Our monetary model is designed to allow us to make the points summarized in the previous section, and to facilitate comparability with the related monetary policy literature (see especially Cochrane (2011) and Atkeson et al. (2010)). Versions of the model have been studied extensively in a series of papers and the variant we consider is closest to the model used in Christiano and Rostagno (2001).[29]

## 3.1 Households

The economy is composed of a large number of identical households. The representative household solves the following problem:

$$\max_{\{c_t,l_t,m_t,d_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\gamma}}{1-\gamma}-\frac{l_t^{1+\psi}}{1+\psi}\right], \quad \gamma>0,\gamma\neq 1,\psi\geq 0 \tag{5}$$

$$s.t. \quad m_t=\bar{R}_t\left(X_t+d_t\right)+m_{t-1}-d_t+W_tl_t-P_tc_t+T_t, \tag{6}$$

$$P_tc_t\leq m_{t-1}-d_t+W_tl_t \tag{7}$$

$$m_{-1} \text{ given,}$$

where $c_t\geq 0$ and $l_t\geq 0$ denote consumption and employment, respectively. Also, $m_{t-1}$ denotes the quantity of currency held by the household at the beginning of period $t$. The right side of equation (6) represents the household's period $t$ sources of currency. The first source of currency takes the form of interest and principle, $\bar{R}_t\left(X_t+d_t\right)$, received on bank deposits, $X_t+d_t$, at the end of period $t$ after goods markets have closed. Here, $X_t$ denotes a government transfer of currency directly into the household's bank deposit at the start of period $t$. Also, $0\leq d_t$ denotes currency deposited by the household into a bank at the start of period $t$ and $\bar{R}_t$ denotes the gross nominal rate of interest. The currency that the household carries over from period $t-1$ is denoted by $m_{t-1}$, so that $m_{t-1}-d_t$ represents the currency that the household chooses to keep on hand at the start of period $t$. Other currency available to the household at the start of period $t$ is the wage bill, $W_tl_t$, which it receives before work begins. The household spends currency, $P_tc_t$, in the goods market, subject to the cash constraint, (7). Finally, after the goods market closes, the household receives $T_t$, which denotes firm profits net of government taxes.[30]

[29]The model was originally constructed by Fuerst (1992), which in turn was inspired by Lucas (1990). See also Christiano (1990), Christiano and Eichenbaum (1992), Chari et al. (1995), Christiano and Eichenbaum (1995) and Christiano et al. (1997).

[30]Nothing substantial depends on our assumption, $\gamma\neq 1$. We discuss this case later.

According to equation (6) and equation (7), the trade off between consumption and leisure offered to the household by the market is a function of the real wage, $W_t/P_t$. Similarly, the tradeoff between consumption in different periods is determined by the real rate of interest, $\bar{R}_t/\bar{\pi}_{t+1}$.

It is easily verified that under these conditions, the first order conditions associated with household employment, deposits and the cash constraint are, respectively,

$$\frac{W_t}{P_t} = c_t^{\gamma} l_t^{\psi}, \tag{8}$$

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \frac{\bar{R}_t}{\bar{\pi}_{t+1}}, \tag{9}$$

$$0 = \left(\bar{R}_t - 1\right)\left(m_{t-1} - d_t + W_t l_t - P_t c_t\right). \tag{10}$$

In equation (10), each expression in braces is non-negative. The transversality condition is:

$$\lim_{j\to\infty} q_j m_j = 0, \tag{11}$$

where $q_j$ is 1 for $j = 0$ and $\left(\prod_{s=0}^{j-1} \bar{R}_s\right)^{-1}$. In Online Appendix A.5, we show that, under the boundedness conditions, the first order conditions and transversality condition are necessary and sufficient for household optimization.

## 3.2 Production

A final output good is produced by a competitive, representative firm using the CES production function with the elasticity of substitution $\kappa > 1$ :

$$Y_t = \left(\int_0^1 Y_{i,t}^{(\kappa-1)/\kappa} di\right)^{\kappa/(\kappa-1)}.$$

Let $p_{i,t}$ denote the price of intermediate good, $Y_{i,t}$, $i \in [0,1]$. The firm takes the price of output, $P_t$, and the prices of the inputs, $p_{i,t}$, $i \in [0,1]$ as given. The first order conditions associated with the $i^{th}$ firm's profit maximization problem is:

$$Y_{i,t} = Y_t \left(\frac{P_t}{p_{i,t}}\right)^{\kappa}. \tag{12}$$

The first order conditions, together with the production function, impose a restriction across the aggregate price index and the price of intermediate goods:

$$P_t = \left[\int_0^1 p_{i,t}^{1-\kappa} di\right]^{\frac{1}{1-\kappa}}. \tag{13}$$

The $i^{th}$ intermediate good, $Y_{i,t}$, is produced by a monopolist with the following production function, $Y_{i,t} = l_{i,t}$. Here, $l_{i,t}$ denotes the labor input employed by the $i^{th}$ firm.

We assume that the $i^{th}$ intermediate good firm must pay its period $t$ wage bill, $W_t l_{i,t}$, at the beginning of the period, before it has obtained its period $t$ receipts. The firm borrows $W_t l_{i,t}$ from the bank and must repay $\bar{R}_t W_t l_{i,t}$ after the goods market closes. At that time, the government provides a subsidy, $\tau_t$, to the firm so that its post-subsidy end-of-period wage bill is $(1-\tau_t)\,\bar{R}_t W_t l_{i,t}$. Since the firm wants to borrow an infinite amount when $\bar{R}_t < 1$, equilibrium requires $\bar{R}_t \geq 1$. The $i^{th}$ firm maximizes end-of-period profits by setting its price as a markup over marginal cost, $(1-\tau_t)\,\bar{R}_t W_t$ :

$$p_{i,t} = \frac{\kappa}{\kappa - 1}(1-\tau_t)\,\bar{R}_t W_t = W_t, \tag{14}$$

for each $i$. The second equality in equation (14) reflects our assumption that $\tau_t$ is selected to neutralize the interest rate and monopoly power distortions in the model. We neutralize the monopoly power distortion to streamline notation and we neutralize the interest distortion to ensure that Cochrane (2011)'s model is a special case of ours (see below).[31] Because the marginal cost for $i$ is the same for all the other firms, all intermediate good firms set the same price, $p_{i,t}$. As a result, by equation (13) the intermediate good firm equilibrium condition is:

$$P_t = W_t. \tag{15}$$

## 3.3 Government Policy

The government's monetary transfer, $X_t$, to households at the start of $t$ satisfies

$$X_t = (\bar{\mu}_t - 1)\,\bar{M}_{t-1}, \quad \bar{\mu}_t = \bar{M}_t/\bar{M}_{t-1}. \tag{16}$$

Here, $\bar{M}_t$ denotes the end-of-period $t$ per capita stock of money and $\bar{\mu}_t$ denotes the gross money growth rate. Monetary policy selects a sequence, $\{\bar{\mu}_t\}_{t=0}^{\infty}$, so that, in equilibrium,

$$\bar{R}_t = \max\left\{1, \bar{R}^*\left(\frac{\bar{\pi}_t}{\bar{\pi}^*}\right)^{\phi}\right\}, \quad \bar{\pi}_{t+1} \equiv \frac{P_{t+1}}{P_t}, \quad \bar{R}^* \equiv \bar{\pi}^*/\beta, \tag{17}$$

where $\bar{\pi}^* \geq 1$ and $\bar{R}^*$ are the desired inflation and interest rate. Here, we assume that $\phi > 1$. In addition, when the lower bound on $\bar{R}_t$ is binding, we assume that the government sets $\bar{M}_t = P_t c_t$, where $c_t$ denotes aggregate consumption.

In addition, the government levies a lump sum tax, $T_t^g$, on households to finance the subsidy in equation (14) to firms. Thus, $T_t^g = \tau_t \bar{R}_t W_t l_t$,where $\tau_t = 1 - (\kappa - 1)/\kappa \bar{R}_t^{-1}$. The government does not purchase any goods, or issue debt.[32]

[31] See Christiano and Rostagno (2001) for an extended analysis of the case, $(1-\tau_t)\,\kappa/(\kappa-1) = 1$.

[32] A number of interesting issues concerning fiscal policy are left out of the analysis. For example, a property of our model is that a non-negative money growth rate rules out a zero interest rate equilibrium. In the presence of government debt, this result is not necessarily true. For further discussion and a defense of the position taken here, see Christiano and Rostagno (2001, Section 2.4).

### 3.4 Market Clearing and Equilibrium

The goods, labor, money and loan market clearing conditions are:

$$c_t = Y_t, \quad \int_0^1 l_{i,t} = l_t, \quad \bar{M}_t = m_t, \quad X_t + d_t = W_t l_t, \tag{18}$$

for $t \geq 0$, respectively. Also, household profits net of taxes are $T_t = \int_i \left[ p_{i,t} Y_{i,t} - \bar{R}_t W_t \left(1 - \tau_t\right) l_{i,t} \right] di - T_t^g$. Equations (12) and (14) imply, using equation (18):

$$c_t = l_t. \tag{19}$$

To ensure that households are identical, we require $m_{-1} = \bar{M}_{-1}$.

Let the time $t$ variables be denoted by:

$$\bar{a}_t = \left(l_t, c_t, P_t, \bar{R}_t, W_t, \bar{\mu}_t, \bar{M}_t, m_t, d_t\right). \tag{20}$$

To simplify notation, we delete reference to $p_{i,t}$ and $l_{i,t}$ because these are simply equal to $P_t$ and $l_t$, respectively, for each $i \in [0, 1]$. We define a competitive equilibrium that starts at date 0 as follows:[33]

**Definition 1.** A *competitive equilibrium under the Taylor rule* is a sequence, $(\bar{a}_t)_{t=0}^{\infty}$, that satisfies, for $t \geq 0$, (i) intermediate good firm optimality; (ii) final good firm optimality; (iii) household optimization, conditional on $m_{-1}$; (iv) government policy; and (v) market clearing.

For purposes of equilibrium characterization, it is useful to note that condition (i) corresponds to equation (14); condition (ii) corresponds to equation (12) and equation (13); condition (iii) corresponds to equation(8) - equation (11); condition (iv) corresponds to equation (16) - equation (17); and condition (v) corresponds to equation (18).

We define competitive equilibrium under alternative monetary policies by obvious adjustments to Definition 1. For example, we can define a *competitive equilibrium under constant money growth*, by replacing equation (16) - equation (17) with a constant value for $\bar{\mu}_t$. Similarly, we can define a *competitive equilibrium under the Taylor strategy.* The Taylor strategy was discussed in the introduction, and will be defined formally in Section 3.6 below.

### 3.5 Properties of Competitive Equilibrium Under the Taylor Rule

We now obtain a dynamic equation that can be used to identify all the equilibria in our model. It turns out that the analysis of equilibrium is greatly simplified by scaling and logging the variables. Later, when we discuss how rational agents coordinate their expectations, the log transform will also be very convenient.[34]

[33]To shorten the definition, we refer to our equilibrium concept as a *competitive* equilibrium. We do this even though some of the agents in our model are not competitive.

[34]See the discussion of Proposition 9 below.

Combining equation (15) and equation (18), we conclude that for all $t \geq 0$, $c_t = l_t = 1$. As a result, the intertemporal Euler equation reduces to the *Fisher equation*:

$$R_t = \pi_{t+1}, \tag{21}$$

where $R_t = \ln\left(\bar{R}_t/\bar{R}^*\right)$, $\pi_{t+1} = \ln\left(\bar{\pi}_{t+1}/\bar{\pi}^*\right)$. Also, $\bar{\mu}^* = \bar{\pi}^*$ is the money growth level in the desired equilibrium. We can also express the Taylor rule in scaled and logged form:

$$R_t = \max\left\{R_l, \phi\pi_t\right\}, \; \bar{M}_t = P_t c_t, \text{ if } R_t = R_l, \tag{22}$$

where $R_l \equiv \ln\left(1/\overline{R}^*\right)$ and $c_t$ denotes aggregate consumption. It is also useful to define the scaled and logged money growth rate:

$$\mu_t \equiv \ln\left(\frac{\bar{\mu}_t}{\bar{\mu}^*}\right). \tag{23}$$

Combining equation (21) and equation (22) we obtain the following first order difference equation:

$$\pi_{t+1} = \max\left\{R_l, \phi\pi_t\right\}. \tag{24}$$

Combining the loan market clearing condition, equation (18), the household Kuhn-Tucker condition, equation (10), and monetary policy in the zero lower bound, we have the equilibrium cash condition:

$$\bar{M}_t = P_t c_t. \tag{25}$$

Also, the transversality condition, taking into account the equilibrium conditions, (18) and the constant consumption level, corresponds to:

$$\lim_{T\to\infty} \beta^T \frac{\bar{M}_{T-1}}{P_T} = 0. \tag{26}$$

Equation (24) is useful for studying the equilibria in the model for the following reason:

**Proposition 1.** *For any sequence, $(\pi_t)_{t=0}^{\infty}$ that satisfies the difference equation, (24), it is possible to construct the other equilibrium variables in such a way that all the conditions for a competitive equilibrium are satisfied.*

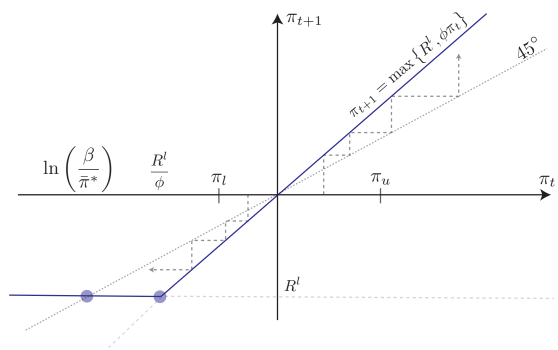
That a model which has an equilibrium characterization of the form in equation (24) has multiple equilibria, is well-known. For completeness, we include a proof in Online Appendix A.4.[35]

Figure 2, a variant on the well-known figure in Benhabib et al. (2001b, Fig. 1), depicts equation (24). From this figure it is easy to see that the model has many equilibria, each indexed by the value of $\pi_0$. The *desired equilibrium* corresponds to $\pi_t = 0$ for $t \geq 0$. Consider the two inflation

[35] Other work that studies models with the equilibrium characterization, equation (24), include Benhabib et al. (2001a), Woodford (2003, chapter 2) and Cochrane (2011). The model underlying these characterizations is an endowment economy, while ours is not. This distinction plays an important role in our analysis.

rates marked in Figure 2 by $\pi_l$ and $\pi_u$. We refer to this interval as the *inflation monitoring range*. Note that, due to the high value of $\phi$, there is exactly one equilibrium, the desired equilibrium, in which inflation is always in the monitoring range, $\pi_t \in [\pi_l, \pi_u]$ for $t \geq 0$. This observation plays an important role in the analysis below.

Figure 2: Fisher Equation and Taylor Rule




## 3.6 Taylor Strategy

The fact that the unique equilibrium with $\pi_t \in [\pi_l, \pi_u]$ is the desired equilibrium for all $t \geq 0$ is an important motivation for the following policy:

**Definition 2.** *Taylor strategy*: if $t = 0$, or if $\pi_s \in [\pi_l, \pi_u]$ for $s \leq t-1$, for $t > 0$, then follow the Taylor rule, (22), with $\phi > 1$ in period $t$. Otherwise, monetary policy sets (scaled and logged) money growth to $\mu_t = \mu + \rho \ln c_t$ for $t \geq 0$, where $\mu \in [\pi_l, \pi_u]$. Here, $\pi_l$ and $\pi_u$ are two constants satisfying

$$\max\left\{\frac{R_l}{\phi}, \ln\frac{1}{\bar{\mu}^*}\right\} < \pi_l \leq 0 \leq \pi_u < \infty. \tag{27}$$

Note our money growth rule responds to (aggregate) consumption. The parameter, $\rho$, is irrelevant for establishing uniqueness of competitive equilibrium. It is important later when we consider off-equilibrium paths (see Section 5 below). We will see in Section 4.2 that the value of $\rho$ plays a role in guaranteeing unique implementation of the desired equilibrium. The restriction on $\mu$ guarantees that if the escape clause is activated, then inflation will be brought to the interior of the monitoring range.

A competitive equilibrium under a Taylor strategy is unique and it is the desired equilibrium. This result is established in three steps. First, we establish that the economy has a unique equilibrium under our money growth rule:

**Lemma 1.** *Suppose $\bar{M}_{-1}$ is given and monetary policy sets $\bar{M}_t = \bar{\mu} c_t^\rho \bar{M}_{t-1}$ for $t \geq 0$, where $\bar{\mu} > 1$. There exists a unique competitive equilibrium with the properties:*

$$\bar{R}_t = \beta^{-1}\bar{\mu} > 1, \quad c_t = 1, \quad \bar{\pi}_{t+1} = \bar{\mu}, \quad \textit{for} \quad t \geq 0, \quad \textit{and} \quad P_0 = \bar{M}_{-1}\bar{\mu}.$$

For the proof, see Online Appendix A.5.

If Lemma 1 were not true and an equilibrium did not exist, it would be impossible to meaningfully ask what would happen if $\pi_t \notin [\pi_l, \pi_u]$, because agents would not know how to form expectations about $t+1$.[36] The second step shows that there is no competitive equilibrium in which $\pi_t \notin [\pi_l, \pi_u]$ :

**Lemma 2.** *Consider the case in which monetary policy is the Taylor strategy defined in Definition 2. An equilibrium has the following property:* $\pi_t \in [\pi_l, \pi_u]$ *for* $t \geq 0$.

The intuition for the proof is straightforward given Figure 2. A formal proof appears in Online Appendix A.5.

The basic result of this section is:

**Proposition 2.** *Suppose monetary policy is governed by the Taylor strategy. The only equilibrium is the desired equilibrium.*

*Proof.* Suppose, to the contrary, that $\pi_0 \neq 0$. From Lemma 2 equilibrium has the property that the monitoring range is never violated, i.e., $\pi_t \in [\pi_l, \pi_u]$. The Taylor rule, (22), the Fisher equation, (21), and $\pi_l > \frac{R^l}{\phi}$, imply that in equilibrium: $\pi_{t+1} = \phi\pi_t$. Evidently, $\pi_0 \neq 0$ implies $\pi_t \notin [\pi_l, \pi_u]$ for some $t$, given $\phi > 1$. This contradicts Lemma 2. We conclude that $\pi_0 = 0$, establishing the proposition. $\square$

The result follows almost immediately from Lemma 2 and Figure 2. The former says that there is no equilibrium with $\pi_t \notin [\pi_l, \pi_u]$. The latter indicates that the only equilibrium with $\pi_t \in [\pi_l, \pi_u]$ is $\pi_t = 0$ for all $t \geq 0$.

Proposition 2 in effect assumes $\phi > 1$ because that is included in our definition of the monetary policy rule (see Definition 2). It is interesting to note that $\phi > 1$ is not necessary for a unique equilibrium in the extreme case, $\pi_l = \pi_u = \mu = 0$. In this case, uniqueness is guaranteed for any $\phi \neq 0$ (this result was shown in Atkeson et al. (2010)). To verify this, simply retrace the proof of Lemma 2, which is the heart of the proof of Proposition 2. We discuss the implications of this proposition for policy and for the Taylor principle in Section 6 below. We refer to this policy as follows:

**Definition 3.** The *zero monitoring range strategy* is a version of the policy in Definition 2 with $\pi_l = \pi_u = \mu = 0$ and $\phi \neq 0$.

Following is a formal statement of the uniqueness result:

**Proposition 3.** *Suppose monetary policy is the zero monitoring range policy. The only equilibrium is the desired equilibrium.*

[36] For our analysis it is convenient that we have a unique equilibrium under the money rule, but we have not investigated whether uniqueness is necessary for our conclusions.

# 4 The Market as a Game

This section transforms the market economy of the previous section into a large, non-cooperative game among the atomistic intermediate good producers. In period $t$, the $i^{th}$ intermediate good firm's payoff is a function of its own action (its price), the history of the economy, $h_{t-1}$, as well as its belief about the price set by other intermediate good firms. As in the example in Section 2, the $i^{th}$ firm's belief about the price set by other firms matters. This is because what the other firms do determines the continuation equilibrium of the economy, which in turn affects the $i^{th}$ firm's payoff. In contemplating the continuation equilibrium, the intermediate good firms think of households as optimizers with clairvoyance about market prices, so that their decisions satisfy first order conditions and clear markets.[37] In addition, each firm understands the monetary policy strategy. The first section below explains how $h_{t-1}$ and the $i^{th}$ firm's belief map - via a continuation equilibrium - into the $i^{th}$ firm's optimal choice of its own price. This mapping defines the firm's best response function. We show that in contemplating the various continuation equilibria associated with alternative beliefs, the firm's best response function depends on the nature of monetary policy. This result may at first seem surprising, since we do not impose any exogenous price setting or other nominal frictions in the model. Monetary policy matters when intermediate good firms form expectations because, as they contemplate different prices set by others, the continuation equilibrium behaves like a sticky price model.

The first subsection below studies the price setting decision of the $i^{th}$ intermediate good producer, as a function of an arbitrary belief and history. The second and third subsections formally define and then discuss the strategy equilibrium of the model.

## 4.1 The Price Decision of the $i^{th}$ Intermediate Good Producer

As explained in Section 3.2, the $i^{th}$ optimizing intermediate good firm wishes to set its price, $p_{i,t}$, to the aggregate wage rate. In the first subsection below we describe the conundrum discussed intuitively in Section 2. The conundrum implies that to make its price decision, the $i^{th}$ firm must form a belief about how other intermediate good firms are setting their date $t$ price. The second subsection defines histories in terms of our scaled variables. The third subsection describes the reasoning that allows the $i^{th}$ firm to combine its knowledge of the past history of the economy with its belief about others' prices to map into a belief about the nominal wage rate, $W_t$. This mapping, which is used to define the $i^{th}$ firm's best response function, is represented analytically in Proposition 4. The key ingredient of the best response function is an $\left(h_{t-1}, \pi^b_{i,t}\right)$ continuation equilibrium. The fourth and fifth subsections provide explicit derivations and economic interpretations of the best response functions.

[37] Recall the discussion about clairvoyance in Section 2.2.3. By assuming that households have clairvoyance we mean that they have rational expectations, i.e., their beliefs correspond to the competitive (continuation) equilibrium.

#### 4.1.1 The Role of Beliefs in a Firm's Price Setting Decision

Intermediate good firms set their prices simultaneously and without coordination. Recall from equation (14) that firms would like to set their price, $p_{i,t}$, equal to the nominal wage rate. For the reasons described in Section 2, it is not possible for the $i^{th}$ firm to literally observe $W_t$ at the time that it sets its price. This is because $W_t$ is determined in an equilibrium in which the state is composed of the observed past history, $h_{t-1}$, as well as the aggregate price level, $P_t$. The aggregate price level is necessary for wage determination in the labor market because, for example, households care about $W_t/P_t$, and not $W_t$ *per se*. That the $i^{th}$ firm cannot observe $P_t$ at the time it chooses $p_{i,t}$ is obvious from equation (13), which indicates that $P_t$ is the consequence of the prices set by all intermediate good firms. The $i^{th}$ firm cannot observe the consequences, $P_t$, of the other firms' actions until *after* those actions have been taken. It follows that when the $i^{th}$ firm sets its price, it must do so based on a belief about how other firms set their price.

We make the following *symmetry assumption* about beliefs: the $i^{th}$ firm believes that all other firms set the same price, and we denote that price by $p^b_{i,t}$. The symmetry assumption is automatically satisfied in a competitive equilibrium. There, each firm's belief about how others set their price is correct and, hence, identical. An implication of our symmetry assumption is that a given belief, $p^b_{i,t}$, maps trivially into a belief about the aggregate price index via equation (13).

#### 4.1.2 History, $h_{t-1}$, and Scaling the Variables

We begin by providing a formal definition of a history, $h_{t-1}$. A history consists of events observed until the end of date $t-1$. Formally,

$$h_{t-1} = \begin{cases} \left(P_{-1}, \bar{M}_{-1}\right) & t = 0 \\ \left(h_{t-2}, a_{t-1}\right) & t \geq 1 \end{cases}.$$

The two variables in $h_{-1}$ are the initial state of the household (e.g., $m_{-1} = \bar{M}_{-1}$) and the lagged price level needed to define period 0 inflation in the Taylor rule. Also,

$$a_t = \left(l_t, c_t, \pi_t, R_t, w_t, \mu_t, \bar{M}_t, m_t, d_t\right). \tag{28}$$

Here, $\pi_t$ corresponds to $P_t$ in $\bar{a}_t$, which is the unscaled version of $a_t$ (see equation 20). We obtain $\pi_t$ from $P_t$ by scaling the latter with $P_{t-1}\bar{\mu}^*$and logging the result. Similarly, $R_t, \mu_t$ correspond to $\bar{R}_t, \bar{\mu}_t$ in $\bar{a}_t$, after scaling and logging. Finally,

$$w_t \equiv \ln\left(\frac{W_t}{P_{t-1}\bar{\mu}^*}\right). \tag{29}$$

Let $p_{i,t}^b$ denote the scaled and logged price that the $i^{th}$ firm believes the other firms set:[38]

$$\pi_{i,t}^b \equiv \ln\left(\frac{p_{i,t}^b}{P_{t-1}\bar{\mu}^*}\right). \tag{30}$$

Let $x_{i,t}$ denote the scaled and logged price set by the $i^{th}$ intermediate good firm:

$$x_{i,t} = \ln\left(\frac{p_{i,t}}{P_{t-1}\bar{\mu}^*}\right). \tag{31}$$

#### 4.1.3 $\left(h_{t-1}, \pi_{i,t}^b\right)$ Continuation Equilibria: Definition and Characterization

A continuation equilibrium associated with $\left(h_{t-1}, \pi_{i,t}^b\right)$ is formally defined as follows:

**Definition 4.** A $\left(h_{t-1}, \pi_{i,t}^b\right)$ *continuation equilibrium* is a sequence, $(a_{i,t+s})_{s=0}^{\infty}$, that satisfies (i) all time $t+s$ equilibrium conditions, $s>0$ and (ii) all time $t$ equilibrium conditions except the period $t$ optimality condition for the intermediate good firm, equation (15).

We use the $i$ subscript on the period $t$ and future $a$'s to highlight that these variables correspond to the beliefs of the $i^{th}$ firm. Past $a$'s (hence, $h_{t-1}$) do not require an $i$ subscript because they are public information. For a list of all the equilibrium conditions in the model, see the discussion after Definition 1. We obviously must place at least mild regularity restrictions on $\pi_{i,t}^b$ for the analysis to be interesting. In particular, the notion that the $i^{th}$ firm might entertain the thought that other firms set a negative or zero price is uninteresting because no continuation equilibrium could exist in that case.[39] This is true, independent of the nature of monetary policy. So, we exclude any belief as unreasonable unless it satisfies:

$$\pi_{i,t}^b > -\infty. \tag{32}$$

We place no restrictions on $h_{t-1}$or $\pi_{i,t}^b$, apart from equation (32). For example, we do not require that the variables in $h_{t-1}$ satisfy the date $t-1$ and earlier equilibrium conditions. But, we do allow configurations of $\left(h_{t-1}, \pi_{i,t}^b\right)$ that correspond to the unique competitive equilibrium under the Taylor strategy.[40]

Most of the discussion below is devoted to the construction of $a_{i,t}$ in the $\left(h_{t-1}, \pi_{i,t}^b\right)$ continuation equilibrium. This is because the mapping from $\left(h_{t-1}, \pi_{i,t}^b\right)$ to $(a_{i,t+s})_{s=1}^{\infty}$ is straightforward due to

[38] It is easy to establish that $\left(h_{t-1}, p_{i,t}^b\right)$ and $\left(h_{t-1}, \pi_{i,t}^b\right)$ span the same information. First, we show that, given $h_{t-1}$ and $p_{i,t}^b$, it is possible to compute $\pi_{i,t}^b$ in equation (30). Note that $P_{t-1} = \bar{\pi}_{t-1} \times \cdots \times \bar{\pi}_0 P_{-1}$ for $t \geq 0$ and that $\bar{\pi}_{t-s}$, $1 \leq s \leq t$, can be recovered from $h_{t-1}$ because $\bar{\pi}_{t-s} = \exp(\pi_{t-s})\,\bar{\mu}^*$ (see the discussion after equation (21)). This establishes that $P_{t-1}$can be recovered from $h_{t-1}$, so that we have our first result. Second, it trivial to verify that, given $h_{t-1}$ and $\pi_{i,t}^b$, one can derive $p_{i,t}^b$.

[39] This corresponds to $\pi_{i,t}^b$ being a complex number or $-\infty$, respectively.

[40] A history would correspond to an equilibrium if (i) the elements of $h_{t-1}$ are composed of allocations and prices up to date $t-1$ in competitive equilibrium under the Taylor strategy; and (ii) $\pi_{i,t}^b = \pi_t$, the inflation rate in this equilibrium. In this case, the continuation equilibrium is simply the date $t+s$, $s \geq 0$ allocations and prices in the unique competitive equilibrium under the Taylor strategy.

Lemma 1 and Proposition 2. In addition, we have a special interest in a particular element of $a_{i,t}$, namely the time $t$ wage rate. We denote the mapping, $F$, from $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ to the scaled and logged wage, $w_{i,t}$, defined in equation (29) by

$$w_{i,t} = F\left(h_{t-1}, \pi_{i,t}^{b}\right). \tag{33}$$

In this notation, if the $i^{th}$ firm believes $\pi_{i,t}^{b}$ then it sets its price as follows (recall equation (14)):

$$x_{i,t} = w_{i,t} = F\left(h_{t-1}, \pi_{i,t}^{b}\right). \tag{34}$$

Thus, in our environment, $F$ can also be interpreted as $i^{th}$ intermediate good firm's best response function. That is, $F\left(h_{t-1}, \pi_{i,t}^{b}\right)$ is the firm's optimal choice of its own price, conditional on other firms setting their price to $\pi_{i,t}^{b}$.

