Singles monotonicity and stability in one-to-one matching problems[1]

Yoichi Kasajima[2] and Manabu Toda[2]

[2]School of Social Sciences, Waseda University

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- Under regional caps, the number of matchings in rural hospitals increased, while the numbers of doctors assigned to their first choice considerably decreased.

$\Rightarrow$ We would like to increase the number of matchings in local hospitals without deceasing the welfare of doctors.

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Question : If we subsidize the applicants to local hospitals, what happens to the outcome of the mechanism? (or if we change the rank-order lists of doctors in favor of a hospital whose quota is vacant, what happens to the outcome?)

Model (one-to-one matching problems)

Let $\mathcal{F}$ and $\mathcal{W}$ be mutually disjoint sets of countably many potential agents. The former is the set of firms and the latter is the set of workers.

The first component of a matching problem is given by a union $F \cup W$ of non-empty finite subsets $F \subset \mathcal{F}$ and $W \subset \mathcal{W}$. For each $a \in F \cup W$,

$$(F \cup W)_a = F \text{ and } (F \cup W)_{-a} = W \text{ if } a \in F,$$

$$(F \cup W)_a = W \text{ and } (F \cup W)_{-a} = F \text{ if } a \in W.$$

In words, $(F \cup W)_a$ is the set of the agents on the same side as a and $(F \cup W)_{-a}$ is the set of agents on the opposite side of a.

Model (one-to-one matching problems)

Each $a \in F \cup W$ has a strict preference ordering $\succ_a$ over the set $(F \cup W)_{-a} \cup \{\phi\}$, where ϕ is the choice of remaining un-matched and the associated weak ordering is denoted by $\succeq_a$.

We define

$$\mathcal{P}_F = \{\succ_a \mid a \in F\}, \quad \mathcal{P}_W = \{\succ_a \mid a \in W\},$$

and $\mathcal{P}_{F \cup W} = \mathcal{P}_F \cup \mathcal{P}_W$. By definition, for each $a \in F \cup W$,

$$\mathcal{P}_{F \cup W} = \mathcal{P}_{(F \cup W)_a} \cup \mathcal{P}_{(F \cup W)_{-a}}.$$

A **matching problem** is a pair $(F \cup W, \mathcal{P}_{F \cup W})$.

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  1. $\mu(a) \in (F \cup W)_{-a} \cup \{\phi\}$
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- A **solution** is a correspondence φ defined on $\mathcal{E}$ satisfying

$$\emptyset \neq \varphi(F \cup W, \mathcal{P}_{F \cup W}) \subset \mathcal{M}(F \cup W)$$

  for each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$.

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- A **single-valued solution** is a function φ defined on $\mathcal{E}$ satisfying $\varphi(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{M}(F \cup W)$ for each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$.

Definitions (Properties of matchings)

- A matching $\mu \in \mathcal{M}(F \cup W)$ is **individually rational** in $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$ if for each $a \in F \cup W$, $\mu(a) \succeq_a \phi$.

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- In a given $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, a pair $(f, w) \in F \times W$ **blocks** a matching $\mu \in \mathcal{M}(F \cup W)$ if $w \succ_f \mu(f)$ and $f \succ_w \mu(w)$.

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Definitions (solutions)

- $\mathcal{S}(F \cup W, \mathcal{P}_{F \cup W})$: the set of all stable matchings in $(F \cup W, \mathcal{P}_{F \cup W})$
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- The $\boldsymbol{F}$**-optimal stable solution** associate with each problem $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$ the matching $\mathcal{S}_F(F \cup W, \mathcal{P}_{F \cup W})$.

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- The $\boldsymbol{W}$**-optimal stable solution** associate with each problem $(F \cup W, \mathcal{P}_{F\cup W}) \in \mathcal{E}$ the matching $\mathcal{S}_W(F \cup W, \mathcal{P}_{F\cup W})$.

Definitions

Definition 1

For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, $h \in F \cup W$, and $a \in (F \cup W)_{-h}$, a preference ordering $\succ_a^h$ on $(F \cup W)_h \cup \{\phi\}$ is a h-***improvement*** over $\succ_a$ if $\succ_a^h$ and $\succ_a$ determine the same ordering on the set $((F \cup W)_h \setminus \{h\}) \cup \{\phi\}$ and $h \succ_a h'$ implies $h \succ_a^h h'$ for each $h' \in (F \cup W)_h$.

