# Efficient and strategy-proof allocation mechanisms in economies with local preference domains*

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**Abstract**

In this paper, we show that in pure exchange economies where the number of goods equals or exceeds the number of agents, any Pareto-efficient and strategy-proof allocation mechanism defined on any local preference domain always allocates the total endowment to some single agent even if the receivers vary.

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## 1 Introduction

Following the seminal work of Hurwicz (1972), the manipulability and efficiency of allocation mechanisms in pure exchange economies have been studied intensively. Zhou (1991) established that any Pareto-efficient and strategy-proof allocation mechanism is dictatorial in exchange economies with two agents having classical (i.e., continuous, strictly monotonic, and strictly convex) preferences. The dictatorship result in two-agent economies has been strengthened by being proven in the domain of restricted preferences.[1]

Compared with the result in two-agent economies, it has been an open question whether Pareto-efficient and strategy-proof allocation mechanisms can be characterized

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[1]See Schummer (1997), Ju (2003), Hashimoto (2008), and Momi (2013a). Nicolò (2004), however, showed a Pareto-efficient, strategy-proof, and non-dictatorial mechanism in the domain of Leontief preferences.

in economies with many agents. In many-agent economies, there actually exist Pareto-efficient, strategy-proof, and non-dictatorial allocation mechanisms as pointed out by Satterthwaite and Sonnenschein (1981) and Kato and Ohseto (2002). A specific feature shared by all known Pareto-efficient and strategy-proof allocation mechanisms is that some single agent receives all goods even if the receivers vary. Such a mechanism is called alternately dictatorial. The natural question to be asked is whether any Pareto-efficient and strategy-proof allocation mechanism is alternately dictatorial.[2]

Recently, Momi (2013b) proved the alternate dictatorship result in three-agent economies. Momi (2017) then proved the result in economies where the number of goods equals or exceeds the number of agents. It is still an open question whether Pareto-efficient and strategy-proof mechanisms can be characterized without such a condition on the numbers of agents and goods. In this paper, we strengthen the alternate dictatorship result in the economies where the number of goods equals or exceeds the number of agents by proving that the result holds even if the mechanism is defined locally in an arbitrarily small neighborhood of a given preference profile.

The contribution of this paper is twofold. First, the characterization of locally defined mechanisms is interesting in inself. It is a probable sitution for us to have some information about the agents' preferences, whcich allows us to narrow our focus to a local preference set, rather than the whole preference set. As far as we are aware, quite a few studies have considered such a situation. For example, the proofs of many of the studies mentioned above, including studies of two-agent econoies, rely on the results obtained for a specific preference profile where all agents have identical preferences, and therefore the proofs cannot be applied to mechanisms that are only defined locally. Second, we believe the results and techniques in this paper would be a first step toward solving the general question without conditions on the numbers of goods and agents. The difficulty we face in this paper is closely related to what we have to overcome to solve the general question. In the rest of this section, we explain this point more in details.

In this paper, we deal with homothetic preferences. A key element of the proof by Momi (2017) is the independence of preferences in the following sense. An efficient allocation given by a Pareto-efficient mechanism uniquely determines the supporting price, and the supporting price vector uniquely determines the direction of possible consumption of each agent with a homothetic preference, which we call the consumption-direction vector of the agent. If these consumption-direction vectors are independent among agents, then there is a unique way to scale these vectors so that they sum up to the total endowment. That is, a supporting price induced by a Pareto-efficient mechanism determines

[2]Some studies have shown the incompatibility of Pareto efficiency and strategy-proofness with allocation restrictions such as individual rationality. See Serizawa (2002), Serizawa and Weymark (2003), and Momi (2013b). Barberà and Jackson (1995) discarded Pareto efficiency and characterized strategy-proof mechanisms satisfying the individual rationality restriction.

the allocation itself when the consumption-direction vectors are independent.

If these consumption-direction vectors are not independent, then there are various ways to scale the vectors to sum up to the total endowment, and hence the allocation is indeterminate. This is the difficulty of the general question, where the number of agents exceeds the number of goods, and hence the independence of the consumption-direction vectors is never expected. Even if the number of goods equals or exceeds the number of agents, whether the independence of the consumption-direction vectors holds for a given preference profile is not obvious at all because the independence depends on not only the preferences but also on the price vector determined by the mechanism. However, as long as we consider the whole preference set, we can tactfully evade this difficulty. As Momi (2017) showed, there exist desirable preferences of agents that ensure independence for any price vectors. In a local preference set, we cannot expect the existence of such desirable preferences. This is the difficulty we face in this paper, and it is similar to that in the general question in the sense that we do not have preferences of agents that ensure the independence of consumption-direction vectors.

The rest of the paper is organized as follows. Section 2 describes the model and results. Section 3 reviews the approach by Momi (2017) and explains the difficulty we face when a mechanism is defined only locally. Section 4 shows some technical results. Section 5 provides the proof of the main result. The Appendix contains proofs of all lemmas and corollaries in Sections 4 and 5.

# 2 Model and results

We consider an economy with $N$ agents, indexed by $\mathbf{N} = \{1, \ldots, N\}$, where $N \geq 2$, and $L$ goods, indexed by $\mathbf{L} = \{1, \ldots, L\}$, where $L \geq 2$. The consumption set for each agent is $R^L_+$. A consumption bundle for agent $i \in \mathbf{N}$ is a vector $x^i = (x^i_1, \ldots, x^i_L) \in R^L_+$. The total endowment of goods for the economy is $\Omega = (\Omega_1, \ldots, \Omega_L) \in R^L_{++}$. An allocation is a vector $\mathbf{x} = (x^1, \ldots, x^N) \in R^{LN}_+$. Thus, the set of feasible allocations for the economy with $N$ agents and $L$ goods is

$$X = \left\{ \mathbf{x} \in R^{LN}_+ : \sum_{i \in \mathbf{N}} x^i \leq \Omega \right\}.$$

A preference $R$ is a complete, reflexive, and transitive binary relation on $R^L_+$. The corresponding strict preference $P_R$ and indifference $I_R$ are defined in the usual way. For any $x$ and $x'$ in $R^L_+$, $xP_Rx'$ implies that $xRx'$ and not $x'Rx$, and $xI_Rx'$ implies that $xRx'$ and $x'Rx$. Given a preference $R$ and a consumption bundle $x \in R^L_+$, the upper contour set of $R$ at $x$ is $UC(x; R) = \{x' \in R^L_+ : x'Rx\}$, and the lower contour set of $R$ at $x$ is $LC(x; R) = \{x' \in R^L_+ : xRx'\}$. We let $I(x; R) = \{x' \in R^L_+ : x'I_Rx\}$ denote the indifference set of $R$ at $x$, and $P(x;R) = \{x' \in R^L_+ : x' P_R x\}$ denotes the strictly preferred set of $R$ at $x$.

A preference $R$ is continuous if $UC(x;R)$ and $LC(x;R)$ are both closed for any $x \in R^L_+$. A preference $R$ is strictly convex on $R^L_{++}$ if $UC(x;R)$ is a strictly convex set in $R^L$ for any $x \in R^L_{++}$. A preference $R$ is monotonic if, for any $x$ and $x'$ in $R^L_+$, $x > x'$ implies that $xRx'$.[3] A preference $R$ is strictly monotonic on $R^L_{++}$ if, for any $x$ and $x'$ in $R^L_{++}$, $x > x'$ implies that $xP_R x'$.[4] A preference $R$ is homothetic if, for any $x$ and $x'$ in $R^L_+$ and any $t > 0$, $xRx'$ implies that $(tx)R(tx')$. A preference $R$ is smooth if for any $x \in R^L_{++}$, there exists a unique vector $p \in S^{L-1}_+ \equiv \{x \in R^L_+ : \| x \| = 1\}$ such that $p$ is the normal of a supporting hyperplane to $UC(x;R)$ at $x$. We call the vector $p$ the gradient vector of $R$ at $x$, and write $p = p(R,x)$. Note that if $R$ is smooth, strictly convex on $R^L_{++}$, and strictly monotonic on $R^L_{++}$, then the gradient vector is positive in the positive orthant: $p(R,x) \in S^{L-1}_{++} \equiv \{x \in R^L_{++} : \| x \| = 1\}$ for any $x \in R^L_{++}$.

We let $\mathcal{R}$ denote the set of preferences that are continuous, strictly convex on $R^L_{++}$, strictly monotonic on $R^L_{++}$, smooth, and homothetic. A preference profile is an $N$-tuple $\mathbf{R} = (R^1, \ldots, R^N) \in \mathcal{R}^N$. We write the subprofile obtained by removing $R^i$ from $\mathbf{R}$ as $\mathbf{R}^{-i} = (R^1, \ldots, R^{i-1}, R^{i+1}, \ldots, R^N)$ and write the profile $(R^1, \ldots, R^{i-1}, \bar{R}^i, R^{i+1}, \ldots, R^N)$ as $(\bar{R}^i, \mathbf{R}^{-i})$. We also write $\mathbf{R}^{-\{i,j\}}$ to denote the subprofile obtained by removing $R^i$ and $R^j$ from $\mathbf{R}$.

A social choice function $f : \mathcal{R}^N \to X$ assigns a feasible allocation to each preference profile in $\mathcal{R}^N$. For a preference profile $\mathbf{R} \in \mathcal{R}^N$, the outcome chosen can be written as $f(\mathbf{R}) = (f^1(\mathbf{R}), \ldots, f^N(\mathbf{R}))$, where $f^i(\mathbf{R})$ is the consumption bundle allocated to agent $i$ by $f$. We let $\mathcal{B} \subset \mathcal{R}^N$ be a subset of $\mathcal{R}^N$. In this paper, we deal with a case where a social choice function is defined on $\mathcal{B}$, or it satisfies desirable properties only on $\mathcal{B}$.

**Definition 1.** A social choice function $f : \mathcal{R}^N \to X$ is strategy-proof on $\mathcal{B} \subset \mathcal{R}^N$ if $f^i(\mathbf{R}) R^i f^i(\bar{R}^i, \mathbf{R}^{-i})$ for any $i \in \mathbf{N}$, any $\mathbf{R} \in \mathcal{B}$, and any $\bar{R}^i \in \mathcal{R}$ such that $(\bar{R}^i, \mathbf{R}^{-i}) \in \mathcal{B}$.

A feasible allocation is Pareto efficient if there is no other feasible allocation that would benefit someone without making anyone else worse off. That is, $\mathbf{x} \in X$ is Pareto efficient for preference profile $\mathbf{R}$ if there exists no $\bar{\mathbf{x}} \in X$ such that $\bar{x}^i R^i x^i$ for any $i \in \mathbf{N}$ and $\bar{x}^j P_{R^j} x^j$ for some $j \in \mathbf{N}$. We say that a social choice function is Pareto efficient if it always assigns a Pareto-efficient allocation.

**Definition 2.** A social choice function $f : \mathcal{R}^N \to X$ is Pareto efficient on $\mathcal{B} \subset \mathcal{R}^N$ if $f(\mathbf{R})$ is Pareto efficient for any $\mathbf{R} \in \mathcal{B}$.

[3]For vectors $x$ and $x'$ in $R^L$, $x > x'$ denote that $x_l \geq x'_l$ for any $l \in \mathbf{L}$ and $x \neq x'$.

[4]Therefore, if $R$ is continuous, strictly convex on $R^L_{++}$, and strictly monotonic on $R^L_{++}$, then $UC(x;R) \subset R^L_{++}$ for any $x \in R^L_{++}$ and the boundary $\partial R^L_+$ is an indifference set.

We say that a social choice function is dictatorial on $\mathcal{B} \subset \mathcal{R}^N$ if there exists an agent who is always allocated the total endowment.

**Definition 3.** A social choice function $f : \mathcal{R}^N \to X$ is dictatorial on $\mathcal{B}$ if there exists $i \in \mathbf{N}$ such that $f^i(\mathbf{R}) = \Omega$ for any $\mathbf{R} \in \mathcal{B}$.

We say that a social choice function is alternately dictatorial if it always allocates the total endowment to some single agent. Note that under an alternately dictatorial social choice function, the identity of the receiver of the total endowment may vary depending on preference profiles.

**Definition 4.** A social choice function $f : \mathcal{R}^N \to X$ is alternately dictatorial on $\mathcal{B} \subset \mathcal{R}^N$ if, for any $\mathbf{R} \in \mathcal{B}$, there exists $i_{\mathbf{R}} \in \mathbf{N}$ such that $f^{i_{\mathbf{R}}}(\mathbf{R}) = \Omega$.

As in previous studies including Serizawa (2002) and Momi (2013b), we introduce the Kannai metric into $\mathcal{R}$ following Kannai (1970), to discuss the continuity in $\mathcal{R}$. For $x \in R^L_+ \setminus 0$, we let $[x]$ denote the ray starting from zero and passing through $x$: $[x] = \{y \in R^L_+ : y = tx, t \geq 0\}$. We define $\mathbf{1} \equiv (1, \dots, 1) \in R^L_+$ so that $[\mathbf{1}]$ denotes the principal diagonal of $R^L_+$. Using these definitions,the Kannai metric $d(R, R')$ for continuous and monotonic preferences $R$ and $R'$ is defined as

$$d(R, R') = \max_{x \in R^L_+} \frac{\| I(x; R) \bigcap [\mathbf{1}] - I(x; R') \bigcap [\mathbf{1}] \|}{1 + \| x \|^2},$$

where $\| \cdot \|$ denotes the Euclid norm in $R^L$. With the Kannai metric, $\mathcal{R}$ is a metric space. See Kannai (1970) for details.