Before formally stating the function, $F$, it is convenient to repeat our restrictions on preferences (see equation (5)) and on the parameters of the monetary policy rule (see Definition 2):

$$\psi \geq 0, \gamma > 0, \gamma \neq 1, \tag{35}$$

$$\phi > 1, \max\left\{\frac{R_l}{\phi}, \ln\frac{1}{\bar{\mu}^*}\right\} < \pi_l \leq 0 \leq \pi_u, \quad \mu \in [\pi_l, \pi_h]. \tag{36}$$

Then,

**Proposition 4.** *If monetary policy is governed by the Taylor strategy (i.e., equations (35) and (36) are satisfied) and the money growth rule parameter, $\rho$, satisfies $\rho < \min\{1, \gamma\}$, then a unique $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium exists, with $F$ given by*

*(i) if $h_{t-1}$ has the property that the inflation monitoring range has never been violated, then $F$ takes the following form:*

$$F\left(h_{t-1}, \pi_{i,t}^{b}\right) = \begin{cases} \left[1 - \frac{\phi}{\gamma}(\gamma + \psi)\right]\pi_{i,t}^{b} & \pi_{i,t}^{b} \in [\pi_l, \pi_u] \\ \pi_{i,t}^{b} - \frac{\gamma+\psi}{\gamma-1}\left[\min\left\{R_l, \phi\pi_{i,t}^{b}\right\} - \mu\right] & \pi_{i,t}^{b} \notin [\pi_l, \pi_u], \end{cases} \tag{37}$$

*(ii) if $h_{t-1}$ has the property that the inflation monitoring range has been violated at least once, then $F$ takes the following form:*

$$F\left(h_{t-1}, \pi_{i,t}^{b}\right) = \begin{cases} \left(1 - \frac{\gamma+\psi}{1-\rho}\right)\pi_{i,t}^{b} + \frac{\gamma+\psi}{1-\rho}\left(\mu + \ln\frac{\bar{M}_{t-1}}{P_{t-1}}\right) & \pi_{i,t}^{b} \in D(h_{t-1}) \\ \left(\frac{\rho+\psi}{\rho-\gamma}\right)\pi_{i,t}^{b} + \frac{\gamma+\psi}{\gamma-\rho}\left[\mu + \ln\frac{\bar{M}_{t-1}}{P_{t-1}} + \mu - R_l\right] & \text{otherwise} \end{cases}, \tag{38}$$

*where*

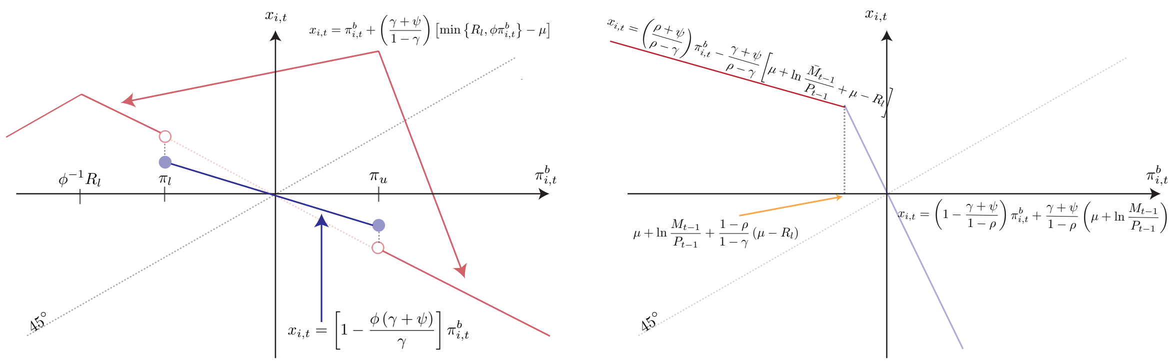
$$D(h_{t-1}) \equiv \left\{\pi_{i,t}^{b} : \frac{1-\gamma}{1-\rho}\left(\mu + \ln\frac{\bar{M}_{t-1}}{P_{t-1}} - \pi_{i,t}^{b}\right) + \mu > R_l\right\} \tag{39}$$

*and $D\left(h_{t-1}\right)$ corresponds to the values of $\pi^b_{i,t}$ having the property that the nominal interest rate in the $\left(h_{t-1}, \pi^b_{i,t}\right)$ continuation equilibrium is positive.*

The proof of Parts (i) and (ii) appear in Sections 4.1.4 and 4.1.5, respectively.[41] The graph of $F$ in equations (37) and (38), for the indicated parameter values, is displayed in Figure 3. The figure

Figure 3: Best Response Function, $\mu = 0, \gamma > 1, \frac{\bar{M}_{t-1}}{P_{t-1}} = 1, \rho = 0$

(a) Taylor Rule Regime (b) Money Growth Regime




Note: panel (a) refers to histories, $h_{t-1}$, in which the inflation monitoring range has never been violated. Panel (b) refers to histories, $h_{t-1}$, in which the inflation monitoring range has been violated at least once.

in each panel indicates that the best response function crosses the 45 degree line exactly once, at the origin.[42] This crossing, or fixed point, corresponds to the unique Nash equilibrium: if every firm had the same Nash belief about what others do, then the actions of all the firms would validate that belief. The existence and uniqueness of the Nash equilibrium in equation (37) does not depend on the values of the parameters, as long as they satisfy equations (35) and (36). Section 4.2 below establishes existence and uniqueness of the fixed point of $F\left(h_{t-1}, \cdot\right)$ for general $h_{t-1}$.

#### 4.1.4 $\left(h_{t-1}, \pi^b_{i,t}\right)$ Continuation Equilibria: $h_{t-1}$ in Which the Monitoring Range Has Never Been Violated

We now consider $F$ for $h_{t-1}$ in which the Taylor rule is in place in period $t$ (see Figure 3a ). Because of our symmetry assumption about beliefs, equation (13) implies that $P_{i,t} = p^b_{i,t}$ in the continuation equilibrium. It follows that the third entry in $a_{i,t}$, which is period $t$ scaled and logged inflation, $\pi_{i,t}$, has the property,

$$\pi_{i,t} = \pi^b_{i,t}. \tag{40}$$

Here, $\pi^b_{i,t}$ is defined in equation (30). Turning to the wage rate, note that,

$$W_{i,t} = P_{i,t}\frac{W_{i,t}}{P_{i,t}} = p^b_{i,t}\left(c_{i,t}\right)^{\psi+\gamma}, \tag{41}$$

[41] As we note in Section B.1 below, we leave one part of the proof for Online Appendix B.

[42] In the case of Panel (b), the unique Nash equilibrium would be at a point different from the origin if $\mu \neq 0$ or $\bar{M}_{t-1}/P_{t-1} \neq 1$.

where, $c_{i,t}$ and $W_{i,t}/P_{i,t}$ denote time $t$ consumption and the real wage in the continuation equilibrium implied by $\left(h_{t-1}, \pi_{i,t}^b\right)$. The second equality in equation (41) uses the fact that $P_{i,t} = p_{i,t}^b$ and substitutes out the real wage, using the household intratemporal Euler equation, equation (8). In addition, equation (41) uses the equilibrium condition that consumption equals labor, equation (19).[43]

After scaling the variables in equation (41) by $P_{t-1}\bar{\mu}^*$, using equation (30) and taking logs, we obtain

$$w_{i,t} = \pi_{i,t}^b + (\gamma + \psi) \ln c_{i,t}. \tag{42}$$

Evidently, to compute $w_{i,t}$ we require $c_{i,t}$.

We consider $\pi_{i,t}^b$ on two segments of the real line. These segments are motivated by the fact that $\pi_{i,t}^b$ maps directly into period $t$ (scaled and logged) inflation in a continuation equilibrium (see equation (40)). We refer to the segment, $\pi_{i,t}^b \notin [\pi_l, \pi_u]$, as *violation of the inflation monitoring range* and refer to the segment, $\pi_{i,t}^b \in [\pi_l, \pi_u]$, as *moderate inflation*.

*Violation of the inflation monitoring range*

Because the inflation monitoring range is violated in period $t$, period $t+1$ is the first period of the money growth regime. According to Definition 2 the gross money growth rate, $\bar{\mu}$, is greater than unity, so that Lemma 1 guarantees existence and uniqueness of $(a_{i,t+s})_{s=1}^{\infty}$. To obtain the remaining elements of $a_{i,t}$ it is useful to note that, according to Lemma 1, $c_{i,t+1} = 1$ and $\bar{M}_{i,t+1} = P_{i,t+1}$. Dividing the latter by the period $t$ equilibrium cash constraint, (25), $\bar{M}_{i,t} = c_{i,t}P_{i,t}$, we obtain[44]

$$\bar{\mu} = \frac{\bar{\pi}_{i,t+1}}{c_{i,t}}. \tag{43}$$

Here, $\bar{\pi}_{i,t+1}$ denotes the gross inflation rate from $t$ to $t+1$ in the continuation equilibrium associated with $\left(h_{t-1}, \pi_{i,t}^b\right)$. Substituting equation (43) into the household's intertemporal Euler equation, (9), we obtain $c_{i,t} = \left(\beta/\bar{\mu}\bar{R}_{i,t}\right)^{1/(1-\gamma)}$. Because $h_{t-1}$ has the property that the inflation monitoring range has not been violated in the past, the Taylor rule, (17), is in operation in period $t$. Substituting the Taylor rule into the latter equation and rearranging, we obtain, after scaling and taking logs,

$$\ln c_{i,t} = \frac{1}{1-\gamma} \left[\min\left\{R_l, \phi\pi_{i,t}^b\right\} - \mu\right], \tag{44}$$

where $\mu$ is the scaled and logged value of $\bar{\mu}$. In equation (44) we used the fact that inflation, $\pi_{i,t}$, in the continuation equilibrium is $\pi_{i,t}^b$. Substituting from equation (44) into equation (42), we obtain:

$$w_{i,t} = \pi_{i,t}^b - \left(\frac{\gamma + \psi}{\gamma - 1}\right) \left[\min\left\{R_l, \phi\pi_{i,t}^b\right\} - \mu\right] = F\left(h_{t-1}, \pi_{i,t}^b\right). \tag{45}$$

[43] Our symmetry assumption on beliefs plays a role in equation (41) by ruling out a price dispersion term that would appear otherwise.

[44] Note that we assume the cash constraint is satisfied as a strict equality, regardless of whether or not the interest rate is at the zero lower bound. When the interest rate is positive, equality reflects reflects household optimality, equation (10). When the interest rate is zero, equality reflects our assumption that in that case, monetary policy sets the aggregate money stock equal to the current nominal value of aggregate consumption.

This completes our discussion of the mapping from $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ to the continuation equilibrium for the case in which the inflation monitoring range is violated for the first time in period $t$. Equations (44) and (45) provide explicit expressions for $c_{i,t}$ and $w_{i,t}$ in the $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium. The other variables in the $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium are straightforward to derive.

There is a point of contact in our environment between models in which (i) prices are flexible and monetary policy is neutral and (ii) sticky price models. In the competitive equilibrium of our model, real variables are determined independent of monetary policy. By contrast, from the perspective of an intermediate good agent contemplating different $\pi_{i,t}^{b}$'s, when $\gamma > 1$ the world behaves in period $t$ like a simple sticky price New (or, old) Keynesian model, in which monetary policy is not neutral. To see this, consider the $i^{th}$ intermediate good firm contemplating a lower (scaled and logged) price set by other firms, $\pi_{i,t}^{b}$, in the range where the zero lower bound on the interest rate is not binding, $\pi_{i,t}^{b} > R_l/\phi$ (see Figure 3a). The firm in effect contemplates how the $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium changes with an exogenous decrease in the aggregate price level, ignoring the equilibrium condition, (15).[45] There are two channels by which the nominal wage in the continuation equilibrium changes in response to a change in $\pi_{i,t}^{b}$. The first channel is an *inflation channel.* In the continuation equilibrium inflation, $\pi_{i,t}$, falls by the same amount as $\pi_{i,t}^{b}$ because of equation (13) and our symmetry assumption on beliefs. The inflation channel is captured by the first term on the right of the equality in equation (45). It reflects that, holding the real wage constant, the nominal wage falls by the same amount, in log terms, as inflation. The *interest rate channel* reflects that a cut in $\pi_{i,t}$ results in a fall in the nominal interest rate, via the Taylor rule. Other things the same, this cut in the interest rate has a direct positive impact on consumption via the Euler equation, (9). This direct impact on consumption of the interest rate cut is amplified by general equilibrium effects when $\gamma > 1$. In particular, the initial rise in consumption raises $\pi_{i,t+1}$ (see equation (43)). This reduces the real interest rate, triggering a secondary positive effect on consumption, which raises $\pi_{i,t+1}$ further, and so on. With $\gamma > 1$ each successive general equilibrium jump in consumption is smaller and the sum of the effects is finite. Thus, a fall in $\pi_{i,t}^{b}$ triggers a jump in $c_{i,t}$.[46] The resulting rise in the demand for labor produces a rise in the equilibrium (scaled and logged) real wage (see the second term after first equality in equation (45)). Because the interest rate effect on the real wage dominates the inflation effect of a fall in $\pi_{i,t}^{b}$, the nominal wage rate rises. The $i^{th}$ firm best-responds by raising the price that it sets. This is why the $i^{th}$ firm's best response function has a negative slope when $\gamma > 1$.[47]

Now suppose that the firm contemplates a reduction in $\pi_{i,t}^{b}$ when the zero lower bound on the interest rate is binding, $\phi \pi_{i,t}^{b} < R_l$. With the nominal interest rate at its lower bound, interest rate

[45]In this section, we allow for arbitrary $\pi_{i,t}^{b}$ and we temporarily ignore any restrictions that optimizing behavior by other agents may imply for $\pi_{i,t}^{b}$. We consider such restrictions in Section 5 below.

[46]The sum of the direct and indirect effects referred to in the text can be expressed as a geometric sum involving $1/\gamma$. It can be verified that the term, $1/(1-\gamma)$ in equation (44) can be interpreted as reflecting the application of the geometric sum formula.

[47]It is easy to verify that when $\gamma > 1$, $1 - \phi(\gamma + \psi)/(\gamma - 1) < 0$. Later, we explain why the model is not interesting if $0 < \gamma \leq 1$. Throughout the analysis, we assume $\psi \geq 0$. We also assume $\phi > 1$ except when we consider consider the special case, $\pi_l = \pi_u = \mu = 0$,as in Proposition 3.

channel described in the previous paragraph is shut down. With only the inflation channel operative, the best response function is parallel to the 45° line (see equation (45)). Reductions in $\pi_{i,t}^b$ when the interest rate is at its lower bound have no impact on real wage, real interest rate, consumption and employment.

We have carefully discussed the case, $\gamma > 1$. When $\gamma < 1$ the $\left(h_{t-1}, \pi_{i,t}^b\right)$ continuation equilibrium is still well defined and the best response function continues to have a unique fixed point at zero. However, the global behavior of the best response function is qualitatively very different. For example, $F\left(h_{t-1}, \pi_{i,t}^b\right)$ is strictly increasing in $\pi_{i,t}^b$ when $\gamma < 1$.[48] In Section 5 we will argue that $\gamma < 1$ is economically uninteresting in this model.

*Moderate inflation*

The analysis of moderate inflation in period $t$ is qualitatively similar to the case just considered when $\gamma > 1$ (again, see Figure 3a). The difference is that here, period $t+1$ inflation in the continuation equilibrium is perfectly anchored at its desired level of zero (thus, equation (45) does not enter the analysis). The reason is that the Taylor rule will be in operation in $t+1$, which implies from Proposition 2 that $\pi_{i,t+1} = 0$ and $c_{i,t+1} = 1$. With $\pi_{i,t+1}$ fixed in this way, the best response function does not exhibit the sensitivity to whether $\gamma$ is greater or less than unity that we see in equation 5.

Substituting $\pi_{i,t+1} = 0$ and $c_{i,t+1} = 1$ into the period $t$ household intertemporal Euler equation, (9), we obtain (after scaling and logging),

$$-\gamma \ln c_{i,t} = R_{i,t} = \phi \pi_{i,t}^b, \tag{46}$$

after making use of the Taylor rule, (17). Substituting into equation (42) and collecting terms, we obtain:

$$w_{i,t} = \left[1 - \frac{\phi}{\gamma}(\gamma + \psi)\right] \pi_{i,t}^b = F\left(h_{t-1}, \pi_{i,t}^b\right). \tag{47}$$

For the range of $\pi_{i,t}^b$'s considered here, the zero lower bound is not encountered. Apart from that, the analysis is qualitatively the same as in the version of previous discussion with $\gamma > 1$. There is a quantitative difference because period $t+1$ inflation is anchored at its desired level when the monitoring range is not violated in the current period. This weakens the interest rare channel by eliminating what we called the general equilibrium effects associated with a change in consumption. Still, we can see in equation (47) that the interest rate channel still dominates the inflation channel, because $\phi(\gamma + \psi)/\gamma > 1$. This completes our discussion of the mapping from $\left(h_{t-1}, \pi_t^b\right)$ to the continuation equilibrium for the moderate inflation case.

[48] One way to understand the dramatic impact of $\gamma$ is to recall our geometric sum argument in the case, $\gamma > 1$. The standard geometric sum formula underlies the coefficient, $(1/\gamma)/(1 - 1/\gamma)$, in equation (45). The geometric sum formula 'works' whether the infinite sum converges or not, but it switches sign from positive to negative when it is misused. In fact, the infinite sum formula was only used in the text to convey intuition. The mathematical computation of the continuation equilibrium did not use that formula. In Section 5.5 we argue that the model is not economically interesting when $\gamma < 1$ (see also Proposition 11, which rules $\gamma < 1$ out).

#### 4.1.5 $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ Continuation Equilibria: $h_{t-1}$ in Which the Monitoring Range Has Been Violated

We now consider histories, $h_{t-1}$, in which the economy is in the money growth regime from time $t$ and into the future (see Figure 3b). The values of $P_{i,t}$ and $\pi_{i,t}$ in the $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium follow immediately from equation (13) and the symmetry assumption on beliefs. Lemma 1 describes the part of the $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium beginning in $t+1$, $(a_{i,t+s})_{s=1}^{\infty}$.[49] In particular, $c_{i,t+1} = 1$ and $P_{i,t+1} = \bar{M}_{i,t}\bar{\mu}$,using the fact that the cash constraint is binding in $t+1$ and using the monetary policy rule (see Lemma 1). Then,

$$\ln c_{i,t+1} = 0,\ P_{i,t+1} = \bar{M}_{i,t}\bar{\mu},\ \pi_{i,t+1} = \mu + \ln \frac{\bar{M}_{i,t}}{P_{i,t}},\ \ln \frac{\bar{M}_{i,t}}{P_{i,t}} = \mu + \rho \ln c_{i,t} - \pi_{i,t}^{b} + \ln \frac{\bar{M}_{t-1}}{P_{t-1}}, \tag{48}$$

where the fourth term is the period $t$ money rule after taking logs and rearranging. We do not include the $i$ subscript on lagged real balances because they are contained in $h_{t-1}$, which is public information.

We suppose that the zero lower bound on the interest rate in the $\left(h_{t-1}, \pi_{i,t}^{b}\right)$ continuation equilibrium is strictly non-binding, i.e., $R_{i,t} > R_l$. Then, the period $t$ cash constraint (10) is binding, which, when combined with the loan market clearing condition (see equation (18)), implies $\ln c_{i,t} = \bar{M}_{i,t}/P_{i,t}$. Substituting out for $\bar{M}_{i,t}/P_{i,t}$ using the monetary policy rule and rearranging, we obtain:

$$\ln c_{i,t} = \frac{1}{1-\rho}\left[\mu + \ln \frac{\bar{M}_{t-1}}{P_{t-1}} - \pi_{i,t}^{b}\right]. \tag{49}$$

Using (49) to substitute out for $\ln c_{i,t}$ in equation (42), we conclude that the (scaled and logged) wage is given by

$$F\left(h_{t-1}, \pi_{i,t}^{b}\right) = \pi_{i,t}^{b} + \frac{\gamma+\psi}{1-\rho}\left(\mu + \ln \frac{\bar{M}_{t-1}}{P_{t-1}} - \pi_{i,t}^{b}\right). \tag{50}$$

We now derive a necessary and sufficient condition on $\pi_{i,t}^{b}$ to guarantee that the zero lower bound on the interest rate is not violated. The household intertemporal Euler equation (9) implies, using (48) and (49),

$$R_{i,t} = \frac{1-\gamma}{1-\rho}\left[\mu + \ln \frac{\bar{M}_{t-1}}{P_{t-1}} - \pi_{i,t}^{b}\right] + \mu.$$

This expression shows that $R_{i,t}$ is strictly greater than $R_l$ if, and only if, $\pi_{i,t} \in D\left(h_{t-1}\right)$ as defined in equation (39).

Now consider the case in which the zero lower bound binds, namely, $\pi_{i,t} \notin D\left(h_{t-1}\right)$. The Euler equation (9) in the zero lower bound is, after rearranging and using equation (48),

[49] Period $t+1$ corresponds to period 0 in the lemma.

$$\ln c_{i,t} = \frac{1}{\gamma - \rho}\left[\mu - \pi^b_{i,t} + \ln \frac{\bar{M}_{t-1}}{P_{t-1}} + \mu - R_l\right]. \tag{51}$$

Using equation (42), the time $t$ wage in the $\left(h_{t-1}, \pi^b_{i,t}\right)$ continuation equilibrium is given by

$$F\left(h_{t-1}, \pi^b_{i,t}\right) = \left(\frac{\rho + \psi}{\rho - \gamma}\right)\pi^b_{i,t} + \frac{\gamma + \psi}{\gamma - \rho}\left[\mu + \ln \frac{\bar{M}_{t-1}}{P_{t-1}} + \mu - R_l\right]. \tag{52}$$

In Online Appendix B.1, we show that the household cash constraint is satisfied in period $t$ in the $\left(h_{t-1}, \pi^b_{i,t}\right)$ continuation equilibrium constructed above.

As in Section 4.1.4, the intuition behind $F$ is very simple, and we describe it in loose terms here. To simplify the discussion, we assume that $\rho = 0$.[50] The 'model' used by the $i^{th}$ intermediate good firm to construct the continuation equilibrium corresponding to a particular belief, $\pi^b_{i,t}$, resembles a standard sticky price model. The model is essentially static, because the firm assumes that starting in period $t+1$ all variables jump to their competitive equilibrium continuation values. We first consider the case in which the zero lower bound on the interest rate is not binding, i.e., $\pi^b_{i,t} \in D\left(h_{t-1}\right)$. Suppose the $i^{th}$ intermediate good firm contemplates a reduction in the (scaled and logged) price set by other firms, $\pi^b_{i,t}$. As in Section 4.1.4, this variation in beliefs operates through two channels. The first channel, the inflation channel, drives the nominal wage down one-for-one in log terms (see the first term after the equality in equation (50)). The second channel operates through the interest rate. With the time $t$ money stock a function of $h_{t-1}$, a fall in the time $t$ aggregate price level raises time $t$ real balances. The household would not be content with its previous deposit decision because that now entails carrying non-interest bearing cash in excess of what is needed for consumption. So, the intermediate good producer imagines the household would respond to the drop in $\pi^b_{i,t}$ by increasing deposits, $d_{i,t}$, in equation (7). This increased supply in the loan market drives the interest rate down and encourages households to raise consumption. Another factor that raises consumption derives from the fact that the period $t+1$ price level is determined by $h_{t-1}$ when $\rho = 0$ (see equation (48)).[51] So, the fall in the period $t$ price level raises $\pi_{i,t+1}$,thus reducing the real interest rate even more. The resulting sharp increase in the demand for labor raises the real wage. The interest rate channel is stronger than the inflation channel when $\gamma > 1$, so a fall in $\pi^b_{i,t}$ raises the nominal wage in the continuation equilibrium (again, see equation (50)). The $i^{th}$ firm best-responds by raising its own price level. Further reductions in $\pi^b_{i,t}$ eventually make the zero lower bound on the interest rate bind. The positive impact of such reductions in $\pi^b_{i,t}$ on consumption are now somewhat smaller because only the inflation channel is operative and the interest rate channel is gone. Thus, reductions in $\pi^b_{i,t}$ in the region of the zero lower bound raise consumption and the wage, but not as strongly as when

[50] This parameter will play an important role in the analysis in the next section, and will be discussed there.

[51] According to equation (48), $P_{i,t+1}$ is a function of $\bar{M}_{i,t}$. When $\rho = 0$ then $\bar{M}_{i,t} = \bar{\mu}\bar{M}_{t-1}$ (see Lemma 1). The object, $\bar{M}_{t-1}$, can be recovered from $h_{t-1}$.

$\pi^b_{i,t} \in D(h_{t-1})$.[52]

Although we think of the money growth rule and the interest rate rule as part of one regime, it is interesting how similarly the economy behaves under each rule. In each case, the mapping from beliefs about inflation to actual price decisions is heavily influenced by an interest rate channel. In this respect, our model provides an example of an argument developed in Taylor (1999). He shows that monetary arrangements that are apparently very different can all have the same property that a rise in the nominal interest rate acts as device to moderate (or, even reverse) a rise in inflation. In his discussion, Taylor (1999) includes not only the types of interest rate and money growth rules discussed here, but also the operation of the specie-flow mechanism under the Gold standard. We return to this theme in Section 6.

## 4.2 Strategy Equilibrium

It is convenient to define the following equilibrium concept:

**Definition 5.** An $h_{t-1}$ *continuation equilibrium* is a sequence, $(a_{t+s})_{s=0}^{\infty}$, that satisfies all time $t+s$ equilibrium conditions, $s \geq 0$.

We now focus on Nash equilibria, in which all intermediate good firms have the same beliefs. So, we can drop the subscript, $i$. A Nash equilibrium belief is a $\pi^b_t$ with the property, $\pi^b_t = F\left(h_{t-1}, \pi^b_t\right)$. We denote such a $\pi^b_t$ by $\pi^b(h_{t-1})$. In principle, $\pi^b(h_{t-1})$ can be empty or it can be a set-valued function of $h_{t-1}$. In the next proposition, we show that $\pi^b(h_{t-1})$ is single-valued for all $h_{t-1}$. In an addition, the following result establishes properties of an $h_{t-1}$ continuation equilibrium, and its relation to an $\left(h_{t-1}, \pi^b_t\right)$ continuation equilibrium:

**Proposition 5.** *If $\gamma > 0$, $\gamma \neq 1$, and $\rho < \min\{\gamma, 1\}$, our model has the following properties:*

*(i) for every history, $h_{t-1}$, there exists a unique continuation equilibrium,*

*(ii) an $h_{t-1}$ continuation equilibrium is an $\left(h_{t-1}, \pi^b_t\right)$ continuation equilibrium iff $\pi^b_t \in \pi^b(h_{t-1})$,*

*(iii) the mapping, $\pi^b(h_{t-1})$, is single-valued for each $h_{t-1}$,*

*(iv) for histories, $h_{t-1}$, in which the inflation monitoring range has never been violated, the unique fixed point of $F$ is $\pi^b_t = 0$.*

[52] The best response function has slope, $-\psi/\gamma$, when the zero lower bound is binding and it has slope $1-(\gamma+\psi)$, when it is not binding. To see that it is flatter when the zero lower bound is binding, we show that $-\psi/\gamma > 1-(\gamma+\psi)$. To see that this is the case, add $\psi/\gamma$ to both sides and collect terms in $\gamma+\psi$, so that the previous inequality is equivalent to $0 > -(\psi+\gamma)\left(1-\frac{1}{\gamma}\right)$, which is true when $\gamma > 1$. The same result is true for the values of $\rho$ allowed in the monetary growth rule (see Definition 2). We explain the economic rationale the restriction on $\rho$ in section 5.5.

For the proof, see Online Appendix B.2.

The above properties of continuation equilibria are crucial for an equilibrium concept that plays an important role in our analysis:

**Definition 6.** A competitive equilibrium is a *Strategy Equilibrium* if the model has the following two properties: (i) for each possible history, $h_{t-1}$, there is a well-defined $h_{t-1}$ continuation equilibrium; and (ii) for each possible $\left(h_{t-1}, \pi_t^b\right)$ with $\pi_t^b > -\infty$, there is a well defined $\left(h_{t-1}, \pi_t^b\right)$ continuation equilibrium.

Our Strategy Equilibrium is substantively not different from the 'sophisticated equilibrium' concept used in Atkeson et al. (2010). It has a different name because the equilibrium objects in Atkeson et al. (2010) are functions while they are sequences here. We use the name, 'strategy equilibrium' even though all the agents in our model are atomistic and they know their decisions have no impact on economic aggregates. Still, the reasoning of our intermediate good firms has a strategic aspect to it. They contemplate different beliefs about the actions of others and the impact that others' actions have on economic aggregates. For an equilibrium to be a strategy equilibrium, it is necessary that the continuation equilibrium associated with each belief is well-defined.

It is easily established Definition 6 is satisfied by our model:

**Proposition 6.** *The Taylor rule equilibrium with an escape clause is a strategy equilibrium for* $\gamma > 0$, $\gamma \neq 1$ *and* $\rho < \min\{\gamma, 1\}$.

*Proof.* The result is immediate by Proposition 4, (i) and Proposition 5, (i). □

It is also interesting to construct the best response function, $F$, under the under the zero monitoring range policy in Definition 3, with $\pi_l = \pi_u = \mu = 0$ and $\phi \neq 0$. In this case, we know from Proposition 3 that the desired equilibrium is the only competitive equilibrium. Consider the case, $h_{t-1}$, in which the inflation monitoring range has never been violated. The best response function, $F$, in this case is the second expression in equation (37) for all $\pi_{i,t}^b$. In the case of $h_{t-1}$ in which there has been a violation of the monitoring range, the function, $F$, is given in equations (38) and (39). This establishes that the equilibrium for this monetary policy is a strategy equilibrium. Formally,

**Proposition 7.** *Suppose (i) monetary policy is governed by the zero monitoring range strategy, and (ii)* $\gamma > 0$, $\gamma \neq 1$ *and* $\rho < \min\{\gamma, 1\}$. *Then, the unique equilibrium is a strategy equilibrium.*

This proposition was established in Atkeson et al. (2010). The economic intuition underlying $F$ in the case of the policy considered here is essentially the same as it is for the case, $\pi_l < \pi_u$ discussed in Sections 4.1.4 and 4.1.5. This is obviously true for histories, $h_{t-1}$, in which there has been a past deviation, because the escape clause is irrelevant in this type of history. It is also true for histories, $h_{t-1}$, in which there never has been a deviation. For those histories, the best response function is the function graphed in Figure 3a, where the middle segment is shrunk to a singleton at the origin. Whether that middle segment is simply a point at the origin, or has positive length, as in Figure 3a, has no significant impact on the underlying economic intuition a described above. So, the policy referred to in Proposition 7 is yet one more example of Taylor (1999)'s observation that seemingly different policy arrangements could nevertheless assign an important role to the interest rate for stabilizing inflation. This observation may at first seem surprising, since Proposition 7 allows for $\phi > 0$ but arbitrarily small. In Section 5, we argue that $\phi < 0$ is not economically interesting.

### 4.3 Observations About the Strategy Equilibrium

The requirement that an equilibrium be a strategy equilibrium places restrictions on the model parameters (see Propositions 6 and 7). We illustrate this with two examples in which the parameter restrictions are not satisfied and the equilibrium fails to be a strategy equilibrium. Our first example considers the log specification of utility, $\gamma = 1$. Our second example considers a particular set of restrictions which in effect convert our economy into an endowment economy. Consistent with the results in Cochrane (2011), the unique competitive equilibrium corresponding to the Taylor strategy is not a strategy equilibrium in the endowment economy case. The reason is that the Taylor strategy implements the desired equilibrium by changing the real interest rate in the event that the economy leaves the desired allocations. In an endowment economy such a policy is not a strategy equilibrium, because there exists no monetary policy that can change the real interest rate. Of course, this logic does not apply to our model because it is a production economy in which it is possible for monetary policy to change the real interest rate.

