Precautionary Saving toward Correlation under Risk and Ambiguity*

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Very Preliminary

Abstract

This paper considers a multivariate situation where there are two-risky attributes. Furthermore, by extending the analyses under risk to the analyses under ambiguity, we analyze the effects of correlation between the two ambiguous variables on the optimal saving as well as the effects of correlation between the two risky variables on the optimal saving. Under plausible conditions, we show that an increase in correlation about ambiguity increases the optimal saving.

Key Words: Correlation; Precautionary saving; Smooth ambiguity model

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1 Introduction

The decision of how much to save and consume is significant for our daily life. In particular, because the future is not foreseeable, considering the role of risk is necessary for analyzing individuals' saving behaviors. In the literature on decisions under risk, one-dimensional risk has been mainly investigated. However, decisions under risk are usually affected by several attributes. For example, not only the level of wealth but also the health condition is crucial in our quality of life. The purpose of this paper is to consider a multivariate situation where there are two-risky attributes. Furthermore, by extending the analyses under risk to the analyses under ambiguity, we analyze the effects of correlation between the two ambiguous variables on the optimal saving as well as the effects of correlation between the two risky variables on the optimal saving. Under plausible conditions, we show that an increase in correlation about ambiguity increases the optimal saving.

The notions of risk and ambiguity (or uncertainty) have been separately investigated in the literature on decision theory.[1] Risk is the situation where decision maker's (DM) beliefs are capture by a unique probability and her preferences are represented by the standard expected utility. Ambiguity is the situation where DM's beliefs are not captured by a unique probability but by a set of probabilities or a non-additive probability, and her preferences are represented by the multiprior-expected utility (MEU) or the Choquet expected utility (CEU). Since Knight (1921) and Ellsberg (1961), the importance of ambiguity has been recognized in the literature. The developments of CEU and MEU have enabled us to investigate DM's behaviors under ambiguity within the frameworks with axiomatic foundations.[2]

This paper adopts the smooth ambiguity model in Klibanoff et al. (2005, 2009). This model enables us to differentiate the DMs' attitudes toward ambiguity from their perception of ambiguity, which implies that the smooth ambiguity model can be considered to be more general than either the MEU or CEU. Furthermore, the

[1] Throughout this paper, we refer to ambiguity and uncertainty are used interchangeably.

[2] MEU and CEU are axiomatized by Gilboa and Schmeidler (1989) and Schmeidler (1989), respectively. See Gilboa (2009), Wakker (2010), or Nishimura and Ozaki (2017).

smooth ambiguity model is more tractable than most of the models analyzing ambiguity. The smooth ambiguity model has been adopted by many researchers. As an application of the smooth ambiguity model, Gollier (2011) analyzes the portfolio allocation problem between a riskless and an ambiguous asset, and analyzes the effects of ambiguity on optimal investment. As an extension of Gollier (2011), Asano and Osaki (2020) consider the portfolio allocation problem between a risky and an ambiguous asset, and analyzes the effects of ambiguity on optimal investment.[3]

Precautionary saving has been investigated in the literature. One of the seminal papers is Leland (1968) that compares the future labor income under certainty with that under risk. Sandmo (1970) and Rothschild and Stiglitz (1971) analyze the effect of randomness of interest rates on the individuals' saving behaviors. Since these seminal papers, a lot of researchers have investigated precautionary saving. For a survey of precautionary saving about theoretical results, see Baidrdi et al. (2020). For empirical results, see Liguilde et al. (2019). Kimball (1990) advances the literature on precautionary saving. Kimball (1990) proposes the notion of prudence that are related to the convexity of marginal utility. In the literature on precautionary saving, risk on labor income and that on interest rate have been studied separately or simultaneously. Although the role of risk on interest rate is important, this paper focuses on labor income.

Above-mentioned papers focus on comparing the saving behavior under certainty with that under risk. A generalization of this approach is to compare one risky situation with another risky situation. The seminal paper on this perspective is Eeckhoudt and Schlesinger (2008). Based on the notion of higher-order risk changes (Nth-order stochastic dominance), Eeckhoudt and Schlesinger (2008) provide a necessary and sufficient condition for the occurrence of precautionary saving. Eeckhoudt and Schlesinger (2008)'s results depend on the signs of derivatives of utility functions. The utility functions commonly adopted in economics has the property of having all odd derivatives positive and all even derivatives negative. This property

[3]See also Asano and Osaki (2021).

