Condorcet Jury Theorem and Cognitive Hierarchy: Theory and Experiments*

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Abstract

An information aggregation problem of the Condorcet Jury Theorem is considered with cognitive hierarchy models in which players best respond holding heterogeneous beliefs on the cognitive level of the other players. Whether the players are aware of the presence of opponents at their own cognitive level turns out to be a key factor for asymptotic properties of the deviation from Nash behavior, and thus for asymptotic efficiency of the group decision. Our laboratory experiments provide evidence for the self-awareness condition. We obtain an analytical result showing that the difference from the standard cognitive hierarchy models arises when the best-reply functions are asymptotically expanding.

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1 Introduction

Since Condorcet's classical work in 1785, mathematical support has been provided for the idea of increasing the accuracy of collective decisions by including more individuals in the process. In his seminal *Essai*, Condorcet considered non-strategic individuals voting to make a decision on a binary issue where each alternative is commonly preferred to the other one in one of the two states of the world. Each individual receives independently an imperfectly informative private signal about the true state of the world and votes accordingly. Under majority rule, the probability of reaching a correct decision monotonically increases with the size of the electorate and converges to certainty in the limit.

Although allowing strategic behavior may imperil the validity of the basic assumptions in the original model, the asymptotic property survives in various circumstances of collective decision making (*e.g.,* Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997)). Studies documented conditions under which the probability of making a right decision increases and converges to certainty as the group size increases, even when strategic players may vote against their signals.

However, assuming complete rationality, especially complete mutual consistency of beliefs, may be too demanding when the strategy space is big and a large number of players are involved. In large games, the collective behavior of strategic players may differ qualitatively from that in small games.[1] This intuition finds support from experimental evidence and information aggregation by a large group is not, to say the least, exempt from it, especially when a major presumption of the strategic models is that voters take into account their probability of being pivotal, as in Downs (1957). As shown by Esponda and Vespa (2014) in their experimental study, the perfectly accurate hypothetical thinking to extract information from others' strategies that is required in most strategic voting models might be too strong as an assumption, even untenable, when it is assumed for all individuals.[2] Battaglini et al. (2008) document in another experimental study an increase in irrational, non-equilibrium play as the size of the electorate increases.[3] Collective performances are correlated across challenges, as demonstrated in Woolley et al. (2010), hence a good knowledge about the behavioral basis in collective decision-making procedures is essential to understanding the more general phenomena that we observe in our societies.

Furthermore, studying strategic thinking in private information games is crucial as they

[1] In her Presidential Speech to the American Political Science Society, Ostrom (1998) called for a behavioral theory of collective action based on models of the boundedly rational individual, pointing to the fact that behavior in social dilemmas is affected by many structural variables, such as the group size. This objective is put forward with an understanding of the "thin" model of rationality as a limiting case of bounded or incomplete rationality, as in Selten (1975).

[2] In their experiment, 50 to 80% of the participants behaved non-strategically when voting is simultaneous, and they find that non-optimal behavior is typically due to difficulty in extracting information from hypothetical events.

[3] As Camerer (2003) Chapter 7 stresses, the effect of group size on behavior in strategic interactions is a persistent phenomenon, especially towards coordination.

are common tools for modeling interactions in bargaining, contracts, matching and financial markets, and political situations, *inter alia*. If strategic naïveté is prevalent, peculiarities due to private information can be more crucial than what is predicted by equilibrium analysis, and thus policy responses may be astray, as noted by Brocas et al. (2014).