First, consider the following proposition:

**Proposition 8.** *Suppose $\gamma = 1$. The competitive equilibrium associated with the Taylor strategy is not a strategy equilibrium.*

*Proof.* It suffices to show that there exists an $\left(h_{t-1}, \pi_t^b\right)$ for which there does not exist a continuation equilibrium. Thus, suppose $h_{t-1}$ is a history in which the monitoring range has never been violated. Consider $\pi_t^b \notin [\pi_l, \pi_u]$. In period $t+1$ the escape clause will be invoked and the economy will switch to the money growth regime. By Lemma 1, we have

$$\ln c_{t+1} = 0. \tag{53}$$

The cash constraint is binding in periods $t$ and $t+1$, so equation (43) is satisfied and

$$\pi_{t+1} = \ln c_t + \mu, \tag{54}$$

after scaling and logging. Taking into account that in the continuation equilibrium, $\pi_t = \pi_t^b \notin [\pi_l, \pi_u]$, it follows that in the scaled and logged Taylor rule,

$$R_t = \phi \pi_t \notin [\pi_l, \pi_u] , \tag{55}$$

because $\phi > 1$. After scaling and logging the household intertemporal Euler equation, (9),

$$\ln c_t = \ln c_{t+1} - [R_t - \pi_{t+1}] \,. \tag{56}$$

Substituting equations (53), (54), into equation (56) we obtain

$$R_t = \mu \in [\pi_l, \pi_u] \,, \tag{57}$$

which contradicts equation (55). □

We infer from this example that the Taylor strategy would not be an effective way to exclude undesired equilibria if $\gamma = 1$. A different policy would be necessary to implement the desired equilibrium in this case.

Another closely related example is the version of the model in which $\gamma > 0$ and in which households supply labor inelastically, $l_t = 1$. In this case, the model economy is effectively an endowment economy, as in Cochrane (2011). By essentially the same logic as in the $\gamma = 1$ case, the Taylor strategy is not a strategy equilibrium.[53] So, using the escape clause to eliminate undesired equilibria in an endowment economy is, once again, uninteresting.[54] If the model economy does not have $\gamma = 1$ or is not an endowment economy, then equations (55) and (56) do not imply a contradiction. A suitable adjustment in $\ln c_t$ can reconcile the two equations. Indeed, Proposition 6 implies that such an adjustment would occur in our model (which is not an endowment economy) with $\gamma \neq 1$. The economics of the adjustment in $\ln c_t$ will be discussed in the next section.

# 5 Rationalizable Implementation

This section derives sufficient conditions and a necessary condition for policy to uniquely implement the desired equilibrium. We begin, in the first subsection, by providing a standard definition of rationalizability and relating it to the concept of iterated deletion discussed in Section 2. The second subsection motivates a refinement of the concept of rationalizability, *robust rationalizability*, which we use in our analysis. The third subsection provides sufficient conditions for rationalizability, as well as a necessary condition. These conditions are the formal representation of the 'leaning against the wind, but not too aggressively' principle discussed in Section 2. To establish our conditions for rationalizability, we must generalize existing results in the literature (see Desgranges (2014)). To our knowledge, the literature does not address models with best response functions like ours. The

[53] The proof by contradiction in the endowment economy requires a minor adjustment to the argument in the text. The argument leading to equation (55) is unchanged. Equation (57) also holds, but in the endowment economy it is derived from the fact, $\ln c_t = \ln l_t = 0$, for all $t$. In particular, $\pi_{t+1} = \mu$ by equation (54) and $R_t = \pi_{t+1}$ by equation (56) implies equation (57).

[54] As noted in the introduction, this observation was first made in Cochrane (2011). The endowment economy assumption is made in Cochrane (2011, p. 574) and our proof by contradiction argument (see Footnote 53) coincides with the argument in Cochrane (2011, p. 584).

best response function in our model lacks differentiability, continuity and has a segment which has slope unity (see Figure 3). The fourth and fifth subsections make use of our rationalizability result to establish sufficient conditions for unique implementation of the desired equilibrium. We also provide a necessary condition. We obtain these results for the Taylor strategy, defined in Definition 2, and also for the zero monitoring range strategy defined in Definition 3.

Our analysis in this paper follows Atkeson et al. (2010) by focusing on one-period deviations from an arbitrary history, $h_{t-1}$. We have extended the analysis to multi-period deviations.[55] For space reasons, that analysis is reported in Online Appendix C.6. Conceptually, the extension to multiperiod deviations is straightforward. However, because of the segmented structure of our best response function, the type of global analysis that we do in the body of the paper is less tractable. So, Online Appendix C.6 proceeds in the style of Evans et al. (2018) by doing the multi-period analysis locally to the unique Nash equilibrium of the relevant game, for arbitrary $h_{t-1}$. We obtain a set of necessary and sufficient conditions for the local, multi-period analog of unique implementation. Those conditions are slightly weaker than the conditions in the one-period setting.[56] We do so for both monetary policies studied in the paper: the Taylor strategy and the zero monitoring range strategy.

## 5.1 Rationalizability

In our environment, intermediate good firms choose an action which is a best response to a belief about the price set by other firms.[57] We are interested in determining what belief would be reasonable, or *rationalizable*, for an individual firm under CK. To this end, it is useful to first define the *set of possible beliefs*, $\Pi_0 = (-\infty, \infty)$. These are the beliefs for which there exists a well defined continuation equilibrium (see the discussion before equation (32)). We also define a *justified* belief:

**Definition 7.** An agent's belief, $\pi_{i,t}^b$, about the actions of others is *justified* if (i) $\pi_{i,t}^b \in \Pi_0$ and (ii) $\pi_{i,t}^b$ is itself a best response to some belief in $\Pi_0$ by everyone else.

Rationalizability has the following recursive definition:

**Definition 8.** We say that a belief, $\pi_{i,t}^b \in \Pi_0$, is *rationalizable* if (i) $\pi_{i,t}^b$ is justified and (ii) the belief to which $\pi_{i,t}^b$ is a best response is also justified.

It is well known that there are two ways to determine the rationalizability of a given belief, *chain of justification* and *iterated deletion*. Both play an important role in our analysis, and so we define them formally here.

[55] We do not appeal to the 'one-shot-deviation principle' in game theory (see e.g., Fudenberg and Tirole (Section 4.2, 1991)) because our setting is different from the environment in which that principle is derived. In that environment there is a large agent who makes a decision which affects the evolution of the economy (see, e.g., Chari and Kehoe (1990)). In our environment all agents are atomistic, so that their own decisions are inconsequential.

[56] See equations (63) and (67) below.

[57] We continue to maintain our symmetry assumption on beliefs.

Given the recursive nature of Definition 8, a belief, $\pi_{i,t}^b$, is rationalizable if it is possible to construct a *chain of justification* for $\pi_{i,t}^b$. To define this chain we first construct a sequence of sets. The first element in the sequence is the singleton, $\pi_{i,t}^b$. The $l^{th}$ element in the sequence, for $l > 1$, is $F^{-l}\left(h_{t-1}, \pi_{i,t}^b\right)$:[58]

$$\left\{\pi_{i,t}^b, F^{-1}\left(h_{t-1}, \pi_{i,t}^b\right), ..., F^{-l}\left(h_{t-1}, \pi_{i,t}^b\right), ...\right\}. \tag{58}$$

A chain of justification for the belief, $\pi_{i,t}^b$, is a sequence of beliefs in equation (58) such that each element belongs to $\Pi_0$. In there exists such a sequence, then $\pi_{i,t}^b$ is rationalizable according to Definition 8. Rationalizability can fail if $F^{-l}\left(h_{t-1}, \pi_{i,t}^b\right) \cap \Pi_0$ is empty for some $l$. This constructive method for verifying that $\pi_{i,t}^b$ is rationalizable is called the chain of justification method.

The iterated deletion method is the one used in the example in the Section 2. It identifies the entire set of rationalizable beliefs. As such, it represents a second method for determining whether a given belief is rationalizable. Suppose that the firm starts out with the set of possible beliefs, $\Pi_0$. Then, $F\left(h_{t-1}, \Pi_0\right)$ denotes the set of best responses associated with all possible beliefs, $\Pi_0$. Consider an element, $x_i \in \Pi_0$ such that $x_i \notin F\left(h_{t-1}, \Pi_0\right)$. The $i^{th}$ firm knows that there is no circumstance in which it would choose $x_i$ and it realizes that others would never choose $x_i$ either. So, the $i^{th}$ firm would drop $x_i$ from the set of possible beliefs, $\Pi_0$. That is, the $i^{th}$ individual would restrict its beliefs about what others do to $\Pi_1 = \Pi_0 \bigcap F\left(\Pi_0\right)$.[59]

This process can be repeated, leading to a sequence, $\Pi_{k+1} = \Pi_k \bigcap F\left(h_{t-1}, \Pi_k\right)$, for $k \geq 0$. Notice that by construction, the sequence is non-increasing, $\Pi_{k+1} \subseteq \Pi_k$ for $k \geq 0$. Consider the set of beliefs, $\pi_{i,t}^b \in \Pi_0$, such that they remain undeleted even as $k \to \infty$:

$$\Pi^*\left(\Pi_0; F\right) = \bigcap_{k=0}^{\infty} \Pi_k. \tag{59}$$

We specify $\Pi_0$ and $F$ as arguments of $\Pi^*$ because we will have occasion to evaluate the operator, $\Pi^*$, for sets other than, $\Pi_0$, and functions other than $F$. We do not include the argument, $h_{t-1}$, in $\Pi_k$ and $\Pi^*$ in order to simplify notation.

As is well known, the set, $\Pi^*\left(\Pi_0; F\right)$, coincides with the set of rationalizable beliefs in the sense of Definition 8. For completeness, we present a proof in Online Appendix C.1.[60]

For each possible $h_{t-1}$ we envision that every intermediate good firm constructs a set of rationalizable beliefs using either the chain of justification or the iterated deletion method, or even a combination of the two. This private mental process used by firms to restrict their beliefs (perhaps to a singleton) is called *eduction* (see Binmore (1987, page 184)).

[58] Here, $x \in F^{-1}\left(h_{t-1}, \pi_{i,t}^b\right)$ if $\pi_{i,t}^b = F\left(h_{t-1}, x\right)$. Also, $F^{-l}\left(h_{t-1}, \pi_{i,t}^b\right)$ is defined similarly for $l \geq 2$.

[59] The reason we define $\Pi_1 = \Pi_0 \bigcap F\left(\Pi_0\right)$ and not simply $\Pi_1 = F\left(\Pi_0\right)$ is because our analysis must be robust to situations in which $F\left(\Pi_0\right) \subsetneq \Pi_0$, in which case $F^2\left(\Pi_0\right)$ is not well defined.

[60] For a textbook treatment, see Mas-Colell et al. (1995, Section 8.C).

## 5.2 Rationalizability with Refinement

Our paper is about designing policy so that agents' expectations are uniquely anchored to the desired equilibrium. As it turns out, a technicality prevents achieving this kind of uniqueness in the model described up to now. The technicality is the assumption implicit in our analysis that intermediate good agents can discriminate between an exact zero and an infinitesimally small number. Intuitively, our refinement assumes that there exists an arbitrarily small positive number, such that agents cannot distinguish between it and zero. With this refinement, there is still a nontrivial policy design problem, but failure of expectations to be anchored only occurs for economically interesting reasons. The version of rationalizability which incorporates our refinement, *robust rationalizability*, is defined below.

To understand the motivation for our refinement, let $h_{t-1}$ be a history in which the inflation monitoring range has never been violated. Consider the rationalizability of a particular belief, $\pi_{i,t}^{b} < R_l/\phi$. This belief lies in the region bounded on the right by the kink point in the best response function where the zero lower bound on the interest rate is binding (see Figure 3a). For any such $\pi_{i,t}^{b}$, there is a unique chain of justification belonging to the set of sequences in equation (58). That chain is constructed by iterating on the inverse of the segment of $F$ that is parallel to the 45° line and has the property that the zero lower bound is binding.[61] If we exponentiate each element in that chain of justification, the resulting sequence corresponds to a sequence of beliefs for the $i^{th}$ intermediate good firm about the (scaled) prices set by others. The limiting belief in that sequence is not justified, because it corresponds to a belief that others set their price to zero. However, technically that limit is never actually reached. For this reason, all beliefs, $\pi_{i,t}^{b} < R_l/\phi$, are in fact rationalizable (see Definition 8). We find this result economically uninteresting. Although the limiting belief is never reached, the sequence of beliefs in the chain of justification does enter any arbitrarily small interval of zero in finite steps. So, rationalizability of beliefs, $\pi_{i,t}^{b} < R_l/\phi$, rests critically on the assumption that agents have the capacity to differentiate infinitesimally small numbers from an exact zero. We drop this assumption and replace it by the assumption that intermediate good firms have an $\varepsilon$ *cognitive impairment*. In particular, we assume that an intermediate good firm mentally records a number smaller than $\varepsilon > 0$ as an exact zero, where $\varepsilon$ can be arbitrarily small.[62]

[61] At least for small values of $l$, $F^{-l}\left(h_{t-1}, \pi_{i,t}^{b}\right)$, generally contains two elements. However, candidate chains of justification constructed using the greater of the two elements inevitably reach a point where no inverse exists. Such candidates are therefore not part of a chain of justification for $\pi_{i,t}^{b}$.

[62] For a recent discussion of limitations on the ability of people to distinguish numbers that are very close to each other, see Woodford (2019). For an early discussion in a decision-theoretic context, see Luce (1956). Interestingly computers have the same cognitive 'problem' because they can only register a finite set of numbers. For example, MATLAB registers positive numbers equal to $10^{-324}$ or less, as an exact zero. Computers also have a symmetric problem with large numbers (in MATLAB, numbers larger than $10^{309}$ are registered as $+\infty$). So, in principle we could adopt a refinement which posits a cognitive impairment in thinking about large numbers. For example, we could suppose that there exists an arbitrarily small $\delta$, with $\delta > 0$, such that $x > 1/\delta$ cannot be distinguished from $1/\delta$. Online Appendix C.3 studies such a refinement and its implications for on- and off- equilibrium paths in our model. In any case, our analysis in the body of this paper does not require any further refinement beyond the one discussed in the text. That is because, for model parameterizations that satisfy necessary and sufficient conditions for unique implementation, our environment places a natural upper bound on beliefs about inflation (see the discussion after

Under the assumption that an intermediate good firm is $\varepsilon$ impaired, the firm (mistakenly) concludes that the chain of beliefs discussed in the previous paragraph does in fact reach zero.[63] With this refinement, the best response function, $F$, changes to $F^{\varepsilon}$:

$$F^{\varepsilon}\left(h_{t-1}, \pi_{i,t}^{b}\right) \equiv \max\left\{F\left(h_{t-1}, \pi_{i,t}^{b}\right), \ln \varepsilon\right\}, \text{for } \pi_{i,t}^{b} \geq \ln \varepsilon, \tag{60}$$

for any $h_{t-1}$. When $\pi_{i,t}^{b} < \ln \varepsilon$, $F^{\varepsilon}\left(h_{t-1}, \pi_{i,t}^{b}\right)$ is not defined.[64] This is because an $\varepsilon$ impaired firm contemplating $\pi_{i,t}^{b} < \ln \varepsilon$ perceives that other firms set their price to zero. The firm's decision problem is not well-defined for such a belief because there does not exist a continuation equilibrium.[65] The rationale for the max operator in equation (60) is as follows. Recall that $F\left(h_{t-1}, \pi_{i,t}^{b}\right)$ corresponds to the (logged and scaled) nominal wage rate in the continuation equilibrium associated with $\left(h_{t-1}, \pi_{i,t}^{b}\right)$. Suppose that $F\left(h_{t-1}, \pi_{i,t}^{b}\right) < \ln \varepsilon$. The firm would in this case perceive the nominal wage to be zero. With a zero marginal cost of production the profit maximizing firm is led to set its price to the lowest positive value. In the absence of the cognitive impairment there is no solution to this problem.[66] However, with our cognitive impairment the firm would choose to set its price to the unique smallest price that it can perceive, which is greater than zero. That price is $\ln \varepsilon$. This explains why $x_i = \ln \varepsilon$ when $F\left(h_{t-1}, \pi_{i,t}^{b}\right) < \ln \varepsilon$.[67]

Let $\Pi_0^{\varepsilon}$ denote the set of $\varepsilon$ *possible beliefs*, $\pi_{i,t}^{b}$, about actions by others which has the property that a continuation equilibrium exists when intermediate good firms are $\varepsilon$ impaired. That is:

$$\Pi_0^{\varepsilon} = [\ln \varepsilon, \infty), \quad 0 < \varepsilon < e^{\pi^{b}(h_{t-1})}. \tag{61}$$

We think of $\varepsilon > 0$ as being arbitrarily small. Here, $\pi^{b}\left(h_{t-1}\right)$ is the unique fixed point of $F\left(h_{t-1}, \cdot\right)$, which is the version of the best response function in which agents have no cognitive impairment (see Proposition 5). Similarly, we define $\varepsilon$ *justifiability* and $\varepsilon$ *rationalizability* analogously to Definitions 7 and 8, respectively.

We now provide the formal definition of robust rationalizability. This definition formalizes the intuition described above, which is that our refinement does not commit to any particular value of $\varepsilon$, other than that $\varepsilon$ is small and positive. We begin by considering a specific value of $\varepsilon$ satisfying

Proposition 11 below). As a result, extending our refinement to large numbers would have no impact on the analysis of rationalizability.

[63] The situation is reminiscent of the literature on the amount of time, $T$, required for the neoclassical model to converge to its steady state. The mathematically correct answer is that convergence never occurs (i.e., there is no finite $T$ large enough so that the capital stock is mathematically equal to its steady state value). Practical analysts understand this, but they also understand that the amount of time needed to get within any $\varepsilon > 0$ region of the steady state is finite. So, in practice, one defines 'time to arrive in steady state' as the amount of time, $T_{\varepsilon}$, needed to enter some specified $\varepsilon$ neighborhood of steady state. Our refinement assumes that the agents in our model adopt this type of logic when examining a sequence of prices that converges to (but does not actually reach) zero.

[64] There are alternative representations of $\varepsilon$ impairment. For example, we could suppose that for $\varepsilon > 0$ but small, agents perceive $0 \leq x \leq \varepsilon$ as $x = \varepsilon$. This does not change our results.

[65] Recall the discussion before equation (32).

[66] Recall equation (12).

[67] Our conclusion implicitly makes use of the quasi-concavity of the intermediate good firm's objective function.

$0 < \varepsilon < e^{\pi^b(h_{t-1})}$. Let $\Pi^*(\Pi_0^\varepsilon, F^\varepsilon)$ denote the associated set of rationalizable beliefs.[68] For notational simplicity, we denote the correspondence from $\varepsilon$ to $\Pi^*(\Pi_0^\varepsilon, F^\varepsilon)$ by $\Pi^*(\varepsilon)$. The set, $\Pi^*(0)$, is the set of rationalizable beliefs in the sense of Definition 8. It is easy to verify that $\Pi^*(\varepsilon)$ is weakly increasing as $\varepsilon$ declines for all $\varepsilon \geq 0$.[69] It follows that the limit, $\Pi^{*,r}$, of $\Pi^*(\varepsilon)$ as $\varepsilon$ goes to zero is given by:

$$\Pi^{*,r} = \bigcup_{0<\varepsilon\leq e^{\pi^b(h_{t-1})}} \Pi^*(\varepsilon). \tag{62}$$

We suppress the dependence of $\Pi^{*,r}$ on $h_{t-1}$ to simplify notation. Rationalizability under our refinement is defined as follows:

**Definition 9.** We say that a belief, $\pi_{i,t}^b$, is *robustly rationalizable* if $\pi_{i,t}^b \in \Pi^{*,r}$, where $\Pi^{*,r}$ is defined in equation (62).

The fact that $\Pi^*(\varepsilon)$ is lower- but not upper- hemicontinuous at $\varepsilon = 0$ provides an alternative perspective on our refinement and explains our use of the adjective, robust, in Definition 9. Consider a robustly rationalizable belief, $\pi_{i,t}^b \in \Pi^{*,r}$. It follows from the weakly increasing property of $\Pi^*(\varepsilon)$, that $\pi_{i,t}^b \in \Pi^*(0)$. Thus, a belief, $\pi_{i,t}^b \in \Pi^{*,r}$, has the robustness property that $\pi_{i,t}^b$ is rationalizable whether or not the $i^{th}$ agent has an $\varepsilon$ cognitive impairment, as long as $\varepsilon$ is small enough. By contrast, consider a belief having the properties, $\pi_{i,t}^b \in \Pi^*(0)$, $\pi_{i,t}^b \notin \Pi^{*,r}$. Such a belief is rationalizable when firms have no cognitive impairment, $\varepsilon = 0$. But, it is not rationalizable for any cognitive impairment, $\varepsilon > 0$, no matter how small $\varepsilon$ is. In this sense, rationalizability of such a belief lacks robustness.

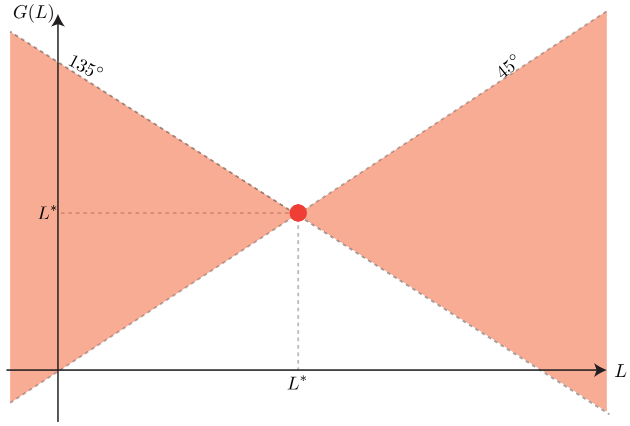
## 5.3 Sufficient and Necessary Conditions for Rationalizability

This section provides a result which plays a central role in the next section, where we establish conditions for unique rationalizability. It is convenient to temporarily use a simpler notation which abstracts from our specific economic setting. Accordingly, suppose the best response of a player in a large game is given by $G$. Let $L$ and $l$ denote the actions, respectively, of the others and of the individual. So, $l = G(L)$. Suppose that the graph of the function is, with one exception, strictly interior to the butterfly-shaped, shaded area in Figure 4a. The exception is the point market with a dot, where the boundaries of the butterfly meet. The boundaries are defined by the dashed lines with angles 45° and 135°. The domain, $\Omega$, of the function $G$ is defined by projecting the butterfly onto the horizontal axis. The dot in the figure corresponds to a Nash equilibrium, a value of $L$ such that $L^* = G(L^*)$.

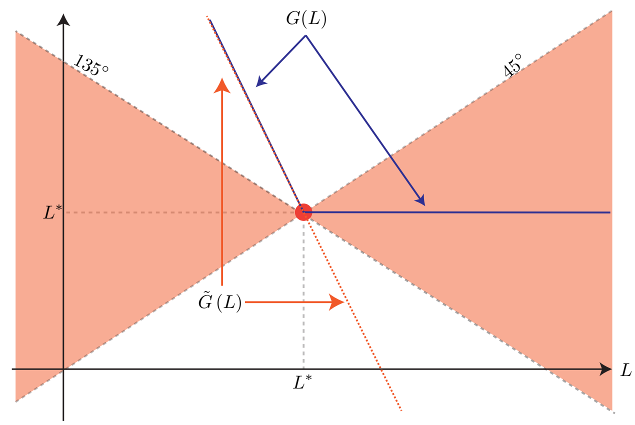
The proposition below provides conditions under which the sequence of sets, $G^k(\Omega)$, $k = 1, 2, ...$ converges to the singleton, $L^*$:

[68] See equation (59) for the definition of the operator, $\Pi^*$.

[69] A simple way to see the weakly increasing result is by induction. Trivially, $x \in \Pi_0^\varepsilon$ implies $x \in \Pi_0^{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Similarly, $F(\Pi_0^\varepsilon) \subseteq F\left(\Pi_0^{\varepsilon'}\right)$. The result follows by repeating this type of argument and using the definition of $\Pi^*$ in equation (59).




(a) The Butterfly




(b) Butterfly and Necessity

**Proposition 9.** *Suppose that a function $G$ satisfies the following properties:*

*(i) the domain of $G$, $\Omega$, is a bounded set on the real line containing $L^*$,*

*(ii) $L^*$ is a fixed point for $G$,*

*(iii) $\sup_{L\in\Omega\setminus\{L^*\}} \frac{|G(L)-L^*|}{|L-L^*|} < 1$.*

*Then we have $\Pi^*(\Omega; G) = \lim_{k\to\infty} G^k(\Omega) = \{L^*\}$.*

Here, $\Pi^*(\Omega; G)$ is defined in equation (59). The role of the boundedness condition in (i) will be explained below. It is easy to see that conditions (ii) and (iii) imply that the fixed point of $G$ is unique. Condition (iii) is the mathematical statement that the function, $G$, lies inside the butterfly.[70] We use the 'sup' operator in condition (iii) because the 'max' operator may not be well defined with the type of best response function, $F$, in our model, because it is not a continuous function.

A formal proof of the proposition appears in Appendix A. In the special case that $\Omega$ is a bounded interval, the intuition is straightforward. In this case, condition (iii) implies $G^k(\Omega) \subseteq G^{k-1}(\Omega)$ for $k = 2, 3, \ldots$ . It is easy to verify that because this sequence of sets is weakly shrinking, it is guaranteed that $\Pi^*(\Omega; G) = \lim_{k\to\infty} G^k(\Omega)$. That the limit is $L^*$ can be verified using the kind of logic used in the discussion of Figure 1a in Section 2.

The sufficient conditions of Proposition 9 are somewhat more general than they might at first appear. In particular, these conditions may not be satisfied for the natural economic representation of the model variables, but they may be satisfied after a suitable transformation. In this case, the

[70] An alternative, equivalent, version of condition (iii) is: $\exists\rho < 1$ such that for all $L \in \Pi_1 \setminus \{L^*\}, |F(L) - L^*| < \rho|L - L^*|$. To see this, suppose first that the previous condition holds. Then $|F(L) - L^*| / |L - L^*| < \rho$. Taking the supremum, we have $\sup_{L\in\Pi_1\setminus\{L^*\}} |F(L) - L^*| / |L - L^*| \leq \rho < 1$. So, (iii) in Proposition 9 holds. Second, suppose that condition (3) holds. Then let $\delta = \sup_{L\in\Pi_1\setminus\{L^*\}} |F(L) - L^*| / |L - L^*|$ and $\rho = (1+\delta)/2 > \delta$. It follows that $|F(L) - L^*| / |L - L^*| \leq \delta < \rho$ for all $L \in \Pi_1$. Therefore $|F(L) - L^*| \leq \delta |L - L^*| < \rho |L - L^*|$ for all $L \in \Pi_1 \setminus \{L^*\}$. The two results establish our equivalence result.

conclusion of Proposition 9 also applies to the original variables.[71] In the case of our monetary model, we have been working with the equilibrium conditions expressed in log form. This transformation was convenient in our analysis of the competitive equilibrium of the model. As we shall see below, the piecewise linearity of $F$ in our model when the variables are expressed in log form makes it easy to apply the sufficient conditions of Proposition 9.

To see that the conditions are sufficient but not necessary, consider the piecewise-linear function in Figure 4b (see the solid line). That function does not lie in the butterfly, but it can easily be verified that $\Pi^*(\Omega; G) = \{L^*\}$ anyway. For the purpose of our analysis, there does exist one particularly useful necessary condition:

**Proposition 10.** *If $G(\cdot)$ has a linear segment passing through $L^*$ then unique rationalizability, $\Pi^*(\Omega; G) = \{L^*\}$, implies that the slope of the linear segment has slope less than unity in absolute value.*

The function, $\tilde{G}(L)$, in Figure 4b does not satisfy the necessary condition of Proposition 10 and so unique implementation does not occur. It is easy to verify that values of $L$ close enough to $L^*$ are all rationalizable.

## 5.4 Implementation

We seek restrictions on the monetary policy parameters which guarantee that private agents uniquely coordinate (or, anchor) their beliefs on the desired equilibrium. If the restrictions are satisfied, then monetary policy achieves *unique implementation.* We begin with the following definition:

**Definition 10.** A competitive equilibrium satisfies *unique implementation* if the model has the following two properties: (i) it is a strategy equilibrium and (ii) for each $h_{t-1}$ there exists a unique *robustly rationalizable* belief, $\pi^b_{i,t}$.

In the first subsection we discuss unique implementation under the Taylor strategy (see Definition 2). The second subsection considers the zero monitoring range policy in Definition 3.

### 5.4.1 Taylor Strategy

We now state the following result:

**Proposition 11.** *Consider the model in which monetary policy is governed by the Taylor strategy (see Definition 2) and the money growth rule parameter, $\rho$, satisfies $\rho < \min\{1, \gamma\}$. A sufficient*

[71] Here is a simple illustration. Let $l = G(L) = L^{-1/2}$, with domain $\Omega = [L_l, L_u]$, where $0 < L_l < L_u$. This function, $G$, has a unique fixed point at $L = 1$. In addition, it is easy to verify that $G^k(\Omega) \to \{1\}$, as $k \to \infty$. However, $G$ does not satisfy condition (iii) of Proposition 9. To see this, note that as $L \to 0$, $G(L) \to \infty$, thus breaking out of the butterfly in Figure 3. However, if $l$ and $L$ are replaced by their logs, then the system is linear, $\ln l = -1/2 \times \ln L$, with domain, $\Omega = [\ln L_l, \ln L_u]$. This system does satisfy the sufficient conditions of 9.

*condition for unique implementation of the unique equilibrium of that model is:*

$$\left|1 - \phi\frac{\gamma + \psi}{\gamma - 1}\right| < 1, \quad 1 - (\gamma + \psi) < \rho < 1 - \frac{\gamma + \psi}{2}. \tag{63}$$

*A necessary condition for unique implementation is:*

$$\left|1 - \phi\frac{\gamma + \psi}{\gamma}\right| < 1, \quad \rho < 1 - \frac{\gamma + \psi}{2}. \tag{64}$$

Before sketching the proof of Proposition 10, we discuss the sufficient conditions in equation (63) and the necessary conditions in equation (64). We stress three observations about the sufficient conditions. First, there is no restriction on the boundaries of the inflation monitoring range $\pi_l$ and $\pi_u$, beyond the (modest) restrictions stated in Definition 2. Second, depending on the values of the parameters associated with the private sector, $\gamma$ and $\psi$, it is possible that there is no value for the monetary policy parameter, $\phi$, that satisfies the sufficient conditions for unique implementation. Note that $\phi > 1$ implies that the first expression in equation (63) is equivalent to:[72]

$$\phi < 2\frac{\gamma - 1}{\gamma + \psi}, \tag{65}$$

which cannot be satisfied if $\gamma < 1$. Equation (65) indicates that the sufficient conditions can also be violated in case $\gamma > 1$ and $\psi$ is large enough. The third interesting feature of the conditions, (63), is the restriction on $\rho$. Condition (65) implies, via the second condition in equation (63), that $\rho < 0$. Thus, a version of the Taylor strategy which switches to a constant money growth rule (i.e., $\rho = 0$) when the escape clause is activated does not satisfy the sufficient conditions for unique implementation. In the next section we provide an example with $\rho = 0$ in which the Taylor strategy fails to anchor expectations.