In short, $\succ_a^h$ is a h-improvement over $\succ_a$ if the order of h is higher in $\succ_a^h$ than in $\succ_a$, while the relative orders among the others stay unchanged.

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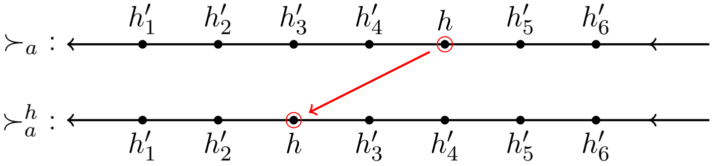

Definitions

Definition 2

For a given preference profile $\mathcal{P}_{F\cup W} = \{\succ_a \mid a \in F \cup W\}$ and $h \in F \cup W$, a preference profile $\mathcal{P}^h_{F\cup W} = \{\succ^h_a \mid a \in F \cup W\}$ is a h-***improvement*** over $\mathcal{P}_{F\cup W}$ if

(1) $\succ^h_a$ is a h-improvement over $\succ_a$ for each $a \in (F \cup W)_{-h}$,

(2) $\succ^h_a = \succ_a$ for each $a \in (F \cup W)_h$.

Singles Monotonicity

Axiom

Own-side singles monotonicity :

For a given $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, suppose that $h \in F \cup W$ satisfies $\mu(h) = \phi$ for each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$. Then, a solution φ satisfies own-side single monotonicity if for each h-improvement $\mathcal{P}^h_{F \cup W}$ over $\mathcal{P}_{F \cup W}$ and each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$, there exists $\nu \in \varphi(F \cup W, \mathcal{P}^h_{F \cup W})$ such that,

$$\mu(a) \succeq_a \nu(a)$$

for each $a \in (F \cup W)_h \setminus \{h\}$.

Suppose that h is single at some problem and every agent on the opposite side of h changes her/his preference in favor of h. The axiom requires that every agent on the same side of h (except h) should not be made strictly better off.

Singles Monotonicity

Axiom

Other-side singles monotonicity :

For a given $(F \cup W, \mathcal{P}_{F\cup W}) \in \mathcal{E}$, suppose that $h \in F \cup W$ satisfies $\mu(h) = \phi$ for each $\mu \in \varphi(F \cup W, \mathcal{P}_{F\cup W})$. Then, a solution φ satisfies other-side singles monotonicity if for each h-improvement $\mathcal{P}^h_{F\cup W}$ over $\mathcal{P}_{F\cup W}$ and each $\nu \in \varphi(F \cup W, \mathcal{P}^h_{F\cup W})$, there exists $\mu \in \varphi(F \cup W, \mathcal{P}_{F\cup W})$ such that,

$$\nu(a) \succeq^h_a \mu(a)$$

for each $a \in (F \cup W)_{-h}$.

Suppose that h is single at some problem and every agent on the opposite side of h changes her/his preference in favor of h. The axiom requires that every agent on the opposite side of h should not be made strictly worse off with respect to the ex-post preference.

Example 1

Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\mathcal{P}_F \cup \mathcal{P}_W$ be given by,

f_1	$w_1 \succ w_2 \succ \phi \succ w_3$
f_2	$w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi$
w_2	$f_1 \succ f_2 \succ \phi \succ f_3$
w_3	$\phi \succ f_3 \succ f_1 \succ f_2$

The F-optimal stable matching $\mathcal{S}_F(F \cup W, \mathcal{P}_{F\cup W})$ is given by

$$\mu_F = \{(f_1, w_2), (f_2, w_1), f_3, w_3\},$$

which is the unique stable matching. Notice that $\mu_F(f_3) = \phi$.

Example 1

f_1	$w_1 \succ w_2 \succ \phi \succ w_3$
f_2	$w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi$
w_2	$f_1 \succ f_2 \succ \phi \succ f_3$
w_3	$\phi \succ f_3 \succ f_1 \succ f_2$

Let $\mathcal{P}_{F\cup W}^{f_3}$ be the f_3-improvement over $\mathcal{P}_{F\cup W}$ defined as follows.

f_1	$w_1 \succ w_2 \succ \phi \succ w_3$
f_2	$w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi$
w_2	$f_1 \succ f_2 \succ \phi \succ f_3$
w_3	$\boldsymbol{f_3} \succ \phi \succ f_1 \succ f_2$