is called mixed risk aversion proposed by Caballé and Pomansky (1996). Since Caballé and Pomansky (1996), the notion of mixed risk aversion has been investigated. From the perspective of analyses of a multivariate utility function, Eeckhoudt et al. (2007) is worth mentioning. Decisions under risk and ambiguity are affected by several attributes. As pointed out by Eeckhoudt et al. (2007), individuals' level of health and the condition of her health play a significant role in our decision makings. By proposing the notions of cross-prudence and cross-temperance that are extensions of correlation aversion by Epstein and Tanny (1980), Eeckhoudt et al. (2007) characterize these notions by cross-derivatives of a utility function.[4]

Another related work is Courbage and Rey (2007) that study precautionary saving motives considering multivariate risks (the future income risk and the background risk), and show that individuals' positive precautionary saving is affected by the correlation about the two risks. Menegatti (2009) provides corrections of some results in Courbage and Rey (2007). Denuit et al. (2011) also investigate the effects of multivariate risks on individuals' optimal choices, and show that positive correlation between the two risks increases the optimal amount of savings. Liu and Menegatti (2019, JRI) analyze the effects of random returns on health and wealth investments, and show that precautionary investments in health and precautionary investment in wealth are determined by the signs of cross-derivatives of utility functions.

Finally, we mention experimental results about a multivariate framework. Based on Eeckhoudt and Schlesinger (2006), Deck and Schlesinger (2010) experimentally test whether subjects are prudent and/or temperate, and find that subjects are prudent but not temperate. Deck and Schlesinger (2014) design experiments to test consistency of higher order risk preferences, and find that risk averse individuals are mixed risk averse and they dislike an increase in risk for every degree n.

[4]Eeckhoudt et al. (2007) show that (i) an individual is correlation averse if and only if $u_{12}(x, y) \leq 0$ for all x, y, (ii) An individual is cross-prudent in health if and only if $u_{112}(x, y) \geq 0$ for all x, y, (iii) An individual is cross-prudent in wealth if and only if $u_{122}(x, y) \geq 0$ for all x, y, and (iv) An individual is cross-temperate if and only if $u_{1122}(x, y) \leq 0$ for all x, y, where $u(x, y)$ is a bivariate utility function of wealth x and health y.

Deck and Schlesinger (2014) also find that risk loving individuals are mixed risk loving and they like an increase in risk for even degrees, but dislike an increase in risk for odd degrees. In the literature on multivariate risk preferences, theoretical models have been investigated by many researchers. However, there is very few empirical research showing the prevalence of correlation aversion, cross-prudence, and cross-temperance. Ebert and van de Kuilen (2015) first experimentally investigate multivariate risk preferences, and observe the prevalence of correlation aversion, cross-prudence, and cross-temperance, but do not observe correlation seeking and cross-imprudence. In the literature of health economics, Attema et al. (2019) study univariate and multivariate risk preferences for health and wealth. While Attema et al. (2019) experimentally find the prevalence of correlation aversion and cross-prudence in health and wealth for multivariate gains, they observe correlation seeking and cross-imprudence for multivariate losses.

2 Optimal saving in the presence of correlation

In this section, we consider a bivariate utility function, and provide a basic framework for analyzing the effects of correlation between two variables on the optimal saving under risk and ambiguity.

Let us consider a simple dynamic model with two dates, $t = 0$ and $t = 1$. An individual enjoys the lifetime time-separable utility from two attributes $(x, y) \in X \times Y \subseteq \mathbb{R}^2$. The first attribute is a financial variable and the second is a non-financial variable. For the sake of exposition, we interpret that the financial variable is wealth and the non-financial variable is health. Another example of the non-financial variable is other's wealth level which is related to consumption externalities. The analysis can be applied to non-financial variables which can be numerically measurable.

Let us denote the bivariate utility function $u : X \times Y \to \mathbb{R}$. We denote $u_{(1,0)}(x, y)$ as $\partial u/\partial x$, $u_{(0,1)}(x, y)$ as $\partial u/\partial y$ and $u_{(1,1)}(x, y)$ as $\partial^2 u/\partial x \partial y$. The same notation can be used for the function $u_{(i,j)}(x, y)$ which stands for $\partial^{i+j}(u)/\partial x^i \partial y^j$. We assume that all higher-order partial and cross derivatives exist if necessary for the analysis. The utility function $u(x, y)$ is increasing and concave in both wealth and health, $u_{(1,0)}(x, y) \geq 0$, $u_{(0,1)}(x, y) \geq 0$ and $u_{(2,0)}(x, y) \leq 0$, $u_{(0,2)}(x, y) \leq 0$. The concavity means risk aversion for wealth and health, respectively. We do not impose any restriction on the sign of $u_{(1,1)}(x, y)$ here. The signs of cross derivatives play a crucial role for the later analyses.