In this paper, we write $B_\epsilon(\bar{R}) \subset \mathcal{R}$ to denote the open ball set of preferences in $\mathcal{R}$, with center $\bar{R}$ and radius $\epsilon > 0$: $B_\epsilon(\bar{R}) = \{R \in \mathcal{R} : d(R, \bar{R}) < \epsilon\}$. For a preference profile $\mathbf{R} = (R^1, \dots, R^N)$, we write $B_\epsilon(\mathbf{R})$ to denote the product set of $B_\epsilon(R^i)$, $i = 1, \dots, N$: $B_\epsilon(\mathbf{R}) = \Pi_{i=1}^N B_\epsilon(R^i) = B_\epsilon(R^1) \times \cdots \times B_\epsilon(R^N)$.

We often write $B^i \subset \mathcal{R}$ to denote an open ball set of agent $i$'s preferences without a specified center or radius and $\mathbf{B} = \Pi_{i=1}^N B^i$ to denote the product of such open ball sets over agents.

It should not cause confusion that we define various open ball sets in a similar manner. For example, we write $B_\varepsilon(\bar{p}) \subset S^{L-1}_{++}$ to denote the open ball set of price vectors $p$: $B_\varepsilon(\bar{p}) = \{p \in S^{L-1}_{++} : \| p - \bar{p} \| < \varepsilon\}$. We write $B_\delta(\bar{y}) \subset R^L$ to denote the open ball set of $L$-dimensional vector $y$: $B_\delta(\bar{y}) = \{y \in R^L : \| y - \bar{y} \| < \delta\}$.

This paper's main result is as follows.

**Theorem.** Assume that $L \geq N$. If a social choice function $f : \mathcal{R}^N \to X$ is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \Pi_{i=1}^N B^i$, then $f$ is alternately dictatorial on $\mathbf{B}$.

## 3 Preliminary results

Momi (2017) proved the alternate dictatorship result for a social choice function $f$ defined on the whole domain $\mathcal{R}^N$: When $L > N$, any Pareto-efficient and strategy-proof social choice function $f : \mathcal{R}^N \to X$ is alternately dictatorial. In this section, referring to this result, we explain the difficulties we face in the case of local preference domains.

We consider a social choice function $f$ that is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \Pi_{i=1}^N B^i$.

As in Momi (2017), we define the option set as follows. For agent $i$, when the other agents' preferences $\bar{\mathbf{R}}^{-i} \in \mathbf{B}^{-i} \equiv \Pi_{j \neq i} B^j$ are fixed, we define the option set, $G^i(\bar{\mathbf{R}}^{-i}) \subset R_+^L$, as the union of the agent's consumption bundles given by $f$ over his preferences in $B^i$:

$$G^i(\bar{\mathbf{R}}^{-i}) = \bigcup_{R^i \in B^i} f^i(R^i, \bar{\mathbf{R}}^{-i}).$$

The key feature of the option set is that, because of the strategy-proofness on $\mathbf{B}$, $f^i(R^i, \bar{\mathbf{R}}^{-i})$ should be the most preferred consumption bundle in $G^i(\bar{\mathbf{R}}^{-i})$ with respect to $R^i \in B^i$.

If $f$ is a Pareto-efficient social choice function and $f(\mathbf{R})$ is a Pareto-efficient allocation, then all agents share the same gradient vector at their consumption, as long as consumption is positive and the gradient vector is well defined. We call this vector the price vector at allocation $f(\mathbf{R})$ and write $p(\mathbf{R}, f) \in S_{++}^{L-1}$.

On the other hand, for a preference $R \in \mathcal{R}$ and a price vector $p \in S_{++}^{L-1}$, we define the consumption-direction vector $g(R, p) \in S_{++}^{L-1}$ as the normalized consumption vector where the gradient vector of $R$ is $p$. Therefore, $g(R^i, p(\mathbf{R}, f))$ is agent $i$'s consumption-direction vector at the preference profile $\mathbf{R}$ under $f$. We simply write $g^i(\mathbf{R}, f) = g(R^i, p(\mathbf{R}, f))$. Agent $i$'s consumption $f^i(\mathbf{R})$ assigned by $f$ should be on the ray $[g^i(\mathbf{R}, f)]$, and we can write $f^i(\mathbf{R}) = \parallel f^i(\mathbf{R}) \parallel g^i(\mathbf{R}, f)$.

Momi (2017) focused on a preference profile $\bar{\mathbf{R}} \in \mathbf{B}$ where the consumption-direction vectors are independent. The role of this independence should be clear. As consumption vectors $f^i(\bar{\mathbf{R}})$, $i = 1, \ldots, N$, are on the rays $[g^i(\mathbf{R}, f)]$, $i = 1, \ldots, N$, respectively, and they sum to the total endowment $\Omega$, the consumption vectors should be determined uniquely if the consumption-direction vectors are independent. Momi (2017) showed that in a neighborhood of $f^i(\bar{\mathbf{R}})$, where the consumption-direction vectors are independent at $\bar{\mathbf{R}}$, the option set $G^i(\bar{\mathbf{R}}^{-i})$ is the $L-1$-dimensional smooth surface of a strictly convex set as drawn in Figure 1 (i).

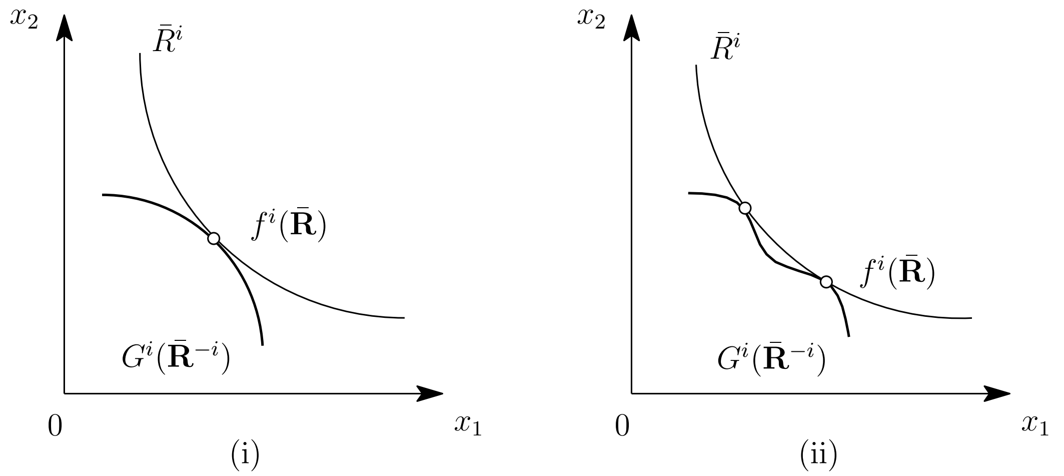


**Figure 1.** The option set

The role of this strict convexity and smoothness is clear. If the option set satisfies such properties, $f^i(R^i, \bar{\mathbf{R}}^{-i})$, which is the most preferred consumption bundle in the option set with respect to $R^i$, is a continuous function of $R^i$. Based on these topological properties of the option set, we can prove the following proposition. See Momi (2017, Proposition 6) for the proof.

**Proposition 1.** Suppose that $f$ is a social choice function that is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \Pi_{i=1}^{N} B^i$. If $g^i(\mathbf{R}, f)$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$, then $f^i(\bar{\mathbf{R}}) \in \{0, \Omega\}$ for any $i \in \mathbf{N}$.

This proposition ensures the alternate dictatorship at a preference profile where the consumption-direction vectors are independent. If the consumption-direction vectors are independent at a preference profile, independence holds for a preference profile in a neighborhood because of the continuity of $f$, and the alternate dictatorship also holds in the neighborhood. However, it is generally difficult to know whether the independence of the consumption-direction vectors holds for a given preference profile. It depends on not only the preference profile $\mathbf{R}$ but also the price vector $p(\mathbf{R}, f)$ determined by the social choice function $f$, whose behavior we do not know. Without the independence of the consumption directions, the option set might be neither strictly convex nor smooth, hence $f$ might not even be a continuous function. Figure 1 (ii) depicts an example of such an option set.

A trick to overcome this difficulty is that there exists a preference profile $\mathbf{R}^* \in \mathcal{R}^N$ that ensures the independence of the consumption-direction for any price vector. Momi (2017) constructed such a preference profile $\mathbf{R}^*$ using Cobb–Douglas utility functions. Then, through preference exchanges between two preference profiles, the alternate dictatorship result at $\mathbf{R}^*$ is extended to any preferecne profile.

However, in a small local domain $\mathbf{B}$, we cannot expect the existence of such a preference profile that ensures the independence of the consumption-direction vectors regardless of the price vectors. This is the difficulty we face in this paper. In an arbitrarily given domain $\mathbf{B}$, we have to find a preference profile that satisfies the condition of Proposition 1.

# 4 Technical results

As mentioned in the previous section, we have to deal with the case where $g^i(\mathbf{R}, f)$, $i = 1, \ldots, N$, are dependent. In the next section, starting from such a preference profile, we construct a preference profile in any neighborhood, where the independence of the consumption-direction vectors holds. In this section, we show some technical results we use for the proof. Through this section, we assume that the social choice function $f$ is Pareto efficient and strategy-proof on a product set $\mathbf{B}$ and we deal with preference profiles in $\mathbf{B}$, although we do not mention them in the lemmas. Proofs of all lemmas and corollaries are in Appendix

For a preference $R \in \mathcal{R}$ and a consumption bundle $x \in R^L_+$, a preference $\bar{R}$ is called a Maskin monotonic transformation (MMT, hereafter) of $R$ at $x$ if $\bar{x} \in UC(x; \bar{R})$ and $\bar{x} \neq x$ implies that $\bar{x} P_R x$. It is well known that if an agent receives $x$ at a preference profile $\mathbf{R}$, strategy-proofness implies that this agent receives the same consumption bundle $x$ when his preference is subject to an MMT at $x$. Note that $\bar{R}$ and $R$ share the same price vector at $x$. As shown in Momi (2013b, Lemma 4), for a preference $R \in \mathcal{R}$ and a consumption bundle $x \in R^L_{++}$, there exists a preference that is an MMT of $R$ at $x$ in any neighborhood of $R$.

As $f^i(R^i, \mathbf{R}^{-i})$ is the most preferred consumption bundle in $G^i(\mathbf{R}^{-i})$ with respect to $R^i \in B^i$, the upper contour set $UC(f^i(R^i, \mathbf{R}^{-i}); R^i)$ intersects with the option set $G^i(\mathbf{R}^{-i})$ at $f^i(R^i, \mathbf{R}^{-i})$. As mentioned in the previous section, this might not be a unique intersection, and then $f^i(R^i, \mathbf{R}^{-i})$ might not be a continuous function of $R^i$. We define $F(R^i; G^i(\mathbf{R}^{-i}))$ as the intersection between the upper contour set of $R^i$ at $f^i(R^i, \mathbf{R}^{-i})$ and the option set: $F(R^i; G^i(\mathbf{R}^{-i})) = UC(f^i(R^i, \mathbf{R}^{-i}); R^i) \bigcap G^i(\mathbf{R}^{-i})$. It is clear from the definition that $f^i(R^i, \mathbf{R}^{-i}) \in F(R^i; G^i(\mathbf{R}^{-i}))$ and $F(R^i; G^i(\bar{\mathbf{R}}^{-i})) \subset I(f^i(R^i, \mathbf{R}^{-i}); R^i)$.

The next lemma insists that if $R^{i\prime}$ is close to $R^i$, then any element of $F(R^{i\prime}; G^i(\mathbf{R}^{-i}))$ is close to $F(R^i; G^i(\mathbf{R}^{-i}))$. For a set $A \subset R^L$, we let $B_\delta(A)$ denote the union of open balls with radius $\delta$ and center $x \in A$: $B_\delta(A) = \bigcup_{x \in A} B_\delta(x)$.

**Lemma 1.** For any $\delta > 0$, there exists $\epsilon > 0$ such that if $R^{i\prime} \in B_\epsilon(R^i)$, then $F(R^{i\prime}; G^i(\mathbf{R}^{-i})) \subset B_\delta(F(R^i; G^i(\mathbf{R}^{-i})))$.

If the consumption $f^i(R^i, \mathbf{R}^{-i})$ is the unique intersection between the upper contour set and the option set, that is, if $f^i(R^i, \mathbf{R}^{-i}) = F(R^i; G^i(\mathbf{R}^{-i}))$, then as $R^{i\prime}$ converges to $R^i$, $F^i(R^{i\prime}; G^i(\mathbf{R}^{-i}))$ converges to $f^i(R^i, \mathbf{R}^{-i})$ as in Lemma 1, and then the consumption $f^i(R^{i\prime}, \mathbf{R}^{-i})$, which is in $F^i(R^{i\prime}; G^i(\mathbf{R}^{-i}))$, also converges to $f^i(R^i, \mathbf{R}^{-i})$, and hence the social choice function $f^i(\cdot, \mathbf{R}^{-i})$ is continuous at $R^i$.

We write $p(R^i; G^i(\mathbf{R}^{-i}))$ to denote the set of gradient vectors at consumption bundles in $F(R^i; G^i(\mathbf{R}^{-i}))$: $p(R^i; G^i(\mathbf{R}^{-i})) = \{p(R^i, x) \in S^{L-1}_{++} : x \in F(R^i; G^i(\mathbf{R}^{-i}))\}$. It is clear from the definition that $p((R^i, \mathbf{R}^{-i}), f) = p(R^i, f^i(R^i, \mathbf{R}^{-i})) \in p(R^i; G^i(\mathbf{R}^{-i}))$.