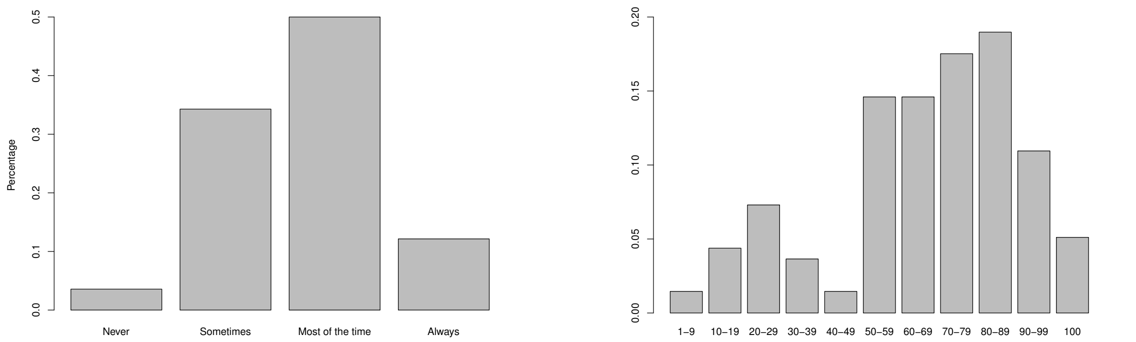
Models of non-equilibrium strategic thinking have been proposed to explain structural deviations from equilibrium thinking in a variety of games. A sizable part of bounded rationality literature is devoted to the models of *cognitive hierarchy*, starting with Nagel (1995) and Stahl and Wilson (1995), which allow heterogeneity among individuals in levels of strategic thinking. In these models, a foundational level of cognitive hierarchy, level-0, represents a strategically naïve initial approach to game, and a level-k player (hereafter Lk, where $k \geq 1$) is assumed to best respond to others with a cognitive hierarchy of level $k-1$. The construction of levels resonates with *rationalizability*, as in Bernheim (1984), due to the fact that the decisions made by a level-k player survive k rounds of iterated elimination of strictly dominated strategies in two-person games.

Closely related, the *Poisson cognitive hierarchy* (Poisson-CH) model is introduced by Camerer et al. (2004), allowing heterogeneity in the beliefs on others' levels. A level-k player best responds to a mixture of lower levels, which is estimated by consistent truncations up to level $k-1$ from a Poisson distribution, for each $k \geq 1$. The relevant Poisson distribution is either obtained from maximum likelihood estimations applied to data, or calibrated from previous estimates. The set of level-1 strategies in the Poisson-CH model (hereafter $CH1$) is exactly the same as that of $L1$. For higher levels, *i.e.,* $k > 1$, Lk and CHk differ. Most notably, strategies in CHk are not rationalizable in general.[4] Experimental studies provide a various sort of evidence that the Poisson-CH model delivers a better fit for explaining the actual behavior of the players in certain games. Common to these models is the assumption that level-k players do not assign any probability to levels higher than k. This assumption emerges from the idea that the cognitive limits among players have indeed a hierarchical structure.

Another assumption shared by these two models is the lack of *self-awareness*. Both models presume that no individual assigns a positive probability to the events in which other players have the same cognitive level. In this paper, we propose a new model, which allows for self-awareness: the *endogenous cognitive hierarchy* (ECH) model builds on the Poisson-CH model (i.e. keeping the partial consistency implied by truncations of the underlying Poisson distribution), by allowing individuals to best respond while holding a belief concerning the cognitive levels of other players that includes the same level as themselves. Therefore, the ECH model maintains the hierarchical structure of cognitive levels in previous models. The aim of this paper is to study the consequences of the presence of the self-awareness condition both theoretically and empirically, and show that a stark contrast is observed in a certain class of games.

There are three reasons why we study the consequences implied by the presence of the

[4] Crawford et al. (2013) provide a fine review of these models and applications.



(a) "When you made decisions, did you think that the other participants in your group used exactly the same reasoning as you did?"

(b) "What is the percentage of the other participants using the same reasoning, according to your estimation?"

