Equilibrium returns with transaction costs

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The aim of the study

How asset returns depend on liquidity ?

- A tractable equilibrium model with transaction costs
- Continuous time
- (Heterogeneous) multiple agents

For maximal tractability, we consider

- Local mean-variance preference
- Quadratic transaction costs

In this talk, we focus on a single asset market with finite horizon; see

Equilibrium returns with transaction costs, *Finance Stoch.* (2018)

for a more general case.

The framework

An asset price dynamics (under business time scale)

$$\mathrm{d}S_t = \mu_t \mathrm{d}t + \mathrm{d}W_t$$

The premium μ_t is to be determined endogenously.

Agents $n = 1, 2, \ldots, N$ receive random endowments

$$\mathrm{d}Y_t^n = a_t^n \mathrm{d}t + \zeta_t^n \mathrm{d}W_t + \mathrm{d}M_t^{\perp,n},$$

where a^n and ζ^n are adapted (square) integrable process and $M^{\perp,n}$ is a square integrable martingale orthogonal to W.

Agent n's wealth $\Pi^n(\varphi)$ with a strategy φ is given by

$$\Pi^n(\varphi)_t = Y_t^n + \int_0^t \varphi_u \mathrm{d}S_u - \lambda \int_0^t \dot{\varphi}_u^2 \mathrm{d}u.$$

Local mean-variance preference

Agent n's problem is to maximize

$$J^n(\varphi) := E[\Pi^n_T(\varphi)] - \frac{\gamma^n}{2} E[\langle \Pi^n(\varphi) \rangle_T].$$

- $\gamma^n > 0$ is Agent n's risk aversion.
- $\langle \Pi^n(\varphi) \rangle$ is the quadratic variation;

$$\langle \Pi^n(\varphi) \rangle_T = \int_0^T |\varphi_u + \zeta^n_u|^2 \mathrm{d}u + \langle M^{\perp,n} \rangle_T.$$

Thus,

$$J^n(\varphi) = \int_0^T E\left[\varphi_t \mu_t - \lambda \dot{\varphi}_t^2 - \frac{\gamma^n}{2} |\varphi_t + \zeta^n_t|^2\right] \mathrm{d}t + E[\langle M^{\perp,n} \rangle_T].$$

Comments

- Without loss of generality, we can assume $M^{\perp,n} = 0$.
- Local mean-variance preference is time consistent.
- Quadratic costs $\lambda\dot{\varphi}^2$ arise from linear market impacts
- Gârleanu and Pedersen
  (J. Finance 2013, J. Econ. Theory 2016)
  – a single rational agent and noise traders
- Sannikov and Skrzypacz (preprint, 2016)
  – market impacts are endogenized

Frictionless case : $\lambda = 0$

$$\text{Maximize} \quad \int_0^T E\left[\varphi_t \mu_t - \frac{\gamma^n}{2}|\varphi_t + \zeta_t^n|^2\right] dt.$$

The solution is

$$\varphi^n = \frac{\mu}{\gamma^n} - \zeta^n. \tag{1}$$

The clearing condition for N agents with noise trader demand ψ:

$$\psi + \sum_{n=1}^{N} \varphi^n = 0.$$

The equilibrium return is therefore

$$\mu = \left(\sum_{n=1}^{N} \frac{1}{\gamma_n}\right)^{-1} \left(-\psi + \sum_{n=1}^{N} \zeta^n\right). \tag{2}$$

The frictional optimizer

Lemma: For given μ, let φ^n be the frictionless optimizer (1). Then, the frictional optimization problem

$$\text{Maximize } J^n(\varphi) = \int_0^T E\left[\varphi_t \mu_t - \lambda \dot{\varphi}_t^2 - \frac{\gamma^n}{2} |\varphi_t + \zeta_t^n|^2\right] \mathrm{d}t$$

has a unique solution $\varphi^{\lambda,n}$, characterized by the FBSDE

$$\begin{aligned} \mathrm{d}\varphi_t^{\lambda,n} &= \dot{\varphi}_t^{\lambda,n} \mathrm{d}t, \quad \varphi_0^{\lambda,n} = 0, \\ \mathrm{d}\dot{\varphi}_t^{\lambda,n} &= \frac{\gamma^n}{2\lambda}(\varphi_t^{\lambda,n} - \varphi_t^n)\mathrm{d}t + \mathrm{d}M_t^n, \quad \dot{\varphi}_T^{\lambda,n} = 0, \end{aligned} \tag{3}$$

where M^n is a square integrable martingale to be determined as part of the solution.