The F-optimal stable matching $\mathcal{S}(F \cup W, \mathcal{P}_{F\cup W}^{f_3})$ is given by

$$\mu_F^{f_3} = \{(f_1, w_1), (f_2, w_2), (f_3, w_3)\}.$$

Example 1

$\mathcal{P}_{F\cup W}$:

f_1	$w_1 \succ w_2 \succ \phi \succ w_3$
f_2	$w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi$
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$$\mu_F = \{(f_1, w_2), (f_2, w_1), f_3, w_3\}$$

$\mathcal{P}^{f_3}_{F\cup W}$:

f_1	$w_1 \succ w_2 \succ \phi \succ w_3$
f_2	$w_2 \succ w_1 \succ \phi \succ w_3$
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w_1	$f_2 \succ f_3 \succ f_1 \succ \phi$
w_2	$f_1 \succ f_2 \succ \phi \succ f_3$
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$$\mu_F^{f_3} = \{(f_1, w_1), (f_2, w_2), (f_3, w_3)\}$$

1. $\mu_F^{f_3}(f) \succ_f \mu_F(f)$ for $f \neq f_3$.
2. $\mu_F(w) \succ_w^{f_3} \mu_F^{f_3}(w)$ for $w = w_1, w_2$ and $\mu_F^{f_3}(w_3) \succ_{w_3}^{f_3} \mu_F(w_3)$.

Example 1

- Since $\mu_F^{f_3}(f) \succ_f \mu_F(f)$ for $f \neq f_3$, every firm other than f_3 is made strictly better off by the f_3-improvement, the F-optimal stable solution $\mathcal{S}_F$ does not satisfy own-side singles monotonicity.
- Since $\mu_F(w) \succ_w^{f_3} \mu_F^{f_3}(w)$ for $w \neq w_3$, there exist workers strictly made worse off by the f_3-improvement, the F-optimal stable solution $\mathcal{S}_F$ does not satisfy other-side singles monotonicity. This also shows that the stable solution $\mathcal{S}$ does not satisfy other-side singles monotonicity.
- The W-optimal stable matching in $(F \cup W, \mathcal{P}_{F \cup W}^{f_3})$ is

$$\mu_W^{f_3} = \{(f_1, w_2), (f_2, w_1), (f_3, w_3)\}.$$

Because μ_F is the unique stable matching in the original problem, it is also the W-optimal stable matching. Since $\mu_F(w) = \mu_W^{f_3}(w)$ for $w \neq w_3$ and $\mu_W^{f_3}(w_3) \succ_{w_3}^{f_3} \mu_F(w_3)$, the W-optimal stable solution does not violate own-side singles monotonicity for the f_3-improvement in Example 1.

single-valued stable solutions

Definition 3

A single-valued solution φ is **stable** if $\varphi(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{S}(F \cup W, \mathcal{P}_{F \cup W})$ for each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$.

Proposition 1

Let φ be a stable single-valued solution. Then, φ satisfies own-side singles monotonicity if and only if it satisfies other-side singles monotonicity.

Proposition 2

There exists no single-valued solution satisfying stability and singles monotonicity.

Let $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ and define $\mathcal{P}_{F \cup W}$ as below.

f_1	$w_1 \succ w_4 \succ w_2 \succ \phi \succ w_3$
f_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$w_2 \succ \phi \succ w_1 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ f_4 \succ f_2 \succ \phi \succ f_3$
w_3	$\phi \succ f_1 \succ f_2 \succ f_3 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

In the problem $(F \cup W, \mathcal{P}_{F \cup W})$,

$$\mu = \{(f_1, w_4), (f_2, w_1), (f_4, w_2), f_3, w_3\}$$

is the unique stable matching. Note that $\mu(f_3) = \phi$.

Let $\mathcal{P}^{f_3}_{F\cup W}$ be the f_3-improvement over $\mathcal{P}_{F\cup W}$ given below.

f_1	$w_1 \succ w_4 \succ w_2 \succ \phi \succ w_3$
f_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$w_2 \succ \phi \succ w_1 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ f_4 \succ f_2 \succ \phi \succ f_3$
w_3	$\boldsymbol{f_3} \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

The problem $(F \cup W, \mathcal{P}^{f_3}_{F\cup W})$ has two stable matchings,

$$\mu^1 = \{(f_1, w_1), (f_2, w_4), (f_3, w_3), (f_4, w_2)\}$$
$$\mu^2 = \{(f_1, w_4), (f_2, w_1), (f_3, w_3), (f_4, w_2)\}$$