When $\hat{R}^i$ is an MMT of $R^i$ at $f^i(R^i, \mathbf{R}^{-i})$, we have $f^i(R^i, \mathbf{R}^{-i}) = f^i(\hat{R}^i, \mathbf{R}^{-i}) = F(R^i, G^i(\mathbf{R}^{-i}))$. Therefore, $F(R^{i\prime}; G^i(\mathbf{R}^{-i}))$ is in a neighborhood of $f^i(R^i, \mathbf{R}^{-i})$ and $p(R^{i\prime}; G^i(\mathbf{R}^{-i}))$ is in a neighborhood of $p((R^i, \mathbf{R}^{-i}), f)$ when $R^{i\prime}$ is close to $\hat{R}^i$. The next lemma considers the case where the other agents' preferences change.

**Lemma 2.** Suppose that $\hat{R}^i$ is an MMT of $R^i$ at $\bar{x}^i$ and the gradient vector at $\bar{x}^i$ is $\bar{p}$. For any $\varepsilon' > 0$, there exists $\phi' > 0$ such that if $f^i(R^i, \mathbf{R}^{-i}) \neq 0$ and $p((R^i, \mathbf{R}^{-i}), f) \in B_{\phi'}(\bar{p})$, then $p(\hat{R}^i; G^i(\mathbf{R}^{-i})) \subset B_{\varepsilon'}(\bar{p})$.

Lemma 2 insists that if $\hat{R}^i$ is an MMT of $R^i$ at $\bar{x}^i$ with price vector $\bar{p}$, and $p((R^i, \mathbf{R}^{-i}), f)$ is sufficiently close to $\bar{p}$, then $p(\hat{R}^i; G^i(\mathbf{R}^{-i}))$ is close to $\bar{p}$, and hence $p((\hat{R}^i, \mathbf{R}^{-i}), f)$ is also close to $\bar{p}$ . In other words, it insists that if $f^i(R^i, \mathbf{R}^{-i})$ is sufficiently close to the ray $[\bar{x}^i]$, then $F(\hat{R}^i; G^i(\mathbf{R}^{-i}))$, the intersection beteen the upper contour set of $\hat{R}^i$ and the option set, is close to $[\bar{x}^i]$, and hence $f^i(\hat{R}^i, \mathbf{R}^{-i})$ is also close to $[\bar{x}^i]$.

The next lemma insists that if the consumption-direction vectors are independent among some agents, then the independence holds after slight changes of their preferences and the price vector. Furthermore, if they share the total endowments among them, then the positive consumption receivers are still such receivers after the changes.

**Lemma 3** Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, where $K < N$, are independent at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N)$. There exist scalars $\bar{\epsilon} > 0$ and $\bar{\varepsilon} > 0$ satisfying the following properties.

(1) $g^i(R^i, p)$, $i = 1, \ldots, K$, are independent for any $R^i \in B_{\bar{\epsilon}}(\bar{R}^i)$ and $p \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$.

(2) Let $\mathbf{R} = (R^1, \ldots, R^N)$ be another preference profile. If $f^j(\bar{\mathbf{R}}) = f^j(\mathbf{R}) = 0$ for $j \geq K+1$, $p(\mathbf{R}, f) \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$, and $R^i \in B_{\bar{\epsilon}}(\bar{R}^i)$, $i = 1, \ldots, K$, then $f^i(\bar{\mathbf{R}}) > 0$ implies $f^i(\mathbf{R}) > 0$, for any $i = 1, \ldots, K$.

In the proof of the theorem, we change the agents' preferences slightly and increase the number of agents whose consumption-direction vectors are independent. In the process, we change the preferences of an agent who receives positive consumption. When agent $i$'s consumption is positive, we exchange the agent's preference $R^i$ with a preference in a neighborhood of an MMT of $R^i$ at the consumption. Then, the price changes only slightly as shown in Lemma 1. However, if we change the preferences of an agent who receives zero consumption, any allocation is possible without violating the strategy-proofness.

The next lemma shows that if the consumption-direction vectors are independent among some agents who are assigned positive consumption, then we can change their preferences slightly such that another agent receives positive consumption and the price vector is sufficiently close to the original price vector.

**Lemma 4.** Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \neq 0$ for any $i = 1, \ldots, K$, at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N)$. For any $\epsilon >$ and $\varepsilon > 0$, there exist $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_\epsilon(\bar{R}^1) \times \cdots \times B_\epsilon(\bar{R}^K)$ and $j \geq K+1$ such that $p((\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}$.

The next corollary is an immediate consequence of Lemma 4.

**Corollary 1.** Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \neq 0$ for any $i = 1, \ldots, K$ at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N)$. There exists $j \geq K + 1$ such that for any $\epsilon >$ and $\varepsilon > 0$, some $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_\epsilon(\bar{R}^1) \times \cdots \times B_\epsilon(\bar{R}^K)$ satisfies $p((\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}$.

The next lemma relaxes the condition of Lemma 4. If the consumption-direction vectors are independent among some agents and one of them is assigned positive consumption, then we can find a slight change in their preferences such that another agent receives positive consumption and the price vector is sufficiently close to the original price vector.

**Lemma 5.** Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \neq 0$ for some $i \leq K$, at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N)$. For any $\epsilon >$ and $\varepsilon > 0$, there exists $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_\epsilon(\bar{R}^1) \times \cdots \times B_\epsilon(\bar{R}^K)$ and $j \geq K+1$ such that $p((\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}$.

Finally, the next corollary is to Lemma 5 as Corollary 1 is to Lemma 4.

**Corollary 2.** Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \neq 0$ for some $i \leq K$ at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N)$. There exists $j \geq K + 1$ such that for any $\epsilon >$ and $\varepsilon > 0$, some $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_\epsilon(\bar{R}^1) \times \cdots \times B_\epsilon(\bar{R}^K)$ satisfies $p((\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}$.

# 5 Proof of the theorem

We first explain how we use CES utility functions to achieve the independence of the consumption-direction vectors and then prove the theorem. All proofs of the lemmas are in Apendix.

Assume that a preference profile $(\bar{R}^1, \ldots, \bar{R}^N)$ and a price vector $\bar{p}$ are given. We write $\bar{g}^i = g(\bar{R}^i, \bar{p})$ to denote agent $i$'s consumption-direction vector for the price $\bar{p}$ and the preference $\bar{R}^i$.

With parameters $\rho < 1$ and $\alpha = (\alpha_1, \ldots, \alpha_L) \in S^L_{++}$, we let $U_{\alpha,\rho} : R^L_+ \to R$ denote the CES utility function defined as

$$U_{\alpha,\rho}(x_1, \ldots, x_L) = (\alpha_1(x_1)^\rho + \cdots + \alpha_L(x_L)^\rho)^{1/\rho} \tag{1}$$

It is straightforward to obtain

$$\frac{\partial U_{\alpha,\rho}}{\partial x}(x) \parallel \left(\alpha_1(x_1)^{\rho-1}, \ldots, \alpha_L(x_L)^{\rho-1}\right)$$

where $y \parallel z$ denote that vector $y \in R^L$ is parallel to vector $z \in R^L$. We set $\alpha^i$ as the parameter such that the gradient vector of $U_{\alpha^i,\rho}$ at $\bar{g}^i$ is parallel to $\bar{p}$.[5]

$$\frac{\partial U_{\alpha^i,\rho}}{\partial x}(\bar{g}^i) \parallel \bar{p}, \tag{2}$$

that is, $\alpha^i$ is parallel to $(\frac{\bar{p}_1}{(\bar{g}^i_1)^{\rho-1}}, \ldots, \frac{\bar{p}_L}{(\bar{g}^i_L)^{\rho-1}})$. Abusing a notation, we let $U_{\alpha^i,\rho}$ denote not only the CES utility function but also the preference represented by the utility function. Because of (2), both the preferences $U_{\alpha^i,\rho}$ and $\bar{R}^i$ have the same gradient vector $\bar{p}$ at $\bar{g}^i$.

It is well known that the CES function defined by (1) converges to a Leontief utility function as $\rho \to -\infty$. Therefore, $U_{\alpha^i,\rho}$ is an MMT of $\bar{R}^i$ at $\bar{g}^i$ when $\rho$ is sufficiently small. We fix $\rho$ such that $U_{\alpha^i,\rho}$ is an MMT of $\bar{R}^i$ at $\bar{g}^i$ for any $i = 1, \ldots, N$. We simply write $U_{\alpha^i}$ with the fixed subscript $\rho$ omitted.

For a given price vector $p = (p_1, \ldots, p_L)$, the consumption-direction vector of $U_{\alpha^i}$, $g(U_{\alpha^i}, p)$ is determined by

$$g(U_{\alpha^i}, p) \parallel ((p_1/\alpha^i_1)^{1/(\rho-1)}, \ldots, (p_L/\alpha^i_L)^{1/(\rho-1)}) \tag{3}$$

by solving $\frac{\partial U_{\alpha^i}}{\partial x}(x) \parallel p$ with respect to $x$.

With a parameter $\delta^i \geq 0$ and a vector $z^i \in R^L$, we define

$$\beta^i(\delta^i, z^i) = ((1/\alpha^i_1)^{1/(\rho-1)}, \ldots, (1/\alpha^i_L)^{1/(\rho-1)}) + \delta^i z^i,$$

[5] The gradient vector of the utility function $U_{\alpha^i,\rho}$ should not be confused with the gradient vector of a preference $R$ we defined in Section 2. If $R$ is the preference represented by the utility function $U_{\alpha^i,\rho}$, then the normalization of $\frac{\partial U_{\alpha,\rho}}{\partial x}(x)$ is the gradient vector of $R$ at $x$.

for each $i = 1, \dots, N$. The next lemma shows that even if the vectors $((1/\alpha^i_1)^{1/(\rho-1)}, \dots, (1/\alpha^i_L)^{1/(\rho-1)})$, $i = 1, \dots, N$, are dependent, we can find the direction vectors $\bar{z}^i$, $i = 1, \dots, N$, with slight changes $\delta^i \leq \bar{\delta}$ in which directions the vectors $\beta^i(\delta^i, z^i)$, $i = 1, \dots, N$, are independent.

**Lemma 6.** There exist $\bar{\delta} > 0$ and $\bar{z}^i \in R^L$, $i = 1, \dots, N$, such that $\beta^i(\delta^i, \bar{z}^i)$, $i = 1, \dots, N$, are positive vectors and independent with any $0 < \delta^i \leq \bar{\delta}$, $i = 1, \dots, N$.

We fix $\bar{z}^i$, $i = 1, \dots, N$, that satisfies Lemma 6. For each $0 \leq \delta^i \leq \bar{\delta}$, we obtain $\hat{\alpha}^i_{\delta^i} = (\hat{\alpha}^i_{\delta^i 1}, \dots, \hat{\alpha}^i_{\delta^i L})$ by solving

$$((1/\hat{\alpha}^i_{\delta^i 1})^{1/(\rho-1)}, \dots, (1/\hat{\alpha}^i_{\delta^i L})^{1/(\rho-1)}) = \beta^i(\delta^i, \bar{z}^i) \tag{4}$$

and define $\alpha^i_{\delta^i} = (\alpha^i_{\delta^i 1}, \dots, \alpha^i_{\delta^i L})$ as the normalization of $\hat{\alpha}^i_{\delta^i}$: $\alpha^i_{\delta^i} = \hat{\alpha}^i_{\delta^i} / \parallel \hat{\alpha}^i_{\delta^i} \parallel$.

Observe that for the preferences $U_{\alpha^i_{\delta^i}}$, $i = 1, \dots, N$, the consumption-direction vectors $g(U_{\alpha^i_{\delta^i}}, p)$, $i = 1, \dots, N$, are independent with any $p$ as long as $0 < \delta^i \leq \bar{\delta}$. The consumption-direction vector $g(U_{\alpha^i_{\delta^i}}, p)$ is parallel to $((p_1/\alpha^i_{\delta^i 1})^{1/(\rho-1)}, \dots, (p_L/\alpha^i_{\delta^i L})^{1/(\rho-1)})$ as in (3), and hence, parallel to $((p_1/\hat{\alpha}^i_{\delta^i 1})^{1/(\rho-1)}, \dots, (p_L/\hat{\alpha}^i_{\delta^i L})^{1/(\rho-1)})$. Therefore, the independence of these vectors among agents is equivalent to the full column rankness of the $L \times N$ matrix

$$\begin{bmatrix} (p_1/\hat{\alpha}^1_{\delta^1 1})^{1/(\rho-1)} & ,\dots, & (p_1/\hat{\alpha}^N_{\delta^N L})^{1/(\rho-1)}) \\ \vdots, & & \vdots \\ (p_L/\hat{\alpha}^1_{\delta^i L})^{1/(\rho-1)} & ,\dots, & (p_L/\hat{\alpha}^N_{\delta^N L})^{1/(\rho-1)} \end{bmatrix},$$

and hence, equivalent to the full column rankness of the matrix

$$\begin{bmatrix} (1/\hat{\alpha}^1_{\delta^1 1})^{1/(\rho-1)} & ,\dots, & (1/\hat{\alpha}^N_{\delta^N L})^{1/(\rho-1)}) \\ \vdots, & & \vdots \\ (1/\hat{\alpha}^1_{\delta^i L})^{1/(\rho-1)} & ,\dots, & (1/\hat{\alpha}^N_{\delta^N L})^{1/(\rho-1)} \end{bmatrix}, \tag{5}$$

which is satisfied because $\hat{\alpha}^i_{\delta^i}$ is set by (4) and $\beta^i(\delta^i, \bar{z}^i)$, $i = 1, \dots, N$, are independent for any $0 < \delta^i \leq \bar{\delta}$, $i = 1, \dots, N$, as in Lemma 6.