The explicit solution

Lemma: The unique solution of (3) is given by

$$\varphi_t^{\lambda,n} = \int_0^t \exp\left\{-\int_s^t F(u)\mathrm{d}u\right\} \bar{\varphi}_s^n \mathrm{d}s,$$

where

$$\bar{\varphi}_t^n = \frac{\gamma^n}{2\lambda}\frac{1}{G(t)}\int_t^T E\left[G(s)\varphi_s^n|\mathcal{F}_t\right]\mathrm{d}s,$$

$$F(t) = \sqrt{\frac{\gamma^n}{2\lambda}}\tanh\left(\sqrt{\frac{\gamma^n}{2\lambda}}(T-t)\right),$$

$$G(t) = \cosh\left(\sqrt{\frac{\gamma^n}{2\lambda}}(T-t)\right).$$

Towards equilibrium

We want to find μ^λ such that the frictional optimizers $\varphi = \varphi^{\lambda,n}$ of $J^n(\varphi)$ for $\mu = \mu^\lambda$, $n = 1, 2, \ldots, N$, satisfy the clearing condition

$$\psi + \sum_{n=1}^{N} \varphi^{\lambda,n} = 0.$$

We add an assumption on the noise trader demand ψ:

$$\begin{aligned} \mathrm{d}\psi_t &= \dot{\psi}_t \mathrm{d}t, \\ \mathrm{d}\dot{\psi}_t &= \mu_t^\psi \mathrm{d}t + \mathrm{d}M_t^\psi, \end{aligned}$$

where μ^ψ is a square integrable adapted process and M^ψ is a square integrable martingale.

Assume $\gamma^1 < \cdots < \gamma^N$ without loss of generality.

Key Lemma

Lemma: There exists a unique solution

$$(\varphi^{\lambda,1}, \dots, \varphi^{\lambda,N-1})$$

of the FBSDE

$$d\varphi_t^{\lambda,n} = \dot{\varphi}_t^{\lambda,n} dt, \quad \varphi_0^{\lambda,n} = 0,$$
$$d\dot{\varphi}_t^{\lambda,n} = \frac{1}{2\lambda}\left(\gamma^n(\varphi_t^{\lambda,n} + \zeta_t^n) - \frac{(\varphi_t^{\lambda} + \zeta_t, \gamma)}{N}\right)dt - \frac{1}{N} d\dot{\psi}_t + dM_t^n \tag{4}$$

with $\dot{\varphi}_T^{\lambda,n} = 0$, $n = 1, 2, \dots, N-1$, where

$$(\varphi_t^{\lambda} + \zeta_t, \gamma) = \sum_{i=1}^{N} (\varphi_t^{\lambda,i} + \zeta_t^i)\gamma^i, \quad \varphi_t^{\lambda,N} = -\sum_{i=1}^{N-1} \varphi_t^{\lambda,i} - \psi_t.$$

The main result

Theorem: Let $\varphi^{\lambda,n}$, $n = 1, 2, \ldots, N$ be the unique solution of FBSDE (4). Let μ be the frictionless equilibrium return (2) and φ^n be Agent n's frictionless optimizer (1) under the frictionless equilibrium. Then, the unique frictional equilibrium return μ^λ is given by

$$\mu^\lambda = \mu + \frac{1}{N}\sum_{n=1}^{N}(\gamma^n - \bar{\gamma})(\varphi^{\lambda,n} - \varphi^n_t) + \frac{2\lambda}{N}\mu^\psi,$$

where

$$\bar{\gamma} = \frac{1}{N}\sum_{n=1}^{N}\gamma_n.$$

Implications on Liquidity Premium

$$LP := \mu^{\lambda} - \mu = \frac{1}{N}\sum_{n=1}^{N}(\gamma^n - \bar{\gamma})(\varphi^{\lambda,n} - \varphi^n_t) + \frac{2\lambda}{N}\mu^{\psi}.$$

- $\mu^{\lambda} = \mu$, that is, LP $= 0$, if
  - $\gamma_n \equiv \bar{\gamma}$ (homogeneous agents) and
  - $\mu^{\psi} = 0$ ($\dot{\psi}$ is driftless).
- The first term of LP is the (sample) covariance between $(\gamma^1, \dots, \gamma^N)$ and $(\varphi^{\lambda,1} - \varphi^1_t, \dots, \varphi^{\lambda,N} - \varphi^N_t)$:
  - positive if more risk averse agents are more excess holders.
  - then, sellers have stronger incentives, agree with higher μ^{λ}.
- $\mu^{\psi} > 0$ (= the convexity of ψ) pushes LP up. Why ?

Multi-asset case

Let the asset price process S be now d dimensional:

$$\mathrm{d}S_t = \mu_t \mathrm{d}t + \sigma \mathrm{d}W_t,$$

where σ is deterministic (and exogenous) with $\Sigma = \sigma\sigma^T$ invertible. The quadratic transaction costs:

$$\lambda \dot{\varphi}^2 \to \dot{\varphi}^T \Lambda \dot{\varphi}$$

with positive definite Λ. Then all $\frac{1}{2\lambda}$ so far should be replaced by $\frac{1}{2}\Lambda^{-1}\Sigma$. The final result; the equilibrium return is given by

$$\mu^\Lambda = \mu + \frac{\Sigma}{N}\sum_{n=1}^{N}(\gamma^n - \bar{\gamma})(\varphi^{\lambda,n} - \varphi_t^n) + \frac{2\Lambda}{N}\mu^\psi.$$

Summary

- Equilibrium return process is determined explicitly,
- under quadratic transaction costs,
- for heterogeneous agents
- with local mean-variance preference.
- The optimal trading strategy is characterized as the unique solution of a FBSDE,
- which admits an explicit expression.
- Positive liquidity premium if more risk averse agents are net sellers.