Figure 1: Responses in the post-experimental questionnaire.

self-awareness condition. First, we show that in a certain class of games, the presence of self-awareness simply matters. A novel finding of this paper is that a lack of the self-awareness condition implies in certain games that the deviation from rational behavior should diverge away without a bound, which we argue is not coherent with the idea of the cognitive hierarchy models.[5] One of the objectives of this paper is to shed light on the asymptotic properties of the deviations, and to understand the role of the assumptions that account for distinct asymptotic properties. Second, a fairly large proportion (96%) of our experimental subjects exhibit a positive degree of self-awareness, according to their responses in the post-experimental questionnaire (Figure 1(a)). Exactly half of them (50%) responded that other players used the same reasoning as they did 'most of the time.' For the participants who gave a positive answer (either 'sometimes', 'most of the time', or 'always'), we also asked their estimation of the percentage among the other members who used the same reasoning. Then, the answer varied (Figure 1(b)). Even though such self-declaration may have a limited statistical validity, it shows that completely ruling out self-awareness may be too strong as an assumption. Lastly, our experimental results provide clear-cut support for including self-awareness (Section 4.6).[6]

[5] Evidence from experimental studies on public good games suggest that as group size increases, individual behavior bears convergent and stabilizing tendencies (see Isaac et al. (1994)).

[6] Colman et al. (2014) point to the observed weak performance of CH in some common interest games. Georganas et al. (2015) conclude that level-k models have a high performance in some games, but not in others. On the other hand, "equilibrium plus noise" models often miss systematic patterns in participants' deviations from equilibrium, which is a main feature of our experimental data.

Related Literature

Gerling et al. (2005) provide an extensive survey on the studies of collective decision-making in committees and the Condorcet Jury Theorem. Palfrey (2016) provides a comprehensive survey on experiments in political economy and particularly in strategic voting. Costinot and Kartik (2007) study voting rules and show that the optimal voting rule is the same when players are sincere, playing according to Nash equilibrium, to level-k, or a mixture of these. Bhattacharya et al. (2013) test experimentally the theoretical predictions about individual behavior and group decisions under costly information acquisition. They find poor support for the comparative statics predictions which are delivered theoretically. Alaoui and Penta (2016) introduce a model of strategic thinking that endogenizes individuals' cognitive bounds as a result of a cost-benefit analysis. Their framework allows individuals to reason about opponents whom they regard as more sophisticated as well. Hanaki et al. (2016) study how the strategic environmental effects depend on the group size in the beauty contest games, finding support for the effects to be present in a large group.

The paper proceeds as follows. We introduce the endogenous cognitive hierarchy model formally in the following section. We furthermore provide a numerical comparison of the individual behavior and the performance of the collective decisions under different specifications of the cognitive hierarchy in a model of information aggregation. In Section 3 we introduce our experimental design that features novelties due to our modeling concerns and signal setup. Section 4 provides the results of the experiment, and the models are compared in terms of the data fit. Section 5 provides theoretical results focusing on linear quadratic games. We present our main theorems, which provide a characterization of games according to the asymptotic properties of the strategic thinking. We conclude by summarizing our findings and presenting further research questions in Section 6. The proofs of the theorems are relegated to Appendix.

2 The Model

Let (N, S, u) be a symmetric normal-form game where $N = \{1, \ldots, n\}$ is the set of players, $S \subset \mathbb{R}$ is a convex set of pure strategies, and $u : S^n \to \mathbb{R}^n$ is the payoff function. Each player forms a belief on the cognitive levels of the other players. Let $g_k(h)$ denote the probability that a k^{th}-level player assigns independently for each of the other players to belong to the h^{th}-level.

In the standard *level-k model* introduced in Nagel (1995), a naïve, nonstrategic behavior is specified as the initial level of cognitive hierarchy (*level*-0, or $L0$). For $k \geq 1$, a level-k (Lk) player holds the belief that all of the other players belong to exactly one level below herself:

$$g_k(h) = \begin{cases} 1 & \text{if } h = k-1, \\ 0 & \textit{otherwise.} \end{cases} \tag{L}$$

In the cognitive hierarchy model introduced in Camerer et al. (2004), each k^{th}-level (CHk) player best responds to a mixture of lower levels. Let $f = (f_0, f_1, \ldots)$ be a distribution over $\mathbb{N}$ which represents the composition of cognitive hierarchy levels. Each k^{th}-level player holds a belief on the distribution of the other players' levels that is a truncation up to one level below herself:

$$g_k(h) = \frac{f_h}{\sum_{m=0}^{k-1} f_m}, \text{ for } 0 \leq h \leq k-1 \text{ and } k \geq 1. \tag{CH}$$

Thus, these two models share the following assumption:

Assumption 1 $g_k(h) = 0$ *for all* $h \geq k$.