From the construction of $U_{\alpha^i_{\delta^i}}$, $i = 1, \dots, N$, the independence of the consumption-direction vectors holds for preference profiles in a neighborhood of $(U_{\alpha^1_{\delta^1}}, \dots, U_{\alpha^N_{\delta^N}})$. Formally, this is stated in the next lemma.

**Lemma 7.** Let $P \subset S^L_{++}$ be a compact price set. There exist functions $\epsilon^i : (0, \bar{\delta}] \to R_{++}$, $i = 1, \dots, N$, such that $g(R^i, p)$, $i = 1, \dots, N$, are independent for any $p \in P$ and any $R^i \in B_{\epsilon^i(\delta^i)}(U_{\alpha^i_{\delta^i}})$ with any $0 < \delta^i \leq \bar{\delta}$.

We construct a preference that is close to $\bar{R}^i$ and is represented by $U_{\alpha^i_{\delta^i}}$ in a neighborhood of $\bar{g}^i$ as follows. See Momi (2017) for more details of the preference construction.

Figure 2 depicts the preferences $\bar{R}^i$ and $U_{\alpha^i_{\delta^i}}$. We let $t > 0$ be a sufficiently small parameter and define

$$A^i_{\alpha^i_{\delta^i}}(t) = co\left(UC(\bar{g}^i; U_{\alpha^i_{\delta^i}}) \bigcup UC(\bar{g}^i + t\bar{p}; \bar{R}^i)\right)$$

where $co(Y)$ denotes the convex hull of a subset $Y \subset R^L$.

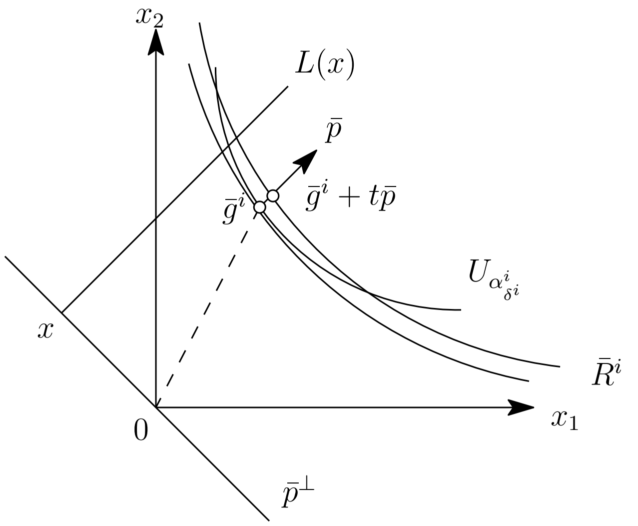


**Figure 2.** The prefrence construction

As the convex set $A^i_{\alpha^i_{\delta^i}}(t)$ is not strictly convex, it cannot be an upper contour set of a preference in $\mathcal{R}$. We let $s > 0$ be a sufficiently small parameter and define

$$B^i_{\alpha^i_{\delta^i}}(t,s) = \bigcup_{x \in \bar{p}^\perp} \left\{(1-s)(\partial A^i_{\alpha^i_{\delta^i}}(t) \bigcap L(x)) + s(I(\bar{g}^i; U_{\alpha^i_{\delta^i}}) \bigcap L(x))\right\},$$

where $L(x)$ is the half line starting from $x$ and extending in the direction of the vector $\bar{p}$: $L(x) = \{y \in R^L | y = x + t\bar{p}, t \geq 0\}$. Note that for $s > 0$, $B^i_{\alpha^i_{\delta^i}}(t,s)$ is the boundary of a strictly convex set. We let $R^i_{\alpha^i_{\delta^i},t,s} \in \mathcal{R}$ denote the preference which has $B^i_{\alpha^i_{\delta^i}}(t,s)$ as its indifference set.

Note that, as long as $t > 0$, the boundary of $A^i_{\alpha^i_{\delta^i}}(t)$ coincides with $B^i_{\alpha^i_{\delta^i}}(t,s)$ in a neighborhood of $\bar{g}^i$, and it is defined by the indifference set of the CES function. That is, the indifference set of $R^i_{\alpha^i_{\delta^i},t,s}$ is equal to that of $U_{\alpha^i_{\delta^i}}$ in a neighborhood of $\bar{g}^i$. Hence, $R^i_{\alpha^i_{\delta^i},t,s}$ inherits the properties of $U_{\alpha^i_{\delta^i}}$. In particular, independence of the consumption-direction vectors holds for prices in a neighborhood of $\bar{p}$.

**Lemma 8.** For any $\epsilon > 0$, there exist $s > 0$, $t > 0$, $\bar{\delta}^i > 0$, $i = 1, \ldots, N$, and $\varepsilon > 0$ satisfying the following properties.

(1) $R^i_{\alpha^i_{\delta^i},t,s} \in B_\epsilon(\bar{R}^i)$, $i = 1, \ldots, N$, for any $0 \leq \delta^i \leq \bar{\delta}^i$.

(2)$g(R^i_{\alpha^i_{\delta^i},t,s}, p)$, $i = 1, \ldots, N$, are independent for any $0 < \delta^i \leq \bar{\delta}^i$ and any $p \in B_\varepsilon(\bar{p})$.

The independence of the consumption-direction vectors holds for preferences in neighborhoods of $R^i_{\alpha^i_{\delta^i},t,s}$, $i = 1, \ldots, N$.

**Lemma 9.** Fix $s > 0$ and $t > 0$. There exist $\bar{\varepsilon} > 0$ and functions $\epsilon^i : (0, \bar{\delta}] \to R_{++}$, $i = 1, \ldots, N$, such that $g(R^i, p)$, $i = 1, \ldots, N$, are independent for any $p \in B_{\bar{\varepsilon}}(\bar{p})$ and any $R^i \in B_{\epsilon^i(\delta^i)}(R^i_{\alpha^i_{\delta^i},t,s})$, $i = 1, \ldots, N$.

Combining this preference construction with the results in the previous section, we prove the theorem. The proof proceeds as follows. Suppose that $\bar{\mathbf{R}}$ is given, where the consumption-direction vectors are dependent and the allocation is not dictatorial. Starting from a group of agents whose consumption-direction vectors are independent, we change their preferences slightly so that another agent is assigned positive consumption as in Corollary 2. Then, we change this agent's preference according to the preference construction explained above so that the consumption-direction vectors among the group of agents including this agent are independent. Repeating this step, we finally have all agents' consumption-direction vectors being independent, which contradicts Proposition 1.

**Proof of the theorem** We let $L \geq N$ and the social choice function $f$ be Pareto efficient and strategy-proof on $\mathbf{B} = \Pi_{i=1}^N B_i$. We suppose that some $j$ satisfies $f^j(\bar{\mathbf{R}}) \notin \{0, \Omega\}$ at some preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and show a contradiction. If $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, N$, are independent, this immediately contradicts Proposition 1. We consider the case where these consumption-direction vectors are dependent.

We write $\bar{p}$ and $\bar{g}^i$ to denote the price vector and agent $i$'s consumption-direction vector at $\bar{\mathbf{R}}$, respectively: $\bar{p} = p(\bar{\mathbf{R}}, f)$ and $\bar{g}^i = g^i(\bar{\mathbf{R}}, f) = g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$.

We fix a scalar $\bar{\epsilon} > 0$ such that $B_{\bar{\epsilon}}(\bar{R}^i) \subset B_i$, for any $i = 1, \ldots, N$. For $\bar{R}^i$, $\bar{g}^i$, $i = 1, \ldots, N$, and $\bar{p}$, we consider the preferences $R^i_{\alpha^i_{\delta^i},t,s}$, $i = 1, \ldots, N$, explained above. Applying Lemma 8 to these preferences with scalar $\bar{\epsilon}$, we have $s$, $t$, $\bar{\delta}^i$, $i = 1, \ldots, N$, and $\varepsilon$ such that

$$R^i_{\alpha^i_{\delta^i},t,s} \in B_{\bar{\epsilon}}(\bar{R}^i),\ i = 1, \ldots, N, \text{ for any } 0 \leq \delta^i \leq \bar{\delta}^i \tag{6}$$

and

$$g^i(R^i_{\alpha^i_{\delta^i},t,s}, p),\ i = 1, \ldots, N, \text{ are independent for any } 0 < \delta^i \leq \bar{\delta}^i \text{ and any } p \in B_\varepsilon(\bar{p}). \tag{7}$$

Furthermore applying Lemma 9, we have $\bar{\varepsilon}$ and $\epsilon^i : (0, \bar{\delta}] \to R_{++}$, $i = 1, \ldots, N$, such that

$$g(R^i, p),\ i = 1, \ldots, N, \text{ are independent for any } R^i \in B_{\epsilon^i(\delta^i)}(R^i_{\alpha^i_{\delta^i},t,s}) \text{ and } p \in B_{\bar{\varepsilon}}(p). \tag{8}$$

We set $\bar{\varepsilon}$ to be less than $\varepsilon$: $\bar{\varepsilon} < \varepsilon$.

For simplicity of notation, we write $R^i_{\delta^i} = R^i_{\alpha^i_{\delta^i},t,s}$ because the parameters other than $\delta^i$ are fixed from now on.

For each $i = 1, \dots, N$, $R^i_0$, that is $R^i_{\delta^i}$ where $\delta^i = 0$, is an MMT of $\bar{R}^i$ at $\bar{g}^i$. Applying Lemma 2 to these preferences, we have a positive function $\bar{\phi}^i : \varepsilon' \mapsto \bar{\phi}(\varepsilon')$ for each $i = 1, \dots, N$, which maps $\varepsilon'$ to $\phi'$ in Lemma 2:

$$\text{If } f^i(\bar{R}^i, \mathbf{R}^{-i}) \neq 0 \text{ and } p((\bar{R}^i, \mathbf{R}^{-i}), f) \in B_{\bar{\phi}^i(\varepsilon')}(\bar{p}), \text{ then } p(R^i_0; G^i(\mathbf{R}^{-i})) \subset B_{\varepsilon'}(\bar{p}) \text{ for any } \mathbf{R}^{-i} \text{ and } \varepsilon' > 0. \quad (9)$$

We select a positive function $\phi$ such that $\phi(\varepsilon') < \min\{\bar{\phi}^1(\varepsilon'), \dots, \bar{\phi}^N(\varepsilon'), \varepsilon'\}$ for any $\varepsilon'$. We write $\phi^{(k)} = \phi \circ \cdots \circ \phi$ to denote the $k$-times operation of $\phi$.

Without loss of generality, we assume that $f^1(\bar{\mathbf{R}}) \notin \{0, \Omega\}$. Replacing $\bar{R}^1$ with $R^1_{\delta^1}$ of any parameter $0 < \delta^1 < \bar{\delta}^1$, we have $f^1(R^1_{\delta^1}, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}$, and there exists another agent receiving positive consumption. As $\delta^1$ changes, the type of agent who receives positive consumption might vary. However, there should exist an agent $j \geq 2$ such that for any $\delta^1$, there exists $\delta^{1\prime} < \delta^1$ satisfying $f^j(R^1_{\delta^{1\prime}}, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}$.

If there exists no such agent, then for each $j \geq 2$ there exists $\delta^1_j$ such that for any $\delta^1 < \delta^1_j$, $f^j(R^1_{\delta^1}, \bar{\mathbf{R}}^{-i}) \in \{0, \Omega\}$. Then, for $\delta^1 < \min\{\delta^1_2, \dots, \delta^1_N\}$, we have $f^j(R^1_{\delta^1}, \bar{\mathbf{R}}^{-i}) \in \{0, \Omega\}$ for any $j \geq 2$. This contradicts that $f^1(R^1_{\delta^1}, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}$ for any parameter $\delta^1$.

Without loss of generality, we assume that agent 2 is such an agent. We let $\Delta^1 = \{\delta^1 | f^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}\}$ denote the set of $\delta^1$ such that agent 2's consumption is not zero at $(R^1_{\delta^1}, \bar{\mathbf{R}}^{-i})$. We consider $\delta^1$ in $\Delta^1$.

If $\delta^1$ in $\Delta^1$ is sufficiently small, then $p(R^1_{\delta^1}; G^1(\bar{\mathbf{R}}^{-i}))$ can be arbitrarily close to $\bar{p}$ because $f^1(\bar{\mathbf{R}}) = f^1(R^1_0, \bar{\mathbf{R}}^{-i})$ is the unique intersection between $UC(f^1(\bar{\mathbf{R}}); R^1_0)$ and $G^1(\bar{\mathbf{R}}^{-i})$, and hence, $F(R^1_{\delta^1}; G^1(\mathbf{R}^{-1}))$ converges to $f^1(R^1_0, \bar{\mathbf{R}}^{-i})$ as $\delta^1 \to 0$ and $R^1_{\delta^1} \to R^1_0$ as shown in Lemma 1. We take $\delta^1$ so that $p(R^1_{\delta^1}; G^1(\bar{\mathbf{R}}^{-i})) \subset B_{\phi^{(N-1)}(\bar{\varepsilon})}(\bar{p})$.