$\mathcal{P}_{F \cup W}$

f_1	$w_1 \succ w_4 \succ w_2 \succ \phi \succ w_3$
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w_3	$\phi \succ f_1 \succ f_2 \succ f_3 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

$\mathcal{P}_{F \cup W}^{f_3}$

f_1	$\boldsymbol{w_1} \succ w_4 \succ w_2 \succ \phi \succ w_3$
f_2	$\boldsymbol{w_4} \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$\boldsymbol{w_3} \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$\boldsymbol{w_2} \succ \phi \succ w_1 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ \boldsymbol{f_1} \succ \phi \succ f_4$
w_2	$f_1 \succ \boldsymbol{f_4} \succ f_2 \succ \phi \succ f_3$
w_3	$\boldsymbol{f_3} \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$f_1 \succ \boldsymbol{f_2} \succ \phi \succ f_3 \succ f_4$

If a stable single-valued solution satisfies own-side singles monotonicity, it selects μ^2 in problem $(F \cup W, \mathcal{P}_{F \cup W}^{f_3})$.

The problem $(F \cup W, \hat{\mathcal{P}}_{F \cup W})$ given below has the unique stable matching $\hat{\mu} = \{(f_1, w_1), (f_2, w_4), (f_3, w_3), f_4, w_2\}$, in which $\hat{\mu}(w_2) = \phi$.

f_1	$w_1 \succ w_4 \succ \phi \succ w_2 \succ w_3$
f_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$\phi \succ w_1 \succ w_2 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ f_4 \succ f_2 \succ \phi \succ f_3$
w_3	$f_3 \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

$$\hat{\mathcal{P}}_{F\cup W}$$

f_1	$w_1 \succ w_4 \succ \phi \succ w_2 \succ w_3$
f_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$\phi \succ w_1 \succ w_2 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ f_4 \succ f_2 \succ \phi \succ f_3$
w_3	$f_3 \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

Let $\hat{\mathcal{P}}^{w_2}_{F\cup W}$ be the w_2-improvement over $\hat{\mathcal{P}}_{F\cup W}$ given below.

f_1	$w_1 \succ w_4 \succ \boldsymbol{w_2} \succ \phi \succ w_3$
f_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$\boldsymbol{w_2} \succ \phi \succ w_1 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ f_4 \succ f_2 \succ \phi \succ f_3$
w_3	$f_3 \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

We can see that $\hat{\mathcal{P}}^{w_2}_{F\cup W} = \mathcal{P}^{f_3}_{F\cup W}$ and μ^2 is selected in $(F \cup W, \hat{\mathcal{P}}^{w_2}_{F\cup W})$.

$$\hat{\mathcal{P}}_{F\cup W}$$

f_1	$w_1 \succ w_4 \succ \phi \succ w_2 \succ w_3$
f_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$
f_3	$w_3 \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$\phi \succ w_1 \succ w_2 \succ w_3 \succ w_4$

w_1	$f_2 \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ f_4 \succ f_2 \succ \phi \succ f_3$
w_3	$f_3 \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$f_1 \succ f_2 \succ \phi \succ f_3 \succ f_4$

$$\hat{\mathcal{P}}^{w_2}_{F\cup W}$$

f_1	$w_1 \succ \boldsymbol{w_4} \succ w_2 \succ \phi \succ w_3$
f_2	$w_4 \succ w_2 \succ \boldsymbol{w_1} \succ \phi \succ w_3$
f_3	$\boldsymbol{w_3} \succ w_1 \succ \phi \succ w_2 \succ w_4$
f_4	$\boldsymbol{w_2} \succ \phi \succ w_1 \succ w_3 \succ w_4$

w_1	$\boldsymbol{f_2} \succ f_3 \succ f_1 \succ \phi \succ f_4$
w_2	$f_1 \succ \boldsymbol{f_4} \succ f_2 \succ \phi \succ f_3$
w_3	$\boldsymbol{f_3} \succ \phi \succ f_1 \succ f_2 \succ f_4$
w_4	$\boldsymbol{f_1} \succ f_2 \succ \phi \succ f_3 \succ f_4$

However, this violates own-side singles monotonicity.

W-singles Monotonicity

In Example 1, we observe that the F-optimal stable solution violates the requirements of singles monotonicity for f_3-improvement, while the W-optimal solution satisfies the requirements.