Assumption 1 enables us to say that what we call "levels" here indeed has a hierarchical structure. To see this, consider $g_k(h)$ in the format of a k-h matrix. Assumption 1 implies that the upper-diagonal entries are all zeros, and thus the remaining non-zero elements have a pyramid structure with strictly lower-diagonal entries. Each level-k player assigns non-zero probabilities only to the levels strictly lower than herself. In that sense, players are assumed to be *overconfident*.[7]

Alternatively, the following assumption can be considered:

Assumption 2 $g_k(h) = 0$ *for all* $h > k$.

Obviously, Assumption 2 is weaker than Assumption 1. As in Assumption 1, zero probability is assigned for all strictly upper-diagonal entries, and thus a hierarchical structure among levels is still preserved. However, the diagonal entries are not restricted to only zero. A level-k player is allowed to assign a non-zero possibility for the other players to have the same level as herself, described as *self-awareness* in Camerer et al. (2004).

In what follows, we formally introduce the *endogenous cognitive hierarchy (ECH) model*, in which Assumption 1, the overconfidence condition, is replaced by Assumption 2, to allow for self-awareness.

2.1 Endogenous Cognitive Hierarchy Model

Fix an integer $K > 0$ which prescribes the highest level considered in the model.[8] In the ECH model, we consider a sequence of mixed strategies $\sigma = (\sigma_0, \ldots, \sigma_K)$, in which for each $k \in \{1, ..., K\}$, $\sigma_k \in \Delta(S)$ is a best reply, assuming that the other players' levels are drawn from the truncation of the underlying distribution f *up to level* k. Note that a sequence of

[7] Camerer and Lovallo (1999) report on experimental evidence for overconfident behavior in the case of the market entry game. When ability is a payoff-relevant variable in a strategic interaction, evidence shows that players tend to be overconfident (see Benoît and Dubra (2011)). On the other hand, Azmat et al. (2016) find an underestimation of students' grades in the absence of feedback.

[8] We assume $f_i > 0$ for all $i \leq K$. For the truncated distribution to be well-defined, it is sufficient to assume $f_0 > 0$, but we restrict ourselves to the cases where all levels are present with a positive probability.

truncated distributions $g = (g_1, \ldots, g_K)$ is uniquely defined from f. As in previous cognitive hierarchy models, we focus on level-symmetric profiles in which all players of the same level use the same mixed strategy.

Definition 1 *A sequence of level-symmetric strategies $\sigma = (\sigma_0, \ldots, \sigma_K)$ is called **endogenous cognitive hierarchy strategies** when there exists a distribution f over $\mathbb{N}$ under which*

$$supp\left(\sigma_k\right) \subset \arg\max_{s_i \in S} \mathbb{E}_{s_{-i}}\left[u\left(s_i, s_{-i}\right)|g_k, \sigma\right], \ \forall k \in \{1, \cdots, K\},$$

where g_k is the truncated distribution induced by f such that

$$g_k\left(h\right) = \frac{f_h}{\sum_{m=0}^{k} f_m} \text{ for } h \in \{0, \cdots, k\}, \tag{ECH}$$

and the expectation over s_{-i} is drawn, for each player $j \neq i$, from a distribution

$$\gamma_k\left(\sigma\right) := \sum_{m=0}^{k} g_k\left(m\right)\sigma_m.$$

We note that Definition 1 is analogous to the definitions used in the standard level-k model and the cognitive hierarchy model. It simply replaces the assumptions on beliefs, (L) and (CH), with (ECH). Building on previous studies (as developed by Camerer et al. (2004)), we maintain the assumption that the underlying distribution f of levels follows a Poisson distribution with coefficient τ:

$$f_k = \frac{\tau^k}{k!}e^{-\tau}.$$

Note that the expectation of the distribution is τ, which thus represents the overall expected level among the players. We discuss in detail how the expected level would change in our experiments as a function of the group size, once we have presented the experimental results in the proceeding sections.