We replace $\bar{R}^2$ with $R^2_0$. Note that $p(R^1_{\delta^1}; G^1(\bar{\mathbf{R}}^{-1})) \in B_{\phi^{(N-1)}(\bar{\varepsilon})}(\bar{p})$ implies that $p((R^1_{\delta^1}, \bar{R}^2, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\phi^{(N-1)}(\bar{\varepsilon})}(\bar{p}) \subset B_{\phi^2(\phi^{(N-2)}(\bar{\varepsilon}))}(\bar{p})$ by the definition of $\phi$ and $\phi^2$. Then, by (9), we have $p(R^2_0; G^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^{(N-2)}(\bar{\varepsilon})}(\bar{p})$.

We then replace $R^2_0$ with $R^2_{\delta^2}$. As $\delta^2 \to 0$ and $R^2_{\delta^2} \to R^2_0$, any element in $F(R^2_{\delta^2}; G^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-\{1,2\}}))$ converges to a consumption bundle in $F(R^2_0; G^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-\{1,2\}}))$ as shown in Lemma 1. Then, any price vector in $p(R^2_{\delta^2}; G^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-\{1,2\}}))$ converges to a price vector in $p(R^2_0; G^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-\{1,2\}}))$. Therefore, we take a value of $\delta^2$ sufficiently small such that $p(R^2_{\delta^2}; G^2(R^1_{\delta^1}, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^{(N-2)}(\bar{\varepsilon})}(\bar{p})$. This, of course, means that $p((R^1_{\delta^1}, R^2_{\delta^2}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\phi^{(N-2)}(\bar{\varepsilon})}(\bar{p})$.

We apply Corollary 2 to agents 1 and 2. Note that, as shown in (7), the consumption-direction vectors of $R^1_{\delta^1}$ and $R^2_{\delta^2}$ are independent at $(R^1_{\delta^1}, R^2_{\delta^2}, \bar{\mathbf{R}}^{-\{1,2\}})$ because $p((R^1_{\delta^1}, R^2_{\delta^2}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\phi^{(N-2)}(\bar{\varepsilon})}(\bar{p}) \subset B_{\bar{\varepsilon}}(\bar{p})$. We have an agent $j \geq 3$ such that for any $\epsilon > 0$ and $\varepsilon > 0$, some $(R^1_{(1)}, R^2_{(1)}) \in B_\epsilon(R^1_{\delta^1}) \times B_\epsilon(R^2_{\delta^2})$ satisfies $p((R^1_{(1)}, R^2_{(1)}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_\varepsilon(p(R^1_{\delta^1}, R^2_{\delta^2}, \bar{\mathbf{R}}^{-\{1,2\}}))$ and $f^j(R^1_{(1)}, R^2_{(1)}, \bar{\mathbf{R}}^{-\{1,2\}}) \notin \{0, \Omega\}$.

Without loss of generality, we assume that agent 3 is such an agent. We set $\bar{\epsilon}^i$, $i = 1, 2$, such that $\bar{\epsilon}^i < \epsilon^i(\delta^i)$ and $B_{\bar{\epsilon}^i}(R^i_{\delta^i}) \subset B_{\bar{\epsilon}}(\bar{R}^i)$. We select $\varepsilon$ such that $B_\varepsilon(p((R^1_{\delta^1}, R^2_{\delta^2}, \bar{\mathbf{R}}^{-\{1,2\}}), f)) \subset B_{\phi^{(N-2)}(\bar{\varepsilon})}(\bar{p})$, select $\epsilon$ such that $B_\epsilon(R^1_{\delta^1}) \times B_\epsilon(R^2_{\delta^2}) \subset B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \times B_{\bar{\epsilon}^2}(R^2_{\delta^2})$, and apply Corollary 2. As a result, we have $(R^1_{(1)}, R^2_{(1)}) \in B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \times B_{\bar{\epsilon}^2}(R^2_{\delta^2})$ such that $p((R^1_{(1)}, R^2_{(1)}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\phi^{(N-2)}(\bar{\varepsilon})}(\bar{p})$ and $f^3(R^1_{(1)}, R^2_{(1)}, \bar{\mathbf{R}}^{-\{1,2\}}) \notin \{0, \Omega\}$.

We replace $\bar{R}^3$ with $R^3_0$. By (9), we have $p(R^3_0; G^3(R^1_{(1)}, R^2_{(1)}, \bar{\mathbf{R}}^{-\{1,2,3\}})) \in B_{\phi^{(N-3)}(\bar{\varepsilon})}(\bar{p})$ as we observed for agent 2.

We replace $R^3_0$ with $R^3_{\delta^3}$, where $\delta^3$ is sufficiently small, such that $p(R^3_{\delta^3}; G^3(R^1_{(1)}, R^2_{(1)}, \bar{\mathbf{R}}^{-\{1,2,3\}})) \in B_{\phi^{(N-3)}(\bar{\varepsilon})}(\bar{p})$ as we observed for agent 2.

We apply Corollary 2 to agents 1,2, and 3. Note that, as shown in (8), the consumption-direction vectors of $R^1_{(1)}$, $R^2_{(1)}$, and $R^3_{\delta^3}$ are independent at $(R^1_{(1)}, R^2_{(1)}, R^3_{\delta^3}, \bar{\mathbf{R}}^{-\{1,2,3\}})$ because $R^1_{(1)} \in B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \subset B_{\epsilon^1(\delta^1)}(R^1_{\delta^1})$, $R^2_{(1)} \in B_{\bar{\epsilon}^2}(R^2_{\delta^2}) \subset B_{\epsilon^2(\delta^2)}(R^2_{\delta^2})$, and $p((R^1_{(1)}, R^2_{(1)}, R^3_{\delta^3}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f) \in B_{\phi^{(N-3)}(\bar{\varepsilon})}(\bar{p}) \subset B_{\bar{\varepsilon}}(\bar{p})$. We have an agent $j \geq 4$ such that for any $\epsilon$ and $\varepsilon$, some $(R^1_{(2)}, R^2_{(2)}, R^3_{(2)}) \in B_\epsilon(R^1_{(1)}) \times B_\epsilon(R^2_{(1)}) \times B_\epsilon(R^3_{\delta^3})$ satisfies $p((R^1_{(2)}, R^2_{(2)}, R^3_{(2)}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f) \in B_\varepsilon(p((R^1_{(1)}, R^2_{(1)}, R^3_{\delta^3}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f))$ and $f^j(R^1_{(2)}, R^2_{(2)}, R^3_{(2)}, \bar{\mathbf{R}}^{-\{1,2,3\}}) \notin \{0, \Omega\}$.

Without loss of generality, we assume that agent 4 is such an agent. We set $\bar{\epsilon}^3$ such that $\bar{\epsilon}^3 < \epsilon^3(\delta^3)$ and $B_{\bar{\epsilon}^3}(R^3_{\delta^3}) \subset B_{\bar{\epsilon}}(\bar{R}^3)$. We select $\varepsilon$ such that $B_\varepsilon(p((R^1_{(1)}, R^2_{(1)}, R^3_{\delta^3}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f)) \subset B_{\phi^{(N-3)}(\bar{\varepsilon})}(\bar{p})$, select $\epsilon$ such that $B_\epsilon(R^1_{(1)}) \times B_\epsilon(R^2_{(1)}) \times B_\epsilon(R^3_{\delta^3}) \subset B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \times B_{\bar{\epsilon}^2}(R^2_{\delta^2}) \times B_{\bar{\epsilon}^3}(R^3_{\delta^3})$, and apply Corollary 2. As a result, we have $(R^1_{(2)}, R^2_{(2)}, R^2_{(2)}) \in B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \times B_{\bar{\epsilon}^2}(R^2_{\delta^2}) \times B_{\bar{\epsilon}^3}(R^3_{\delta^3})$ such that $p((R^1_{(2)}, R^2_{(2)}, R^2_{(2)}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f) \in B_{\phi^{(N-3)}(\bar{\varepsilon})}(\bar{p})$ and $f^4(R^1_{(2)}, R^2_{(2)}, R^3_{(2)}, \bar{\mathbf{R}}^{-\{1,2,3\}}) \notin \{0, \Omega\}$.

We replace $\bar{R}^4$ with $R^4_0$, and replace $R^4_0$ with $R^4_{\delta^4}$, where $\delta^4$ is sufficiently small, in a similar way. Applying Corollary 2 to agents $1, \ldots, 4$, we obtain a preference subprofile $(R^1_{(3)}, \ldots, R^4_{(3)}) \in B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \times \cdots \times B_{\bar{\epsilon}^4}(R^4_{\delta^4})$ where $\bar{\epsilon}^4$ satisfies $\bar{\epsilon}^4 < \epsilon^4(\delta^4)$ and $B_{\bar{\epsilon}^4}(R^4_{\delta^4}) \subset B_{\bar{\epsilon}}(\bar{R}^4)$ such that the price vector at $(R^1_{(3)}, \ldots, R^4_{(3)}, \bar{\mathbf{R}}^{-\{1,\ldots,4\}})$ is in $B_{\phi^{(N-4)}(\bar{\varepsilon})}(\bar{p})$ and an agent $j \geq 5$ receives positive consumption at $(R^1_{(3)}, \ldots, R^4_{(3)}, \bar{\mathbf{R}}^{-\{1,\ldots,4\}})$.

We repeat this process. Finally, we have $(R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)})$ in $B_{\bar{\epsilon}^1}(R^1_{\delta^1}) \times \cdots \times B_{\bar{\epsilon}^{N-1}}(R^{N-1}_{\delta^{N-1}})$, where $\bar{\epsilon}^i$, $i = 1, \ldots, N-1$, satisfies $\bar{\epsilon}^i < \epsilon^i(\delta^i)$ and $B_{\bar{\epsilon}^i}(R^i_{\delta^i}) \subset B_{\bar{\epsilon}}(\bar{R}^i)$, such that $p((R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, \bar{R}^N), f) \in B_{\phi(\bar{\varepsilon})}(\bar{p})$ and $f^N(R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, \bar{R}^N) \notin \{0, \Omega\}$. Replacing $\bar{R}^N$ with $R^N_0$, we have $p(R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, R^N_0) \in B_{\bar{\varepsilon}}(\bar{p})$ because of (9) and $f^N(R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, R^N_0) \notin \{0, \Omega\}$. Replacing $R^N_0$ with $R^N_{\delta^N}$ where $\delta^N$ is sufficiently small, we still have $p((R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, R^N_{\delta^N}), f) \in B_{\bar{\varepsilon}}(\bar{p})$ and $f^N(R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, R^N_{\delta^N}) \notin \{0, \Omega\}$.

However, the consumption-direction vectors $g^i(R^i_{(N-2)}, p)$, $i = 1, \ldots, N-1$, and $g(R^N_{\delta^N}, p)$ are independent at $p = p(R^1_{(N-2)}, \ldots, R^{N-1}_{(N-2)}, R^N_{\delta^N})$ because $R^i_{(N-2)} \in B_{\bar{\epsilon}^i}(R^i_{\delta^i})$, $i = 1, \ldots, N-1$, and $p \in B_{\bar{\varepsilon}}(\bar{p})$. This contradicts Proposition 1. ■

# A Appendix

**Proof of Lemma 1.** We fix $\delta$ arbitrarily. We let $\{\epsilon_n\}_{n=0}^{\infty}$ be a decresing sequence of scalars converging to 0: $\epsilon_n < \epsilon_{n'}$ for $n > n'$ and $\epsilon_n \to 0$ as $n \to \infty$. We assume that for any $n$, there exist some $R^i_{(n)} \in B_{\epsilon_n}(R^i)$ and $x_{(n)} \in F(R^i_{(n)}; G^i(\mathbf{R}^{-i}))$ satisfying $x_{(n)} \notin B_\delta(F(R^i; G^i(\mathbf{R}^{-i}))$, and we show a contradiction.

Since $x_{(n)} I_{R^i_{(n)}} f^i(R^i_{(n)}, \mathbf{R}^{-i}) R^i_{(n)} f^i(R^i, \mathbf{R}^{-i})$, we have $x_{(n)} \in UC(f^i(R^i, \mathbf{R}^{-i}); R^i_{(n)})$. Since $x_{(n)} \in G^i(\mathbf{R}^{-i})$, it should not be preferred over $f^i(R^i, \mathbf{R}^{-i})$ with respect to $R^i$, that is, $x_{(n)} \in LC(f^i(R^i, \mathbf{R}^{-i}); R^i)$. Therefore, $x^i_{(n)}$ is in the intersection $UC(f^i(R^i, \mathbf{R}^{-i}); R^i_{(n)}) \bigcap LC(f^i(R^i, \mathbf{R}^{-i}); R^i)$. As $n \to \infty$, we have $R^i_{(n)} \to R^i$, and hence this intersection converges to some set in the indifference set $I(f^i(R^i, \mathbf{R}^{-i}); R^i)$.

Note that $\{x_{(n)}\}_{n=0}^{\infty}$ has an convergent subsequence $\{x_{(n_k)}\}_{k=0}^{\infty}$ where $n_k < n_{k'}$ for $k > k'$ because the set of feasible allocations is compact. We let $\hat{x} = \lim_{k\to\infty} x_{(n_k)}$ denote the limit. Thus, we have $\hat{x} \in I(f^i(R^i, \mathbf{R}^{-i}); R^i)$.