The individual faces the future income risk and the health risk. Two types of risks, which are called "good" and "bad," are involved in both the future income risk and the health risk. As for the future income risk, the random variables $\tilde{\epsilon}_B$ and $\tilde{\epsilon}_G$ occur with probability p and $1-p$. As for the future health risk, the random variables $\tilde{\delta}_B$ and $\tilde{\delta}_G$ occur with probability q and $1-q$. We assume that all future risks, $\tilde{\epsilon}_G$, $\tilde{\epsilon}_B$, $\tilde{\delta}_G$ and $\tilde{\delta}_B$, are mutually independent. The terms of "good" and "bad" mean that the individual prefers "good" to "bad", that is, $E[u(x+\tilde{\epsilon}_G, y)] \geq E[u(x+\tilde{\epsilon}_B, y)]$ and $E[u(x, y+\tilde{\delta}_G)] \geq E[u(x, y+\tilde{\delta}_B)]$ for all x, y.

There are four possible combinations between the future income risk and the health risk. The following stands for the combinations and their probabilities:

- $\tilde{\epsilon}_B$ and $\tilde{\delta}_B$ with probability kpq;
- $\tilde{\epsilon}_G$ and $\tilde{\delta}_B$ with probability $(1-kp)q$;
- $\tilde{\epsilon}_B$ and $\tilde{\delta}_G$ with probability $p(1-kq)$;
- $\tilde{\epsilon}_G$ and $\tilde{\delta}_G$ with probability $1-p-q+kpq$.

The bad future income risk occurs with probability $kpq+p(1-kq)$. Other probabilities of the future income and the health risks can be calculated in a similar way. A value of k taking a positive value is chosen so that all probabilities are non-negative and less than unity. This value k can capture the correlation between the future income risk and the health risk. When the value of k is unity, the future income risk and the health risk are independent. For example, if $k = 1$, then the probability of the bad future income risk, $kpq + p(1-kq)$, turns out to be p. A value of k

greater (less) than unity indicates a positive (negative) correlation. The correlation is increasing in k.

The individual earns the sure income w and is endowed with the sure health condition h in both $t = 0$ and $t = 1$. We omit h in the second attribute because it is not explicitly related to the analysis. In addition to the sure income and health, the individual faces the income and health risks at $t = 1$. The individual must decide saving at $t = 0$ to transfer wealth from $t = 0$ to $t = 1$. The negative saving is the amount of borrowing from the future income for the current consumption. We assume that the net rate of return is equal to zero. We also assume no time discounting. Because of our simplest setting, we can focus on the effect of risk on saving decisions. The individual determines the level of saving to maximize the lifetime time-separable utility from wealth and health:

$$U(s) = u(w - s) + kpqE[u(w + s + \tilde{\epsilon}_B, \tilde{\delta}_B)] + (1 - kp)qE[u(w + s + \tilde{\epsilon}_G, \tilde{\delta}_B)] \\ + p(1 - kq)E[u(w + s + \tilde{\epsilon}_B, \tilde{\delta}_G)] + (1 - p - q + kpq)E[u(w + s + \tilde{\epsilon}_G, \tilde{\delta}_G)]. \tag{1}$$

The first-order condition for (1) is

$$U'(s) = -u'(w - s^*) + kpqE[u_{(1,0)}(w + s^* + \tilde{\epsilon}_B, \tilde{\delta}_B)] + (1 - kp)qE[u_{(1,0)}(w + s^* + \tilde{\epsilon}_G, \tilde{\delta}_B)] \\ + p(1 - kq)E[u_{(1,0)}(w + s^* + \tilde{\epsilon}_B, \tilde{\delta}_G)] + (1 - p - q + kpq)E[u_{(1,0)}(w + s^* + \tilde{\epsilon}_G, \tilde{\delta}_G)] = 0. \tag{2}$$

Beucase $U(s)$ is concave by $u_{(2,0)} \leq 0$, the second-order condition for a maximum is satisfied. For simplicity, we assume that the optimal saving is interior, $-w \leq s^* \leq w$, and is unique.

3 Stochastic dominance

In this section, we introduce the notion of stochastic dominance to represent the "good" and "bad" future income and health risks. Stochastic dominance is a partial order to compare two random variables. Let us consider two random variables $\tilde{x}$ and $\tilde{y}$ with the cumulative distribution functions F and G which are defined over bounded support $[a, b]$. We note that this notation is used for the exposition of stochastic dominance in this section and you do not get confused the notation x and y which is used for wealth and health.