2.2 Condorcet Jury Theorem

A group of n individuals makes a binary collective decision $d \in \{-1, 1\}$. The true state of the world is also binary, $\omega \in \{-1, 1\}$, with a common prior of equal probabilities. The payoff is a function of the realized state and the collective decision as follows:

$$u(d, \omega) = \begin{cases} 0 & \text{if } d \neq \omega, \\ q & \text{if } d = \omega = 1, \\ 1-q & \text{if } d = \omega = -1, \end{cases}$$

with $q \in (0,1)$ for all individuals.[9] Each individual $i \in \{1, \cdots, n\}$ receives a private signal $t_i \in T$, distributed independently conditional on the true state ω. A collective decision is made by the majority rule. Upon receiving signal t_i, individual i casts a vote $v_i \in \{-1, 1\}$, and the collective decision is determined by the sign of $\sum_i v_i$.

We do not restrict the signal space T to be binary.[10] We assume $T \subset \mathbb{R}$ so that T is an ordered set, and we assume that the commonly known distribution satisfies the monotone likelihood ratio property, that is, the posterior distribution $\Pr[\omega = 1|t_i]$ is monotonically increasing in t_i.

A strategic Condorcet Jury Theorem claims that asymptotic efficiency is obtained among the rational individuals with homogeneous preferences and costless information acquisition, as described above. More precisely, it claims that, under the Nash equilibrium behavior, the probability of making a right decision converges to one as n goes to infinity. In the following subsection we ask whether the asymptotic, collective efficiency would be obtained under the cognitive hierarchy models in which individuals may show systematic deviations from the Nash behavior.

2.3 Asymptotic efficiency: a numerical example

In order to underline the differences implied by different behavioral assumptions, we provide numerical computation results using the game described above. The model parameters are chosen so that the game coincides exactly with the one with asymmetric payoffs in our experiments (Section 3).[11] Our aim here is to highlight through these computations the behavioral consequences caused by the self-awareness condition.

Four different behavioral specifications are compared: Nash equilibrium (NE), the standard level-k model (L), the Poisson cognitive hierarchy model (CH), and the Poisson endogenous cognitive hierarchy model (ECH).

Figure 2 shows the probability of making a correct decision as a function of the group size. Convergence to one in Nash equilibrium reflects the strategic Condorcet Jury Theorem, as a

[9]The assumption of symmetric prior is without loss of generality, since we allow the payoffs of the two types of right decisions to be heterogeneous. Although the preferences are often represented by a loss function for wrong decisions in the standard CJT models, we equivalently use a gain function for right decisions in accordance with our experiment which awards positive points to the right decisions, rather than subtract points for wrong decisions.

[10]Even though many CJT models assume the binary signal space, we believe that it is not the right assumption for many information aggregation problems. Even under the binary state space, beliefs are continuous and thus there are uncountably many ways to update the prior belief. The binary signal space can accommodate only two ways of Bayesian update, which is far from being innocuous in many situations.

[11]More precisely, we set $q = 9/11$ and $T = \{0, 1, \cdots, 10\}$, where the signal distribution follows 10 random draws from 100 cards with 60 of the right color. The logistic error term is taken as the average of estimated values: $\varepsilon = 2.01$. The average of the estimated Poisson parameter values is used for the CH and ECH models: $\tau = 5.34$. For L, we additionally assume that each player's level is drawn from the Poisson distribution with the same parameter value τ.

benchmark for the cognitive hierarchy models under our consideration. The ECH model also shows high efficiency as n increases. On the other hand, the quality of the group decision is disastrous both in the L model and in the CH model. The probability of making the correct decision converges to 0.5 in a large group, which is as bad as pure noise.