Now, consider the two necessary conditions in equation (64). The first and second of these conditions correspond to Proposition 10 applied to the Taylor rule and the money growth rule, respectively, in the Taylor strategy. Given that $\phi, \gamma, \psi$ are positive, the first condition in equation 64 is equivalent to

$$\phi < \frac{2}{1 + \psi/\gamma}. \tag{66}$$

Note that $\psi$ and $\gamma$ play similar, though opposite, roles in determining the upper bound on $\phi$. With an increase in $\psi$, or a decrease in $\gamma$, a given value of $\phi$ corresponds to a more aggressive response in the Taylor rule to inflation. To see this, suppose a firm contemplates a higher belief about the prices set by others. In the continuation equilibrium, this implies a higher level of the interest rate, dependent on the given value of $\phi$. For a fixed value of $\gamma$, this implies a particular drop in consumption (see

[72]To see this, note that the first condition in equation (63) implies $-1 < 1 - \left(\frac{\gamma+\psi}{\gamma-1}\right)\phi < 1$. Subtracting 1 from each term and multiplying the result by $-1$, we obtain $2 > \left(\frac{\gamma+\psi}{\gamma-1}\right)\phi > 0$. Given $\phi > 1 > 0$, the second inequality in the last expression implies $\gamma > 1$, so that the first condition in equation (63) coincides with equation (65).

equation (46)). This drop in consumption produces a greater fall in marginal cost, the higher is $\psi$ (see equation (47)). Similarly, for a fixed value of $\psi$, a lower value of $\gamma$ increases intertemporal substitution in consumption. This leads to a larger drop in consumption and reduces marginal cost by more (see equations (46) and (47)). This is why a higher value of $\psi/\gamma$ increases the likelihood that a given $\phi$ leans against the wind too aggressively.

The upper bound in equation (66) may appear surprisingly low, given the values of $\psi$ and $\gamma$ often used in practice. There is reason to think that in the type of models that work well empirically, this upper bound might be higher. For example, Online Appendix E presents a version of our model with Calvo-style price-setting frictions. In that model, only a subset of the intermediate good firms adjust their price in a given period. When the number of such firms is small, then the upper bound on $\phi$ is higher. The intuition for this resembles the intuition for $\gamma$ and $\psi$ just described. In particular, when a price setter considers higher prices set by other price setters, the impact on aggregate inflation (hence, on policy) is smaller, the smaller is the number of price setters. With fewer price setters (more stickiness in prices), policy is less aggressive for given value of $\phi$, so that an upper bound like the one in equation (66) is higher.[73]

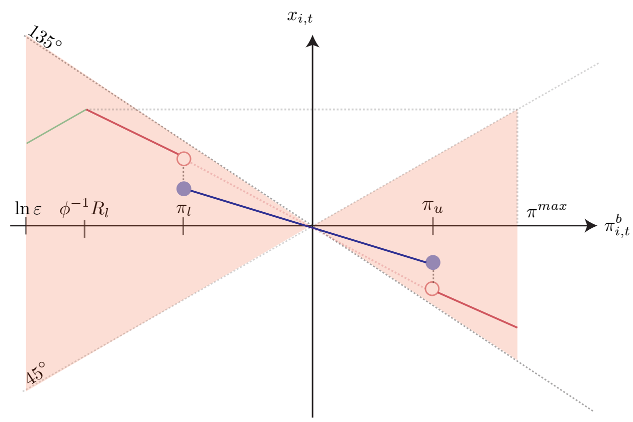
To prove Proposition 11, we first consider the type of history, $h_{t-1}$, in which the inflation monitoring range has never been violated. Because $\phi$ in the monetary policy rule is positive, the conditions (65) in Proposition 11 require $\gamma > 1$. Proposition 6 then implies that the equilibrium is a strategy equilibrium, so that part (i) of Definition 10 is satisfied. The key result used to establish part (ii) of Definition 10 is Proposition 9, with $G(\cdot) = F^{\varepsilon}(h_{t-1}, \cdot)$ ,for a specific, small value of $\varepsilon > 0$. To meet condition (i) of Proposition 9, we must identify a suitable bounded set, $\Omega$. We define $\Omega$ as the result of the first iteration in the iterated deletion method: $\Omega = \Pi_0^{\varepsilon} \cap F(h_{t-1}, \Pi_0^{\varepsilon})$ .[74] That $\Omega$ is bounded below follows from the lower bound on $\Pi_0^{\varepsilon}$ (see equation (61)). The set, $\Omega$, is also bounded above, given the parameter restrictions, (63). This is because there is a maximal value of $F(h_{t-1}, \cdot)$ (see, for example, Figure 4). It is straightforward to verify that the parameter restrictions also imply that $F^{\varepsilon}(h_{t-1}, \cdot)$ lies inside the butterfly in Figure 4a. That is, conditions (ii) and (iii) of Proposition 9 are satisfied, where the unique fixed point of $F^{\varepsilon}(h_{t-1}, \cdot)$ is $L^* = \pi^b(h_{t-1})$ . It follows from Proposition 9 that $\Pi^*(\Omega; F^{\varepsilon}) = \left\{\pi^b(h_{t-1})\right\}$ for each $\varepsilon > 0$. Since $\Pi^*(\varepsilon) = \Pi^*(\Omega; F^{\varepsilon})$ it follows that the set of robustly rationalizable beliefs (see Definition 9), $\Pi^{*,r}$, is composed of the singleton, $\left\{\pi^b(h_{t-1})\right\}$. It follows that part (ii) of Definition 10 is satisfied. We conclude that $\pi^b(h_{t-1})$ is the only robustly rationalizable belief.

Now consider the type of history, $h_{t-1}$, in which the inflation monitoring range has been violated at least once. Lemma 18 in Online Appendix C.4 establishes that equation (63) implies $F\left(h_{t-1}, \pi_{i,t}^b\right)$ is bounded above and satisfies the conditions in Proposition 9. Thus, unique rationalizability is established for this type of history, so that Proposition 11 holds.

[73] See equation (151) in Online Appendix E.

[74] See the discussion in the paragraph before equation (59) for a demonstration that a rational firm would ignore any belief that is in $\Pi_0^{\varepsilon}$, but not in $\Omega$.

Figure 4: Best Response Function, $F^{\varepsilon}$, with $\mu = 0$




Notes: The graph displays $F^{\varepsilon}\left(h_{t-1}, \pi^b_{i,t}\right)$ for $\pi^b_{i,t} \in \Omega$, where $\Omega$ is the horizontal segment from $\ln\varepsilon$ to $\pi^{max}$ and $\pi^{max} = \max_{x\in\Pi^{\varepsilon}_0} F^{\varepsilon}\left(h_{t-1}, x\right)$ and $\Pi^{\varepsilon}_0$ is defined in equation (61). Also, $F^{\varepsilon}\left(h_{t-1}, \pi^b_{i,t}\right)$ is constructed using $F$ and $\varepsilon$ according to equation (60). The shaded area corresponds the shaded area in Figure 4a. Also, $h_{t-1}$ corresponds to a history in which the inflation monitoring range has never been violated.

### 5.4.2 Zero Monitoring Range Policy

It is also of interest to consider the zero monitoring range policy in Definition 3, with $\pi_l = \pi_u = \mu = 0$ and $\phi \neq 0$. Consider the case, $h_{t-1}$, in which the inflation monitoring range has never been violated. The best response function for this type of history is the function graphed in Figure 4, with the middle segment shrunk to a singleton at the origin. Alternatively, it is the second element of $F$ in equation (37) for all $\pi^b_{i,t}$. So, for this type of $h_{t-1}$, $|1 - (\gamma + \psi)\,\phi/(\gamma - 1)| < 1$ guarantees that the best response function lies in the butterfly. This in turn is sufficient to guarantee unique implementation in the sense of Definition 10. In the case of $h_{t-1}$ in which there has been a violation of the monitoring range, the result in Proposition 11 applies. We summarize these observations in the form of a proposition as follows:

**Proposition 12.** *Suppose monetary policy is the zero monitoring range policy (see Definition 3). Unique equilibrium of the model satisfies unique implementation if and only if*

$$|1 - (\gamma + \psi)\,\phi/(\gamma - 1)| < 1, \quad \rho < 1 - \frac{\gamma + \psi}{2}. \tag{67}$$

*then unique equilibrium of the model satisfies unique implementation.*

Note that unique implementation requires $\gamma \neq 1$ and $\phi \neq 0$. Interestingly, negative or small positive values of $\phi$ are consistent with unique implementation. The logic behind unique implementation of the above policy is essentially the same as it is for the policy considered in Proposition 11. The policy has the effect that if a particular intermediate good producer believes others set high prices, then - via the interest rate channel - the continuation equilibrium is such that that agent's best response is to post low prices. What is crucial for unique implementation is that variations in beliefs generate relatively small responses. Formally, the responses must be small enough that the best response function lies inside the butterfly. This result is guaranteed when the model parameters satisfy equation (67).

## 5.5 Discussion of Proposition 11

We now discuss cases where the necessary and sufficient conditions in Proposition 11 are not satisfied, so that unique implementation fails. We describe a baseline set of parameter values which satisfies the conditions of Proposition 11. We then consider perturbations in which the conditions are not satisfied. In the perturbations, the competitive equilibrium is still unique and it is a strategy equilibrium. This discussion goes to the core message of the paper: the finding that an equilibrium is unique and that it is a strategy equilibrium is not sufficient for unique implementation. For a policy to satisfy unique implementation requires stronger restrictions on policy parameters.

Our baseline parameter values are:

$$\psi = 1/2, \gamma = 6, \phi = 1.5, \rho = -2, \mu = 0$$

The value of $\rho$ implies that if consumption goes up by 1 percent, then monetary policy in the money growth regime reduces the money stock by 2 percent. Table 1 summarize the restrictions required to guarantee various equilibrium properties. The baseline parameters satisfy all properties in Table 1. We now perturb the value of each parameter so that unique implementation fails.

Table 1: Parameter Restrictions

(a) Taylor strategy

| | $\phi$ | $\rho$ | $\gamma$ | $\psi$ |
|---|---|---|---|---|
| Uniqueness of competitive equilibrium (CE) | $\phi > 1$ | No Restriction | $\gamma > 0$ | $\psi \geq 0$ |
| CE is a strategy equilibrium | None | $\rho < \min\{1, \gamma\}$ | $\gamma \neq 1$ | None |
| CE is uniquely implemented | $\phi < 2\frac{\gamma-1}{\gamma+\psi}$ | $\rho < -\psi$ | $\gamma > 2 + \psi$ | |

(b) Zero monitoring range policy

| | $\phi$ | $\rho$ | $\gamma$ | $\psi$ |
|---|---|---|---|---|
| Uniqueness of competitive equilibrium (CE) | $\phi > 0$ | No Restriction | $\gamma > 0$ | $\psi \geq 0$ |
| CE is a strategy equilibrium | None | $\rho < \min\{1, \gamma\}$ | $\gamma \neq 1$ | None |
| CE is uniquely implemented | $0 < \frac{\gamma+\psi}{\gamma-1}\phi < 2$ | $\rho < 1 - \frac{\gamma+\psi}{2}$ | | |

Note: in each table, the first row indicates parameter restrictions used to establish uniqueness of equilibrium for our model (Proposition 2); subsequent rows indicate incremental restrictions to obtain the results in the first column (these correspond to Propositions 6 and 11).

Consider $\phi$ first. Given our baseline model parameters, the upper bound on $\phi$ is 1.53. The impact on the best response function of a $\phi$ larger than this upper bound can be seen in Figure 5a. Note that this change causes the best response function to have a slope that is steeper $-1$. With a higher value of $\phi$ an increase in $\pi_{i,t}^b$ is associated with a bigger increase in $R_{i,t}$ in the continuation equilibrium, which in turn results in a bigger drop in $c_{i,t}$ and, hence, the wage. The result is that the $i^{th}$ firm's best response function no longer lies interior to the butterfly, and so it violates our sufficient condition for unique robust rationalizability (see Proposition 9). Intuitively, a high value of $\phi$ has the consequence that the $i^{th}$ firm's action is more sensitive to variations in its beliefs, a phenomenon that makes unique (robust) rationalizability less likely. Consistent with our discussion in Section 2, when $\phi$ is too large, the 'leaning against the wind, but not too aggressively' principle is violated. To see that unique rationalizability does not hold, consider a belief, $\pi_{i,t}^b \neq 0$, that lies inside the monitoring range (see Figure 5a). Because the best response function has a slope steeper than $-1$, the inverse of $F$ has a slope less than unity in absolute value. Thus, a chain of justification can be constructed for the posited $\pi_{i,t}^b$ which converges into the Nash equilibrium (i.e., the origin in the figure). Since the Nash equilibrium is justified, the entire chain of beliefs is justified and so the posited $\pi_{i,t}^b$ is rationalizable.[75] We conclude that when $\phi$ lies above its upper bound, then rationality and CK are not sufficient for agents to coordinate on the unique competitive equilibrium. When $\phi$ is too large, inflation expectations are not anchored.

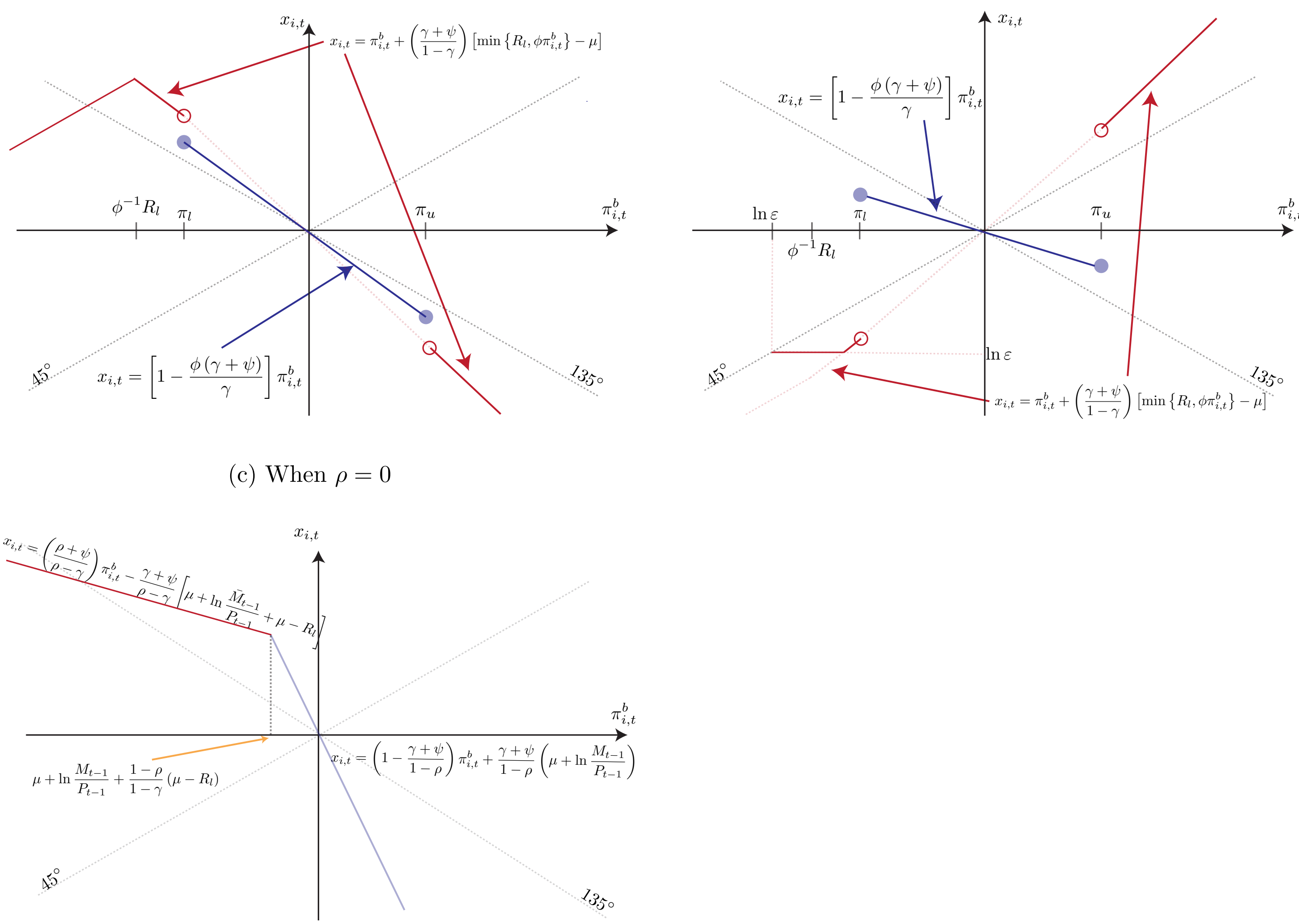
Next we consider a perturbation in $\gamma$. Figure 5b displays what happens if $\gamma < 1$ but close to unity. There is a segment of the best response function that lies interior to the butterfly, but the best response dramatically leaves the butterfly outside the inflation monitoring interval, $[\pi_l, \pi_u]$. The reason for this very sharp change in the best response function is the presence of $1 - \gamma$ in the denominator of $F$ outside the monitoring range. The switch in the value of $\gamma$ relative to unity reflects the switch in the sign of the response of equilibrium consumption to a change in the nominal interest rate. With $\gamma < 1$ a rise in $\pi_{i,t}^b$ drives $R_{i,t}$ up, but the real rate down, and so induces households to increase consumption (see equation (44)). This in turn raises the wage rate and induces the $i^{th}$ intermediate good firm to increase its price by even more than the rise in $\pi_{i,t}^b$ (see equation (45)). So, the Taylor strategy violates the leaning against the wind principle discussed in Section 2 and unique rationalizability fails. I also fails because there is a second Nash equilibrium.[76] The failure of

[75] Note that the inverse of $F$ actually has two elements. In the text we focused on the upper element, while the other element has to do with the region to the left of the zero lower bound, $R_l/\phi$. For rationalizability, one need only find one chain of justification (see Definition 8), so that nothing was lost in the text when we ignored the lower element in the inverse of $F$.

[76] This example is reminiscent of the example in the introduction to Kocherlakota (2018), which motivates his analysis. Essentially, his example is version of the example in our Figure 1a, in which the best response function cuts the 45° line from below. Suppose we apply our cognitive refinement for a specific $\varepsilon > 0$ but small. In this case, the best response function becomes horizontal for expected inflation, $\pi_i^b$, at or below $\ln \varepsilon$ (see equation (60) and also the horizontal segment in Figure 5b). The horizontal segment extends from the best response function to the 45° line, so that the cognitive gives rise to a second Nash equilibrium. If we introduce an analogous cognitive impairment in perceiving large numbers (see Online Appendix C.3), then a similar horizontal segment would appear for sufficiently high inflation. This change would give rise to a third Nash equilibrium. In this example the cognitive impairment has the effect of compactifying the space of expected inflation. As in Kocherlakota (2018)'s discussion, this compactification increases the number of Nash equilibria. This effect on the number of Nash equilibria reflects that a best response

the leaning against the wind principle in this example was discussed informally in Subsection 2.2.2 above.

Figure 5: Best Response Functions with Perturbations on Baseline Parameter Values




(a) Best Response Function $\phi$ is big

(b) Best Response Function $\gamma < 1$ but $\gamma$ close to unity

(c) When $\rho = 0$

Next, we turn to rationalizability in histories, $h_{t-1}$, in which the inflation monitoring range has been violated at least once. According to Table 1 the money growth rule must have a 'leaning against the wind' feature, $\rho < -\psi$. Since $\psi \geq 0$, there is no way that $\rho = 0$ can satisfy the sufficient conditions for implementation. To see why this is the case consider Figure 5c. The slope of the best response function, $F$, reflects the assumptions, $\rho = 0$ and $\psi > 0$. The best response of intermediate good

function cutting the 45° line from below violates the butterfly condition, (iii) in Proposition 9. For example, when the best response function cuts the 45° from above, so the butterfly condition is satisfied, then it is easily verified that our refinement and even its extension to large numbers does not affect the number of Nash equilibria. Kocherlakota (2018) argues that analysis in models where the number of Nash equilibria is not robust to compactification can be very misleading. We reach a similar conclusion: analysis of the competitive equilibrium of a model in which the butterfly condition is not satisfied is misleading because there is no basis for the notion that the equilibrium would actually occur. For that equilibrium to occur it is necessary for the butterfly condition to be satisfied.

producers is very sensitive to variations in their beliefs. We discussed the case, $\rho = 0$, in Section 4.1.5. There, we explained that when a firm contemplates a low value of $\pi^b_{i,t}$, the associated continuation equilibrium has a low real interest rate, thus stimulating consumption and raising marginal cost. This is why the best response function has a negative slope. As it turns out, when $\rho = 0$ that slope is steeper than $-1$. This is too steep to allow iterated deletion to do its work and so is not compatible with rationalizability. By making $\rho < 0$, the problem is fixed. As $c_{i,t}$ rises with a low $\pi^b_{i,t}$ then the period $t$ money stock is reduced and this mitigates the decline in the real rate, flattening the best response function. When $\rho < -\psi$ the best response lies inside the butterfly diagram and we obtain unique implementation.

# 6 The Taylor Strategy Versus Two Alternatives

We have identified parameter restrictions, equation (63), under which three apparently different monetary policy regimes uniquely implement[77] desired inflation: (i) the Taylor strategy - $\phi > 1$ and $\pi_l \leq \pi_u$; (ii) the zero monitoring range strategy - $\phi \neq 0$ and $\pi_l = \pi_u = \mu = 0$;[78] and (iii) a pure money growth rule (see Lemma 1). We explain how these three different policy strategies illustrate the observation in Taylor (1999) that many apparently different policy arrangements, including the Gold Standard and money growth rules, can be thought of as interest rate rules which stabilize inflation by raising the interest rate in case inflation rises.[79] According to Taylor (1999), the important question is which interest rate policy works best. In the deterministic model described thus far, each of our three policies performs equally well because it supports the same desired allocations.

In this section we introduce two modifications which, while preserving much of the tractability of the existing model, allows us to also incorporate considerations that make the model more realistic. With these changes, the model suggests that the best policy among those considered here is the Taylor strategy, with $\pi_l < \pi_u$. We make the argument in two steps. First, we point out that zero monitoring range policy, (ii), has a knife edge property: remaining in the Taylor rule regime is not robust to arbitrarily small trembles by an arbitrarily small number of intermediate good firms. Without further changes, this is of no importance in our existing model because the monetary growth rule to which the economy switches under trembles also uniquely implements the desired equilibrium. Our second change is to introduce money demand shocks which, since at least the classic paper of Poole (1970), are an important motivation for interest rate rules. The idea is that the interest rate rule implicit in a money growth rule implies excessive, welfare-reducing volatility in the interest rate when there

[77] Here, by implementation we mean Definition 10.

[78] Notice unique implementation is consistent with $\phi < 0$ under policy (ii). However, in this case unique implementation requires $\gamma < 1$ (see equation (63)). Perhaps surprisingly, the Taylor logic works in this case as well, though it does so via the real interest rate, not the nominal rate. This can be seen by retracing the logic in the subsection, 'Violation of the inflation monitoring range' in Section 4.1.4, for the case, $\phi < 0$ and $\gamma < 1$.

[79] Recall the discussion of the interest rate channel in the case of the Taylor strategy (see the discussion after equation (45)); in the case of the zero monitoring rate strategy (see the discussion after Proposition 7); and in the case of the pure money growth rule (see the discussion after equation (52)).

are substantial shocks to money demand. As we show below, our model with money demand shocks captures the Poole (1970) idea. A policy of the type, (i), is welfare superior to a money growth rule like (iii) or to (ii) when there is the possibility of a tremble.

## 6.1 Trembles

We consider a history, $h_{t-1}$, in which no deviation from the inflation monitoring range has occurred. We imagine that in period $t$ a tiny subset (mass) of firms make a tiny mistake when they set their prices. When such mistakes happen, then inflation jumps out of the monitoring range under policy (ii), triggering a regime shift to the money growth rule.

Formally, suppose that intermediate good firms form their beliefs as they do in previous sections, without imagining the possibility of trembles. They arrive at a unique set of beliefs by iterated deletion, as in Section 5.[80] However, when the time comes for firms to post their period $t$ price, a small fraction of them make a tiny mistake. Say, the hand of each of these firms trembles as it endeavors to write its price the menu. As in Section 4.1.1, let $p_{i,t}^b$ denote the $i^{th}$ firm's expectation of how other firms post their price. Then the optimal price chosen by $i^{th}$ firm is

$$p_{i,t} = p_{i,t}^b \left(c_{i,t}\right)^{\gamma+\psi} \upsilon_{i,t}, \tag{68}$$

where $c_{i,t}$ denotes consumption in the continuation equilibrium associated with $p_{i,t}^b$. Suppose $p_t^b$ is the unique belief that survives iterated deletion, so that all firms, $i \in [0,1]$, adopt it. Then, each firm (not being aware that a tremble might occur) resolves to set its own price to $p_t^b$. This being the Nash belief, it follows that $c_{i,t} = 1$ in the continuation equilibrium imagined by each of the $i$ firms.[81] Next, suppose that there is a tremble in the form of the additional variable, $\upsilon_{i,t}$, on the right side of equation (68). That tremble is a unit mean random variable which is drawn independently by each firm. The distribution of $\upsilon_{i,t}$ has two parameters, $J_t, \delta_t \in [0,1]$. With probability $1 - J_t$, the $i^{th}$ firm does not tremble at all, so that $\upsilon_{i,t} \equiv 1$. With probability $J_t$ the $i^{th}$ firm draws $\upsilon_{i,t}$ from a uniform distribution with support, $[1-\delta_t, 1+\delta_t]$. We can allow both $J_t > 0$ and $\delta_t > 0$ to be arbitrarily close to zero.

According to equation (13), the actual price index in the $\left(h_{t-1}, p_t^b\right)$ continuation equilibrium is determined as follows:

$$P_t = \left[\int_0^1 \left(p_t^b \upsilon_{i,t}\right)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}} = p_t^b \exp\left(\kappa_t\right) \tag{69}$$

where

$$\exp\left(\kappa_t\right) = \left[1 - J_t + J_t \frac{\left(1+\delta_t\right)^{2-\varepsilon} - \left(1-\delta_t\right)^{2-\varepsilon}}{2\delta_t\left(2-\varepsilon\right)}\right]^{\frac{1}{1-\varepsilon}}.$$

[80] We assume that the parameter restrictions required for Proposition 11 are satisfied.

[81] As was discussed earlier, $c_{i,t} = 1$ in the continuation equilibrium associated with the Nash equilibrium belief. See equation (46)

As expected, $\kappa_t \to 0$ as $J_t \to 0$ or $\delta_t \to 0$. Divide both sides of equation (69) by $\bar{\mu}^* P_{t-1}$ take logs and make use of the definition of $\ln\left(p_t^b/\left(\bar{\mu}^* P_{t-1}\right)\right)$ in equation (30) to obtain:

$$\pi_t = \pi_t^b + \kappa_t, \tag{70}$$

where $\pi_t^b = 0$, the uniquely rationalizable belief.[82] Since realized inflation at date $t$ is different from zero, $\pi_t = \kappa_t \neq 0$. Since the inflation monitoring range is violated, the tremble pushes the economy into the money growth regime.[83]

## 6.2 Money Demand Shocks

In the model as it is set up now, the regime shift does not matter from a welfare standpoint. The same desired equilibrium outcomes occur whether the economy is following the Taylor rule or the money growth rule. But, if we assume money demand shocks are realized after firms set their prices, then there is a substantial loss in switching from the Taylor rule to the money growth rule. The assumption that the money demand shocks are realized after firms set their prices is a way to capture the notion that money demand shocks operate at a higher frequency than price changes. The model is a variant of the sticky price model in Christiano et al. (1997), where time $t$ prices are predetermined when time $t$ shocks are realized.

We introduce money demand shocks, $\nu_t$, by inserting them into the cash constraint, equation (7):

$$P_t c_t \leq \left(m_{t-1} + W_t l_t - d_t\right) \exp\left(\nu_t\right).$$

We assume $\nu_t$ is a mean-zero stochastic process, distributed independently over time. With this change in the model, it is no longer analytically tractable. As a result, we study a log-linear approximation of the model in a neighborhood of the desired equilibrium. We redo all the analysis in previous sections, using local versions of the concepts of equilibrium, rationalizability and implementation studied in the previous sections. The results are similar and are provided in Online Appendix D. We establish the following:

**Proposition 13.** *Consider the version of the model with trembles (see Section 6.1) and with sufficiently small shocks, $\nu_t$. Welfare under the zero monitoring range strategy (Definition 3) is lower than welfare under the Taylor strategy with $\pi_l < \pi_u$.*

As discussed in the introduction to this section, this result is in effect a formalization of the finding in Poole (1970).

[82] See (iv), Proposition 5.

[83] Atkeson et al. (2010) find that their model is robust to trembles. That is because in their model the mapping from intermediate good prices to the aggregate price index is linear.

# 7 Concluding Remarks

This paper analyzes a monetary policy strategy in which an interest rate rule is adopted in normal times and switches to a monetary growth rule if inflation gets out of hand. We present a model and describe sufficient conditions, as well as some necessary conditions, under which the inflation expectation of each agent is firmly anchored at its desired level. The concept of implementation that we use takes into account that agents take an active role in forming their expectations.

To derive sufficient conditions for expectations to be anchored requires that our analysis be global and so tractability required that we limit ourselves to a simple model. In our model we identified a principle - leaning against wind, but not too aggressively - which guarantees that expectations are anchored. We conjecture that this principle generalizes to the type of models used in empirical and policy analysis. Our analysis of the necessary conditions for expectations to be anchored in a simple New Keynesian model with price-setting frictions lends support to our conjecture.[84] Since the methods used to identify necessary conditions are based on linear approximations in a neighborhood of steady state, they can be generalized to the kinds of models used in practice.

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[84] For details, see Online Appendix E.

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# A Appendix: Proof for Proposition 9

In order to prove Proposition 9, we use the following lemma.

**Lemma 3.** *Suppose that $F$ satisfies the conditions in Proposition 9. Then for any set $C \subset X_1$,*

$$\sup_{x\in C} |F(x)| < \sup_{x\in C} |x| . \tag{71}$$

*Proof.* Let $\delta = \sup_{x\in X_1\setminus\{0\}} \frac{|F(x)|}{|x|} < 1$. Then for all $x \in C$,$|F(x)| \leq \delta |x|$. Taking the supremum of the right-hand-side, we have for all $x \in C$,$|F(x)| \leq \delta \sup_{x\in C} |x|$. Now taking the supremum of the left-hand-side, we have

$$\sup_{x\in C} |F(x)| \leq \delta \sup_{x\in C} |x| < \sup_{x\in C} |x| .$$

The last inequality holds because $\delta < 1$. We establish equation (71). □

The following is a proof for Proposition 9.

*Proof.* Since $0 \in X_n$ for all $n \geq 1$, $0 \in \lim_{n\to\infty} X_n = \bigcap_{n=1}^{\infty} X_n$. So it suffices to show that there exists a decreasing sequence of $\{Y_n\}_{n=1}^{\infty}$ such that $X_n \subset Y_n$, and $Y_n \to \{0\}$. We define recursively $Y_n$ as follows. Set $Y_1$ as follows: $Y_1 = [-\alpha_1, \alpha_1]$, where $\alpha_1 = \sup_{x\in Y_1} |x|$. For all $n \geq 2$, set $Y_n = [-\alpha_n, \alpha_n]$ ,where

$$\alpha_n = \sup_{x\in Y_{n-1}} |f(x)| \tag{72}$$

Then it is trivial that $X_n \subset Y_n$ for all $n$. Also notice that $\alpha_n$ is also expressed as follows:

$$\alpha_n = \sup\{|x| ; x \in Y_n\} = \sup\{|f(x)| ; x \in Y_{n-1}\} . \tag{73}$$

We show that $\alpha_n$ is strictly decreasing and converging to 0.