Definition 4

For a given $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, suppose that $w \in W$ satisfies $\mu(w) = \phi$ for each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$. Then, a solution φ satisfies own-side W-singles monotonicity if for each w-improvement $\mathcal{P}^{w}_{F \cup W}$ over $\mathcal{P}_{F \cup W}$ and each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$, there exists $\nu \in \varphi(F \cup W, \mathcal{P}^{w}_{F \cup W})$ such that

$$\mu(a) \succeq_a \nu(a)$$

for each $a \in W \setminus \{w\}$.

W-singles Monotonicity

Definition 5

For a given $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, suppose that $w \in W$ satisfies $\mu(w) = \phi$ for each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$. Then, a solution φ satisfies other-side W-singles monotonicity if for each w-improvement $\mathcal{P}^{w}_{F \cup W}$ over $\mathcal{P}_{F \cup W}$ and each $\nu \in \varphi(F \cup W, \mathcal{P}^{w}_{F \cup W})$, there exists $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$ such that

$$\nu(a) \succeq^{w}_{a} \mu(a)$$

for each $a \in F$.

Definition 6

A solution φ is W-**singles monotonic** if it satisfies both own-side and other-side W-singles monotonicity.

F-singles Monotonicity

Definition 7

For a given $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, suppose that $f \in F$ satisfies $\mu(f) = \phi$ for each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$. Then, a solution φ satisfies own-side F-singles monotonicity if for each f-improvement $\mathcal{P}^f_{F \cup W}$ over $\mathcal{P}_{F \cup W}$ and each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$, there exists $\nu \in \varphi(F \cup W, \mathcal{P}^f_{F \cup W})$ such that

$$\mu(a) \succeq_a \nu(a)$$

for each $a \in F \setminus \{f\}$.

F-singles Monotonicity

Definition 8

For a given $(F \cup W, \mathcal{P}_{F\cup W}) \in \mathcal{E}$, suppose that $f \in F$ satisfies $\mu(f) = \phi$ for each $\mu \in \varphi(F \cup W, \mathcal{P}_{F\cup W})$. Then, a solution φ satisfies other-side F-singles monotonicity if for each f-improvement $\mathcal{P}^f_{F\cup W}$ over $\mathcal{P}_{F\cup W}$ and each $\nu \in \varphi(F \cup W, \mathcal{P}^f_{F\cup W})$, there exists $\mu \in \varphi(F \cup W, \mathcal{P}_{F\cup W})$ such that

$$\nu(a) \succeq^f_a \mu(a)$$

for each $a \in W$.

Definition 9

A solution φ is F-**singles monotonic** if it satisfies both own-side and other-side F-singles monotonicity.

Remark

A solution φ satisfies own-side singles monotonicity if and only if it satisfies own-side W-singles and F-singles monotonicity. A solution satisfies other-side singles monotonicity if and only if it satisfies other-side W-singles and F-singles monotonicity.

Proposition 3

Let φ be a stable single-valued solution. Then, φ satisfies own-side W-singles monotonicity if and only if it satisfies other-side W-singles monotonicity.

Proposition 4

Let φ be a stable single-valued solution. Then, φ satisfies own-side F-singles monotonicity if and only if it satisfies other-side F-singles monotonicity.

Theorem 1

The F-optimal stable solution $\mathcal{S}_F$ satisfies W-singles monotonicity.

Proof of Theorem 1

The Blocking Lemma

Let μ_F be the F-optimal matching and μ an individually rational matching. If the set

$$F' \equiv \{f' \in F \mid \mu(f') \succ_{f'} \mu_F(f')\} \neq \emptyset,$$

there exists a blocking pair (f, w') of μ such that $f \in F \setminus F'$ and $w' \in \mu(F')$.

Proof of Theorem 1

Proof.