The distribution function F dominates the distribution function G in the sense of first-order stochastic dominance (FSD) if $F(z) \leq G(z)$ for all $z \in [a, b]$. If the random variables $\tilde{x}$ and $\tilde{y}$ have the distribution functions F and G, we take the liberty to say that a random variable $\tilde{x}$ dominates a random variable $\tilde{y}$ in the sense of FSD. The same goes for other notions of stochastic dominance. Applying FSD to the future income risk, the individual with $u_{(1,0)} \geq 0$ prefer the good future income risk $\tilde{\epsilon}_G$ to the bad one $\tilde{\epsilon}_B$. Formally, the following two conditions are equivalent:

- $\tilde{\epsilon}_G$ dominates $\tilde{\epsilon}_B$ in the sense of FSD;
- $E[u(x + \tilde{\epsilon}_G, y)] \geq E[u(x + \tilde{\epsilon}_B, y)]$ for $u_{(1,0)} \geq 0$.

The same argument can be applied to the future health risk. The individual with $u_{(0,1)} \geq 0$ prefer the good future health risk $\tilde{\delta}_G$ to the bad one $\tilde{\delta}_B$, that is, $E[u(x + \tilde{\epsilon}_G, y)] \geq E[u(x + \tilde{\epsilon}_B, y)]$ for $u_{(0,1)} \geq 0$.

For the distribution functions F and G on $[a, b]$, let us define $F_1(z) = F(z)$ and $G_1(z) = G(z)$, and define $F_{n+1} = \int_a^z F_n(t)dt$ and $G_{n+1} = \int_a^z G_n(t)dt$ for all $z \in [a, b]$ and for all $n = 2, 3, \ldots, N-1$. Following Jean (1980) and Ingersoll (1987), the distribution function F dominates the distribution function G in the sense of Nth-order stochastic dominance (NSD) if $F_N(z) \leq G_N(z)$ for all $z \in [a, b]$ and $F_n(b) \leq G_n(b)$ for all $n = 1, 2, \ldots, N-1$. The following result is well-known in the literature. For example, see Ingersoll (1987).

- $\tilde{\epsilon}_G$ dominates $\tilde{\epsilon}_B$ in the sense of NSD;
- $E[u(x+\tilde{\epsilon}_G, y)] \geq E[u(x+\tilde{\epsilon}_B, y)]$ for any function u such that $(-1)^{n+1}u_{(n,0)} \geq 0$ for $n = 1, 2, \ldots, N-1$.

The individual prefers the future good income risk to the bad one which are ranked by second-order stochastic dominance because we assume that $u_{(1,0)} \geq 0$ and $u_{(2,0)} \leq 0$. For third-order stochastic dominance, we need to assume $u_{(3,0)} \geq 0$ in addition to $u_{(2,0)} \leq 0$ so that the individual prefer good future income risk to bad one. Because the positive third-order derivative is called prudence, we call $u_{(3,0)} \geq 0$ as prudence for wealth. The same argument can be applied to the future health risk. The following two conditions are equivalent:

- $\tilde{\delta}_G$ dominates $\tilde{\delta}_B$ in the sense of NSD;
- $E[u(x, y+\tilde{\delta}_G,)] \geq E[u(x, y+\tilde{\delta}_B)]$ for any function u such that $(-1)^{n+1}u_{(0,n)} \geq 0$ for $n = 1, 2, \ldots, N-1$.

Following the terminology coined by Caballé and Pomansky (1996), the second condition is called mixed risk aversion. In other words, the investor is called mixed risk aversion if the signs of successive derivatives of the utility function have alternate signs, with all positive odd derivatives and all negative even derivatives. As shown by Brockett and Golden (1987), the utility functions commonly adopted in economics have the property of having all odd derivatives positive and all even derivatives negative.[5]

We introduce another stochastic dominance relation which is called an increase in N-th degree risk. Following Ekern (1980), the distribution function G has more Nth-degree risk than the distribution function F if $F_{N-1}(z) \leq G_{N-1}(z)$ for all $z \in [a, b]$ and $F_n(b) = G_n(b)$ for all $n = 1, 2, \ldots, N-1$. This indicates that the first $N-1$th moments of F and G coincide. Ekern (1980) shows that the following conditions are equivalent:

[5] A real-valued function $u(x)$ on $(0, \infty)$ is *complete monotone* if its derivatives $u^n(x)$ of all orders exist and $(-1)^n u^n(x) \geq 0$ for all $x > 0$ and all $n = 0, 1, 2, \ldots$. A real-valued, continuous utility function u defined on $[0, \infty)$ exhibits *mixed risk aversion* if it has a completely monotone first derivative on $(0, \infty)$ and $u(0) = 0$. As also pointed out by Pratt and Zeckhauser (1987), a majority of utility functions analyzed in applied work have completely monotone first derivatives. For example, if a class of utility functions u is the class of hyperbolic absolute risk aversion (HARA) with $-u''(x)/u'(x) = 1/(a + bx)$ for $a > 0$ and $b > 0$, then it is mixed risk averse. See Caballé and Pomansky (1996, p.490) in detail.

- $\tilde{\delta}_B$ has more Nth-degree risk than $\tilde{\delta}_G$;
- $E[u(x + \tilde{\epsilon}_G, y)] \geq E[u(x + \tilde{\epsilon}_B, y)]$ for $(-1)^{N+1}u_{(N,0)} \geq 0$.

The same argument can be applied to the case of health risk. We provide some example which is known in the literature. An increase in risk by Rothchild and Stiglitz (1970) corresponds to a second-degree increase in risk. Rothchild and Stiglitz (1970) show that any increases in risk can be obtained by a series of mean-preserving spread. This means that the means coincide for two random variables which are ranked by an increase in risk. The risk averse investor dislikes any increases in risk. An increase in downside risk aversion introduced by Menezws et al. (1980) corresponds to a third-degree increase in risk. The Means and variances coincide for two random variables which are ranked by an increase in downside risk. The prudent investor dislikes any increases in downside risk.

4 Precautionary saving toward correlation under risk

In this section, we examine how correlation influences the optimal saving. In our setup, this means that we examine how the optimal saving changes in k which is the parameter to represent correlation. Before stating the main result, we prepare the following lemma.

Lemma 1. *Let us consider the payoff function $f(\epsilon, \delta)$ and define*

$$\begin{aligned} E[f(\tilde{\epsilon}, \tilde{\delta})] &= kpqE[f(\tilde{\epsilon}_B, \tilde{\delta}_B)] + (1 - kp)qE[f(\tilde{\epsilon}_G, \tilde{\delta}_B)] \\ &\quad + p(1 - kq)E[f(\tilde{\epsilon}_B, \tilde{\delta}_G)] + (1 - p - q + kpq)E[f(\tilde{\epsilon}_G, \tilde{\delta}_G)]. \end{aligned}$$

Suppose that

- *$\tilde{\epsilon}_G$ dominates $\tilde{\epsilon}_B$ in the sense of NSD;*
- *$\tilde{\delta}_G$ dominates $\tilde{\delta}_B$ in the sense of MSD.*

If $(-1)^{(n+m)} f_{(n,m)} \geq (\leq) 0$ *for all* $n = 1, 2, \ldots, N$ *and* $m = 1, 2, \ldots, M$*, then* $E[f(\tilde{\epsilon}, \tilde{\delta})]$ *increases (decreases) in* k*.*

Proof We will prove the case that $E[f(\tilde{\epsilon}, \tilde{\delta})]$ increases in k, because the opposite case can be proven in a similar way.

By a simple calculation, we have the following:

$$\frac{\partial E[f(\tilde{\epsilon}, \tilde{\delta})]}{\partial k} = E[f(\tilde{\epsilon}_B, \tilde{\delta}_B)] - E[f(\tilde{\epsilon}_G, \tilde{\delta}_B)] - E[f(\tilde{\epsilon}_B, \tilde{\delta}_G)] + E[f(\tilde{\epsilon}_G, \tilde{\delta}_G)].$$

This leads to the following:

$$\text{sgn}\left(\frac{\partial E[f(\tilde{\epsilon}, \tilde{\delta})]}{\partial k}\right) \geq 0 \Leftrightarrow E[f(\tilde{\epsilon}_G, \tilde{\delta}_G)] - E[f(\tilde{\epsilon}_G, \tilde{\delta}_B)] \geq E[f(\tilde{\epsilon}_B, \tilde{\delta}_G)] - E[f(\tilde{\epsilon}_B, \tilde{\delta}_B)]. \tag{3}$$