Suppose that $\alpha_n$ is not strictly decreasing. So there exists $k$ such that $0 < \alpha_k \leq \alpha_{k+1}$. From equation (72), $\alpha_k \leq \sup_{x\in[-\alpha_k,\alpha_k]} |f(x)|$. Equation (71) in Lemma (71) implies that

$$\alpha_k \leq \sup_{x\in[-\alpha_k,\alpha_k]} |f(x)| < \sup_{x\in[-\alpha_k,\alpha_k]} |x| = \alpha_k,$$

which is a contradiction. Therefore $\alpha_k > \alpha_{k+1}$.

A decreasing sequence bounded by below has a limit $\alpha_k \downarrow \alpha^*$. Now we show that $\alpha^* = 0$. Suppose that $\alpha^* > 0$. Then again equation (71) in Lemma (71) implies that

$$\alpha^* = \sup_{x\in[-\alpha^*,\alpha^*]} |f(x)| < \sup_{x\in[-\alpha^*,\alpha^*]} |x| = \alpha^*,$$

which is a contradiction again. Thus, $\alpha^*$ is 0. Thus we conclude that $X_n \subset Y_n \downarrow \{0\}$ so that $X_n \to \{0\}$ as $n \to \infty$. □

# B Appendix: Proof of Proposition 11

*Proof.* That the equilibrium is a strategy equilibrium follows from the fact that conditions (i) and (ii) of the Definition of a strategy equilibrium (see Definition 6) are guaranteed by Proposition 5. This establishes part (i) of Definition 10. It remains to establish part (ii) of that definition. We restrict the domain of $F(h_{t-1}, \cdot)$ to $\Pi_1^{\varepsilon}$. This domain satisfies condition (i) of Proposition 9. By (iii) of Proposition 5, the fixed point, $\pi(h_{t-1})$, of $F(h_{t-1}, \cdot)$ exists and is unique, so that (ii) of Proposition 9 is satisfied. Online Appendix C.4 formally establishes that conditions (63) guarantees that condition (iii) of Proposition 9 is satisfied for any $h_{t-1}$. It is easy to see this informally for histories, $h_{t-1}$, in which there has never been a violation of the inflation monitoring range. In this case, the graph of $F$ is strictly interior to the butterfly graph except at the unique Nash equilibrium (see Figure 4). With the three conditions of Proposition 9 satisfied, we conclude that $F^k(h_{t-1}, \Pi_1^{\varepsilon}) \to \pi(h_{t-1})$ as $k \to \infty$. The object on the right of (iii) in Proposition 16, $\Pi_1^{\varepsilon} \bigcap \left(\bigcap_{k=1}^{\infty} F^k(h_{t-1}, \Pi_1^{\varepsilon})\right)$ ,is the set containing the singleton $\pi(h_{t-1})$. Condition (iii) of that proposition then implies that $\Pi^*(\Pi_1^{\varepsilon}) = \{\pi(h_{t-1})\}$ for each $\pi(h_{t-1}) > \varepsilon > -\infty$. It follows trivially that

$$\bigcup_{-\infty < \varepsilon \leq \pi(h_{t-1})} \Pi^*(\Pi_1^{\varepsilon}) = \{\pi(h_{t-1})\}.$$

Also, $\pi(h_{t-1})$ is rationalizable because it is a Nash equilibrium, $\pi(h_{t-1}) = F^k(h_{t-1}, \pi(h_{t-1}))$. So, conditions (i) and (ii) of Definition 10 are verified. This establishes that conditions (i) and (ii) of Definition 10 are satisfied. The result follows. □

# Online Appendix to 'Anchoring Inflation Expectations'

Lawrence J. Christiano and Yuta Takahashi

For convenience, the title of each section below has the form, 'Appendix: XX', where XX corresponds to the title of a manuscript section.

# A Appendix: Competitive Equilibrium

In this section we establish results for Section 3 in the paper. The first three subsections below derive various properties of the competitive equilibrium. In the first subsection we describe restrictions on market prices that are necessary for an equilibrium to exist. Moreover, for the household problem to be well defined in an equilibrium, it is necessary that prices have the property that all budget-feasible choices of the household result in finite discounted utility. The second subsection takes these restrictions as given and establishes conditions on household choices that are necessary and sufficient for their optimization problem.

## A.1 Equilibrium Restrictions

We derive six properties of the set of equilibrium prices. To describe the first property, for convenience, we repeat the representative household's objective and constraints:

$$\lim_{T\to\infty}\sum_{t=0}^{T}\beta^{t}\left[\frac{c_t^{1-\gamma}}{1-\gamma}-\frac{l_t^{1+\psi}}{1+\psi}\right],\quad \gamma>0,\gamma\neq1,\psi>0 \tag{74}$$

$$s.t.\quad m_t\leq\bar{R}_t\left(X_t+d_t\right)+m_{t-1}-d_t+W_tl_t-P_tc_t+T_t, \tag{75}$$

$$P_tc_t\leq m_{t-1}-d_t+W_tl_t \tag{76}$$

$$m_{-1}\text{ given,}$$

A *budget-feasible* sequence, $(c_t,l_t,m_t,d_t)_{t=0}^{\infty}$, has the property that, given $\left(P_t,\bar{R}_t,W_t,T_t,X_t\right)_{t=0}^{\infty}$ and $m_{-1}$, it satisfies the flow budget constraints (75), the cash constraints (76), and the following non-negativity constraints

$$0\leq\left(c_t,l_t,m_t,d_t\right),\ l_t\leq N. \tag{77}$$

Here, $N>0$ denotes the household's endowment of time.

We define a *feasible* sequence, $(c_t,l_t,m_t,d_t)_{t=0}^{\infty}$, as a budget-feasible sequence for which the limit of partial sums in equation (74) is finite. The household problem is to maximize equation (74) over the set of feasible sequences. Our first property of prices is that the feasible set of sequences is non-empty. If this were not so, then there would not be an equilibrium.

We briefly discuss why we restrict ourselves to a subset of the set of budget-feasible sequences. In the case, $\gamma > 1$ restricting ourselves to the set of feasible sequences is done without loss of generality. In this case, the elements in the partial sum in equation (74) are all negative. As a result, the sequence of partial sums either converges to a finite, non-positive number or to $-\infty$. We can ignore sequences which generate $-\infty$ utility because the household would always prefer one of the feasible sequences. In case of $\gamma < 1$ the limit of partial sums in equation (74) is bounded above and below. The reason is that we only consider equilibrium prices. In an equilibrium it must be that $c_t = l_t > 0$ and $l_t \leq N$, so that period utility is bounded above and below.

Our second property of prices is that $P_t > 0$ for all $t$. If in any period $P_t \leq 0$, then no feasible sequence is optimal for the household. For any sequence of consumptions, higher consumption would also be feasible and it would generate higher utility. Third, consider the nominal interest rate. In many models, $\bar{R}_t < 1$ cannot be an equilibrium because households could in that case be able to turn the loan market into a money pump, allowing unbounded consumption in $t+1$. In our model households cannot borrow, so their optimization problem may be well-defined for $\bar{R}_t < 1$. However, as discussed in Section 3.2, the fact that firms borrow implies $\bar{R}_t \geq 1$ in any equilibrium. If the $\bar{R}_t < 1$ in some period, then it is the intermediate good firm that can turn the loan market into a money pump in that period. With $\bar{R}_t < 1$ the firm problem has no solution. For any high level of borrowing it would earn even more profits by borrowing more. After paying off the extra principle and interest on any borrowing, it would have extra money left over which it could send as additional profits to its owner. Fourth, $W_t > 0$ is necessary for an equilibrium. If the condition were not met, then for any level of employment the firm could earn more profits by hiring more workers.

Fifth, it will be useful for us to use the concept of a date 0 price of a time $t$ good, $q_t$, defined in the following way:

$$q_t = \begin{cases} 1 & t = 0 \\ \prod_{s=1}^{t} \bar{R}_s^{-1} & t \geq 1 \end{cases}. \tag{78}$$

Because $\bar{R}_t > 1$ it follows that $q_t > 0$ for all $t$. Taking into account equation (77) we also know that $q_t m_t \geq 0$ for each $t$. Sixth, firm optimization implies $P_t = W_t$ and the resource constraint and technology imply $c_t = l_t$. Summarizing, we have

$$W_t, P_t > 0,\ P_t = W_t,\ c_t = l_t,\ \bar{R}_t \geq 1,\ q_t m_t \geq 0, \tag{79}$$

for each $t$.

## A.2 Restatement of the Household Problem

We find it convenient to reformulate the household problem in the canonical form used by Stokey et al. (1989). This allows us, in the next section, to easily apply the arguments in Stokey et al. (1989)

and Kamihigashi (2002) to obtain the necessary and sufficient conditions for household optimality. Of course, we solve the household problem under the assumption that prices correspond to those in an equilibrium.

We impose (obviously, without loss of generality) the flow budget constraint, (6), as a strict equality. We use that equation to solve for $l_t$ :

$$l_t = \frac{m_t - \left(\bar{R}_t\left(X_t + d_t\right) + m_{t-1} - d_t + T_t - P_t c_t\right)}{W_t}, \tag{80}$$

for all $t \geq 0$. We denote the date $t$ choice variables by the $3 \times 1$ column vector $x_t$ :

$$x_t = \left(c_t, m_t, d_t\right)',$$

We denote the period utility function in the reformulated system by the function, $F$ :

$$F\left(x_{t-1}, x_t\right) = u\left(c_t, \frac{m_t - \bar{R}_t\left(X_t + d_t\right) - m_{t-1} + d_t - T_t + P_t c_t}{W_t}\right). \tag{81}$$

Note that $F$ is concave and differentiable in $\left(x_{t-1}, x_t\right)$ since $u$ is has these properties in $c$ and $l$.

Obviously, the function, $F\left(x_{t-1}, x_t\right)$, is not only determined by the household choice variables $\left(x_{t-1}, x_t\right)$, but also by the other variables, $\left(P_t, \bar{R}_t, W_t, T_t, X_t\right)$ and $m_{-1}$, which are beyond the control of the household. To keep the notation simple, we do not make the dependence of $F$ on the latter variables explicit.

We now consider the various inequality constraints in the model. By equation (80), the upper and lower bound constraints on labor imply:

$$0 \leq m_t - \left(\bar{R}_t\left(X_t + d_t\right) + m_{t-1} - d_t + T_t - P_t c_t\right) \leq W_t N, \tag{82}$$

for all $t \geq 0$. These inequalities are not binding on the households at equilibrium prices. Equilibrium conditions guarantee that the first inequality is strict at equilibrium prices. To see this, recall from equation (79) that, in an equilibrium, $P_t = W_t$. So, equality of the household marginal rate of substitution with the real wage implies $c_t^{\gamma} l_t^{\psi} = W_t / P_t = 1$. From this it follows that $c_t$ and $l_t$ are strictly greater than zero. The second inequality in equation (82) is strict because we do not restrict the magnitude of $N$.

We express the household's cash constraint (after substituting out for $W_t l_t$ from equation 80) as well as the restrictions, $m_t \geq 0$ and $d_t \geq 0$,respectively, as follows:

$$A_t x_t \geq b_t. \tag{83}$$

Here, $A_t$ and $b_t$ are sequences taken as given by the household:

$$A_t = \begin{bmatrix} 0 & 1 & -\bar{R}_t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; b_t = \begin{bmatrix} \bar{R}_t X_t + T_t \\ 0 \\ 0 \end{bmatrix}.$$

In any equilibria, $b_t$ becomes positive, so that we assume that $b_t \geq 0$. We do not include the restriction, $c_t \geq 0$, in equation (83) because we showed above that it is non-binding at equilibrium prices. Another inequality in the model is the last condition in equation (79):

$$q_t \tilde{A}_t x_t \geq 0, \; \tilde{A}_T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \tag{84}$$

The date 0 problem of the household given equilibrium prices is:

$$\max_{\{x_t\}_{t=0}^{\infty} \in \mathcal{C}} \quad \lim\nolimits_{T \to \infty} \quad \sum_{t=0}^{T} \beta^t F\left(x_{t-1}, x_t\right), \tag{85}$$

subject to the restriction in equation (83). Here, $\mathcal{C}$ denotes the set of sequences for which the limit of partial sums is finite. We assume the set, $\mathcal{C}$, is non-empty for otherwise there would be no equilibrium. The set, $\mathcal{C}$, is a function of the equilibrium prices, but to keep the notation simple we do not make that dependence explicit.

## A.3 Necessary and Sufficient Conditions for Household Optimality

In this section we discuss the necessary and sufficient conditions for household optimization at equilibrium prices. That the first order conditions (see equations (8), (9) and (87)) are part of the necessary and sufficient conditions is easy to verify and we do not discuss that here. Instead, we focus on the sufficiency and necessity of a transversality condition.

The first order conditions are:

$$F_2\left(x_{t-1}, x_t\right) + \mu_t \times A_t + \beta F_1\left(x_t, x_{t+1}\right) = 0, \tag{86}$$

$$\text{diag}\left(\mu_t\right)\left(A_t x_t - b_t\right) = 0, \tag{87}$$

$$\mu_t \geq 0, \tag{88}$$

plus equation (83). Here, $F_1$ and $F_2$ denote $1 \times 3$ row vectors of derivatives with respect to the first and second arguments, respectively. Also, $\mu_t$ is the $1 \times 3$ row vector of multipliers on the cash constraint, $A_t x_t \geq b_t$. $\text{diag}\left(\mu_t\right)$ is the $3 \times 3$ square matrix whose diagonal elements are $\mu_t$.

The transversality condition is:

$$\lim_{T \to \infty} \beta^T F_1\left(x_{T-1}, x_T\right) x_{T-1} = 0. \tag{89}$$

To interpret (89) in terms of our underlying model, note

$$\beta^T F_1(x_{T-1}, x_T)\, x_{T-1} = \beta^T \frac{-u_{l,T}}{W_T} m_{T-1} = \frac{c_0^{-\gamma} \bar{R}_T}{P_0 R_0} q_{T-1} m_{T-1} \geq 0. \tag{90}$$

We let $u_{l,t}$ and $u_{c,t}$ denote the partial derivatives of household utility with respect to $l_t$ and $c_t$, respectively. The first equality in equation (90) uses the definition of $F$ in equation (81). The second equality uses the labor first order condition, $u_{l,T} = -u_{c,T} W_T / P_T$, the intertemporal first order condition,

$$\beta \frac{u_{c,t}}{P_t} \times \frac{P_{t-1}}{u_{c,t-1}} = \bar{R}_{t-1},$$

and the definition of $q_t$ in equation (78) for $t = 1, 2, ..., T$ . The inequality in equation 90 follows from equation (79). According to equation (90), equation (89) requires that the present value of money should be zero in the limit as $T \to \infty$.

### A.3.1 Sufficiency of Transversality Condition

According to the following proposition, if a budget-feasible sequence, $\{x_t\}_{t=0}^{\infty}$, satisfies (86), (87) and (88) for $t \geq 0$ and the transversality condition, (89), then that sequence solves the household problem in that there is no other feasible sequence that generates higher utility.

**Proposition 14.** *Suppose prices correspond to a competitive equilibrium and let $\{x_t^*\}_{t=0}^{\infty}$ and $\{\mu_t\}_{t=0}^{\infty}$ denote a sequence of decisions and multipliers associated with the household problem. If the decisions and multipliers satisfy (86), (87), (88), and (89), then the sequence, $\{x_t^*\}_{t=0}^{\infty}$, solves the household problem, (85).*

*Proof.* The rest of the proof style follows that of Theorem 4.15 in Stokey et al. (1989). Let $\{x_t\}_{t=0}^{\infty}$ denote an arbitrary feasible sequence. Define

$$D_T = \sum_{t=0}^{T} \beta^t \left[ F\left(x_{t-1}^*, x_t^*\right) - F\left(x_{t-1}, x_t\right) \right].$$

We wish to show that

$$\lim_{T \to \infty} D_T \geq 0.$$

First, concavity of $F$ implies:

$$F\left(x_{t-1}, x_t\right) \leq F_1\left(x_{t-1}^*, x_t^*\right)\left(x_{t-1} - x_{t-1}^*\right) + F_2\left(x_{t-1}^*, x_t^*\right)\left(x_t - x_t^*\right).$$

Then,

$$D_T \geq \sum_{t=0}^{T} \beta^t \left[ F_1\left(x_{t-1}^*, x_t^*\right)\left(x_{t-1}^* - x_{t-1}\right) + F_2\left(x_{t-1}^*, x_t^*\right)\left(x_t^* - x_t\right) \right].$$

Collecting terms

$$D_T \geq F_1\left(x_{-1}^*, x_0^*\right)\left(x_{-1}^* - x_{-1}\right) + \sum_{t=0}^{T-1} \beta^t \left[F_2\left(x_{t-1}^*, x_t^*\right) + \beta F_1\left(x_t^*, x_{t+1}^*\right)\right]\left(x_t^* - x_t\right) + \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right).$$

or, after using equation (86):

$$D_T \geq F_1\left(x_{-1}^*, x_0^*\right)\left(x_{-1}^* - x_{-1}\right) + \sum_{t=0}^{T-1} \beta^t \left[-\mu_t A_t \left(x_t^* - x_t\right)\right] + \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right).$$

It is easy to show that

$$\sum_{t=0}^{T-1} \beta^t \left[-\mu_t A_t \left(x_t^* - x_t\right)\right] = \sum_{t=0}^{T-1} \beta^t \left[\left(-\mu_t \left(A_t x_t^* - b_t\right) + \mu_t \left(A_t x_t - b_t\right)\right)\right].$$

By the complementary slackness condition, (87), $\mu_t\left(A_t x_t^* - b_t\right) = 0$ for all $t$. Also since $\{x_t\}_{t=0}^{\infty}$ is budget-feasible, $A_t x_t - b_t \geq 0$ for all $t \geq 0$. Thus, combining with the fact that $\mu_t \geq 0$ and $x_{-1}^* = x_{-1}$, we get

$$D_T \geq \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right).$$

Again using equation (86) again, we have

$$\begin{aligned} D_T &\geq \beta^T F_2\left(x_{T-1}^*, x_T^*\right)\left(x_T^* - x_T\right) \\ &= \beta^T \left(-\beta F_1\left(x_T^*, x_{T+1}^*\right) - \mu_T A_T\right)\left(x_T^* - x_T\right) \\ &\geq -\beta^{T+1} F_1\left(x_T^*, x_{T+1}^*\right) x_T^* + \beta^{T+1} F_1\left(x_T^*, x_{T+1}^*\right) x_T. \end{aligned}$$

The transversality condition, (89), and feasibility of $x_t$ imply that, respectively,

$$\lim_{T\to\infty} \beta^T F_1\left(x_{T-1}^*, x_T^*\right) x_T^* = 0, \quad \lim_{T\to\infty} \beta^{T+1} F_1\left(x_T^*, x_{T+1}^*\right) x_T \geq 0.$$

We conclude that

$$\lim_{T\to\infty} D_T \geq 0.$$

$D_T$ is the difference between two partial sums and the regularity conditions in (i) guarantee that those partial sums converge. This establishes our result. □

### A.3.2 Necessity of Transversality Condition

We turn to the necessity of 89. We establish the result using the argument in Kamihigashi (2002). We show that his argument applies almost without change, even though he works with a version of the Stokey et al. (1989) model, while ours is a monetary model with a cash constraint.

**Proposition 15.** *Suppose prices correspond to a competitive equilibrium and* $\{x_t^*\}_{t=0}^{\infty}$ *is a solution to the household problem, (85). Then,* $\{x_t^*\}_{t=0}^{\infty}$ *satisfies (89).*

*Proof.* Consider a class of perturbations, $\{x_t(\lambda, T)\}_{t=0}^{\infty}$, on the optimal path, $\{x_t^*\}_{t=0}^{\infty}$:

$$x_t(\lambda, T) = \begin{cases} x_t^* & t \leq T \\ \lambda x_t^* & t > T \end{cases},$$

where $\gamma \leq \lambda < 1$. Also, $\gamma$ is a scalar, $0 < \gamma < 1$, having the property that $\{x_t(\lambda, T)\}$ is a feasible interior sequence with finite discounted utility value for each $\lambda \in [\gamma, 1)$. Feasibility is trivial and imposes no restriction on $\gamma$, since $A_t x_t^* \geq b_t$ implies $A_t x_t(\lambda, T) \geq b_t$ for all $t \geq 0$, and each $\lambda \in [\gamma, 1)$. Similarly, the natural debt limit, (84), also places no restriction on $\gamma$. To see this, note that

$$\lim_{T\to\infty} q_t \tilde{A}_t x_t(\lambda, T) = \lambda \lim_{T\to\infty} q_t \tilde{A}_t x_t^* \geq b_t.$$

That we can always choose a value of $\gamma$ (perhaps very near to unity) so that $\{x_t(\lambda, T)\}_{t=0}^{\infty}$ has finite discounted utility follows from continuity of $F$.

Since $\{x_t^*\}$ is optimal, it follows that:

$$\infty > \sum_{t=0}^{\infty} \beta^t F\left(x_{t-1}^*, x_t^*\right) \geq \sum_{t=0}^{\infty} \beta^t F\left(x_{t-1}(\lambda, T), x_t(\lambda, T)\right). \tag{91}$$

Kamihigashi (2002)'s argument exploits a property of concavity. In particular, the fact that the slope of a concave function, $f : [\gamma, 1] \to \mathbb{R}$, is declining implies

$$\frac{f(1) - f(\lambda)}{1-\lambda} \leq \frac{f(1) - f(\gamma)}{1-\gamma}, \tag{92}$$

for $\lambda \in [\gamma, 1)$ (see Kamihigashi (2002), Lemma 3). Rearranging 91, we obtain

$$\begin{aligned} \beta^{T+1} \frac{F\left(x_T^*, \lambda x_{T+1}^*\right) - F\left(x_T^*, x_{T+1}^*\right)}{1-\lambda} &\leq \sum_{t=T+1}^{\infty} \beta^{t+1} \frac{F\left(x_t^*, x_{t+1}^*\right) - F\left(\lambda x_t^*, \lambda x_{t+1}^*\right)}{1-\lambda} \\ &\leq \sum_{t=T+1}^{\infty} \beta^{t+1} \frac{F\left(x_t^*, x_{t+1}^*\right) - F\left(\gamma x_t^*, \gamma x_{t+1}^*\right)}{1-\gamma}, \end{aligned} \tag{93}$$

where the second inequality is an application of (92). Note

$$
\begin{aligned}
&\beta^{T+1}\frac{F\left(x_T^*, \lambda x_{T+1}^*\right)-F\left(x_T^*, x_{T+1}^*\right)}{1-\lambda} \\
&\overset{\lambda\uparrow 1}{\to} -\beta^{T+1}F_2\left(x_T^*, x_{T+1}^*\right)x_{T+1}^* \\
&=\beta^{T+2}F_1\left(x_{T+1}^*, x_{T+2}^*\right)x_{T+1}^*+\beta^{T+1}\mu_{T+1}\left(A_{T+1}x_{T+1}^*-b_{T+1}\right)+\beta^{T+1}\mu_{T+1}b_{T+1} \\
&=\beta^{T+2}F_1\left(x_{T+1}^*, x_{T+2}^*\right)x_{T+1}^*+\beta^{T+1}\mu_{T+1}b_{T+1}\geq 0.
\end{aligned}
$$

Here, the limit result follows from differentiability of $F$, the equality uses (90) and the weak inequality reflects feasibility.

Driving $\lambda \to 1$ in (93) and using the latter result, we obtain,

$$
0\leq \beta^{T+2}\underbrace{F_1\left(x_{T+1}^*, x_{T+2}^*\right)x_{T+1}^*}_{\geq 0}+\beta^{T+1}\underbrace{\mu_{T+1}b_{T+1}}_{\geq 0}\leq \sum_{t=T+1}^{\infty}\beta^{t+1}\frac{F\left(x_t^*, x_{t+1}^*\right)-F\left(\gamma x_t^*, \gamma x_{t+1}^*\right)}{1-\gamma}.
$$

As $T \to \infty$, the term on the right converges to zero since the both infinite sums are finite when summed over all $t \geq 0$. This establishes (89). $\square$

## A.4 Proposition 1

*Proof.* For each $\pi_0 > 0$, there is a sequence of inflation, $\{\pi_t\}_{t=1}^{\infty}$, that satisfies (22) in which inflation explodes to $\infty$. The interest rate associated with such a sequence is greater than unity for each $t$, so that (25) is binding, $\bar{M}_T/P_T = c_T = 1$, for all $T$. Trivially, the transversality condition, (26), is satisfied.

$$
\beta^T\frac{\bar{M}_{T-1}}{P_T}=\beta^T\frac{1}{\pi_T}\to 0.
$$

Similarly, a sequence that satisfies (24) with $\pi_0 < 0$ converges to $\ln\left(\beta/\bar{\pi}^*\right)$, so that actual gross inflation converges to $\beta$. To see that this sequence satisfies (26) note that $\pi_t$ converges in finite time to its fixed point, $\pi_t = \ln\left(\beta/\bar{\pi}^*\right)$. Suppose convergence occurs at $t = \bar{t} \geq 0$. Then, for $T > \bar{t}$, $R_T = 1$ so (9) implies $P_T = \beta^{T-\bar{t}}P_{\bar{t}}$. Setting $\bar{M}_T = \beta^{T-\bar{t}}\bar{M}_{\bar{t}}$ for all $T > \bar{t}$ so that $\bar{M}_T/P_T = \bar{M}_{\bar{t}}/P_{\bar{t}}$, a constant $\geq c_T$ for all $T > \bar{t}$, we have that the cash constraint, (25), is satisfied and

$$
\lim_{T\to\infty}\beta^T\frac{\bar{M}_{T-1}}{P_T}=\lim_{T\to\infty}\frac{\beta^{T-t}}{\underbrace{P_T/P_t}_{=1}}\beta^t\frac{\bar{M}_{T-1}}{P_t}=\frac{\beta^t}{P_t}\lim_{T\to\infty}\bar{M}_{T-1}>0
$$

so that (26) is satisfied too. The uniqueness result is stated in Proposition 1.

Appendix: $\square$

### A.5 Taylor Strategy

Following is a proof of Lemma 1. For convenience, we restate Proposition 1 here:

**Lemma 4.** *Suppose $\bar{M}_{-1}$ is given and monetary policy sets $\bar{M}_t = \bar{\mu}c_t^{\rho}\bar{M}_{t-1}$ for $t \geq 0$,where $\overline{\mu} > 1$. There exists a unique competitive equilibrium with the properties:*

$$\bar{R}_t = \beta^{-1}\bar{\mu} > 1, \quad c_t = 1, \quad \textit{for} \quad t \geq 0, \quad \textit{and} \quad P_0 = \bar{M}_{-1}\bar{\mu}.$$

*Proof.* First, equations (8), (15) and (19) imply $c_t = 1$, so that monetary policy sets money growth to $\bar{\mu}$. Second, we show that in any equilibrium, $\bar{R}_t > 1$ for all $t$. Suppose, to the contrary, that $\bar{R}_t = 1$ for some $t \geq 0$. Then it follows that $\bar{R}_{t+1} = 1$. To see this, note that $\bar{\pi}_{t+1} = \beta$ by equation (9) so that $P_{t+1} < P_t$. Note also that $\bar{M}_{t+1}/\bar{M}_t = \bar{\mu} > 1$. Combining equations (10), (16), (18) and $c_t = 1$ we have $\bar{M}_t \geq P_t$. The results, $P_{t+1} < P_t$ and $\bar{M}_{t+1}/\bar{M}_t > 1$ then imply (using $c_{t+1} = 1$) $\bar{M}_{t+1} > P_{t+1}$. The complementary slackness condition, (10), then implies that $\bar{R}_{t+1} = 1$. By induction, we conclude that $\bar{R}_t = 1$ implies $\bar{R}_{t+s} = 1$ and $\bar{\pi}_{t+s+1} = \beta$ for $s \geq 0$. Finally,

$$q_T = q_t,$$

for all $T \geq t$ (see the equation after (11)). Note that for any fixed $t$,

$$q_T\bar{M}_T = q_t\bar{M}_T \to \infty,$$

because $q_t$ is fixed and $\bar{M}_T$ is increasing in $T$. This contradicts the transversality condition (11), so we conclude that in any equilibrium, $\bar{R}_t > 1$ for all $t \geq 0$.

Third, we show that with $\bar{R}_t > 1$, for all $t \geq 0$, the equilibrium conditions uniquely determine all variables. Equation (25) implies that the cash constraint binds, $\bar{M}_t - P_t = 0$, for each $t \geq 0$, so $\bar{\pi}_{t+1} = \bar{\mu}$ for all $t \geq 0$. Fourth, the Fisher equation, (9), implies

$$\bar{R}_t = \beta^{-1}\bar{\pi}_{t+1} = \frac{\bar{\mu}}{\beta} > 1.$$

The cash constraint, $c_0 = 1$ and $\bar{R}_0 > 1$ implies

$$P_0 = \bar{M}_0 \Longrightarrow P_0 = \bar{M}_{-1}\frac{\bar{M}_0}{\bar{M}_{-1}} = \bar{M}_{-1}\bar{\mu}.$$

We have established the results desired. □

Following is a proof of Lemma 2:

*Proof.* Suppose not, so that there exists an equilibrium with $\pi_T \notin [\pi_l, \pi_u]$ where $T \geq 0$ is the first date in which the monitoring range is violated. Consider the case, $\pi_T > \pi_u$. The Taylor rule implies

$R_T = \phi\pi_T > \pi_u$. Using Lemma 1, it is easy to verify that $\pi_{T+1} = \mu \leq \pi_u$, so that[85]

$$R_T - \pi_{T+1} > 0, \tag{94}$$

violating the Fisher equation, (21). This contradicts the assumption of equilibrium.

Next, consider the case $\pi_T < \pi_l$. Then, $R_T = \max\left\{R^l, \phi\pi_T\right\} \leq 0$. But, $R^l < \phi\pi_l < \pi_l$, so $R_T < \pi_l$. Also, it is easy to verify $\pi_{T+1} \geq \mu \geq \pi_l$, so that[86]

$$R_T - \pi_{T+1} < 0, \tag{95}$$

violating the Fisher equation, (21). This contradicts the assumption of equilibrium. This establishes the result sought. □

# B Appendix: The Market as a Game

## B.1 Proposition 4

Here, we we consider the cash constraint in the $\left(h_{t-1}, \pi^b_{i,t}\right)$ continuation equilibrium in Proposition (4). The proposition considers two types of histories, $h_{t-1}$ : those in which the inflation monitoring range has never been violated (see Part (i)) and those in which the inflation monitoring range has been violated at least once (Part (ii)). The cash constraint is always satisfied as an equality in Part (i) by the design of our monetary policy rule. However, our construction of the $\left(h_{t-1}, \pi^b_{i,t}\right)$ continuation equilibrium (see Section 4.1.5) in the second type of history ignored the time $t$ cash constraint. This is not an issue when the equilibrium interest rate is positive. But, needs to be verified in the case that the interest rate is zero. We do that here.