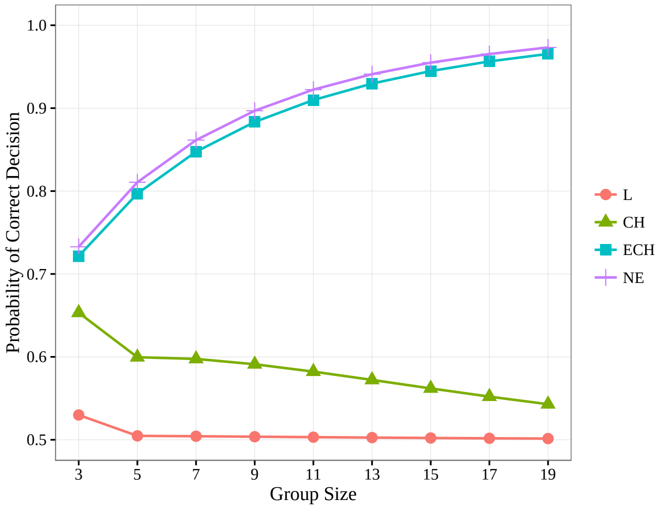


Figure 2: Probability of correct group decision as a function of group size.

The key difference in asymptotic properties between CH and ECH stems from the self-awareness condition. With the presence of self-awareness, each player in the ECH model chooses the optimal behavior which maximizes the expected utility, given that the other players may have the same level. As is shown more analytically later in Section 5, the distance of the ECH strategies from the Nash equilibrium does not diverge away even in a large group, since the strategy at each level is obtained as a best reply to the other players' strategies, which may match each other with a positive probability. A highest-level player in the ECH model is thus capable of choosing his strategy fully rationally, correcting the biases caused by the lower levels. Without self-awareness, every single player fails to hold the correct belief on the levels of all players in the group.

Figure 3 depicts level-1 and level-2 strategies under the L, CH, and ECH models, together with the Nash equilibrium strategies, as a function of the group size. In the L model, strategies hit the boundary for both level-1 and level-2. In the CH model, level-1 is the same as in the L model, while the level-2 strategy is decreasing, meaning a divergence from the Nash behavior. In the ECH model, strategies are increasing in both level-1 and level-2, in accordance with the Nash behavior. Moreover, the convergence of the ECH2 strategy is remarkably quick, so that it visibly coincides with the Nash equilibrium.

These figures demonstrate a stark contrast among the behavioral assumptions under our consideration. In the following sections, we show the results from our stylized laboratory experiment which provides evidence for our scrutiny of the models.

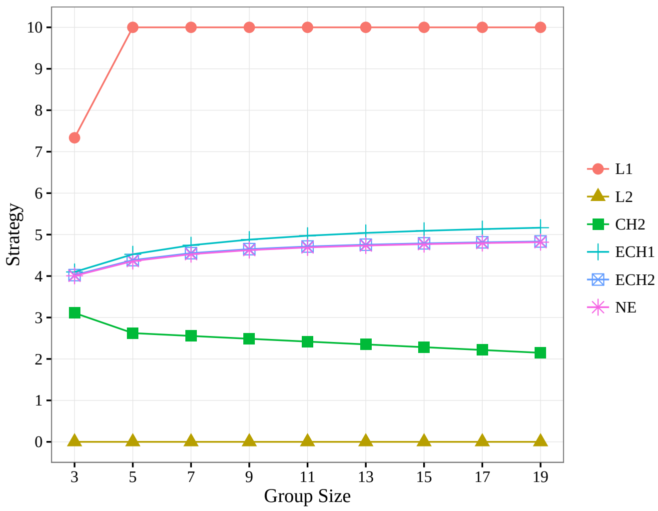


Figure 3: Level-1 and level-2 strategies in the L, CH, and ECH models with the NE strategy. Note that $CH1 = L1$.

3 Experimental Design

All of our computerized experimental sessions were held at the Ecole Polytechnique Experimental Economics Laboratory in November and December 2013.[12] In total we had 140 actual participants in 7 sessions, in addition to the pilot sessions with more than 60 participants. In each session, 20 participants took part in 4 phases (together with a short trial phase) which lasted about one hour in total. Earnings were expressed in experimental currency units (ECUs) and exchanged for cash, to be paid immediately following the session. Participants earned an average of about 21 Euros, including a default 5 Euros for participation. Complete instructions and details can be found in our online appendix.[13] The instructions pertaining to the entire experiment were read aloud at the beginning of each session. Before each phase, the changes from the previous phase were read aloud, and an information sheet providing the relevant details of the game was distributed. These sheets were exchanged with the new ones before each phase.