It suffices to show othe-side W-singles monotonicity. Let $\mu_F(w) = \phi$ for some $w \in W$ and let μ_F^w be the F-optimal matching in $(F \cup W, \mathcal{P}^w_{F\cup W})$, where $\mathcal{P}^w_{F\cup W}$ is a w-improvement over $\mathcal{P}_{F\cup W}$. Because $\mu_F(f) \neq w$ for each $f \in F$, μ_F is individually rational in $(F \cup W, \mathcal{P}^w_{F\cup W})$. Suppose

$$F' \equiv \{f' \in F \mid \mu_F(f') \succ^w_{f'} \mu^w_F(f')\} \neq \emptyset.$$

By the Blocking Lemma, there exists a pair (f, w') such that $f \in F \setminus F'$ and $w' \in \mu_F(F')$, and $w' \succ^w_f \mu_F(f)$ and $f \succ_{w'} \mu_F(w')$. Since $w' \in \mu_F(F')$, $w' \neq w$ and obviously $\mu_F(f) \neq w$. Then, $w' \succ_f \mu_F(f)$, implying (f, w') block μ_F, which is a contradiction. Hence, $F' = \emptyset$ and $\mu^h_F(f) \succeq^w_f \mu_F(f)$ for each $f \in F$. This shows other-side W-singles monotonicity. □

Theorem 2

The W-optimal stable solution $\mathcal{S}_W$ satisfies F-singles monotonicity.

	own F-S.MON	other F-S.MON	own W-S.MON	other W-S.MON
$\mathcal{S}_F$	$-$	$-$	$+$	$+$
$\mathcal{S}_W$	$+$	$+$	$-$	$-$
	F-S.MON		W-S.MON	
$\mathcal{S}_F$	$-$		$+$	
$\mathcal{S}_W$	$+$		$-$	

Observation

The F-optimal stable solution is not the unique single-valued stable solution satisfying W-singles monotonicity and the W-optimal stable solution is not the unique single-valued stable solution satisfying F-singles monotonicity.

For the stable solution $\mathcal{S}$, we may obtain the following result.

Theorem 3

The stable solution $\mathcal{S}$ satisfies own-side F-singles (W-singles) monotonicity and hence own-side singles monotonicity.

Proof.

Let $(F \cup W, \mathcal{P}_{F\cup W}) \in \mathcal{E}$ and suppose that $\mu(f) = \phi$ for each $\mu \in \mathcal{S}(F \cup W, \mathcal{P}_{F\cup W})$ and $\mathcal{P}^f_{F\cup W}$ is an f-improvement over $\mathcal{P}_{F\cup W}$. Let μ_W and μ^f_W be the W-optimal stable matchings in $(F \cup W, \mathcal{P}_{F\cup W})$ and in $(F \cup W, \mathcal{P}^f_{F\cup W})$, respectively. Because the W-optimal stable matching is the worst for each firm among stable matchings and the W-optimal stable solution satisfies own-side F-singles monotonicity, we have

$$\mu(a) \succeq_a \mu_W(a) \succeq_a \mu^f_W(a)$$

for each $a \in F \setminus \{f\}$, which shows own-side F-singles monotonicity of $\mathcal{S}$. By the same arguments, $\mathcal{S}$ satisfies own-side W-singles monotonicity. □

	own S.MON		other S.MON	
$\mathcal{S}$	$+$		$-$	
	own F-S.MON	own W-S.MON	other F-S.MON	other W-S.MON
$\mathcal{S}$	$+$	$+$	$-$	$-$

Axiomatizations of the stable solution

Axiom

Weak unanimity : For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, if there exits a matching $\mu \in \mathcal{M}(F \cup W)$ such that for each $a \in F \cup W$ and each $b \in (F \cup W)_{-a} \cup \{\phi\}$, $\mu(a) \succ_a b$, then $\varphi(F \cup W, \mathcal{P}_{F \cup W}) = \{\mu\}$.

Definition 10

For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, each $h \in \mathbb{F} \cup \mathbb{W} \setminus (F \cup W)$, and each $a \in (F \cup W \cup \{h\})_{-h}$, a preference ordering $\succ'_a$ is a h-extension of $\succ_a$ if

1. $\succ'_a$ is a strict preference ordering over the set $(F \cup W \cup \{h\})_{-a} \cup \{\phi\}$,
2. for each $h', h'' \in (F \cup W)_{-a} \cup \{\phi\}$, $h' \succ_a h''$ implies $h' \succ'_a h''$.

Definition 11

For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$ and each $h \in \mathbb{F} \cup \mathbb{W} \setminus (F \cup W)$, a problem $(F \cup W \cup \{h\}, \mathcal{P}'_{F \cup W \cup \{h\}})$ is a h-extension of $(F \cup W, \mathcal{P}_{F \cup W})$ if

1. each preference ordering in $\mathcal{P}'_{(F \cup W \cup \{h\})_{-h}}$ is a h-extension of its corresponding preference ordering in $\mathcal{P}_{(F \cup W)}$,
2. each preference ordering in $\mathcal{P}'_{(F \cup W \cup \{h\})_h \setminus \{h\}}$ is equal to its corresponding ordering in $\mathcal{P}_{(F \cup W)}$.