When

$$(-1)^{N+1} \frac{\partial^N \{E[f(\epsilon, \tilde{\delta}_G)] - E[f(\epsilon, \tilde{\delta}_B])\}}{\partial \epsilon^N} \geq 0, \tag{4}$$

we have that

$$E[f(\tilde{\epsilon}_G, \tilde{\delta}_G)] - E[f(\tilde{\epsilon}_G, \tilde{\delta}_B)] \geq E[f(\tilde{\epsilon}_B, \tilde{\delta}_G)] - E[f(\tilde{\epsilon}_B, \tilde{\delta}_B)]$$

because $\tilde{\epsilon}_G$ dominates $\tilde{\epsilon}_B$ in the sense of NSD. We can prove (3) by determining the condition in which (4) holds. The condition (4) can be rewritten:

$$(-1)^{(N+1)} E[f_{(N,0)}(\epsilon, \tilde{\delta}_G)] \geq (-1)^{(N+1)} E[f_{(N,0)}(\epsilon, \tilde{\delta}_B)]. \tag{5}$$

The inequality (5) holds for

$$(-1)^{M+1} \frac{\partial^N \{(-1)^{N+1} f_{(N,0)}(\epsilon, \delta)\}}{\partial \delta^M} = (-1)^{N+M} f_{(N.M)}(\epsilon, \delta) \geq 0$$

because $\tilde{\epsilon}_G$ dominates $\tilde{\epsilon}_B$ in the sense of NSD. Thus, the proof is complete. (Q.E.D.)

A similar result can be obtained for Nth-degree risk.

Lemma 2. *For the payoff function $f(\epsilon, \delta)$, define*

$$E[f(\tilde{\epsilon}, \tilde{\delta})] = kpqE[f(\tilde{\epsilon}_B, \tilde{\delta}_B)] + (1-kp)qE[f(\tilde{\epsilon}_G, \tilde{\delta}_B)] \\ + p(1-kq)E[f(\tilde{\epsilon}_B, \tilde{\delta}_G)] + (1-p-q+kpq)E[f(\tilde{\epsilon}_G, \tilde{\delta}_G)].$$

Suppose that

- *$\tilde{\epsilon}_B$ has more Nth -degree risk than $\tilde{\epsilon}_G$;*
- *$\tilde{\delta}_B$ has more Mth -degree risk than $\tilde{\delta}_G$.*

If $(-1)^{(N+M)} f_{(N,M)} \geq (\leq) 0$, then $E[f(\tilde{\epsilon}, \tilde{\delta})]$ is increasing (decreasing) in k.

Let us consider that the future income risk is ranked by NSD and the future health risk is ranked by MSD, that is,

- $\tilde{\epsilon}_G$ dominates $\tilde{\epsilon}_B$ in the sense of NSD;
- $\tilde{\delta}_G$ dominates $\tilde{\delta}_B$ in the sense of MSD.

Setting $f(\epsilon, \delta) = u_{(1,0)}(w + s + \epsilon, \delta)$, Lemma 1 states that

$$E[v(s,k)] = kpqE[u(w+s+\tilde{\epsilon}_B, \tilde{\delta}_B)] + (1-kp)qE[u(w+s+\tilde{\epsilon}_G, \tilde{\delta}_B)] \\ + p(1-kq)E[u(w+s+\tilde{\epsilon}_B, \tilde{\delta}_G)] + (1-p-q+kpq)E[u(w+s+\tilde{\epsilon}_G, \tilde{\delta}_G)]. \tag{6}$$

is increasing in k for $(-1)^{N+M} u_{(N+1,M)}(x, y) \geq 0$. Here, $E[v]$ represents the expected utility at $t = 1$.

Let us consider two individuals who are identical except for values of k, k_L, k_H with $k_L \leq k_H$. The value of k represents the correlation between the future income risk and health risk. Note that the correlation is increasing in k. The individual who bears k_H in mind considers higher correlation than the individual who bears k_L in mind. We denote s_L and s_H as the optimal saving under k_L and k_H, respectively.

Suppose that $(-1)^{n+m}u_{(n+1,m)}(x,y) \geq (\leq)0$. We obtain the following inequality:

$$\begin{aligned} &V_s(s_k, k_L) = -u'(w - s_L) + E[v_s(s_L, k_L)] = 0 \\ &\leq (\geq) - u'(w - s_L) + E[v_s(s_L, k_H)] = V_s(s_k, k_H) \\ &\Leftrightarrow s_L \leq (\geq) s_H \end{aligned} \tag{7}$$

The inequality follows from Lemma 1 where $f(\epsilon, \delta)$ is set $u_{(1,0)}(w + s + \epsilon, \delta)$. Now, we can summarize the above argument into the following proposition.