Observe that $\hat{x} \in G^i(\mathbf{R}^{-i})$. Since $x_{(n_k)} \in G^i(\mathbf{R}^{-i})$, we have $\hat{x} \in \overline{G^i(\mathbf{R}^{-i})}$, where $\overline{G^i(\mathbf{R}^{-i})}$ is the closure of $G^i(\mathbf{R}^{-i})$. This implies $\hat{x} \in G^i(\mathbf{R}^{-i})$, as proved by Momi (2017, Lemma 3).

Thus we have $\hat{x} \in F(R^i; G^i(\mathbf{R}^{-i}))$ and this contradits that $x_{(n_k)} \notin B_\delta(F(R^i; G^i(\mathbf{R}^{-i})))$. ■

**Proof of Lemma 2.** We fix $\varepsilon'$ arbitarily. We suppose that for any $\phi'$, there exists $\mathbf{R}^{-i}$ such that $f^i(R^i, \mathbf{R}^{-i}) \neq 0$, $p((R^i, \mathbf{R}^{-i}), f) \in B_{\phi'}(\bar{p})$, and $p(\hat{R}^i; G^i(\mathbf{R}^{-i})) \notin B_{\varepsilon'}(\bar{p})$, and we show a contradiction.

Since $p(\hat{R}^i; G^i(\mathbf{R}^{-i}))$ is not included in $B_{\varepsilon'}(\bar{p})$, there exists $\hat{x}^i \in F(\hat{R}^i; G^i(\mathbf{R}^{-i}))$ such that $p(\hat{R}^i; \hat{x}^i) \notin B_{\varepsilon'}(\bar{p})$. Since $p(\hat{R}^i; \bar{x}^i) = \bar{p}$, $p(\hat{R}^i; \hat{x}^i) \notin B_{\varepsilon'}(\bar{p})$ implies that the ray $[\hat{x}^i]$ is away from the ray $[\bar{x}^i]$. Formally, we measure the distance between rays by the distance between their intersections with $S^L_{++}$, that is, we define the distance between two rays $[y]$ and $[z]$, where $y, z \in R^L_{++}$, by the distance between $y/\parallel y \parallel$ and $z/\parallel z \parallel$.

Note that, even if $\hat{x}^i$ is different from $f^i(\hat{R}^i, \mathbf{R}^{-i})$, they are indifferent with respect to $\hat{R}^i$. Therefore, we have $\hat{x}^i \hat{R}^i f^i(R^i, \mathbf{R}^{-i})$ because of strategy-proofness. On the other hand, since $\hat{x}^i \in G^i(\mathbf{R}^{-i})$, we have $f^i(R^i, \mathbf{R}^{-i}) R^i \hat{x}^i$. Thus, we have $\hat{x}^i \in LC(f^i(R^i, \mathbf{R}^{-i}); R^i) \bigcap UC(f^i(R^i, \mathbf{R}^{-i}); \hat{R}^i)$.

By taking a sufficiently small value of $\phi'$, we have $p((R^i, \mathbf{R}^{-i}), f)$ arbitrarily close to $\bar{p}$. Then, the ray $[f^i(R^i, \mathbf{R}^{-i})]$ is arbitrarily close to $[\bar{x}^i]$. Then, for any $x \in LC(f^i(R^i, \mathbf{R}^{-i}); R^i) \bigcap UC(f^i(R^i, \mathbf{R}^{-i}); \hat{R}^i)$, the ray $[x]$ is arbitrarily close to the ray $[\bar{x}^i]$, and hence the ray $[\hat{x}^i]$ should be arbitrarily close to $[\bar{x}^i]$. This is a contradiction. ■

**Proof of Lemma 3.** Note that $g^i(\bar{\mathbf{R}}, f) = g(R^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, K$, and they are independent. As $p \to p(\bar{\mathbf{R}}, f)$, we have $g(\bar{R}^i, p) \to g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$. Therefore, there exists

$\varepsilon'$ such that $g(\bar{R}^i, p)$, $i = 1, \ldots, K$, are independent for any $p \in \overline{B_{\varepsilon'}(p(\bar{\mathbf{R}}, f))}$, where $\overline{B_{\varepsilon'}(p(\bar{\mathbf{R}}, f))}$ denotes the closure of $B_{\varepsilon'}(p(\bar{\mathbf{R}}, f))$.

For each $p \in \overline{B_{\varepsilon'}(p(\bar{\mathbf{R}}, f))}$, $g(R^i, p) \to g(\bar{R}^i, p)$ as $R^i \to \bar{R}^i$. Therefore, there exists $\epsilon_p$ such that $g(R^i, p)$, $i = 1, \ldots, K$, are independent for any $R^i \in B_{\epsilon_p}(\bar{R}^i)$, $i = 1, \ldots, K$. We define $\epsilon'$ as the minimum of $\epsilon_p$ as $p$ moves over $\overline{B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))}$: $\epsilon' = \min_{p \in \overline{B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))}} \epsilon_p$. It is clear that these $\epsilon'$ and $\varepsilon'$ satisfies Lemma 3 (1).

We consider $R^i \in B_{\epsilon'}(\bar{R}^i)$, $i = 1, \ldots, K$, and $p(\mathbf{R}, f) \in B_{\varepsilon'}(p(\bar{\mathbf{R}}, f))$. Then, $g^i(\mathbf{R}, f)$, $i = 1, \ldots, K$, are independent.

Note that $f^j(\mathbf{R}) = f^j(\bar{\mathbf{R}}) = 0$ for $j \geq K + 1$, means that $\sum_{i=1}^K f^i(\mathbf{R}) = \Omega$ and $\sum_{i=1}^K f^i(\bar{\mathbf{R}}) = \Omega$.

We define $\bar{\alpha}^i$, $i = 1, \ldots, K$, as $f^i(\mathbf{R}) = \bar{\alpha}^i g^i(\mathbf{R}, f)$. Since $f^i(\bar{\mathbf{R}}) \in [g^i(\bar{\mathbf{R}}, f)]$, $i = 1, \ldots, K$, and $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, are independent, these $\bar{\alpha}^i$'s are a unique solution of $\sum_{i=1}^K \alpha^i g^i(\bar{\mathbf{R}}, f) = \Omega$.

Since $g^i(\bar{\mathbf{R}}, f) = g(\bar{R}^i, p(\mathbf{R}, f))$ and $g^i(\mathbf{R}, f) = g(R^i, p(\mathbf{R}, f))$, $g^i(\mathbf{R}, f)$ converges to $g^i(\bar{\mathbf{R}}, f)$ as $R^i$ and $p(\mathbf{R}, f)$ converge to $\bar{R}^i$ and $p(\bar{\mathbf{R}}, f)$, respectively. Therefore the scalars $\alpha^i(\mathbf{R}, f)$, $i = 1, \ldots, K$ satisfying $\sum_{i=1}^K \alpha^i(\mathbf{R}, f) g^i(\mathbf{R}, f) = \Omega$, are determined uniquely and $\alpha^i(\mathbf{R}, f)$ converges to $\bar{\alpha}^i$, for any $i = 1, \ldots, K$, as $g^i(\mathbf{R}, f)$ converges to $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$. Then, $f^i(\mathbf{R}) = \alpha^i(\mathbf{R}, f) g^i(\mathbf{R}, f)$ converges to $f^i(\bar{\mathbf{R}})$, $i = 1, \ldots, K$. Thus $f^i(\bar{\mathbf{R}}) > 0$ implies $f^i(\mathbf{R}) > 0$ ■

**Proof of Lemma 4.** We select $\epsilon$ and $\varepsilon$ arbitrarily. If $f^j(\bar{\mathbf{R}}) \neq 0$ for some $j \geq K + 1$, the lemma holds. We suppose that $f^j(\bar{\mathbf{R}}) = 0$ for any $j \geq K + 1$. We let $\bar{\epsilon} > 0$ and $\bar{\varepsilon} > 0$ be scalars that support Lemma 3 with respect to $\bar{\mathbf{R}}$. We define $\tilde{\epsilon} = \min\{\epsilon, \bar{\epsilon}\}$ and $\tilde{\varepsilon} = \{\varepsilon, \bar{\varepsilon}\}$. Note that, because of Lemma 3, at any $\mathbf{R}$ such that $R^i \in B_{\tilde{\epsilon}}(\bar{R}^i)$, $i = 1, \ldots, K$, and $p(\mathbf{R}, f) \in B_{\tilde{\varepsilon}}(p(\bar{\mathbf{R}}, f))$, if $f^j(\mathbf{R}) = 0$ for $j \geq K + 1$, then $f^i(\mathbf{R})$, $i = 1, \ldots, K$, are all positive and independent.

We let $\hat{R}^i \in B_{\tilde{\epsilon}}(\bar{R}^i)$ be an MMT of $\bar{R}^i$ at $f^i(\bar{\mathbf{R}})$ for each $i = 1, \ldots, K$.

First, we observe that the lemma holds if $f^j(R^1, \ldots, R^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \neq 0$ for some $j \geq K + 1$ and $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \ldots, K$.

For any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \ldots, K$, we let $S(R^1, \ldots, R^K) = \sharp\{i \in \{1, \ldots, K\} : R^i = \hat{R}^i\}$ denote the number of agents whose preference is $\hat{R}^i$, and let $\bar{S}$ denote the minimum of $S(R^1, \ldots, R^K)$ such that some consumer $j \geq K + 1$ has positive consumption with the preference profile: $\bar{S} = \min\{S(R^1, \ldots, R^K) : f^j(R^1, \ldots, R^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \neq 0 \text{ for some } j \geq K+1 \text{ and } R^i \in \{\bar{R}^i, \hat{R}^i\}, i = 1, \ldots, K\}$. Without loss of generality, by relabeling the consumer indexes if necessary, we assume that $f^j(\hat{R}^1, \ldots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \ldots, \bar{R}^N) \neq 0$ for some $j \geq K + 1$. We observe that the lemma holds at $(\hat{R}^1, \ldots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \ldots, \bar{R}^N)$. It is clear that the preference subprofile $(\hat{R}^1, \ldots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \ldots, \bar{R}^K)$ is in $B_\epsilon(\bar{R}^1) \times \cdots \times B_\epsilon(\bar{R}^K)$. We observe that the price vector at the preference profile is in $B_\varepsilon(p(\bar{\mathbf{R}}, f))$.

From the definition of $\bar{S}$, we have $f^j(R^1, \ldots, R^{\bar{S}-1}, \bar{R}^{\bar{S}}, \ldots, \bar{R}^N) = 0$ for any $j \geq K+1$

and any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, \bar{S}-1$. Since $f^i(\bar{\mathbf{R}})$, $i = 1, \dots, K$, are all positive and independent and $\hat{R}^i$ is a MMT of $\bar{R}^i$ at $f^i(\bar{\mathbf{R}})$, we have $p((R^1, \dots, R^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$ and $f^i(R^1, \dots, R^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N) = f^i(\bar{\mathbf{R}})$ for any $i = 1, \dots, K$, and any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, \bar{S}-1$. In particular, we have $f^{\bar{S}}(\hat{R}^1, \dots, \hat{R}^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N) = f^{\bar{S}}(\bar{\mathbf{R}})$ and $p((\hat{R}^1, \dots, \hat{R}^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$. Since $\hat{R}^{\bar{S}}$ is an MMT of $\bar{R}^{\bar{S}}$ at $f^{\bar{S}}(\bar{\mathbf{R}})$, we have $p((\hat{R}^1, \dots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$ as desired.

Next, we consider the case where $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) = 0$ for any $j \geq K+1$ and any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, K$. Note that $f^i(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) = f^i(\bar{\mathbf{R}})$ for $i = 1, \dots, K$, and $p((R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$ for any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, K$, because $f^i(\bar{\mathbf{R}})$, $i = 1, \dots, K$, are all positive and independent and $\hat{R}^i$ is an MMT of $\bar{R}^i$ at $f^i(\bar{\mathbf{R}})$, $i = 1, \dots, K$.

For each $i = 1, \dots, K$, we consider a function $\bar{\phi}^i : \varepsilon' \mapsto \bar{\phi}^i(\varepsilon')$ that maps $\varepsilon'$ to $\phi'$ in Lemma 2 with respect to $\bar{R}^i$ and $\hat{R}^i$. Furthermore we define a function $\phi^i : \varepsilon' \to \phi(\varepsilon')$ as $\phi^i(\varepsilon') = \min\{\bar{\phi}^i(\varepsilon'), \varepsilon'\}$. Then, of course, for any $\varepsilon$ and any $\mathbf{R}^{-i}$, if $f^i(\bar{R}^i, \mathbf{R}^{-i}) \neq 0$ and $p((\bar{R}^i, \mathbf{R}^{-i}), f) \in B_{\phi^i(\varepsilon)}(p(\mathbf{R}, f))$, then $p(\hat{R}^i; G^i(\mathbf{R}^{-i})) \in B_{\varepsilon}(p(\mathbf{R}, f))$ because of Lemma 2.

Starting from the preference profile $\bar{\mathbf{R}}$, we replace $\bar{R}^i$ with preferences in a neighborhood of $\hat{R}^i$ for $i = 1, \dots, K$, as follows.