According to Proposition (4), the zero lower bound is binding for $\pi^b_{i,t} \notin D\left(h_{t-1}\right)$, or, more explicitly,

$$\frac{1-\gamma}{1-\rho}A_{i,t} + \mu \leq R_l, \tag{96}$$

where

$$A_{i,t} \equiv \mu + \ln\frac{\bar{M}_{t-1}}{P_{t-1}} - \pi^b_{i,t}.$$

For convenience, we repeat the relevant equilibrium conditions derived in the text. Equation (48) combines the period $t+1$ equilibrium allocations as well as the period $t$ and $t+1$ money growth rule

[85] The (unsurprising) result, $\pi_{T+1} = \mu$, follows from four observations. Observation #1 (see Definition 2) is that the escape clause is invoked in period $T+1$. Observation #2 is that observation #1 and Lemma 1 imply $P_{T+1} = \mu\bar{M}_T$. Observation #3 is that the unscaled interest rate, $\bar{R}_T$, exceeds unity. The latter can be seen from: the definition after (21), $\bar{R}_T \equiv \bar{R}^* \exp\left(R_T\right)$; our assumption, $\bar{R}^* > 1$; and our assumptions $\phi, \pi_T > 0$ which imply $R_T = \phi\pi_T > 0$. Observation #4 is that observation #3, (25) and the equilibrium condition, $c_t = 1$, imply $P_T = \bar{M}_T$. The result follows from observations #2 and #4.

[86] The result, $\pi_{T+1} \geq \mu$, is straightforward. As in Footnote (85), Lemma 1 implies $P_{T+1} = \mu\bar{M}_T$. Combining equation (15) and equation (18), $c_t = 1$ for all $t \geq 0$. So the cash constraint, (25), implies $\bar{M}_T \geq P_T$. The result follows.

to obtain:

$$\pi_{i,t+1} = \mu + \rho \ln c_{i,t} + A_{i,t}. \tag{97}$$

The household Euler equation in the zero lower bound, making use of the fact, $\ln c_{i,t+1} = 0$, is

$$\pi_{i,t+1} = R_l + \gamma \ln c_{i,t}, \tag{98}$$

where $R_l$ denotes the scaled and logged interest rate at the zero lower bound (see Section 3.5). Combining the loan market clearing condition, $W_t l_t = X_t + d_t$, (see equation (18)) with the household cash constraint, equation (7), we obtain:

$$\ln \frac{\bar{M}_{i,t}}{P_{i,t}} \geq \ln c_{i,t} \tag{99}$$

Finally, the money growth rule is, expressed in terms of real balances,

$$\ln \frac{\bar{M}_{i,t}}{P_{i,t}} = \rho \ln c_{i,t} + A_t. \tag{100}$$

Equations (97), (98) and (100) are the three equations used in the text to solve for the three unknowns, $\bar{M}_{i,t}, c_{i,t}, \pi_{i,t+1}$. These are solved given $A_{i,t}, P_{i,t}$ implied by $\left(h_{t-1}, \pi^b_{i,t}\right)$ which have the property that the equilibrium nominal rate of interest is at its lower bound (i.e., $A_{i,t}$ satisfies equation (96). Here, we verify that in the equilibrium constructed in in the text, the household cash constraint, (99) is satisfied when $\rho < \min\{\gamma, 1\}$.

Combining equation (97) and (98), we can solve for $\ln c_{i,t}$:

$$\ln c_{i,t} = \frac{\mu - R_l + A_{i,t}}{\gamma - \rho}. \tag{101}$$

Also, after substituting out for $\rho \ln c_{i,t} + A_t$ in equation (97) from equation (100) we obtain $\pi_{i,t+1} = \mu + \ln \frac{\bar{M}_{i,t}}{P_{i,t}}$, which, when combined with (98) yields

$$\gamma \ln c_{i,t} = \ln \frac{\bar{M}_{i,t}}{P_{i,t}} + \mu - R_l \geq \ln c_{i,t} + \mu - R_l,$$

where the weak inequality uses (99). Collecting terms, we have

$$(\gamma - 1) \ln c_{i,t} \geq \mu - R_l. \tag{102}$$

Substituting out for $\ln c_{i,t}$ in (102) using (101), we obtain,

$$(\gamma - 1) \frac{\mu - R_l + A_{i,t}}{\gamma - \rho} \geq \mu - R_l. \tag{103}$$

This establishes that the equilibrium conditions, (97), (98), (97) and the cash constraint, (99) imply equation (103). It is easily verified that if equation (103) and the equilibrium conditions, (97), (98), (97), are satisfied, then the cash constraint, (99) is satisfied. We conclude that equation (103) is equivalent to the cash constraint, given our three equilibrium conditions. For this reason, in this appendix we refer to (103) as the cash constraint.

With this characterization result in hand, we ask whether the cash constraint, (103), is satisfied at the equilibrium values of $c_{i,t}$ and $\pi_{i,t+1}$ derived in the text, for $A_{i,t}$ such that the nominal interest rate is at its lower bound (i.e., that satisfy (96)), given our restrictions on $\gamma$ and $\rho$.

Consider first the case, $\gamma > 1$ and $\rho < 1$. The interest rate being at its lower bound places the following bound on $A_{i,t}$ (see equation (96)):

$$A_{i,t} \geq \frac{1-\rho}{1-\gamma}\left(R_l - \mu\right). \tag{104}$$

The coefficient on $A_{i,t}$ in the cash constraint, (103), is positive so the term on the left of that inequality is minimized by minimizing the value of $A_{i,t}$. Putting $A_{i,t}$ at the minimum consistent with the interest being at its zero lower bound (see equation (104)), we obtain

$$(\gamma - 1)\frac{\mu - R_l + A_{i,t}}{\gamma - \rho} \geq (\gamma - 1)\frac{\mu - R_l + \frac{1-\rho}{1-\gamma}\left(R_l - \mu\right)}{\gamma - \rho} = \mu - R_l.$$

We conclude that the cash constraint, (103), is satisfied.

Now consider the case, $\gamma < 1$ and $\rho < \gamma$. The bound on $A_{i,t}$ implied by the interest rate being at its lower bound is (e.g., equation (96)) is

$$A_{i,t} \leq \frac{1-\rho}{1-\gamma}\left(R_l - \mu\right). \tag{105}$$

The coefficient on $A_{i,t}$ in the cash constraint, (103), is negative, so the term on the left of that inequality is minimized by maximizing the value of $A_{i,t}$. Putting it at the maximum consistent with the interest rate being at its lower bound is (see equation (105)), we obtain:

$$(\gamma - 1)\frac{\mu - R_l + A_{i,t}}{\gamma - \rho} \geq (\gamma - 1)\frac{\mu - R_l + \frac{1-\rho}{1-\gamma}\left(R_l - \mu\right)}{\gamma - \rho} = \mu - R_l.$$

We conclude that the cash constraint, (103), is satisfied. This establish the result we set out to prove.

## B.2 Proof of Proposition (5)

Result (i) follows from Lemma 1 and Proposition 2. Now consider (ii). Suppose we have a $\pi_t^b$ that has the fixed point property in (ii). The only thing that distinguishes an $h_{t-1}$ continuation equilibrium and an $\left(h_{t-1}, \pi_t^b\right)$ continuation equilibrium (see Definition 4) is that the latter does not necessarily satisfy the time $t$ intermediate good equilibrium condition, $P_t = W_t$ (see equation (15)). This difference can be described in terms of $F$. Suppose $\pi_t^b$ is an arbitrary belief about how other firms set their (scaled and logged, as in equation (30)) prices. In the continuation equilibrium associated with $\left(h_{t-1}, \pi_t^b\right)$, the (scaled and logged) aggregate price index is simply $\pi_t^b$ itself. The (scaled and logged) wage rate in the $\left(h_{t-1}, \pi_t^b\right)$ continuation equilibrium is $F\left(h_{t-1}, \pi_t^b\right)$ (see equation (33)). So, a fixed point of $F$ corresponds to a sequence, $(a_{t+j})_{j=0}^{\infty}$, that satisfies all equilibrium conditions in periods $t$,$t+1,\ldots$. Hence the sequence is an $h_{t-1}$ continuation equilibrium. Now consider the converse in part (ii). Let $(a_{t+j})_{j=0}^{\infty}$ denote an $h_{t-1}$ continuation equilibrium. By definition, all period $t, t+1, \ldots$ equilibrium conditions (including equation (15)) are satisfied. Let $\pi_t$ denote period $t$ inflation in this equilibrium and set $\pi_t^b = \pi_t$. Because equation (15) is satisfied, it follows that $\pi_t^b = F\left(h_{t-1}, \pi_t^b\right)$. This establishes (ii).

Consider (iii). First, we establish existence of a fixed point for $F$, for any $h_{t-1}$. According to part (i) there is a unique $h_{t-1}$ continuation equilibrium. By the converse proof of part (ii) above, it then follows that there exists a fixed point for $F$. To see that this fixed point is unique, suppose it is not. If there were two fixed points of $F$ for some history, $h_{t-1}$, each corresponds to a distinct $h_{t-1}$ continuation equilibrium. This contradicts (i). Finally, consider (iv). Condition (iii) implies there exists a unique fixed point of $F$, and condition (ii) implies that the $\left(h_{t-1}, \pi_t^b\right)$ continuation equilibrium associated with that fixed point coincides with the $h_{t-1}$ continuation equilibrium. Since, by Proposition 2, inflation in the $h_{t-1}$ continuation equilibrium is $\pi_t = 0$, it follows that the same is true in the $\left(h_{t-1}, \pi_t^b\right)$ continuation equilibrium. But, $\pi_t^b = \pi_t$ in the latter equilibrium, establishing (iv).

# C Appendix: Rationalizable Implementation

## C.1 Iterated Deletion and Rationalizability

Let $L$ denote an action taken by others and let $l = F(L)$ denote the individual's best response to $L$. In terms of this notation, Definition 7 is expressed as follows. Consider the following sequence of sets:

$$\left\{F^{-k}(L)\right\}_{k=0}^{\infty}. \tag{106}$$

Here, $F^{-1}(L)$ denotes the set of beliefs, $\Omega$, for which $L$ is a best response. That is, $F(\Omega) = L$. Also,

$$F^{-2}(L) \equiv F^{-1}(\Omega).$$

The set, $F^{-2}(L)$, is the set of beliefs which justify the actions corresponding to each element in $\Omega$. The sets, $F^{-n}(L)$, for $n > 2$ are defined similarly. Finally, $F^{-0}(L) \equiv \{L\}$. If, for all $n$, $F^{-n}(L) \neq \{\emptyset\}$, then there exists at least one sequence of numbers in equation (106) that is a chain of justified beliefs that supports $L$. If so, then according to Definition 7, we say that $L$ is rationalizable.

Under the method of iterated deletion, the agent's initial candidate set of beliefs corresponds to the set of justified beliefs, which we denote by $\Pi_1$. Then $F(\Pi_1)$ denotes the set of best responses associated with some justified belief. Suppose there is a value of $l \in \Pi_1$ which is not a best response to any $L \in \Pi_1$, so that there no circumstances in which the individual would choose $l$. Under CK, the individual would also expect that others would never choose such an action, and so it would be deleted from the set of candidate beliefs. Specifically, the individual would restrict its beliefs about what others do to $\Pi_2 = \Pi_1 \bigcap F(\Pi_1)$. [87]

This process can be repeated, leading to a sequence, $\Pi_{k+1} = \Pi_k \bigcap F(\Pi_k)$, for $k = 1, 2, \ldots$ . Notice that by construction, the sequence is non-decreasing, $\Pi_k \subseteq \Pi_{k-1}$ for $k = 2, \ldots$ . Consider the set of elements, $L \in \Pi_1$, such that they remain undeleted even as $k \to \infty$:

$$\Pi^*(\Pi_1) = \bigcap_{k=1}^{\infty} \Pi_k. \tag{107}$$

So, the set, $\Pi^*(\Pi_1)$, is composed of the beliefs in $\Pi_1$ that survive iterated deletion. We will also have occasion to apply $\Pi^*$ to sets different from the set of justified beliefs, $\Pi_1$. When the argument of $\Pi^*(\cdot)$ is $\Pi_1$, we economize on notation by dropping the argument. Following is a characterization of the relationship between $\Pi^*$ and the set of rationalizable beliefs:

**Proposition 16.** *The following hold:*

*(i) $L$ is rationalizable if and only if $L \in \Pi^*$;*

*(ii) if there exists a set $\Pi$ having the property, $F(\Pi) = \Pi$, then each element in $\Pi$ is rationalizable;*

*(iii) $\Pi^*(\Pi) \subseteq \Pi \bigcap \left(\bigcap_{k=1}^{\infty} F^k(\Pi)\right)$ for any set, $\Pi$, that is a subset of the domain of $F$.* [88]

The proof is provided in the next section.

Result (i) is useful because sometimes it is more convenient to study rationalizability by chain of justification and at other times iterated deletion is more convenient. Part (ii) of Proposition 16 provides another characterization of rationalizability that is useful for us below. Part (iii) will be very useful for us below. It implies that if the object on the right of '$\subseteq$' is a singleton when $\Pi = \Pi_1$, then $\Pi^*(\Pi_1)$ is a singleton too, and we have (by part (i)), unique rationalizability. [89]

## C.2 Proof of Proposition 16

*Proof.* (i) (Sufficiency) Suppose, to the contrary, that $L \notin \Pi^*$ but $L$ is rationalizable according to Definition 8. By rationalizability, $L \in \Pi_1$. The fact, $L \notin \Pi^*$, implies by equation (107), that there exists a $k$ such that $L \notin \Pi_k$ but $L \in \Pi_{k-1}$. Since $L$ is rationalizable, $F^{-k}(L)$ is not empty and is a

[87] If $C$ is a set, then $g(C) = \{y : y = g(x), \text{for } x \in C\}$. Also, $A \bigcap B$ means $\{y : y \in A, y \in B\}$.
[88] Here $F^2(A) \equiv F(F(A))$, and $F^k(A)$ is recursively as follows: $F^k(A) = F\left(F^{k-1}(A)\right)$ for all $k \geq 2$.
[89] If $F(\Pi) \subseteq \Pi$ then, the '$\subseteq$' can be replaced by an equality.

subset of $\Pi_1$. It then follows that $L \in F^k(\Pi_1)$. In particular, there exists a sequence of $\{L'_s\}_{s=1}^{k-1}$ such that

$$L = F\left(L'_{k-1}\right), \quad L'_s = F\left(L'_{s-1}\right), \quad L'_1 \in \Pi_1,$$

for all $s \leq k-1$. Note that $L_s \in \Pi_s$ for all $s \leq k-1$ by construction. So, $L \in \Pi_k$, a contradiction.

(Necessity) Suppose that $L \in \Pi^*$, but is not rationalizable. If follows by Definition 7 that there does not exist a chain of justified beliefs that supports $L$. This means that there exists a $k \geq 1$ such that $F^{-k}(L) = \{\emptyset\}$ and $F^{-s}(L) \neq \{\emptyset\}$ for all $s \leq k-1$. It follows that $F^{k-1}(L) \notin \Pi_1$ and, if $k > 1$, $F^s(L) \in \Pi_1$ for all $s \leq k-1$. So, $L \notin \Pi_k$, a contradiction.

(ii) Consider an arbitrary $L_1 \in \Pi$, where $F(\Pi) = \Pi$. There exists a $L_2 \in \Pi$ which justifies $L_1$. By induction, we can construct a sequence, $\{L_k\}_{k=1}^{\infty}$, recursively. Therefore $L_1$ is rationalizable.

(iii) The claim immediately holds since $\Pi_{k+1} \subset F(\Pi_k)$ for all $k \geq 1$. $\square$

## C.3 Impact of Refinement on Competitive Equilibria

In the body of the paper we show that our refinement has the effect of trimming economically uninteresting off-equilibrium paths under the Taylor rule with an escape clause. Could it be that the refinement trims all undesired competitive equilibria, eliminating the need for the escape clause in the first place? Here, we show that the answer is 'no'. The set of equilibria with the Taylor rule and no escape clause is essentially the same with or without our refinement. Section 3.5 indexes the equilibria by the initial gross rate of inflation, $\bar{\pi}_0 > 0$.[90] The refinement eliminates a minuscule set of equilibria by trimming those with $\bar{\pi}_0 < \bar{\pi}^* \varepsilon$ for arbitrarily small $\varepsilon > 0$.[91]

But, what if we extend our refinement so that intermediate good firms have a similar cognitive impairment which prevents them from being able to distinguish large numbers? Suppose we adopt a cognitive impairment which results in a cap on the price level. Such an impairment would not be interesting because it would rule out all equilibria in which inflation is positive, such as our desired equilibrium. An alternative approach results in an arbitrarily high, but finite, cap, $\bar{\pi}^{cap}$, on inflation. We explore this approach in detail and conclude that this refinement does not trim a substantial number of equilibria and thus is not a substitute for the escape clause.

One might suppose that a cap on inflation would trim competitive equilibria in which the initial inflation rate, $\bar{\pi}_0$, is larger than its desired value, $\bar{\pi}^*$.[92] A rationale for that conjecture might be the argument in Section 3.5 which shows that, absent any refinement and absent the escape clause, all equilibria with $\bar{\pi}_0 > \bar{\pi}^*$ lead to exploding inflation rates. We argue that such a conjecture is wrong. In fact there are many equilibria with $\bar{\pi}_0 > \bar{\pi}^*$. The only equilibria that are ruled out are the ones with $\bar{\pi}_0 > \bar{\pi}^{cap}$. This does not significantly trim the number of equilibria since $\bar{\pi}^{cap}$ can be arbitrarily large.

[90] Obviously, there can be no equilibrium with $\bar{\pi}_0 \leq 0$, since $\bar{\pi}_0 = P_0/P_{-1}$, where $P_t$ denotes the price of consumption in period $t$.

[91] Recall, $\bar{\pi}^*$ denotes the gross inflation rate in the desired equilibrium.

[92] A cap on inflation, $\bar{\pi}^{cap}$, would have no impact on our analysis of rationalizability in off-equilibrium paths because we find that there is a natural upper bound on inflation on those paths.

With the refinement and no escape clause there remains an equilibrium corresponding to $\bar{\pi}^* < \bar{\pi}_0 \leq \bar{\pi}^{cap}$. Each equilibrium exhibits 'exploding inflation' in the sense that in each equilibrium inflation converges to the very high number, $\bar{\pi}^{cap}$. The key reason that inflation does not literally explode to infinity is that when the cognitive impairment becomes binding, the Euler equation ceases to be the Fisher equation. We proceed now to make these observations formal.

In our analysis, only the intermediate good firms have a cognitive impairment. All other agents solve their equilibrium conditions exactly. A cognitive impairment may prevent the intermediate good firm from setting its period $t$ price equal to the actual period $t$ wage rate.[93] We assume that there is an arbitrarily small value of $\delta$, where $\delta$ lies in the interval, $(0,1)$, such that when the intermediate good firm is confronted with a wage rate, $W_t$, with the property, $W_t > (P_{t-1}\bar{\pi}^*)/\delta$, then it actually perceives the wage to be $(P_{t-1}\bar{\pi}^*)/\delta$. Absent our refinement, optimization leads firms to set $P_t = W_t$ (see equation (15)). With our refinement, the equation characterizing firm optimization, (15), is replaced by the following complementary slackness-type condition:

$$(W_t - P_t)(\ln(1/\delta) - \pi_t) = 0,\ W_t \geq P_t,\ \ln(1/\delta) \geq \pi_t. \tag{108}$$

Here, recall that $\pi_t$ is the logged and scaled inflation rate, $\pi_t = \ln(P_t/(P_{t-1}\bar{\pi}^*))$. In terms of scaled and logged inflation, the inflation cap is the third term in equation (108).

The representative household's Euler equation for employment, equation (8), together with the resource constraint, $c_t = l_t$, corresponds to

$$\frac{W_t}{P_t} = c_t^{\gamma+\psi},\ \gamma > 0,\ \psi \geq 0. \tag{109}$$

Given $P_t > 0$, equations (108) and (109) are equivalent to:

$$(c_t - 1)\left(\ln\frac{1}{\delta} - \pi_t\right) = 0,\ c_t \geq 1,\ \ln(1/\delta) \geq \pi_t. \tag{110}$$

The intertemporal Euler equation, equation (9), combined with the Taylor rule, equation (17), is:

$$\ln(c_{t+1}/c_t) = \frac{1}{\gamma}(\phi\pi_t - \pi_{t+1}). \tag{111}$$

We now consider the set of equilibria for the model. As in Section 3 equilibria can be indexed by $\pi_0$. Obviously, the equilibria associated with $\pi_0 \leq 0$ are the same with or without the refinement. Also, equilibria associated with $\pi_0 > \ln(1/\delta)$ no longer exist. This is a small set of equilibria, since $\delta > 0$ can be arbitrarily small. We now consider the equilibria associated with $\pi_0 \in (0, \ln(1/\delta)]$. In particular, we construct a sequence,$\{c_t, \pi_t\}_{t=0}^{\infty}$, corresponding to each $\pi_0 \in (0, \ln(1/\delta)]$ which

[93] Recall, we are studying the competitive equilibrium, in which it is assumed (recall the clairvoyance assumption discussed at the end of Section 15) that the firm directly observes the actual equilibrium wage rate. The cognitive impairment results in a possible distinction between the wage rate that is observed and the wage rate that is perceived.

satisfies equations (111) and (110) for $t \geq 0$.[94] After that we show that each of these sequences is an equilibrium because it satisfies the transversality condition of the household, equation (11).

Consider $t = 0$. Since $0 < \pi_0 \leq \ln(1/\delta)$, equation (110) is satisfied by setting $c_0 = 1$. Compute $\pi_t$ for $t = 1, ..., T + 1$ using

$$\pi_t = \phi^t \pi_0,$$

where $t = T + 1$ is the first date when $\pi_t > \ln(1/\delta)$.[95] Since $\pi_0 > 0$ and $\phi > 1$, a finite $T \geq 1$ exists. Then, the computed $\pi_t$, along with $c_t = 1$, for $t = 0, ..., T$, satisfy equations (111) and (110).

Now consider equation (111) for $t = T$, setting $\pi_{T+1} = \ln(1/\delta)$:

$$\ln(c_{T+1}) = \frac{1}{\gamma}(\phi\pi_T - \ln(1/\delta)) > 0, \tag{112}$$

so that $c_{T+1} > 1$. Then, for $t > T + 1$, set $\pi_t = \ln(1/\delta)$ and

$$\frac{c_{t+1}}{c_t} = (1/\delta)^{\frac{1}{\gamma}(\phi-1)}. \tag{113}$$

with cap on price level. With $\{c_t, \pi_t\}_{t=0}^{\infty}$ computed in this way, equations (110) and (111) are satisfied for all $t$.

To verify the transversality condition, note that for $j > 1$ equations (112) and (113) imply:

$$c_{T+j} = c_{T+1}(1/\delta)^{(j-1)\frac{1}{\gamma}(\phi-1)}.$$

In the transversality condition, equation (11), we have that $m_j = P_j c_j$ and $q_j$ is proportional to $\beta^j u'(c_j)/P_j$, where $u'(c_j)$ denotes the marginal utility of consumption. Note that $q_j m_j$ is proportional to $\beta^T c_T u'(c_T) = \beta^T c_T^{1-\gamma}$,thus ensuring that $\beta^T c_T u'(c_T) \to 0$ if $\gamma \geq 1$. We have established the following result:

**Proposition 17.** *Suppose $\gamma \geq 1$ and that the intermediate good firm has the cognitive impairment defined in this subsection. There exists an equilibrium corresponding to each $\pi_0 \in (-\infty, \ln(1/\delta)]$. The equilibrium associated with each $-\infty < \pi_0 \leq 0$ coincides with the corresponding equilibrium without the refinement (see Proposition 1). The constructed equilibrium associated with each $0 < \pi_0 \leq \ln(1/\delta)$ has an inflation rate that converges to $\ln(1/\delta)$ in finite steps.*

With this cognitive impairment there are multiple equilibria, including equilibria with very high inflation, as in the version of the model with no refinement and no escape clause. The nature of the equilibria are someone different, however, even if the number of equilibria is roughly the same. Most importantly, without the refinement all the equilibria generate the same amount of welfare. With the refinement welfare associated with high inflation is very low. This is because consumption growth is

[94] Equilibria in which $\pi_0 > -\ln\delta$ are ruled out by (110).

[95] Here, we drop the dependence of $T$ on $\delta$ and $\pi_0$ to simplify notation.

eventually greater than unity and eventually becomes extraordinarily high (see equation (113)). This is a large deviation from the desired equilibrium, which is first best.

## C.4 Rationalizability When The Monitoring Range Has been Violated

**Proposition 18.** *Consider an arbitrary history $h_{t-1}$ in which the monitoring range has violated. If condition (63) is satisfied, then there exists a unique robustly rationalizable belief.*

*Proof.* Fix any arbitrary small $0 < \varepsilon \leq e^{\pi^b(h_{t-1})}$. We show that for any such $\varepsilon$, the conditions for Proposition 9 are satisfied for some set. Then Proposition 9 implies that $\Pi^*(\varepsilon)$ is a singleton for all $0 < \varepsilon \leq e^{\pi^b(h_{t-1})}$. Since the Nash belief, $\pi^b(h_{t-1})$, is included in $\Pi^*(\varepsilon)$, we conclude that there exists a unique robustly rationalizable belief.

In order to show that the three conditions for Proposition 4 are satisfied, we first show that $F\left(h_{t-1}, \pi^b_{i,t}\right)$ has the following properties: (i) $F\left(h_{t-1}, \pi^b_{i,t}\right)$ is increasing in $\pi^b_{i,t}$ for all $\pi^b_{i,t} \leq \bar{\pi}^b(h_{t-1})$ and decreasing otherwise where $\bar{\pi}^b(h_{t-1})$ is a unique value satisfying the following linear equation,

$$\frac{1-\gamma}{1-\rho}\left(\mu + \ln \frac{\bar{M}_{t-1}}{P_{t-1}} - \pi^b_{i,t}\right) + \mu = R_l;$$

(ii) the slop of $F\left(h_{t-1}, \pi^b_{i,t}\right)$ is less than 1 in its absolute value.

In order to show (i), the conditions (63) imply that $\gamma > 1$ and $\rho < 1$. Therefore if $\pi^b_{i,t} \leq \bar{\pi}^b(h_{t-1})$, then $\pi^b_{i,t} \notin D(h_{t-1})$. Otherwise, $\pi^b_{i,t} \in D(h_{t-1})$. Consider $\pi^b_{i,t} \leq \bar{\pi}^b(h_{t-1})$. The slope of $F\left(h_{t-1}, \pi^b_{i,t}\right)$ is $(\rho + \psi)/(\rho - \gamma)$, which is given by (ii) in Proposition 4. Since $\rho < \gamma$ from the assumption, $\rho - \gamma < 0$. From the last condition in (63), we have

$$\rho + \psi < 1 - \frac{\gamma + \psi}{2} + \psi = 1 - \frac{\gamma - \psi}{2}. \tag{114}$$

The first condition in (63) implies that

$$2 + \psi < \gamma. \tag{115}$$

So, combining the above two equations (114) and (115), we obtain

$$\rho + \psi < 1 - \frac{\gamma - \psi}{2} < 1 - \frac{2 + \psi - \psi}{2} = 0.$$

Therefore, $(\rho + \psi)/(\rho - \gamma)$ is positive. Now consider $\pi^b_{i,t} > \bar{\pi}^b(h_{t-1})$. The slope of $F\left(h_{t-1}, \pi^b_{i,t}\right)$ is $\left(1 - \frac{\gamma+\psi}{1-\rho}\right)$, which is given by (ii) in Proposition 4. It is easy to show that the slope is negative since $1 - (\gamma + \psi) < \rho$.

Now we prove (ii). It is immediate that $(\rho + \psi)/(\rho - \gamma) < 1$ since

$$(\rho + \psi) > (\rho - \gamma),$$

and $\rho - \gamma < 0$. It is easy to show that $1 - \frac{\gamma+\psi}{1-\rho}$ is bigger than $-1$ if $\rho < 1 - \frac{\gamma+\psi}{2}$.

The two properties (i) and (ii) imply that $\pi^{\max} = \max_{\pi \in [\ln \varepsilon, \infty)} F(h_{t-1}, \pi) < \infty$. Thus intermediate firms can disregard any beliefs $\pi^b_{i,t} > \pi^{max}$. Set $\Omega = [\ln \varepsilon, \pi^{\max}]$. Then all the conditions in Proposition 9 are satisfied. So we conclude that for all $0 < \varepsilon < \exp(\pi(h_{t-1}))$,

$$\Pi^* (\varepsilon) = \pi (h_{t-1}).$$

Therefore $\Pi^{r,*} = \pi (h_{t-1})$, which is desired. □

## C.5 Local Rationalizability

In this section, we derive $F^{(N)}(h_{t-1}, \pi^b)$ for the history in which the monitoring range has been violated once and show that $DF^{(N)}$ is a lower triangular matrix whose diagonal elements are $1 - (\gamma + \psi) / (1 - \rho)$.

Consider first $h_{t-1}$ in which there has been a violation in the monitoring range. First notice that at the Nash equilibrium belief, $\pi^b(h_{t-1}) = \left(\mu + \ln \frac{\bar{M}_{t-1}}{P_{t-1}}, 0, \cdots, 0\right)$, the zero lower bound does not bind. This comes from the fact that the interest rates at the equilibrium belief are given by $R_{i,t+j} = \mu$ for all $j \in \{0, \cdots, N-1\}$. Therefore, we can choose a small neighborhood of $\pi^b(h_{t-1})$ so that for any $\pi^b \in \Pi$, the zero lower bound does not bind.

For $\pi^b \in \Pi$, the money growth rule (see Definition 2) is in place in periods $t+j$, for all $j \geq 0$. We know from Lemma 1 that $\ln c_{i,t+N} = \pi_{i,t+N} = 0$. The period $t+j$ wage is the simple function of $\pi^b_{i,t+j}$ and $\ln c_{i,t+j}$ given in equation (42). We obtain $\ln c_{i,t+j}$ simply by iterating forward on the equilibrium cash constraint (25) with the given initial real balance, $\ln \frac{M_{t-1}}{P_{t-1}}$. Iterating in this way yields:

$$\ln c_{i,t+j} = \frac{1}{1-\rho} \left[ \mu - \pi^b_{i,t+j} + \sum_{s=0}^{j-1} \left( \mu_{t+s} \left( \pi^b_{i,t}, \cdots, \pi^b_{i,t+s} \right) - \pi^b_{i,t+s} \right) + \ln \frac{M_{t-1}}{P_{t-1}} \right],$$

where $\mu_{t+s}(\pi_t, \cdots, \pi_{t+s})$ is recursively defined given the initial real balance $\ln \frac{M_{t-1}}{P_{t-1}}$:

$$\mu_{t+s}(\pi_t, \cdots, \pi_{t+s}) = \mu + \frac{\rho}{1-\rho} \left[ \mu - \pi^b_{i,t+j} + \sum_{k=0}^{s-1} \left( \mu_{t+k} \left( \pi^b_{i,t}, \cdots, \pi^b_{i,t+k} \right) - \pi^b_{i,t+k} \right) + \ln \frac{M_{t-1}}{P_{t-1}} \right],$$

with the understanding that the summation is defined as zero if $s - 1 < 0$. So, we have for all $t$

$$F^{(N)}_j (h_{t-1}, \pi^b) = \pi^b_{t+j-1} + \frac{\gamma + \psi}{1 - \rho} \left[ \mu - \pi^b_{i,t+j} + \sum_{s=0}^{j-1} \left( \mu_{t+s} \left( \pi^b_{i,t}, \cdots, \pi^b_{i,t+s} \right) - \pi^b_{i,t+s} \right) + \ln \frac{M_{t-1}}{P_{t-1}} \right].$$

It is clear that $F^{(N)}_j (h_{t-1}, \pi^b)$ is a linear function of $\pi^b$ and only depends on the past and current inflation. So $DF^{(N)}$ is a lower triangular matrix. It is trivial show that the diagonal elements are $1 - (\gamma + \psi) / (1 - \rho)$. We establish the desired.