Proposition 1. *Suppose that the future income risk is ranked by NSD and the future health risk is ranked by MSD, that is,*

- $\tilde{\epsilon}_G$ *dominates* $\tilde{\epsilon}_B$ *in the sense of NSD;*
- $\tilde{\delta}_G$ *dominates* $\tilde{\delta}_B$ *in the sense of MSD.*

If $(-1)^{n+m}u_{(n+1,m)}(x,y) \geq (\leq)0$ *holds, then the optimal saving is increasing (decreasing) in* k*.*

Let us consider the special case of $N = M = 1$. In this case, if $u_{(2,1)}(x,y) \geq (\leq)0$, then the optimal saving is increasing in k which is the parameter to represent correlation. This notion is called cross prudence (imprudence) in health by Eeckhoudt et al. (2007). We give an interpretation of the condition on the sign of cross derivatives and provide an intuition of this proposition in the next section.

5 Ambiguous correlation

In the former sections, we consider the case of risk. In this section, we introduce ambiguity into correlation through the parameter k. We suppose that a plausible set of k is a set $(k_1, k_2, \ldots, k_\Theta)$. Without loss of generality, k_θ is arranged in an ascending order, $k_1 < k_2 < \ldots < k_\Theta$. The individual attaches subjective probability q_θ to the parameter of k_θ for $\theta = 1, 2, \ldots, n$. We assume that the individual follows the recursive smooth ambiguity model by Klibanoff et al. (2009). Define an increasing and concave second-order utility ϕ whose variable is expected utility. The concavity of ϕ captures ambiguity aversion. Given s, the objective function is written as

$$V(s) = u(w-s) + \phi^{-1}(\sum_{\theta=1}^{\Theta} q_\theta \phi(E[v(s,k)]). \tag{8}$$

Recall that

$$\begin{aligned} E[v(s,k)] &= kpqE[u(w+s+\tilde{\epsilon}_B, \tilde{\delta}_B)] + (1-kp)qE[u(w+s+\tilde{\epsilon}_G, \tilde{\delta}_B)] \\ &\quad + p(1-kq)E[u(w+s+\tilde{\epsilon}_B, \tilde{\delta}_G)] + (1-p-q+kpq)E[u(w+s+\tilde{\epsilon}_G, \tilde{\delta}_G)]. \end{aligned}$$

The first-order condition for (8) is

$$V'(s^*) = -u'(w-s^*) + \sum_{\theta=1}^{\Theta} q_\theta \frac{\phi'(E[v(s^*,k_\theta)])}{\phi'(\phi^{-1}(\sum_\theta q_\theta(\phi(E[v(s^*,k_\theta)]))))} E[v_s(s^*,k_\theta)] = 0. \tag{9}$$

The second-order condition is easily verified by the concavity of u and ϕ.

We let k_O define $k_O = \sum_\theta q_\theta k_\theta$. The level of optimal saving is denoted s^O under k_O. To examine the effect of ambiguous correlation on the optimal saving, we evaluate (9) at s^O

$$\begin{aligned} V'(s^O) &= -u'(w-s^O) + \sum_\theta \frac{\phi'(E[v(s^O,k_\theta)])}{\phi'(\phi^{-1}(\sum_\theta q_\theta(\phi(E[v(s^O,k_\theta)]))))} E[v_s(s^O,k_\theta)] \\ &= -u'(w-s^O) + \sum_\theta q_\theta \frac{\phi'(E[v(s^O,k_\theta)])}{\phi'(\phi^{-1}(\sum_\theta q_\theta(\phi(E[v(s^O,k_\theta)]))))} \sum_\theta q_\theta E[v_s(s^O,k_\theta)] \\ &\quad + \frac{Cov(\phi'(E[v(s^O,k_\theta)], E[v_s(s^O,k_\theta)])}{\phi'(\phi^{-1}(\sum_\theta q_\theta(\phi(E[v(s^O,k_\theta)]))))} \end{aligned} \tag{10}$$

Following Osaki and Schlesinger (2014), it holds that

$$\frac{\phi'(E[v(s^O,k)])}{\phi'(\phi^{-1}(\sum_\theta q_\theta(\phi(E[v(s^O,k)]))))}$$

is more (less) than unity when ϕ exhibits decreasing absolute (increasing) ambiguity aversion which define that $-\phi''/\phi'$ is a decreasing (increasing) function. We can determine the sign of (10) by the sign of covariance in the third term of (10).

We assume for now that the future income risk is ordered by NSD and the future health risk is ordered by MSD, respectively. Similar to the proof of Lemma 1, we can show the following Lemma.