Definition 12

For each $(F, W) \in \mathcal{F} \times \mathcal{W}$, each $\mu \in \mathcal{M}(F \cup W)$, and each $h \in \mathbb{F} \cup \mathbb{W} \setminus (F \cup W)$, let $\mu_{+h} \in \mathcal{M}(F \cup W \cup \{h\})$ be such that

1. for each $a \in F \cup W$, $\mu_{+h}(a) = \mu(a)$,
2. $\mu_{+h}(h) = \phi$.

Axiom

Null player invariance : For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, each $h \in \mathbb{F} \cup \mathbb{W} \setminus (F \cup W)$, and each h-extension $(F \cup W \cup \{h\}, \mathcal{P}'_{F \cup W \cup \{h\}})$ of $(F \cup W, \mathcal{P}_{F \cup W})$ in which h is unacceptable for each $a \in (F \cup W \cup \{h\})_{-h}$, we have $\{\mu_{+h} \mid \mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})\} = \varphi(F \cup W \cup \{h\}, \mathcal{P}'_{F \cup W \cup \{h\}})$.

Definition 13

For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, each $\mu \in \mathcal{M}(F \cup W)$, each $F' \subset F$ with $F' \neq \emptyset$, and each $W' \subset W$ with $W' \neq \emptyset$, a problem $(F' \cup W', \mathcal{P}'_{F' \cup W'})$ is a **reduced problem of** $(F \cup W, \mathcal{P}_{F \cup W})$ **at** μ if for each $a \in F' \cup W'$,

1. if $\mu(a) \neq \emptyset$, then $\mu(a) \in (F' \cup W')_{-a}$
2. agent a's preference ordering in $\mathcal{P}'_{F' \cup W'}$ is the restriction of agent a's preference ordering in $\mathcal{P}_{F \cup W}$ onto $(F' \cup W')_{-a} \cup \{\phi\}$.

We also define $\mu_{F' \cup W'} \in \mathcal{M}(F' \cup W')$ is the the restriction of μ to the set $F' \cup W'$.

Axiom

Consistency: For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$ and each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$, if $(F' \cup W', \mathcal{P}'_{F' \cup W'})$ is a reduced problem of $(F \cup W, \mathcal{P}_{F \cup W})$ at μ, then

$$\mu_{F' \cup W'} \in \varphi(F' \cup W', \mathcal{P}'_{F' \cup W'}).$$

Definition 14

For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$, each $\mu \in \mathcal{M}(F \cup W)$, and each $a \in F \cup W$, let $L(\mu, \succ_a) = \{b \in (F \cup W)_{-a} \cup \{\phi\} \mid \mu(a) \succeq_a b\}$.

For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$ and each $\mu \in \mathcal{M}(F \cup W)$, a preference profile $\mathcal{P}'_{F \cup W} = \{\succ'_a \mid a \in F \cup W\}$ **is obtained by a monotonic transformation of** $\mathcal{P}_{F \cup W}$ **at** μ if for each $a \in F \cup W$,

$$L(\mu, \succ_a) \subseteq L(\mu, \succ'_a).$$

Axiom

Maskin invariance: For each $(F \cup W, \mathcal{P}_{F \cup W}) \in \mathcal{E}$ and each $\mu \in \varphi(F \cup W, \mathcal{P}_{F \cup W})$, if $\mathcal{P}'_{F \cup W}$ is obtained by a monotonic transformation of $\mathcal{P}_{F \cup W}$ at μ, then

$$\mu \in \varphi(F \cup W, \mathcal{P}'_{F \cup W}).$$

Theorem 4

The stable solution is the unique solution satisfying weak unanimity, null player invariance, own-side singles monotonicity, and consistency.

Theorem 5

The stable solution is the unique solution satisfying weak unanimity, null player invariance, own-side singles monotonicity, and Maskin invariance.

Remark

All axioms in Theorems 4 and 5 are mutually independent.

Related literature

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- Characterization of the "deferred acceptance rule": **Kojima and Manea (2010), Morrill (2013), Ehlers and Klaus (2014), Chen (2017)**