## C.6 Multiple Shots

Our strategy equilibrium concept allows the $i^{th}$ intermediate good firm to entertain a wide range of beliefs about what other firms do in period $t$ when it makes its own period $t$ price decision. But, conditional on a given belief about what other firms do in period $t$, only one sequence of future actions by all intermediate good firms (including the $i^{th}$ firm itself) is contemplated. These are the actions in the unique $\left(h_{t-1}, \pi_t^b\right)$ continuation equilibrium. In effect, the $i^{th}$ firm treats the other firms in period $t$ as players in a large period $t$ game, but all players (including the $i^{th}$ firm itself) are treated as robots satisfying equilibrium conditions (or, playing Nash) in future periods.

A natural question is whether we obtain the same results when we consider $N-$period beliefs, captured by the $N \times 1$ vector, $\overrightarrow{\pi}_{i,t}^b \equiv \left[\pi_{i,t}^b, \pi_{i,t+1}^b, ..., \pi_{i,t+N-1}^b\right]'$. The vector, $\overrightarrow{\pi}_{i,t}^b$, contains the beliefs of the $i^{th}$ intermediate good firm about the prices set by the other intermediate good firms in periods $t+s$, $s=0,...,N-1$, for $\infty > N > 0$. In this sense, the analysis up to now has only addressed $N=1$ period beliefs. For the $i^{th}$ firm to determine its best response when $N>1$ requires that it compute an $\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^b\right)$ continuation equilibrium. This is a sequence, $(a_{i,t+s})_{s=0}^{\infty}$, that satisfies all time $t+s$ equilibrium conditions, $s \geq 0$, not including the firm optimality conditions, equation (15), in periods $t+s$, for $s=0,...,N-1$. The variables in the continuation equilibrium that are of interest to the $i^{th}$ firm are $(w_{i,t},...,w_{i,t+N-1}) = F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^b\right)$. The firm's best response is the analog of (34), $(x_{i,t},...,x_{i,t+N-1}) = F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^b\right)$, where $x_{i,t+s}$ is the (scaled and logged) price set by the firm in period $t+s$. The function, $F^{(N)}$ inherits the piecewise linear structure that it has for $N=1$ (see Proposition 4). With $N=1$, $F^{(N)}$ is composed of 5 linear segments. As $N$ increases, the type of global analysis we were able to do in the $N=1$ case becomes unwieldy. Instead, we proceed as in Evans et al. (2018) by doing a local analysis of implementation.

Let $\Pi\left(h_{t-1}, \varepsilon\right)$ denote an $\varepsilon$-cylinder around the Nash equilibrium, $\overrightarrow{\pi}^b\left(h_{t-1}\right)$, defined by $\overrightarrow{\pi}^b\left(h_{t-1}\right) = F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}^b\left(h_{t-1}\right)\right)$. Thus, $\Pi\left(h_{t-1}, \varepsilon\right) = \overrightarrow{\pi}^b\left(h_{t-1}\right) + \left\{\overrightarrow{\pi}^b \in \mathbb{R}^N : \left|\pi_i^b\right| \leq \varepsilon, i=1,...,N\right\}$, where $\varepsilon > 0$, but small.[96] We will show that the $N \times 1$ vector-valued function, $F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^b\right)$, is linear for $\overrightarrow{\pi}_{i,t}^b \in \Pi\left(h_{t-1}, \varepsilon\right)$ and has a unique zero for any $h_{t-1}$. We will exploit this linearity property to establish what we call local rationalizability.

Consider first $h_{t-1}$ in which there never has been a violation in the monitoring range. For $\overrightarrow{\pi}_{i,t}^b \in \Pi\left(h_{t-1}, \varepsilon\right)$, the Taylor rule is in place in periods $t+j$, for $j \geq 0$. We know from Proposition (2) that $\ln c_{i,t+N} = \pi_{i,t+N} = 0$. The period $t+j$ wage is the simple function of $\pi_{i,t+j}^b$ and $\ln c_{i,t+j}$ given in equation (42). We obtain $\ln c_{i,t+j}$ by iterating backward on the household Euler equation, with the given initial conditions for $c_{i,t+N}$ and $\pi_{i,t+N}$. Iterating in this way yields:

$$\ln c_{i,t+j} = -\frac{1}{\gamma}\sum_{k=j}^{N-2}\left(\phi\pi_{i,t+k}^b - \pi_{i,t+k+1}^b\right) - \frac{\phi}{\gamma}\pi_{i,t+N-1}^b,$$

[96] For $h_{t-1}$ in which the monitoring range has never been violated, we can choose $\varepsilon = \min\left\{-\pi_l, \pi_u\right\}$. Otherwise, we can choose the $\varepsilon$- cylinder so that the zero lower bound is never encountered. (See equation (39) for the restrictions on $\pi_t^b$ when $N=1$.) For both types of $h_{t-1}$, $F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^b\right)$ in our model is a linear function.

where we define the summation as zero when $j > N-2$. We have

$$F_j^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^{b}\right) = \pi_{t+j-1}^{b} - \left(\frac{\gamma+\psi}{\gamma}\right)\left[\sum_{k=j-1}^{N-2}\left(\phi\pi_{t+k}^{b} - \pi_{t+k+1}^{b}\right) + \phi\pi_{t+N-1}^{b}\right],$$

where $j = 1$ is the top element of the $N \times 1$ vector, $F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^{b}\right)$. Evidently, the function is linear in $\overrightarrow{\pi}_{i,t}^{b}$ and $F^{(N)}$ has the following representation:

$$F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^{b}\right) = DF^{(N)} \times \overrightarrow{\pi}_{i,t}^{b},$$

where $DF^{(N)}$ denotes an $N \times N$ upper diagonal matrix with $1 - \left(\frac{\gamma+\psi}{\gamma}\right)\phi$ along its diagonal. Equation (63) implies that $\left|1 - \left(\frac{\gamma+\psi}{\gamma}\right)\phi\right| < 1$. With its eigenvalues inside the unit circle, $DF^{(N)}$ is a convergent matrix[97], so that $\left[DF^{(N)}\right]^k\left(\Pi\left(h_{t-1}, \varepsilon\right)\right) \to \{0\}$ as $k \to \infty$.[98] We adopt the idea of local rationalizability in Evans et al. (2018) by assuming that $\Pi\left(h_{t-1}, \varepsilon\right)$ is the set of justified beliefs and that this is common knowledge.[99] In that case, it follows trivially by the convergence property of $DF^{(N)}$ that $\overrightarrow{\pi}_{i,t}^{b} = 0$ is the unique locally rationalizable belief. In this sense, we obtain a local version of unique implementation.

In the case where $h_{t-1}$ contains at least one violation of the inflation monitoring range, it is also possible to construct $F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^{b}\right)$ locally for $\overrightarrow{\pi}_{i,t}^{b}$ in an $\varepsilon$-cylinder, $\Pi\left(h_{t-1}, \varepsilon\right)$, about the Nash equilibrium. The Nash equilibrium, $\overrightarrow{\pi}^{b}\left(h_{t-1}\right)$, exists and is unique because of the uniqueness of an $h_{t-1}$ continuation equilibrium (see Ii) of Proposition 5). Our linear characterization requires only that the cylinder, $\Pi\left(h_{t-1}, \varepsilon\right)$, be sufficiently 'narrow' that it excludes inflation rates low enough to avoid the zero lower bound on the interest rate. In this case, the $N \times 1$ vector-valued function, $F^{(N)}$, can be written $F^{(N)}\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^{b}\right) = DF^{(N)} \times \left[\overrightarrow{\pi}_{i,t}^{b} - \overrightarrow{\pi}^{b}\left(h_{t-1}\right)\right]$. The $N \times N$ matrix, $DF^{(N)}$, is lower triangular with the scalar, $1 - (\gamma+\psi)/(1-\rho)$, in every entry on its diagonal. So, $DF^{(N)}$ is a convergent matrix if $\left|1 - (\gamma+\psi)/(1-\rho)\right| < 1$. This condition is satisfied under the parameter restrictions, equation 63, sufficient for unique implementation (see Proposition 11).

We summarize the above results as follows. Consider first the analog of unique implementation for arbitrary $N$:

**Definition 11.** Let $\Pi\left(h_{t-1}, \varepsilon\right)$ be an $\varepsilon$- cylinder about the unique Nash equilibrium. Suppose it is CK that intermediate good firms only consider beliefs in the set, $\Pi\left(h_{t-1}, \varepsilon\right)$. A competitive equilibrium satisfies *locally unique N-shot implementation* if there exists $\varepsilon > 0$ such that: (i) there is a continuation equilibrium for each $h_{t-1}$ and for each $\left(h_{t-1}, \overrightarrow{\pi}_{i,t}^{b}\right)$ such that $\overrightarrow{\pi}_{i,t}^{b} \in \Pi\left(h_{t-1}, \varepsilon\right)$ and (ii) for each $h_{t-1}$, $\Pi^*\left(\Pi\left(h_{t-1}, \varepsilon\right)\right) = \left\{\overrightarrow{\pi}^{b}\left(h_{t-1}\right)\right\}$, where $\Pi^*$ is defined in 107.

[97] A convergent matrix, $T$, is a matrix that has the property, $\lim_{k\to\infty} T^k = 0$. A matrix, $T$, is convergent iff its largest eigenvalue is less than unity in absolute value (see Meyer (2000, p. 617)).

[98] $\left[DF^{(N)}\right](\Pi)$ is defined as a set $\left\{DF^{(N)}x; x \in \Pi\right\}$. $\left[DF^{(N)}\right]^k(\Pi)$ is defined as recursively. $\left[DF^{(N)}\right]^k(\Pi) = \left[DF^{(N)}\right]\left(\left[DF^{(N)}\right]^{k-1}\Pi\right)$ for all $k \geq 2$.

[99] Evans et al. (2018) refer to their idea as 'local eductive stability'.

Condition (i) corresponds to the requirement that the equilibrium locally be a strategy equilibrium (see Definition 6). Condition (ii) requires that only the Nash equilibrium, $\overrightarrow{\pi}^b(h_{t-1})$, survives iterated deletion of beliefs.

We now state the following result:

**Proposition 19.** *Consider the model in which monetary policy is governed by the Taylor strategy (see Definition 2). Then the competitive equilibrium satisfies locally unique N-shot implementation if and only if*

$$\left|1-\left(\frac{\gamma+\psi}{\gamma}\right)\phi\right|<1, \quad \left|1-\left(\frac{\gamma+\psi}{1-\rho}\right)\right|<1. \tag{116}$$

*Suppose monetary policy is governed by the zero monitoring range policy (see Definition 3). Then the necessary and sufficient conditions for locally unique N-shot implementation are the second condition in equation (116) and the following:*

$$\left|1-\left(\frac{\gamma+\psi}{\gamma-1}\right)\phi\right|<1.$$

The first condition in equation (3) pertains to histories, $h_{t-1}$, when the inflation monitoring range has never been violated. The condition applies to the slope of the best response function in a small neighborhood around the unique Nash equilibrium. When $\pi^l<\pi^u$, that slope is different than when $\pi^l=\pi^u$ (see, for example, Figure 5a). This is the reason for the different sufficiency conditions across the two monetary policies considered in the Proposition 19.

# D Appendix: The Taylor Strategy Versus Two Alternatives

This section describes the stochastic version of the model studied in the main text. The global equilibrium analysis of this model under the Taylor strategy is more tedious than the analysis of the deterministic model. While the equilibrium conditions in the deterministic model are linear in the log of the variables, this is not the case in the stochastic version of the model. So, we work with a version of the equilibrium conditions that have been log-linearized about the unique equilibrium allocations in the deterministic version of the model. These allocations are also the allocations in the locally unique stochastic version of the model.[100] As in the analysis of Poole (1970), the Taylor rule has the effect of preventing the money demand shock from having an impact on allocations. Under the money growth rule, consumption and employment are a function of the realization of the money demand shock. However, inflation is not a function of that shock (prices are set before the money

[100] In the deterministic model, the off-equilibrium path in which monetary policy is the money growth rule can have a constant term, $\mu$, that lies anywhere in a specific interval (see Definition 2). This means that we are somewhat flexible in what the inflation rate and interest rate are in these off-equilibrium paths. In the stochastic model, we set $\mu=0$, so that the inflation rate in the money growth regime is the same as it is in the interest rate regime. This ensures the accuracy of our log-linear approximation.

demand shock is realized) and has the consequence that much of the best response function analysis in the main text goes through with relatively little modification.

The first section below in effect repeats all the analysis done in the paper for the stochastic model. However, our various concepts are modified to reflect that our analysis is local to the Taylor strategy equilibrium in the deterministic model. Thus, Section 3 below defines and analyzes a *linear competitive equilibrium* under the Taylor strategy. We establish local uniqueness of that equilibrium. Section 4 transforms the equilibrium into a game and describes the model best response functions. We establish that the linear competitive equilibrium is a *local strategy equilibrium.* Finally, the third subsection below establishes unique local implementation of the equilibrium. All this analysis pertains to what we call one-period deviations in the introduction to Section 5. The extension to multi-period (or, multi-shot) deviations discussed in Online Appendix C.6 applies equally to the stochastic model discussed here. That is because, as we show below, the properties of the best response function, $F^{\nu}$, in the stochastic model are locally identical to the best response function, $F$, in the model without shocks. The main reason for this is that prices are set before the realization of the current period money demand shock, which is itself independent over time.

Finally, the results for trembles in Section 6.1 applies equally to the stochastic model studied here. We evaluate welfare before $t = 0$ when the equilibrium starts up. When a tremble occurs, the aggregate price level is perturbed as in equation (69)). Under the zero monitoring range strategy, $\pi_l = \pi_u$, so that a tremble, no matter how small, induces a reversion to the constant money growth regime. In the Taylor strategy with $\pi_l < \pi_u$, a tremble, if sufficiently small (or, if $\pi_u - \pi_l$ is sufficiently big), will never trigger the escape clause. Under the Taylor strategy, the equilibrium is first best with $c_t = 1$ for all $t$. Under the money growth regime, consumption fluctuates stochastically in response to the money demand shock (see equation (122)). As a result, in the version of the model with trembles, the zero monitoring range strategy is welfare inferior to the Taylor rule strategy with $\pi_l < \pi_u$. This establishes We state the result for trembles in Proposition 13.

## D.1 Money Demand Shocks

We introduce a shock to money demand, $\nu_t$, that is realized after intermediate good firms set their price:[101]

$$P_t c_t \leq (m_{t-1} + W_t l_t - d_t) \exp(\nu_t) . \tag{117}$$

All other time $t$ variables are determined after the realization of $\nu_t$. We modify our definition of a history, $h_{t-1}$, to include the record of all shocks up to, and including, period $t-1$. For reasons of tractability, we limit ourselves to an analysis of the log-linearized equilibrium conditions and we ignore the non-negativity constraint on the interest rate.[102] The equilibrium conditions, after log linearizing in the neighborhood of the desired equilibrium, are:

[101] The model is a variant of the sticky price model in Christiano et al. (1997), where time $t$ prices are predetermined when time $t$ shocks are realized.

[102] Our style of analysis in this section follows the approach in Atkeson et al. (2010).

$$\ln c_t = E_t \ln c_{t+1} - \frac{1}{\gamma}\left(R_t - E_t \pi_{t+1}\right) \tag{118}$$

$$0 = E_{t-1} \ln c_t \tag{119}$$

$$\ln c_t = \nu_t + \ln \frac{\bar{M}_t}{P_t} \tag{120}$$

$$R_t = \phi \pi_t. \tag{121}$$

Here, $E_t$ denotes the conditional expectation based on shocks up to period $t$. The first equation is the log-linearized version of the household's intertemporal optimality condition (9).[103] The second equation combines the intermediate good firm's optimality condition (hence, the presence of $E_{t-1}$) with the household's intra-temporal optimality condition, (8), and the resource constraint, equation (19).[104] The third equation is the equilibrium version of the cash constraint, (117), that takes into account loan market clearing (see equation (18)). In our analysis in this section, we impose (120) as a strict equality. The last equation is the version of the Taylor rule, (22), which ignores the lower bound constraint on $R_t$.

As in Section 3.6, when the government conducts its monetary policy based on the money growth rule, equation (121) is replaced with the money growth rule in Definition 2.

Monetary policy in the log-linearized economy is defined as a version of Definition 2 in which we ignore the non-negativity constraint on the interest rate:

**Definition 12.** The Taylor strategy in the linearized economy is: (i) for histories, $h_{t-1}$, in which $\pi_s \in [\pi_l, \pi_u]$ for each $s \leq t-1$ the period $t$ rule is given by (121) with $\phi > 1$ and (ii) for the other histories the money growth rule is the one in Definition 3.6.

We define a linear competitive equilibrium as follows:

**Definition 13.** A linear competitive equilibrium with an escape clause is an allocation $a = (a_t)_{t=0}^{\infty}$ such that (i) $a$ satisfies equations (118), (119), (120); (ii) monetary policy is given in Definition 12; (iii) there exists a constant $C > 0$ such that $var(\pi_t) < C$ for all $t \geq 0$.

By $var(\pi_t)$ we mean the variance of $\pi_t$ conditional on information available in period $t = 0$. Condition (iii) in Definition 13 guarantees that the variance of all other real variables is also bounded.

### D.1.1 Uniqueness of Competitive Equilibrium with Taylor Rule under Escape Clause

The analysis proceeds as follows. We begin by establishing a version of Lemma 1 which (as before) helps to guarantee that inflation expectations are anchored in the long run. In particular, under the

[103] Log-linearization in this equation entails two approximation errors. First, we replace $\ln E_t\left(1/c_{t+1}\right)$ by $E_t \ln\left(1/c_{t+1}\right)$ after taking the log of equation (118). Second, we replace $\ln E_t\left(1/\bar{\pi}_t\right)$ with $E_t \ln\left(1/\bar{\pi}_t\right)$. We also scale inflation before logging, but that involves no approximation error.

[104] As in equation (118), the only approximation in equation (119) is that we replace $\ln E_{t-1} c_t$ with $E_{t-1} \ln c_t$ .

money growth rule in Lemma 1, there exists a unique linear competitive equilibrium. The equilibrium allocations are similar those in Lemma 1, with the obvious exception that we have a money demand shock.

**Lemma 5.** *Suppose $\bar{M}_{-1}$ is given and the monetary rule is $\bar{M}_t = c_t^{\rho}\bar{\mu}\bar{M}_{t-1}$ for $t \geq 0$. There exists a unique linear competitive equilibrium with the properties:*

$$R_t = \mu + \frac{\rho - \gamma}{1-\rho}\nu_t, \quad \ln c_t = \frac{1}{1-\rho}\nu_t, \quad \pi_{t+1} = \mu + \frac{\rho}{1-\rho}\nu_t, \quad \textit{for} \quad t \geq 0, \quad \textit{and} \quad P_0 = \bar{M}_{-1}\bar{\mu}. \tag{122}$$

Recall (see equation (23)) that $\mu$ is scaled and logged $\bar{\mu}$. The uniqueness proof is straightforward and we include it here.[105]

*Proof.* Combining the cash constraint (120) with equation (119), $E_{t-1}\ln\left(\bar{M}_t/P_t\right) = 0$. Rearranging the latter and taking into account that $P_t$ is contained in the $t-1$ information set, we have

$$\ln P_t = E_{t-1}\ln\left(c_t^{\rho}\bar{\mu}\bar{M}_{t-1}\right) = \ln\bar{\mu} + \ln\bar{M}_{t-1}. \tag{123}$$

The object in parentheses uses monetary policy to substitute out for $\bar{M}_t$. The second equality in equation (123) uses equation (119). First differencing equation (123) and using the monetary policy rule, we obtain

$$\pi_{t+1} = \mu + \rho\ln c_t \tag{124}$$

for all $t \geq 0$. Consumption is determined by combining equation (120), the monetary policy rule and equation (123):

$$\ln c_t = \nu_t + \ln\frac{\bar{M}_t}{\bar{M}_{t-1}\bar{\mu}} = \nu_t + \rho\ln c_t = \frac{1}{1-\rho}\nu_t. \tag{125}$$

Finally, the expression for $R_t$ in equation (122) is determined by combining equations (119), (118) and (124). Uniqueness follows by construction. □

Next, we prove that in any linear competitive equilibrium with an escape clause, the inflation monitoring range is not violated:

**Lemma 6.** *Suppose monetary policy is given by Definition 12. Any linear competitive equilibrium has the property that $\pi_t \in [\pi_l, \pi_u]$ for all $t \geq 0$.*

[105] The proof is simpler than the proof of Lemma 1 because that proof respects the lower bound constraint on the nominal interest rate. To establish uniqueness, we have to consider the possibility that the lower bound is binding and the cash constraint is not satisfied as an equality. To rule out this kind of equilibrium the proof of Lemma 1 uses the transversality condition of the household (see equation (11)). When we ignore the lower bound on the interest rate, as we do here, the transversality condition is not needed to establish uniqueness of the equilibrium under the money growth rule. Interestingly, as long as the money demand shock, has a suitable lower bound, the non-negativity constraint on the nominal interest rate, $R_t \geq R_l$, is satisfied in the unique equilibrium.

The proof appears in Online Appendix D.2 and essentially follows the proof of Lemma 3.6. Lemma 6 is crucial for establishing the analog of Proposition 2:

**Proposition 20.** *Suppose monetary policy is given by Definition 12. The only linear competitive equilibrium is the desired equilibrium. Money growth in this equilibrium is* $\bar{M}_t = \exp\left(-\nu_t + \nu_{t-1}\right)\bar{M}_{t-1}\bar{\mu}$.

Given Lemma 6, the proof is straightforward and is given in Online Appendix D.3. That proof makes use of the variance condition, (ii) in Definition 13.

Comparing $c_t$ in the money growth equilibrium (see Lemma 5) with $c_t$ in the equilibrium in which monetary policy is the interest rate rule in Definition 12, we can see the Poole (1970) result. Under the money growth rule, consumption fluctuates inefficiently. By contrast, the interest rate rule supports the desired allocations (i.e., $c_t = 1$ and $\pi_t = 0$, independent of $\nu_t$), which are first-best in our environment. Not surprisingly, there exists a money growth rule that can achieve the same result. Indeed, Proposition 20 describes a money growth rule for which the only equilibrium is the desired allocations. However, that rule requires observing the money demand shocks. The advantage of the interest rate rule is that it does not require observing $\nu_t$ and nevertheless is able to cancel the welfare-reducing effects of $\nu_t$ on $c_t$.

### D.1.2 Local Strategy Equilibrium

Now we consider a local strategy equilibrium, which is the analog of Definition 6.

**Definition 14.** Let $\Pi\left(h_{t-1},\varepsilon\right)$ be an $\varepsilon$- neighborhood of the unique Nash equilibrium. A linear competitive equilibrium is a *local strategy equilibrium* if there exists $\varepsilon > 0$ such that there is a linear continuation equilibrium for each $h_{t-1}$ and for each $\left(h_{t-1},\pi^b_{i,t}\right)$ such that $\pi^b_{i,t} \in \Pi\left(h_{t-1},\varepsilon\right)$.

As in Section 4, we show that for each $h_{t-1}$ and $\pi^b_{i,t}$, there exists a continuation equilibrium. First consider $h_{t-1}$ in which the monitoring range has never been violated. We construct the $\left(h_{t-1},\pi^b_{i,t}\right)$ continuation equilibrium for $\pi^b_{i,t} \in [\pi_l,\pi_u]$. The $i^{th}$ intermediate good firm wishes to set its price to the expected value of the wage. As before, the realized time $t$ nominal rate can be expressed as the product of the real wage and the aggregate price level. The (scaled) aggregate price level is $\exp\left(\pi^b_{i,t}\right)$ by our symmetry assumption on beliefs and the price equation, (13). Households observe the realized time $t$ real wage at the time that they make their $c_t$ and $l_t$ decisions. It follows that the realized period $t$ real wage is $c^{\gamma+\psi}_{i,t}$, after using the resource constraint, equation (19). So, the (scaled) price set by the $i^{th}$ firm is set as follows:

$$\exp\left(x_{i,t}\right) = E_{t-1}\exp\left(\left(\gamma+\psi\right)\ln c_{i,t}\right)\exp\left(\pi^b_{i,t}\right).$$

After linearizing this expression in terms of $x_{i,t}$, $\ln c_{i,t}$ and $\pi^b_{i,t}$ around their values in the desired equilibrium, we obtain:

$$x_{i,t} = \left(\gamma+\psi\right)E_{t-1}\ln c_{i,t} + \pi^b_{i,t}. \tag{126}$$

The mapping from $\pi_{i,t}^b$ to $c_{i,t}$ is obtained by an argument similar to the one used in Section 4. According to the monetary policy rule, the Taylor rule (121) is operative at date $t$. Moreover, the Taylor strategy will be in place in period $t+1$ as well. According to Proposition (20), the unique equilibrium beginning in $t+1$ has the property, $c_{i,t+1} = 1$, $\pi_{i,t+1} = 0$ regardless of the realization of $\nu_t$ and $\nu_{t+1}$. Combining these facts with the intertemporal Euler equation, (118), and the Taylor rule, (121), the mapping from $\pi_{i,t}^b$ to $\ln c_{i,t}$ is simply $\ln c_{i,t} = -\frac{\phi}{\gamma}\pi_{i,t}^b$, so that after substitution into equation (126) we obtain

$$F^\nu\left(h_{t-1}, \pi_{i,t}^b\right) = \left[1 - \frac{\phi\left(\gamma+\psi\right)}{\gamma}\right]\pi_{i,t}^b. \tag{127}$$

Here, $F^\nu$ denotes the segment of the $i^{th}$ firm's best response function for the indicated $h_{t-1}$ and for $\pi_{i,t}^b \in [\pi_l, \pi_u]$. The superscript, $\nu$, indicates that this is the best response function in the linearized economy. This segment of the best response function coincides with the same segment in the (non-linearized) version of our economy without $\nu$ shocks (compare equation (127) with the $\pi_{i,t}^b \in [\pi_l, \pi_u]$ segment of $F$ in equation (37)). More broadly, it is easy to verify that $F^\nu\left(h_{t-1}, \pi_{i,t}^b\right) = F\left(h_{t-1}, \pi_{i,t}^b\right)$ for all $h_{t-1}$ and all $\pi_{i,t}^b$, except when the zero lower bound on the interest rate is binding in the non-linearized model. It is easy to verify that $F^\nu$ corresponds to the best response function in Proposition 4 in which the min operator is replaced by $\phi R_t$ and the set, $D$, is replaced by the real line. In terms of Figure 3, $F^\nu$ is $F$ in which the 'hook' to the left of the origin is pushed off to $-\infty$. We summarize these findings as follows:

**Proposition 21.** *In the linearized economy with monetary policy given in Definition 12, the unique linear competitive equilibrium is a local strategy equilibrium with best response function, $F^\nu$, for the following two types of history, $h_{t-1}$:*

*(i) if $h_{t-1}$ has the property that the inflation monitoring range has never been violated, then $F^\nu$ takes the following form:*

$$F^\nu\left(h_{t-1}, \pi_{i,t}^b\right) = \begin{cases} \left[1 - \frac{\phi}{\gamma}\left(\gamma+\psi\right)\right]\pi_{i,t}^b & \pi_{i,t}^b \in [\pi_l, \pi_u] \\ \left[1 - \phi\frac{\gamma+\psi}{\gamma-1}\right]\pi_{i,t}^b + \frac{\gamma+\psi}{\gamma-1}\mu & \pi_{i,t}^b \notin [\pi_l, \pi_u] \end{cases}, \tag{128}$$

*(ii) if $h_{t-1}$ has the property that the inflation monitoring range has been violated at least once, then $F^\nu$ takes the following form:*

$$F^\nu\left(h_{t-1}, \pi_{i,t}^b\right) = \left(1 - \frac{\gamma+\psi}{1-\rho}\right)\pi_{i,t}^b + \frac{\gamma+\psi}{1-\rho}\left(\mu + \ln\frac{\bar{M}_{t-1}}{P_{t-1}}\right), \tag{129}$$

*(iii) the set of fixed points of $F^\nu$, $\{x : x = F^\nu\left(h_{t-1}, x\right)\}$, coincides with $\pi^b\left(h_{t-1}\right)$ in part (iii) of Proposition 5.*

We established (i) above. For a proof of (ii), see Section D.4 below. Part (iii) follows from the observation that $F = F^\nu$ in a neighborhood of the unique Nash equilibrium of the version of the model without the money demand shock.

### D.1.3 Unique Implementation

The definition of unique implementation suited to the framework in this Online Appendix is:

**Definition 15.** A linear competitive equilibrium satisfies *unique local implementation* if the model has the following two properties: (i) it is a local strategy equilibrium and (ii) there exits $\varepsilon$ such that for each $h_{t-1}$, $\Pi^*\left(\Pi\left(h_{t-1},\varepsilon\right)\right)=\left\{\pi^b\left(h_{t-1}\right)\right\}$, where $\Pi\left(h_{t-1},\varepsilon\right)$ is an $\varepsilon$- neighborhood of the unique Nash equilibrium, $\pi^b\left(h_{t-1}\right)$.

This definition of unique local implementation is essentially the same as the concept of 'locally strongly eductive stability' in Evans et al. (2018, p. 831).