We first replace $\bar{R}^1$ with $\hat{R}^1$. This replacement does not change the price vector. We let $\epsilon^1$ be a sufficiently small scalar such that $B_{\epsilon^1}(\hat{R}^1) \subset B_{\tilde{\epsilon}}(\bar{R}^1)$ and

$$p(R^{1\prime}; G^1(\bar{\mathbf{R}}^{-1})) \subset B_{\phi^2 \circ \cdots \circ \phi^K(\tilde{\varepsilon})}(p(\bar{\mathbf{R}}, f)) \tag{10}$$

for any $R^{1\prime} \in B_{\epsilon^1}(\hat{R}^1)$. Note that $f^1(\bar{\mathbf{R}})$ is the unique intersection of $UC(f^1(\bar{\mathbf{R}}); \hat{R}^1)$ and $G^1(\mathbf{R}^{-1})$. Therefore, if $R^{1\prime}$ is sufficiently close to $\hat{R}^1$, then $f(R^{1\prime}, \bar{\mathbf{R}}^{-1})$ and $p(R^{1\prime}; G^1(\bar{\mathbf{R}}^{-1}))$ are sufficiently close to $f^1(\bar{\mathbf{R}})$ and $p(\bar{\mathbf{R}}, f)$, respectively, as discussed after Lemma 1. Thus, a value of $\epsilon^1$ satisfying the desired properties exists.

The lemma holds if $f^j(R^{1\prime}, \bar{\mathbf{R}}^{-1}) \neq 0$ for some $j \geq K+1$ with some $R^{1\prime} \in B_{\epsilon^1}(\hat{R}^1)$. We suppose that $f^j(R^{1\prime}, \bar{\mathbf{R}}^{-1}) = 0$ for any $j \geq K+1$ and any $R^{1\prime} \in B_{\epsilon^1}(\hat{R}^1)$. Then, $f^i(R^{1\prime}, \bar{\mathbf{R}}^{-1})$, $i = 1, \dots, K$, are all positive and independent because $R^{1\prime} \in B_{\epsilon^1}(\hat{R}^1) \subset B_{\tilde{\epsilon}}(\bar{R}^1)$ and $p((R^{1\prime}, \bar{\mathbf{R}}^{-1}), f) \in B_{\phi^2 \circ \cdots \circ \phi^K(\tilde{\varepsilon})}(p(\bar{\mathbf{R}}, f))) \subset B_{\tilde{\varepsilon}}(p(\bar{\mathbf{R}}, f))$.

Then, we replace $R^2$ with $\hat{R}^2$. By (10) and Lemma 2, we have

$$p(\hat{R}^2; G^2(R^{1\prime}, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^3 \circ \cdots \circ \phi^K(\tilde{\varepsilon})}(p(\bar{\mathbf{R}}, f)) \tag{11}$$

for any $R^{1\prime} \in B_{\epsilon^1}(\hat{R}^1)$. We let $\epsilon^2$ be sufficiently small such that $B_{\epsilon^2}(\hat{R}^2) \subset B_{\tilde{\epsilon}}(\bar{R}^2)$ and

$$p(R^{2\prime}; G^2(R^{1\prime}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \subset B_{\phi^3 \circ \cdots \circ \phi^K(\tilde{\varepsilon})}(p(\bar{\mathbf{R}}, f)) \tag{12}$$

for any $R^{2\prime} \in B_{\epsilon^2}(\hat{R}^2)$ and any $R^{1\prime} \in B_{\epsilon^1}(\hat{R}^1)$. Note that a sufficiently small value of $\epsilon^2$ ensures (12) because of (11).

The lemma holds if $f^j(R^{1\prime}, R^{2\prime}, \bar{\mathbf{R}}^{-\{1,2\}}) \neq 0$ for some $j \geq K+1$ and some $(R^{1\prime}, R^{2\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$. We suppose that $f^j(R^{1\prime}, R^{2\prime}, \bar{\mathbf{R}}^{-\{1,2\}}) = 0$ for any $j \geq K+1$ and any $(R^{1\prime}, R^{2\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$. Then $f^i(R^{1\prime}, R^{2\prime}, \bar{\mathbf{R}}^{-\{1,2\}})$, $i = 1, \ldots, K$, are all positive and independent because $(R^{1\prime}, R^{2\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2) \subset B_{\bar{\epsilon}}(\bar{R}^1) \times B_{\bar{\epsilon}}(\bar{R}^2)$ and $p((R^{1\prime}, R^{2\prime}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\phi^3 \circ \cdots \circ \phi^K(\bar{\varepsilon})}(p(\bar{\mathbf{R}}, f)) \subset B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$.

Then, we replace $\bar{R}^3$ with $\hat{R}^3$. By (12) and Lemma 2, we have

$$p(\hat{R}^3; G^3(R^{1\prime}, R^{2\prime}, \bar{\mathbf{R}}^{-\{1,2,3\}})) \subset B_{\phi^4 \circ \cdots \circ \phi^K(\bar{\varepsilon})}(p(\bar{\mathbf{R}}, f))$$

for any $(R^{1\prime}, R^{2\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$. We let $\epsilon^3$ be a sufficiently small scalar such that $B_{\epsilon^3}(\hat{R}^3) \subset B_{\bar{\epsilon}}(\bar{R}^3)$ and

$$p(R^{3\prime}; G^3(R^{1\prime}, R^{2\prime}, \bar{\mathbf{R}}^{-\{1,2,3\}})) \subset B_{\phi^4 \circ \cdots \circ \phi^K(\bar{\varepsilon})}(p(\bar{\mathbf{R}}, f))$$

for any $R^{3\prime} \in B_{\epsilon^3}(\hat{R}^3)$ and any $(R^{1\prime}, R^{2\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$.

The lemma holds if $f^j(R^{1\prime}, \ldots, R^{3\prime}, \bar{\mathbf{R}}^{-\{1,2,3\}}) \neq 0$ for some $j \geq K+1$ and some $(R^{1\prime}, \ldots, R^{3\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times \cdots \times B_{\epsilon^3}(\hat{R}^3)$. We suppose that $f^j(R^{1\prime}, \ldots, R^{3\prime}, \bar{\mathbf{R}}^{-\{1,2,3\}}) = 0$ for any $j \geq K+1$ and any $(R^{1\prime}, \ldots, R^{3\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times \cdots \times B_{\epsilon^3}(\hat{R}^3)$. Then, $f^i(R^{1\prime}, \ldots, R^{3\prime}, \bar{\mathbf{R}}^{-\{1,2,3\}})$, $i = 1, \ldots, K$, are still all positive and independent because $(R^{1\prime}, \ldots, R^{3\prime}) \in B_{\bar{\epsilon}}(\hat{R}^1) \times \cdots \times B_{\bar{\epsilon}}(\hat{R}^3)$ and $p((R^{1\prime}, \ldots, R^{3\prime}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f) \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$.

We repeat this process. Finally, we replace $\bar{R}^K$ with $\hat{R}^K$ and have

$$p(\hat{R}^K; G^K(R^{1\prime}, \ldots, R^{K-1\prime}, \bar{R}^{K+1}, \ldots, \bar{R}^N)) \subset B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$$

for any $(R^{1\prime}, \ldots, R^{K-1\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times \cdots \times B_{\epsilon^{K-1}}(\hat{R}^{K-1})$. We let $\epsilon^K$ be a sufficiently small scalar such that $B_{\epsilon^K}(\hat{R}^K) \subset B_{\bar{\epsilon}}(\bar{R}^K)$ and

$$p(R^{K\prime}; G^K(R^{1\prime}, \ldots, R^{K-1\prime}, \bar{R}^{K+1}, \ldots, \bar{R}^N)) \subset B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$$

for any $R^{K\prime} \in B_{\epsilon^K}(\hat{R}^K)$ and any $(R^{1\prime}, \ldots, R^{K-1\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times \cdots \times B_{\epsilon^{K-1}}(\hat{R}^{K-1})$. There should exist some preference profile $(R^{1\prime}, \ldots, R^{K\prime}) \in B_{\epsilon^1}(\hat{R}^1) \times \cdots \times B_{\epsilon^K}(\hat{R}^K) \subset B_{\epsilon}(\bar{R}^1) \times \cdots \times B_{\epsilon}(\bar{R}^K)$ such that $f^j(R^{1\prime}, \ldots, R^{K\prime}, \bar{R}^{K+1}, \ldots, \bar{R}^N) \neq 0$ for some $j \geq K+1$. Otherwise, $f(\cdot, \ldots, \cdot, \bar{R}^{K+1}, \ldots, \bar{R}^N)$ becomes a social choice function in the economy with agents $i = 1, \ldots, K$, that is Pareto efficient, strategy-proof, and non-alternately dictatorial on $B_{\epsilon^1}(\hat{R}^1) \times \cdots \times B_{\epsilon^K}(\hat{R}^K)$. This contradicts Proposition 1. This ends the proof of Lemma 4. ■

**Proof of Corollary 1.** Contrary to the statement of the corollary, we suppose that for each $j \geq K+1$ there exist $\epsilon^j$ and $\varepsilon^j$ such that no $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_{\epsilon^j}(\bar{R}^1) \times \cdots \times B_{\epsilon^j}(\bar{R}^K)$ satisfies $p((\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N), f) \in B_{\varepsilon^j}(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}$. We let $\bar{\epsilon} = \min\{\epsilon^{K+1}, \ldots, \epsilon^N\}$ and $\bar{\varepsilon} = \min\{\varepsilon^{K+1}, \ldots, \varepsilon^N\}$. Then, there is $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_{\bar{\epsilon}}(\bar{R}^1) \times \cdots \times B_{\bar{\epsilon}}(\bar{R}^K)$ satisfies $p((\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N), f) \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$

and $f^j(\tilde{R}^1, \ldots, \tilde{R}^K, \bar{R}^{K+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}$ for any $j \geq K+1$. This contradicts Lemma 4. ■

**Proof of Lemma 5.** We select $\epsilon$ and $\varepsilon$ arbitrarily. If $f^j(\bar{\mathbf{R}}) \neq 0$ for some $j \geq K+1$, the lemma holds. We suppose that $f^j(\bar{\mathbf{R}}) = 0$ for any $j \geq K+1$. We let $\bar{\epsilon} > 0$ and $\bar{\varepsilon} > 0$ be scalars that support Lemma 3 with respsect to $\bar{\mathbf{R}}$. We define $\tilde{\epsilon} = \min\{\epsilon, \bar{\epsilon}\}$ and $\tilde{\varepsilon} = \{\varepsilon, \bar{\varepsilon}\}$.

Without loss of generality, we assume that $f^1(\bar{\mathbf{R}}) \notin \{0, \Omega\}$. Then, there exists some $2 \leq i \leq K$ who is assigned positive consumption at $\bar{\mathbf{R}}$. Without loss of generality , we assume that $f^i(\bar{\mathbf{R}}) \neq 0$ for $i = 2, \ldots, M_1$, where $2 \leq M_1 \leq K$. When we apply Lemma 4 to agents $i = 1, \ldots, M_1$, there exist $(\tilde{R}^1, \ldots, \tilde{R}^{M_1}) \in B_{\tilde{\epsilon}}(\bar{R}^1) \times \cdots \times B_{\tilde{\epsilon}}(\bar{R}^{M_1})$ and $j \geq M_1 + 1$ such that

$$p((\tilde{R}^1, \ldots, \tilde{R}^{M_1}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f) \in B_{\tilde{\varepsilon}}(p(\bar{\mathbf{R}}, f)) \tag{13}$$

and

$$f^j(\tilde{R}^1, \ldots, \tilde{R}^{M_1}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}. \tag{14}$$

If (14) holds with some agent $j \geq K+1$, then the lemma holds. We assume that

$$f^j(\tilde{R}^1, \ldots, \tilde{R}^{M_1}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N) = 0$$

for any $j \geq K+1$ and any $(\tilde{R}^1, \ldots, \tilde{R}^{M_1}) \in B_{\tilde{\epsilon}}(\bar{R}^1) \times \cdots \times B_{\tilde{\epsilon}}(\bar{R}^{M_1})$ satisfying (13).

We apply Corollary 1 to agents $i = 1, \ldots, M_1$. There exists $j \geq M_1 + 1$ such that for any $\epsilon_1$ and $\varepsilon_1$, some $(R^1_{(1)}, \ldots, R^{M_1}_{(1)}) \in B_{\epsilon_1}(\bar{R}^1) \times \cdots \times B_{\epsilon_1}(\bar{R}^{M_1})$ satisfies

$$p((R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f) \in B_{\varepsilon_1}(p(\bar{\mathbf{R}}, f)) \tag{15}$$

and

$$f^j(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}. \tag{16}$$

Without loss of generality, we assume that agents $j = M_1 + 1, \ldots, M_2$, where $M_1 + 1 \leq M_2 \leq K$, are such agents, and set a sufficiently small $\epsilon_1 < \tilde{\epsilon}$ and $\varepsilon_1 < \tilde{\varepsilon}$, and we fix $(R^1_{(1)}, \ldots, R^{M_1}_{(1)}) \in B_{\epsilon_1}(\bar{R}^1) \times \cdots \times B_{\epsilon_1}(\bar{R}^{M_1})$ satisfying (15) and (16).

Remember Lemma 4 and our choice of the scalars $\epsilon_1$ and $\varepsilon_1$. First, $g^i((R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f)$, $i = 1, \ldots, K$, are independent because $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \ldots, K$, are independent, $R^i_{(1)} \in B_{\epsilon_1}(\bar{R}^i) \subset B_{\tilde{\epsilon}}(\bar{R}^i)$, $i = 1, \ldots, M_1$, and $p((R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f) \in B_{\varepsilon_1}(p(\bar{\mathbf{R}}, f)) \subset B_{\tilde{\varepsilon}}(p(\bar{\mathbf{R}}, f))$. Furthermore $f^i(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N) \neq 0$ for any $i = 1, \ldots, M_1$, because $f^i(\bar{\mathbf{R}}) \neq 0$ for $i = 1, \ldots, M_1$. As a result, $f^i(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N)$, $i = 1, \ldots, M_2$, are all positive and independent.