Lemma 3. *Suppose that*

- $\tilde{\epsilon}_G$ *dominates* $\tilde{\epsilon}_B$ *in the sense of NSD;*
- $\tilde{\delta}_G$ *dominates* $\tilde{\delta}_B$ *in the sense of MSD.*

When the individual is mixed correlation averse (seeking), that is, $(-1)^{(n+m)}u_{(n,m)} \geq (\leq) 0$ *for all* $n = 1, 2, \ldots, N$ *and* $m = 1, 2, \ldots, M$*, then* $E[v(s,k)]$ *is increasing (decreasing) in* k*.*

Let us consider that u exhibits mixed correlation aversion or seeking. From Lemmas 1 and 3, the signs of $\partial E[v(s^O, k)]/\partial k$ and $\partial E[v_s(s^O, k)]/\partial k$ are different. When u exhibits mixed correlation aversion (seeking), $E[v(s^O, k)]$ is decreasing (increasing) in k, but $E[v_s(s^O, k)]$ is increasing (decreasing) in k. Because ϕ' is decreasing, the covariance is positive when u exhibits mixed correlation aversion or seeking. Combining the above argument, we have the following proposition:

Proposition 2. *Suppose that the future income risk is ranked by NSD and the future health risk is ranked by MSD, that is,*

- $\tilde{\epsilon}_G$ *dominates* $\tilde{\epsilon}_B$ *in the sense of NSD;*
- $\tilde{\delta}_G$ *dominates* $\tilde{\delta}_B$ *in the sense of MSD.*

If ϕ *exhibits decreasing absolute ambiguity aversion and* u *exhibits mixed correlation aversion or seeking, then ambiguous correlation raises the optimal saving.*

The same result can be obtained for the case where the future income risk and the future health risk are replaced with Nth-degree risk and Mth-degree risk.

Lemma 4. *Suppose that*

- $\tilde{\epsilon}_B$ *increases* $\tilde{\epsilon}_G$ *in* N*thtdegree risk;*
- $\tilde{\delta}_B$ *increases* $\tilde{\delta}_G$ *in* M*th-degree risk.*

When $(-1)^{(N+M)}u_{(N,M)} \geq (\leq)0$*,* $E[v(s,k)]$ *is increasing (decreasing) in* k*.*

Let us consider that u exhibits mixed correlation aversion or seeking.

Proposition 3. *Suppose that the future income risk is ranked by* N*th-degree risk and the future health risk is ranked by* M*th-degree risk, that is,*

- $\tilde{\epsilon}_B$ *increases* $\tilde{\epsilon}_G$ *in* N*th-degree risk;*
- $\tilde{\delta}_B$ *increases* $\tilde{\delta}_G$ *in* M*th-degree risk.*

If ϕ *exhibits decreasing absolute ambiguity aversion and* u *satisfies* $(-1)^{(N+M)}u_{(N,M)} \geq (\leq)0$ *and* $(-1)^{(N+M)}u_{(N+1,M)} \geq (\leq)0$*, then ambiguous correlation raises the optimal saving.*

We note that mixed correlation aversion (seeking) is a sufficient condition to hold $(-1)^{(N+M)}u_{(N,M)} \geq (\leq)0$ and $(-1)^{(N+M)}u_{(N+1,M)} \geq (\leq)0$. Berger and Bosetti (2020) found experimental evidence on decreasing absolute ambiguity aversion.

We consider a special case of $N = M = 1$ and relate experimental evidences of the preference of bivariate utility to the above proposition. Even though we can find empirical and experimental analyses for correlation aversion in the framework of bivariate utility which is determined by wealth and health, higher-order risk preference of bivariate utility, e.g. cross prudence and cross temperance, is new to the literature. In a recent experiment, Attema et al. (2019) examined higher-order risk preference of bivariate utility. They observed that most subjects exhibit correlation aversion in the gain domain and correlation seeking in the loss domain. However, they can observe both cross prudence and imprudence in health in both gain and loss domains. For example, when subjects exhibit correlation aversion and cross prudence in health, ambiguous correlation raises the optimal saving. However, for cross imprudence in health, the result is opposite, and ambiguous correlation lowers the optimal saving. When a non-financial variable is other's wealth, Ebert and van de Kulien (2015) observed the majority of subjects exhibit correlation seeking, but observed both cross prudence and imprudence in other's wealth. Ambiguous correlation lowers the optimal saving for cross prudent subjects, but raises it for cross imprudent subjects.

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