## D.2 Lemma 6

*Proof.* Suppose not, so that there exists an equilibrium with $\pi_T\notin[\pi_l,\pi_u]$ where $T\geq 0$ is the first date in which the monitoring range is violated. The Taylor rule implies $R_T=\phi\pi_T\notin[\pi_l,\pi_u]$. The money rule will be followed in period $T+1$ and Lemma 5 implies $P_{T+1}=\bar{M}_T\bar{\mu}$. Dividing by $P_T$ and using the period $T$ cash constraint, we obtain $\pi_{T+1}=\mu+\ln c_T$. By the period $T$ Euler equation, (118) we have, using (119) and (121),

$$\ln c_T=-\frac{1}{\gamma}\left(R_T-\pi_{T+1}\right)=-\frac{1}{\gamma}\left(\phi\pi_T-\pi_{T+1}\right).$$

Substituting $\pi_{T+1}=\mu+\ln c_T$ into the above equation,

$$\ln c_T=-\frac{1}{\gamma}\left(\phi\pi_T-\mu-\ln c_T\right).$$

Taking the expectation as of time $T-1$, using (119) and the facts, $\mu\in[\pi_l,\pi_u]\,,\phi>1:$

$$\pi_T=\frac{\mu}{\phi}\in[\pi_l,\pi_u]\,,$$

a contradiction. This establishes the result sought. □

## D.3 Proposition 20

*Proof.* Suppose, to the contrary, that $\pi_t\neq 0$ for some $t$. From Lemma 2 equilibrium has the property that the monitoring range is never violated, i.e., $\pi_t\in[\pi_l,\pi_u]$. The Taylor rule, (121), the first order conditions by the representative household, (118) and firm optimality, (119), imply that in equilibrium:

$$-\gamma\ln c_t=\phi\pi_t-\pi_{t+1}. \tag{130}$$

Taking the expectation as of date $t-1$, we obtain

$$\phi \pi_t = E_{t-1} \pi_{t+1}.$$

We can equivalently write this as

$$\pi_t = \phi \pi_{t-1} + \varepsilon_{t-1},$$

where $\{\varepsilon_t\}_{t=0}^{\infty}$ independent over time, with $E_{t-1}\varepsilon_t = 0$ and $\varepsilon_t \perp \pi_{t-s}$, $s \geq 0$. Then, after repeated substitution, we obtain

$$\pi_t = \phi^t \pi_0 + \sum_{s=1}^{t} \phi^{s-1} \varepsilon_{t-s}.$$

Then the variance of $\pi_t$ is

$$var(\pi_t) = \sum_{s=1}^{t} \phi^{s-1} var(\varepsilon_{t-s}).$$

If $var(\varepsilon_s) \neq 0$ for some $s$, then the variance of $\pi_t$ goes up without bound since $\phi > 1$. So, in any local equilibrium, $\varepsilon_t = 0$ for all $t \geq 0$.

$$\pi_t = \phi^t \pi_0. \tag{131}$$

Then, if $\pi_t \neq 0$ the inflation monitoring range will be violated eventually since $\phi > 1$. But, that would contradict Lemma 6. So, $\pi_0 = 0$. Equation (131) implies that $\pi_t = 0$ for all $t = 0$. Therefore the nominal interest rate $R_t$ is zero for all $t$. Also, equation (130) implies that the consumption $c_t$ is zero for all $t \geq 0$. Therefore, the desired allocation is the unique equilibrium under the Taylor strategy. □

## D.4 Proof for Proposition 21

*Proof.* (i) Consider $\pi_{i,t}^b$ such that it lies inside of the monitoring range. Suppose that $\pi_{i,t}^b \notin [\pi_l, \pi_u]$. Then according to Lemma 6, the price level at $t+1$ is $\ln P_{t+1} = \ln \bar{M}_t + \ln \bar{\mu}$. Differencing (123) and using the cash constraint (120), we obtain

$$\pi_{t+1} = \mu + \ln c_t - \nu_t. \tag{132}$$

Substituting this expression into the Euler equation (118), we get : $\ln c_t = \frac{1}{1-\gamma}\left[\phi \pi_{i,t}^b - \mu + \nu_t\right]$.So the substituting the equation into equation (126),

$$F\left(h_{t-1}, \pi_{i,t}^b\right) = \left[1 - \phi \frac{\gamma + \psi}{\gamma - 1}\right] \pi_{i,t}^b + \frac{\gamma + \psi}{\gamma - 1} \mu.$$

Now we prove (ii). Combining the monetary policy and the cash constraint (126),

$$\ln c_t = \frac{1}{1-\rho}\left[\nu_t + \mu - \pi_{i,t}^b + \ln \frac{\bar{M}_{t-1}}{P_{t-1}}\right].$$

Substituting this equation into equation (126), we obtain $F\left(h_{t-1}, \pi_{i,t}^b\right)$ for the histories in which the monitoring range has been violated in the past:

$$F\left(h_{t-1}, \pi_{i,t}^b\right) = \left[1 - \frac{\gamma+\psi}{1-\rho}\right]\pi_{i,t}^b + \frac{\gamma+\psi}{1-\rho}\left[\mu + \ln\frac{\bar{M}_{t-1}}{P_{t-1}}\right].$$

□

# E Appendix: Conclusion

Here, we consider a simple version of the standard New Keynesian model with Calvo-style price setting frictions. We start by describing the non-linear equations of the model and we derive the best response functions for intermediate good firms in the game representation of the model. We analyze the stochastic version of the model in a small neighborhood of its interior nonstochastic steady state. We derive an upper bound on $\phi$ that is necessary for the desired equilibrium to be uniquely implementable (the bound is the analog of the first expression in equation (64)). We show that this upper bound is higher, the smaller is the set of marginal price setters. This substantiates a claim made after equation (66) and in the conclusion of the main text. Our analysis provides a roadmap to derive analogous necessary conditions in a wide range of specifications of the New Keynesian model.

We establish our results by working with the equilibrium conditions and the best response functions linearized around non-stochastic equilibrium. Our logic is that if an equilibrium fails to be uniquely rationalizable conditional on beliefs being within an arbitrarily small neighborhood of the equilibrium, then unique rationalizability (hence, unique implementation) fails for the underlying nonlinear system. Since our analysis is local, we have no need to specify our escape clause and the inflation monitoring range must simply be an interval that contains the target inflation rate in its interior.[106] We must assume (though we do not display) that there exists an escape strategy which guarantees global uniqueness of the equilibrium that we study.

## E.1 Households

The representative household solves:

[106] Thus, we do not consider the zero monitoring range policy in Definition (3).

$$
\begin{aligned}
\max \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \\
\text{s.t.} \quad & P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + T_t,
\end{aligned} \tag{133}
$$

where $B_t$ denotes the beginning-of-period $t$ stock of bonds, and $C_t, N_t, W_t, P_t, T_t$ denote consumption, employment, the nominal wage rate, the nominal price level, and the profits net of government transfers and taxes, respectively. The exogenous disturbance to preferences, $\tau_t$, had the following law of motion:

$$
\tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}.
$$

The first order optimality conditions are:

$$
\begin{aligned}
\frac{1}{C_t} &= \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \\
e^{\tau_t} C_t N_t^{\varphi} &= \frac{W_t}{P_t}.
\end{aligned}
$$

where $\bar{\pi}_t \equiv \frac{P_t}{P_{t-1}}$.

## E.2 Firms

### E.2.1 Aggregate Output

Final goods are produced by a representative, competitive firm using the following production function:

$$
Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}.
$$

Profit maximization leads to the following demand for $Y_{i,t}$

$$
Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^{\varepsilon}
$$

where $P_{i,t}$ is the price of $Y_{i,t}$. Substituting the demand curve back into the production function and rearranging, we obtain the following restriction across prices:

$$
P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}. \tag{134}
$$

### E.2.2 Intermediate Good Firm

The intermediate good,$Y_{i,t}$, is produced by a monopolist using the following production function:

$$Y_{i,t} = e^{a_t} N_{i,t},$$

where

$$\Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a,\ \Delta a_t \equiv a_t - a_{t-1}.$$

The monopolist chooses its price and output subject to the demand curve and the Calvo price friction:

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1-\theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases} . \tag{135}$$

With probability $1-\theta$ the firm chooses its price, denoted by $\tilde{P}_t$, optimally. We denote the real pre-subsidy marginal cost by $s_t$:

$$s_t = \frac{\frac{W_t}{P_t}}{e^{a_t}} = e^{\tau_t} C_t N_t^{\varphi} / e^{a_t},$$

after using the household's intratemporal Euler equation. The post-tax marginal cost is $(1-\nu)\, s_t$, so that the firm's objective is:

$$E_t^i \sum_{j=0}^{\infty} \beta^j\ \upsilon_{t+j} \left[ P_{i,t+j} Y_{i,t+j} - P_{t+j} (1-\nu)\, s_{t+j} Y_{i,t+j} \right],$$

where $\upsilon_{t+j}$ denotes the Lagrange multiplier on household budget constraint. Here, the superscript, $i$, on the expectation operator is meant to signal that the expectation is over the idiosyncratic price shock, in addition to aggregate variables. As is customary with the standard, simple New Keynesian model, we set $\nu$ to eliminate the monopoly distortion in steady state:

$$\frac{\varepsilon}{\varepsilon - 1} (1-\nu) = 1.$$

Let $\widetilde{P}_t$ denote the price set by the $1-\theta$ firms who optimize at time $t$. These firms are referred to as the 'marginal price setters'. The expected value of future profits are the sum of two parts: future states in which price is still $\widetilde{P}_t$, so $\widetilde{P}_t$ matters, and future states in which the price is not $\widetilde{P}_t$, so $\widetilde{P}_t$ is irrelevant. In this way we can express the intermediate good firm objective as follows

$$\begin{aligned} & E_t^i \sum_{j=0}^{\infty} \beta^j \upsilon_{t+j} \left[ P_{i,t+j} Y_{i,t+j} - P_{t+j} (1-\nu)\, s_{t+j} Y_{i,t+j} \right] \\ &= \overbrace{E_t \sum_{j=0}^{\infty} (\beta\theta)^j\ \upsilon_{t+j} \left[ \tilde{P}_t Y_{i,t+j} - P_{t+j} (1-\nu)\, s_{t+j} Y_{i,t+j} \right]}^{Z_t} + \chi_t, \end{aligned}$$

where $Z_t$ is the present value of future profits over all future states in which the firm's price is $\tilde{P}_t$ and $X_t$ is the present value over all other states, so $d\chi_t/d\widetilde{P}_t = 0$. Also, the absence of the $i$ superscript on $E_t$ indicates that the expectation only involves aggregate variables, because the expectation over idiosyncratic variables is captured explicitly by the presence of $\theta^j$ in the objective.

Substitute out demand curve in the firm's objective, and ignoring $\chi_t$,

$$\begin{aligned} & E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j}(1-\nu) s_{t+j} Y_{i,t+j}\right] \\ &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[\tilde{P}_t^{1-\varepsilon} - P_{t+j}(1-\nu) s_{t+j} \tilde{P}_t^{-\varepsilon}\right]. \end{aligned}$$

Take the FONC with respect to $\tilde{P}_t$ :

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[(1-\varepsilon)\left(\tilde{P}_t\right)^{-\varepsilon} + \varepsilon P_{t+j}(1-\nu) s_{t+j} \tilde{P}_t^{-\varepsilon-1}\right] = 0,$$

so that

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - s_{t+j}\right] = 0,$$

using the fact that the subsidy cancels the monopoly markup. Substitute out for the Lagrange multiplier:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \overbrace{\frac{u'(C_{t+j})}{P_{t+j}}}^{=\upsilon_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - s_{t+j}\right] = 0.$$

and using the assumed log-form of utility, as well as the resource constraint, $Y_{t+j} = C_{t+j}$, we obtain:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - s_{t+j}\right] = 0,$$

$$\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t},$$

$$\bar{\pi}_t \equiv \frac{P_t}{P_{t-1}}, \quad X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}} & j \geq 1 \\ 1 & j = 0 \end{cases} \tag{136}$$

$$X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}} \quad j > 0. \tag{137}$$

Solving the above expression:

$$\tilde{P}_t = \frac{P_t e^{\tau_t} p_t^* N_t^{1+\varphi} + P_t E_t \sum_{j=1}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}}$$

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}, \tag{138}$$

say. Consider the numerator term:

$$\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} s_{t+j} \\
&= s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left( \overbrace{\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1}}^{=X_{t,j}, \quad \because (137)} \right)^{-\varepsilon} s_{t+j} \\
&= s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} s_{t+1+j} \\
&\overbrace{=}^{\text{law of iterated expectations}} s_t + \beta\theta E_t E_{t+1} \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} s_{t+1+j} \\
&= s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \overbrace{E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} s_{t+1+j}}^{=K_{t+1}}
\end{aligned}$$

so, we have,

$$K_t = s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}(1) \tag{139}$$

Similarly, for $F_t$:

$$F_t = 1 + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \ (2). \tag{140}$$

## E.3 Aggregate Restrictions

Evaluating equation x(134) using the assumption, equation (135):

$$P_t = \left( (1-\theta) \tilde{P}_t^{1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}},$$

or, after rearranging,

$$\tilde{p}_t = \left[ \frac{1 - \theta (\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}.$$

Let

$$Y_t^* \equiv \int_0^1 Y_{i,t} di.$$

Substituting out from the demand curve for $Y_{i,t}$, we obtain:

$$Y_t^* = Y_t \int_0^1 \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di = Y_t P_t^{\varepsilon} \left( P_t^* \right)^{-\varepsilon},$$

where $P_t^* = \left( \int_0^1 P_{i,t}^{-\varepsilon} di \right)^{\frac{-1}{\varepsilon}}$. We conclude

$$Y_t = p_t^* e^{a_t} N_t,$$

where

$$p_t^* = \left( \frac{P_t^*}{P_t} \right)^{\varepsilon}, \; P_t^* = \left( \int_0^1 P_{i,t}^{-\varepsilon} di \right)^{-\frac{1}{\varepsilon}}.$$

The variable, $p_t^*$, is the distortion implied by price dispersion originally derived by Yun (1996).

Applying the Calvo result to $P_t^*$:

$$P_t^* = \left[ (1-\theta) \tilde{P}_t^{-\varepsilon} + \theta \left( P_{t-1}^* \right)^{-\varepsilon} \right]^{\frac{-1}{\varepsilon}}$$

After rearranging,

$$\begin{aligned} p_t^* &\equiv \left( \frac{P_t^*}{P_t} \right)^{\varepsilon} = \left[ (1-\theta) \tilde{p}_t^{-\varepsilon} + \theta \frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \\ &= \left( (1-\theta) \left[ \frac{1-\theta \left( \bar{\pi}_t \right)^{\varepsilon-1}}{1-\theta} \right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta \frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right)^{-1} \qquad (4) \end{aligned}$$

## E.4 Linearized Equilibrium Conditions

Collecting the 7 (non-linear) equilibrium conditions in our 7 endogenous variables:

$$\begin{aligned} & K_t = s_t + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} \; (1), s_t \equiv \frac{e^{\tau_t} C_t N_t^{\varphi}}{A_t} \\ & F_t = 1 + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \; (2), \quad \frac{K_t}{F_t} = \left[ \frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} \quad (3) \\ & p_t^* = \left[ (1-\theta) \left( \frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \quad (4) \\ & \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \; (5), \quad C_t = p_t^* e^{a_t} N_t \; (6) \\ & \frac{R_t}{R} = \left( \frac{\bar{\pi}_t}{\bar{\pi}} \right)^{\phi} \; (7), \end{aligned}$$

where the last equation is the monetary policy rule. We now log-linearize these equations about nonstochastic steady state to obtain the standard three equation NK model.

It is worth noting that when $\theta = 0$, the allocations and prices in the equilibria of the model considered here coincides with the allocations and prices in the equilibria of the model considered in the main manuscript (see Section 3). This is so, even though the underlying models are very different.[107] In the policy rule, (7), $\bar{\pi}$ denotes the policy maker's inflation target and $R$ denotes the nominal interest rate in non-stochastic steady state. As in the standard presentation of the three equation NK model, we set $\bar{\pi} = 1$, the steady state is given by:

$$\begin{aligned}
K &= \frac{1}{1-\beta\theta} \text{ (1)},\ \ s = 1,\ F = \frac{1}{1-\beta\theta} \text{ (2)} \\
\frac{K}{F} &= 1 \text{ (3)}, \\
p^* &= 1 \\
1 &= \beta \exp\left(-\Delta a\right) R \text{ (5)} \\
c &= N,
\end{aligned} \tag{141}$$

where $c_t = C_t/A_t$ and absence of a time subscript indicates non-stochastic steady state.

Let $\hat{x}_t \equiv \left(dx_t\right)/x$, where $dx_t$ denotes a small deviation of $x_t$ from its steady state value. Then, log linearizing equation (139) around steady state,

$$\hat{K}_t = \hat{s}_t \left(1 - \beta\theta\right) + \beta\theta \left(\varepsilon \hat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right).$$

We conclude (after applying the same manipulations to the other price optimality conditions):

$$\hat{K}_t = \left(1-\beta\theta\right)\hat{s}_t + \beta\theta E_t\left(\varepsilon \hat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right) \text{ (a)} \tag{142}$$

$$\hat{F}_t = \beta\theta E_t\left(\left(\varepsilon - 1\right)\hat{\bar{\pi}}_{t+1} + \hat{F}_{t+1}\right) \text{ (b)} \tag{143}$$

$$\hat{K}_t = \hat{F}_t + \frac{\theta}{1-\theta}\hat{\bar{\pi}}_t. \text{ (c)} \tag{144}$$

Substitute out for $\hat{K}_t$ in equation (142) using equation (144) and then substitute out for $\hat{F}_t$ from equation (143) to obtain:

[107] In the model in this appendix, we are not specific about what gives rise to money demand. One possibility, which is consistent with all the equilibrium conditions described here, is that real balances enter additively in the utility function and money accumulation is explicitly included in the household's budget constraint, equation (133). At this level, the model is obviously quite different from the one in the body of the paper, despite their being observationally equivalent in terms of equilibrium prices, interest rates, employment and output.

$$\beta\theta E_t\left(\left(\varepsilon-1\right)\hat{\bar{\pi}}_{t+1}+\hat{F}_{t+1}\right)+\frac{\theta}{1-\theta}\hat{\bar{\pi}}_t=\left(1-\beta\theta\right)\hat{s}_t+\beta\theta E_t\left(\varepsilon\hat{\bar{\pi}}_{t+1}+\hat{F}_{t+1}+\frac{\theta}{1-\theta}\hat{\bar{\pi}}_{t+1}\right).$$

Collecting terms, we get :

$$\hat{\bar{\pi}}_t=\frac{\left(1-\theta\right)\left(1-\beta\theta\right)}{\theta}\hat{s}_t+\beta E_t\hat{\bar{\pi}}_{t+1}$$

Now, we go for $\hat{s}_t$. First, we rewrite $s_t$ as follows:

$$\begin{aligned}
s_t &= \frac{\exp\left(\tau_t\right)C_tN_t^{\varphi}}{A_t} \\
&= \frac{\exp\left(\tau_t\right)}{\left(p_t^*\right)^{\varphi}}\left(\frac{C_t}{A_t}\right)^{1+\varphi} \qquad \because N_t=C_t/\left(p_t^*A_t\right) \\
&= \frac{1}{\left(p_t^*\right)^{\varphi}}\left(\frac{C_t}{A_t\exp\left[-\frac{\tau_t}{1+\varphi}\right]}\right)^{1+\varphi} \\
&= \frac{1}{\left(p_t^*\right)^{\varphi}}X_t^{1+\varphi},
\end{aligned}$$

where $X_t$ is the ratio of actual consumption to what is easily verified to be its first-best level, $C_t^*$:

$$X_t=\frac{C_t}{C_t^*}=\frac{C_t}{A_t}\exp\left(\frac{\tau_t}{1+\varphi}\right)=c_t\exp\left(\frac{\tau_t}{1+\varphi}\right). \tag{145}$$

Since $\bar{\pi}=1$,

$$\hat{p}_t^*=\theta\hat{p}_{t-1}^*.$$

So, under the assumption that the economy started up long ago, to a first order approximation (around the efficient steady state), we get $\hat{p}_t^*=0$. Then,

$$\hat{s}_t=\left[\frac{1+\varphi}{s\left(p^*\right)^{\varphi}}X^{1+\varphi}\right]x_t=\left(1+\varphi\right)x_t$$

where $x_t$ denotes the output gap, $x_t\equiv\hat{X}_t$. The output gap here is the log deviation of actual output relative to its first-best level, $A_t\exp\left[-\frac{\tau_t}{1+\varphi}\right]$. Our calculation takes into account that in non-stochastic steady state, actual output and its first best level coincide. In this way, we have derived the Phillips curve:

$$\hat{\bar{\pi}}_t=\frac{\left(1-\theta\right)\left(1-\beta\theta\right)}{\theta}\left(1+\varphi\right)x_t+\beta E_t\hat{\bar{\pi}}_{t+1}$$

Note that $\hat{\bar{\pi}}_t=\frac{\bar{\pi}_t-\bar{\pi}}{\bar{\pi}}=\pi_t$,

$$\pi_t=\frac{\left(1-\theta\right)\left(1-\beta\theta\right)}{\theta}\left(1+\varphi\right)x_t+\beta E_t\pi_{t+1}. \tag{146}$$

We now derive the NK IS curve. The 'interest rate', $R_t^*$, when consumption is at its first best solves:

$$1 = R_t^* E_t \beta \frac{C_t^*}{C_{t+1}^*} = R_t^* E_t \beta \frac{A_t \exp\left[-\frac{\tau_t}{1+\varphi}\right]}{A_{t+1} \exp\left[-\frac{\tau_{t+1}}{1+\varphi}\right]} = R_t^* E_t \beta \frac{1}{\exp\left[\Delta a_{t+1} - \frac{\tau_{t+1}-\tau_t}{1+\varphi}\right]}.$$

Multiplying the intertemporal Euler equation of the household by $A_t$:

$$E_t\left[\frac{1}{c_t} - \beta \frac{1}{c_{t+1}\exp\left(\Delta a_{t+1}\right)} \frac{R_t}{\bar{\pi}_{t+1}}\right] = 0(5),$$

where $c_t \equiv C_t/A_t$. Alternatively,

$$E_t\left[\frac{1}{X_t} - \beta \frac{1}{X_{t+1}\exp\left(\Delta a_{t+1} - \frac{\tau_{t+1}-\tau_t}{1+\varphi}\right)} \frac{R_t}{\bar{\pi}_{t+1}}\right] = 0(5),$$

After log-linearizing (5):

$$E_t\left[1 - x_t - \left(1 - x_{t+1} + \hat{R}_t - \hat{\bar{\pi}}_{t+1} - \left[\Delta a_{t+1} - \Delta a - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right]\right)\right] = 0,$$

where the $d'$s are dropped from $\tau_t$ because it has the same steady state at each date. Then,

$$E_t\left[x_t - \left(x_{t+1} - \left(\hat{R}_t - \hat{\bar{\pi}}_{t+1}\right) + \left[\Delta a_{t+1} - \Delta a - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right]\right)\right] = 0. \tag{147}$$

Now, we log-linearize the equation that defines $R_t^*$:

$$\hat{R}_t^* = E_t\left[\Delta a_{t+1} - \Delta a - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right]$$

Substituting this into equation (147)

$$x_t = E_t\left(x_{t+1} - \left(\hat{R}_t - \pi_{t+1} - \hat{R}_t^*\right)\right). \tag{148}$$

The Taylor rule is:

$$\frac{R_t}{R} = \left(\frac{\bar{\pi}_t}{\bar{\pi}}\right)^{\phi},$$

which, after log-linearization about nonstochastic steady state is:

$$\hat{R}_t = \phi \pi_t. \tag{149}$$

Equations (146), (148) and (149) represent the three equations of the standard simple NK model.

## E.5 Best Response Function of an Individual, Marginal Price Setter

We now discuss the decision of the $i^{th}$ individual marginal price setter. This price setter is one of the measure, $1-\theta$, of marginal price setters. Each marginal price setter makes its price decision, $\tilde{P}_{i,t}$, independently and without communicating with the others. Let $z_{i,t}$ denote $\tilde{P}_{i,t}/P_{t-1}$. As in the discussion in the paper, the $i^{th}$ firm forms a belief, $z_{i,t}^b$, about what the other marginal price setters are doing (after scaling by $P_{t-1}$). Conditional on $z_{i,t}^b$ and $h_{t-1}$, the firm can compute an $\left(h_{t-1}, z_{i,t}^b\right)$ continuation equilibrium. The variables, $K_{i,t}$ and $F_{i,t}$ in that continuation equilibrium are required for it to evaluate the terms on the right of its optimality condition, equation (138). Let $\bar{\pi}_{i,t}$ denote the value of $P_t/P_{t-1}$ in the $\left(h_{t-1}, z_{i,t}^b\right)$ continuation equilibrium. In this notation, the firm's optimality condition, equation (138), is:

$$z_{i,t} = \frac{K_{i,t}}{F_{i,t}} \bar{\pi}_{i,t}.$$

Log-linearizing around steady state:

$$\hat{z}_{i,t} = \hat{K}_{i,t} - \hat{F}_{i,t} + \pi_{i,t}.$$

Substituting the linearized expressions for $\hat{K}_t$ and $\hat{F}_t$, equations (142) and (143), we obtain

$$\hat{z}_{i,t} = (1-\beta\theta)\,\hat{s}_{i,t} + \beta\frac{\theta}{1-\theta}E_t\pi_{t+1} + \pi_{i,t}.$$

As before, the subscript, $i$, indicates the $i^{th}$ price setter's $\left(h_{t-1}, z_{i,t}^b\right)$ continuation equilibrium. Note that $\pi_{t+1}$ does not have a subscript, $i$. This reflects that we only consider one-shot deviations and that after log-linearizing around an undistorted steady state, our linearized model has no state variables. As a result, $x_{t+j}$ and $\pi_{t+j}$ are not functions of $j$, $j>0$. Also, we have

$$\begin{aligned}
\hat{s}_{i,t} &= (1+\varphi)\,x_{i,t} \\
x_{i,t} &= E_t\left(x_{i,t+1} - \left(\hat{R}_{i,t} - \pi_{t+1} - \hat{R}_t^*\right)\right).
\end{aligned}$$

There is no $i$ subscript on $\hat{R}_t^*$ because that is common knowledge. It follows that

$$\begin{aligned}
\hat{z}_{i,t} &= \pi_{i,t} + (1-\beta\theta)(1+\varphi)\,E_t\left(x_{i,t+1} - \left(\hat{R}_{i,t} - \pi_{t+1} - \hat{R}_t^*\right)\right) + \beta\frac{\theta}{1-\theta}E_t\pi_{t+1} \\
&= \pi_{i,t}\left[1-(1-\beta\theta)(1+\varphi)\,\phi\right] + (1-\beta\theta)(1+\varphi)\,E_t\left(x_{t+1} + \pi_{t+1} + \hat{R}_t^*\right) + \beta\frac{\theta}{1-\theta}E_t\pi_{t+1}
\end{aligned}$$

Inflation is related to what the other marginal price setters are doing based on the cross price restriction:

$$\bar{\pi}_{i,t} = \left((1-\theta)\left(z_{i,t}^b\right)^{1-\varepsilon} + \theta\right)^{\frac{1}{1-\varepsilon}}.$$

Log-linearizing this expression around $\bar{\pi}_{i,t} = z^b_{i,t} = 1$, we obtain

$$\pi_{i,t} = (1-\theta)\,\hat{z}^b_{i,t}.$$

So, we conclude that the best response function denoted by $f$, has the following form:

$$\hat{z}_{i,t} = [1-(1-\beta\theta)(1+\varphi)\phi](1-\theta)\,\hat{z}^b_{i,t} + \underbrace{(1-\beta\theta)(1+\varphi)E_t\left(x_{t+1}+\pi_{t+1}+\hat{R}^*_t\right)+\beta\frac{\theta}{1-\theta}E_t\pi_{t+1}}_{=\xi_t} = f\left(\hat{z}^b_{i,t},\xi_t\right). \tag{150}$$

Let $\pi_t$ denote the Nash equilibrium of the best response function, equation (150). It is a general result that the competitive equilibrium coincides with the Nash equilibrium when the economy is represented as a game. So, we can verify our calculations by solving equation (150) for $\hat{z}_{i,t} = \hat{z}^b_{i,t} = \pi^{Nash}_t$ and verifying that $\pi^{Nash}_t = \pi_t$, where $\pi_t$ satisfies the Phillips curve, equation (146). A simple chain-of-justification argument verifies the standard result that the Nash equilibrium (e.g., the competitive equilibrium) is rationalizable. We now inquire whether there exists another rationalizable equilibrium in an arbitrarily small neighborhood, $D(\pi_t;\varepsilon)$, of $\pi_t$, where $D(\pi_t;\varepsilon) = \{x : |\pi_t - x| < \varepsilon\}$ and $\varepsilon > 0$ but sufficiently small. We say that the equilibrium satisfies *locally unique rationalizability* if the set, $\Pi^*(D(\pi_t;\varepsilon);f)$, is a singleton, $\pi_t$, for all competitive equilibrium values of $\pi_t$ and all $t$, for $\varepsilon$ sufficiently small.[108] The correspondence, $\Pi^*$, is defined in equation (59). In the present case locally unique rationalizability coincides with the requirement that the slope of the best response function, $f$, in terms of $\hat{z}^b_{i,t}$ is less than unity, in absolute value. From equation (150), that coefficient is $[1-(1-\beta\theta)(1+\varphi)\phi](1-\theta)$. Trivially, this coefficient is less than unity. So locally unique rationalizability obtains if and only if

$$[1-(1-\beta\theta)(1+\varphi)\phi](1-\theta) > -1,$$

which is equivalent to:

$$\frac{1}{(1-\theta\beta)(1+\varphi)}\left(1+\frac{1}{1-\theta}\right) > \phi. \tag{151}$$

A qualitative feature of this expression is interesting from the perspective of our 'leaning against the wind, but not too aggressively', principle. In particular, the upper bound on $\phi$ is less restrictive, the bigger is $\theta$. When the individual marginal price setter considers what other do, what matters is their impact on the aggregate price index, which is small when their number is small. This is captured by the expression, $(1-\theta)$, in front of the inequality in equation (151).

## E.6 The Taylor Rule Works Well in Normal Times

Here, we substantiate the claim made in Footnote (**??**) and repeated in Section 7, that the Taylor rule has good operating characteristics ('works well') in a neighborhood of the unique equilibrium local

[108] We do not include $\xi_t$ in the definition because the Phillips makes $\xi_t$ a simple function of $\pi_t$.

to the interior steady state. After log-linearization, the equilibrium conditions reduce to three: the IS curve, equation (148); the monetary policy rule, equation (149); and the Phillips curve, equation (146). For simplicity, we display these equations here. The IS curve is:

$$x_t = E_t x_{t+1} - [r_t - E_t \pi_{t+1} - r_t^*],$$

where $x_t$ denotes the log-difference between equilibrium and output and output at its first-best level. Also, $r_t^*$ corresponds to $\hat{R}_t^*$ above and

$$r_t^* = E_t a_{t+1} - a_t,$$

where $a_t$ is an exogenous shock with the following law of motion:

$$\Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t, \quad \Delta a_t = a_t - a_{t-1},\ \rho \in [0,1)$$

The Phillips curve is:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t,\ \beta \in (0,1),$$

where $\kappa > 0$. Finally, the Taylor rule is:

$$r_t = \phi \pi_t,$$

where $\phi > 1$ and $r_t$ corresponds to $\hat{R}_t$ above.

It is easy to verify that the locally unique equilibrium has the form:

$$\begin{aligned} r_t - E_t \pi_{t+1} &= \psi \Delta a_t, \\ x_t &= \psi (1 - \beta\rho) / [\kappa (\phi - \rho)] \Delta a_t, \\ \pi_t &= [\psi / (\phi - \rho)] \Delta a_t \end{aligned}$$

where

$$\psi \equiv \rho [(1 - \beta\rho)(1 - \rho) / (\kappa (\phi - \rho)) + 1]^{-1}.$$

Evidently, for $\phi$ sufficiently large, $\psi$ is close to $\rho$ and $r_t - E_t \pi_{t+1} \simeq r_t^*$, $\pi_t \simeq 0$ and $x_t \simeq 0$. We conclude that based on the analysis of equilibrium alone, a big value of $\phi$ works well in the sense that the policy stabilizes the equilibrium around first best. It is easy to verify that this stabilization result also holds when the technology process is replaced by $a_t = \rho a_{t-1} + \varepsilon_t$ or when the shock is instead a stationary disturbance to labor supply.