Here, we let $\bar{\epsilon}_1$ and $\bar{\varepsilon}_1$ be scalars such that they support Lemma 4 with respect to $(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N)$ and satisfy $B_{\bar{\epsilon}_1}(R^i_{(1)}) \subset B_{\epsilon}(\bar{R}^i)$ for $i = 1, \ldots, M_1$,

$B_{\bar{\epsilon}_1}(\bar{R}^i) \subset B_\epsilon(\bar{R}^i)$ for $i = M_1+1, \ldots, K$, and $B_{\bar{\varepsilon}_1}(p(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f) \subset B_\epsilon(p(\bar{\mathbf{R}}, f))$.

We apply Lemma 4 to agents $i = 1, \ldots, M_2$. There exist $(\tilde{R}^1, \ldots, \tilde{R}^{M_2}) \in B_{\bar{\epsilon}_1}(R^1_{(1)}) \times \cdots \times B_{\bar{\epsilon}_1}(R^{M_1}_{(1)}) \times B_{\bar{\epsilon}_1}(\bar{R}^{M_1+1}) \times \cdots \times B_{\bar{\epsilon}_1}(\bar{R}^{M_2})$ and $j \geq M_2+1$ such that

$$p((\tilde{R}^1, \ldots, \tilde{R}^{M_2}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N), f) \in B_{\bar{\varepsilon}_1}(p((R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f)) \quad (17)$$

and

$$f^j(\tilde{R}^1, \ldots, \tilde{R}^{M_2}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}. \quad (18)$$

If (18) holds with some $j \geq K+1$, the lemma holds. We assume that

$$f^j(\tilde{R}^1, \ldots, \tilde{R}^{M_2}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N) = 0.$$

for any $j \geq K+1$ and any $(\tilde{R}^1, \ldots, \tilde{R}^{M_2}) \in B_{\bar{\epsilon}_1}(R^1_{(1)}) \times \cdots \times B_{\bar{\epsilon}_1}(R^{M_1}_{(1)}) \times B_{\bar{\epsilon}_1}(\bar{R}^{M_1+1}) \times \cdots \times B_{\bar{\epsilon}_1}(\bar{R}^{M_2})$ satisfying (17).

We apply Corollary 1 to agents $i = 1, \ldots, M_2$. There exists $j \geq M_2+1$ such that for any $\epsilon_2$ and $\varepsilon_2$, some $(R^1_{(2)}, \ldots, R^{M_2}_{(2)}) \in B_{\epsilon_2}(R^1_{(1)}) \times \cdots \times B_{\epsilon_2}(R^{M_1}_{(1)}) \times B_{\epsilon_2}(\bar{R}^{M_1+1}) \times \cdots \times B_{\epsilon_2}(\bar{R}^{M_2})$ satisfies

$$p((R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N), f) \in B_{\varepsilon_2}(p((R^1_{(1)}, \ldots, \tilde{R}^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f)) \quad (19)$$

and

$$f^j(R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N) \notin \{0, \Omega\}. \quad (20)$$

Without loss of generality, we assume that agents $j = M_2+1, \ldots, M_3$, where $M_2+1 \leq M_3 \leq K$, are such agents. We set $\epsilon_2 < \bar{\epsilon}_1$ and $\varepsilon_2 < \bar{\varepsilon}_1$ , and we fix $(R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}) \in B_{\epsilon_2}(R^1_{(1)}) \times \cdots \times B_{\epsilon_2}(R^{M_1}_{(1)}) \times B_{\epsilon_2}(\bar{R}^{M_1+1}) \times \cdots \times B_{\epsilon_2}(\bar{R}^{M_2})$ satisfying (19) and (20).

Remember Lemma 4 and our choice of the scalars $\epsilon_2$ and $\varepsilon_2$. First, $g^i((R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N), f)$, $i = 1, \ldots, K$, are independent because $g^i(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N)$, $i = 1, \ldots, K$, are independent, $R^i_{(2)} \in B_{\epsilon_2}(R^i_{(1)}) \subset B_{\bar{\epsilon}_1}(\bar{R}^i)$, $i = 1, \ldots, M_1$, $R^i_{(2)} \in B_{\epsilon_2}(\bar{R}^i) \subset B_{\bar{\epsilon}_1}(\bar{R}^i)$, $i = M_1+1, \ldots, M_2$, and $p(R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N) \in B_{\varepsilon_2}(p((R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f)) \subset B_{\bar{\varepsilon}_1}(p((R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N), f))$. Furthermore, $f^i(R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N) \neq 0$ for any $i = 1, \ldots, M_2$, because $f^i(R^1_{(1)}, \ldots, R^{M_1}_{(1)}, \bar{R}^{M_1+1}, \ldots, \bar{R}^N) \neq 0$ for $i = 1, \ldots, M_2$. As a result, $f^i(R^1_{(2)}, \ldots, \tilde{R}^{M_2}_{(2)}, \bar{R}^{M_2+1}, \ldots, \bar{R}^N)$, $i = 1, \ldots, M_3$, are all positive and independent.

We repeat this process. In each step, we have at least one additional agent who receives positive consumption at the preference profile in the $\epsilon$-neighborhood of $\bar{\mathbf{R}}$, where the price vector is in the $\varepsilon$-neighborhood of $p(\bar{\mathbf{R}})$. Finally, some agent $j \geq K+1$ should receive positive consumption. This ends the proof of Lemma 5. ■

**Proof of Corollary 2.** Contrary to the statement of the corollary, we suppose that for each $j \geq K+1$ there exist $\epsilon^j$ and $\varepsilon^j$ such that no $(\tilde{R}^1, \ldots, \tilde{R}^K) \in B_{\epsilon^j}(\bar{R}^1) \times \cdots \times B_{\epsilon^j}(\bar{R}^K)$

satisfies $p((\tilde{R}^1,\dots,\tilde{R}^K,\bar{R}^{K+1},\dots,\bar{R}^N),f) \in B_{\varepsilon^j}(p(\bar{\mathbf{R}},f))$ and $f^j(\tilde{R}^1,\dots,\tilde{R}^K,\bar{R}^{K+1},\dots,\bar{R}^N) \notin \{0,\Omega\}$. We let $\bar{\epsilon} = \min\{\epsilon^{K+1},\dots,\epsilon^N\}$ and $\bar{\varepsilon} = \min\{\varepsilon^{K+1},\dots,\varepsilon^N\}$. Then, there is $(\tilde{R}^1,\dots,\tilde{R}^K) \in B_{\bar{\epsilon}}(\bar{R}^1)\times\cdots\times B_{\bar{\epsilon}}(\bar{R}^K)$ satisfies $p((\tilde{R}^1,\dots,\tilde{R}^K,\bar{R}^{K+1},\dots,\bar{R}^N),f) \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}},f))$ and $f^j(\tilde{R}^1,\dots,\tilde{R}^K,\bar{R}^{K+1},\dots,\bar{R}^N) \notin \{0,\Omega\}$ for any $j \geq K+1$. This contradicts Lemma 5. ■

**Proof of Lemma 6.** We write $a^i = ((1/\alpha^i_1)^{1/(\rho-1)},\dots,(1/\alpha^i_L)^{1/(\rho-1)})$. First of all, it is clear that the lemma holds when $a^i$, $i=1,\dots,N$, are independent. If $a^i$, $i=1,\dots,N$, are independent, $\beta^i(\delta,z^i)$, $i=1,\dots,N$, are positive and independent with any vectors $\bar{z}^i$, $i=1,\dots,N$, when $\delta^i$, $i=1,\dots,N$, are sufficiently small.

When they are dependent, $\bar{z}^i$, $i=1,\dots,N$, can be obtained as follows, for example. Without loss of generality we assume that the vectors $a^1,\dots,a^N$ span $S$-dimensional space, where $S<N$ and $a^1,\dots,a^S$ are independent. We set $\bar{z}^i = a^i$ for $i=1,\dots,S$. We let $\langle a^1,\dots,a^S\rangle$ denote the $S$-dimensinal vector space spanned by $a^1,\dots,a^S$ and let $\langle a^1,\dots,a^S\rangle^\perp$ denote the orthogonal complement space of $(L-S)$-dimensions, We select $\bar{z}^i$, $i=S+1,\dots,N$, so that they are independent vectors in $\langle a^1,\dots,a^S\rangle^\perp$. When $\delta^i$ is sufficiently small $a^i+\delta^i\bar{z}^i$ is positive for any $i=1,\dots,N$. Their independence is clear from the construction. ■

**Proof of Lemma 7.** In the proof of Lemma 6, we set $\bar{z}^i$ so that $a^i+\delta^i\bar{z}^i$, $i=1,\dots N$, are independent. It is clear that any vector $z^i$ sufficiently close to $\bar{z}^i$ sustains the independence, and Lemma 6 holds with $z^i$ instead of $\bar{z}^i$. That is, there exists $\bar{t}$ such that $\beta^i(\delta^i,z^i)$, $i=1,\dots,N$ are independent for any $0<\delta^i\leq\bar{\delta}$ and any $z^i \in B_{\bar{t}}(\bar{z}^i)$, $i=1,\dots,N$.

For each $\delta^i$ and $z^i \in B_t(\bar{z}^i)$, we obtain $\alpha^i_{\delta^i z^i} = (\alpha^i_{\delta^i z^i 1},\dots,\alpha^i_{\delta^i z^i L})$ by solving

$$((1/\hat{\alpha}^i_{\delta^i z^i 1})^{1/(\rho-1)},\dots,(1/\hat{\alpha}^i_{\delta^i z^i L})^{1/(\rho-1)}) = \beta^i(\delta^i,z^i) \tag{21}$$

and define $\alpha^i_{\delta^i z^i} = (\alpha^i_{\delta^i z^i 1},\dots,\alpha^i_{\delta^i z^i L})$ as the normalization of $\hat{\alpha}^i_{\delta^i z^i}$: $\alpha^i_{\delta^i z^i} = \hat{\alpha}^i_{\delta^i z^i}/\parallel \hat{\alpha}^i_{\delta^i z^i}\parallel$. Then, as in the case of $U_{\alpha^i_{\delta^i}}$, with any price $p$, the consumption-direction vectors of $U_{\alpha^i_{\delta^i z^i}}$, $i=1,\dots,N$, are independent for any $0<\delta^i<\bar{\delta}$ and any $z^i \in B_{\bar{t}}(\bar{z}^i)$ because of the independence of $\beta(\delta^i,z^i)$, $i=1,\dots,N$.

We define $G^i(\delta^i,\bar{t},p) = \bigcup_{z^i\in B_{\bar{t}}(\bar{z}^i)} g^i(U_{\alpha^i_{\delta^i z^i}},p)$ as the set of the consumption-direction vectors of $U_{\alpha^i_{\delta^i z^i}}$ at $p$ while $z^i$ moves over $B_{\bar{t}}(\bar{z}^i)$.

As $R^i \to U_{\alpha^i_{\delta^i}}$, we have $g(R^i,p) \to g^i(U_{\alpha^i_{\delta^i}},p)$. Therefore, for each $p$ and $0<\delta^i\leq\bar{\delta}$, there exists $\epsilon^i(p,\delta^i)$ such that if $R^i \in B_{\epsilon^i(p,\delta^i)}(U_{\alpha^i_{\delta^i,\rho}})$, then $g^i(R^i,p) \in G^i(\delta^i,\bar{t},p)$. Finally, we define $\epsilon^i(\delta^i) = \min_{p\in P}\epsilon(p,\delta^i)$. ■

**Proof of Lemma 8.** When $\delta^i=0$, $R^i_{\alpha^i_0,t,s}$ is an MMT of $\bar{R}^i$ at $\bar{g}^i$ and $R^i_{\alpha^i_0,t,s} \to \bar{R}^i$ as $t\to 0$ and $s \to 0$. We fix $s$ and $t$ such that $R^i_{\alpha^i_0,t,s} \in B_\epsilon(\bar{R}^i)$ for any $i=1,\dots,N$.

Since $R^i_{\alpha^i_{\delta^i},t,s} \to R^i_{\alpha^i_0,t,s}$ as $\delta^i \to 0$, we select $\bar{\delta}^i$ such that $R^i_{\alpha^i_{\bar{\delta}^i},t,s} \in B_\epsilon(\bar{R}^i)$. Then we have $R^i_{\alpha^i_{\delta^i},t,s} \in B_\varepsilon(\bar{R}^i)$ for any $0\leq\delta^i\leq\bar{\delta}^i$.

For any $0 < \delta^i \leq \bar{\delta}^i$,the consumption-direction vectors of $R^i_{\alpha^i_{\delta^i},t,s}$, $i = 1,\ldots,N$, are independent for prices in a neighborhood of $\bar{p}$. That is there exists $\varepsilon$ such that $g(R^i_{\alpha^i_{\delta^i},t,s}, p)$, $i = 1,\ldots,N$, are independent for $p \in B_\varepsilon(\bar{p})$. ■

**Proof of lemma 9.** Let $\epsilon^i$ be the same as in Lemma 7 and set $\bar{\varepsilon}$ such that $g(R^i_{\alpha^i_{\delta^i},t,s}, p) = g(U_{\alpha^i_{\delta^i}}, p)$, $i = 1,\ldots,N$, for $p \in B_{\bar{\varepsilon}}(\bar{p})$. Then, Lemma 9 follows Lemma 7. ■

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