We employed a within-subject design where each participant played all 4 phases consecutively in a session. Each phase contained 15 periods of play, and thus each participant played for a total of 60 periods under a direct-response method. Since the question of our research relates to the strategic aspects of group decisions, our experiment was presented to participants as an abstract group decision-making task where neutral language was used to avoid any reference to voting or elections of any sort.

In the beginning of each period, the computer randomly formed groups of participants,

[12] We utilized a z-Tree program (see Fischbacher (2007)) and a website for participant registration, both developed with technical assistance from Sri Srikandan, to whom we are very much grateful.

[13] The online appendix can be found at http://sites.google.com/site/ozkesali.

of a size which was commonly known and predetermined for each phase (either $n = 5$, 9, or 19).[14] Then, a box was shown to each participant with one hundred squares (to be referred as *cards* from now on), all colorless (gray in *z-Tree*). At the same time, the unknown true color of the box for each group was determined randomly by the computer. The participants were informed that the color of the box would be either blue or yellow, with equal probability. It was openly announced that the blue box contained 60 blue and 40 yellow cards, whereas the yellow box contained 60 yellow and 40 blue cards.

After confirming their agreements to proceed to the next screen, 10 cards drawn by the computer with random locations in the box were shown to the participants, this time with the true colors. These draws were independent among all participants but were drawn from the same box for those participants who are in the same group. Having observed the 10 randomly-drawn cards, the participants were required to choose either blue or yellow by clicking on the corresponding button. Then, the decision for the group was reached by majority rule, which was resolute every time, since we only allowed an odd number as the group size and abstention was not allowed. Once all participants in a group had made their choices, the true color of the box, the number of members who chose blue, the number of members who chose yellow, and the earnings for that period were revealed on the following screen. A new period started after everyone confirmed.

In one of the four phases, the group size was set at $n = 5$ and the payoffs were symmetric. Each participant earned 500 ECUs for any correct group decision (i.e., a blue decision when true color of the box was blue, or a yellow decision when true color of the box was yellow). In the case of an incorrect decision, no award was earned. In the other three phases, each treatment differed only in the size of the groups (5, 9, or 19) where asymmetric payoffs were fixed. The correct group decision when the true color of the box was blue awarded each participant in the group with 900 ECUs, whereas the correct group decision when the true color of the box was yellow awarded them with 200 ECUs.[15] Lastly, we implemented a random-lottery incentive system where the final payoffs at each phase were determined by the payoffs from a randomly-drawn period.[16]

Let us underline that the asymmetry introduced in the remuneration was the primary source of deviation from the Nash equilibrium behavior in our experiment. As seen later, it is not surprising that an informative strategy (i.e. voting for the choice favored by the signal),

[14]In the phase with $n = 9$, two groups of 9 randomly-chosen members were formed at each period. Having 20 participants in total in each session, 2 randomly-chosen participants were 'on wait' during the period. The same method is applied in the phase with $n = 19$. A group of 19 was formed and thus one randomly-chosen participant waited.

[15]We also conducted pilot sessions with the reward of 800 ECUs and 300 ECUs. As there was less of a marked contrast in the observed deviations from Nash behavior, we decided to run the rest of the sessions with the rewards 900 ECUs and 200 ECUs.

[16]Subjects were told both verbally and through info sheets that in the case where the lottery picked a period for remuneration in which a participant has been waiting, the payoff in that phase for this participant was set at 500 ECUs, which is about the average of the winning points.

or one close to it, is employed by a large majority of the participants under symmetric awards (see histogram in Figure 4 (a)). When it is commonly known that one of the alternatives may provide a larger award, in addition to the change of the symmetric Nash equilibrium shifting towards the ex ante preferable alternative, each individual's behavior may shift, and furthermore, such shifts may be heterogeneous across individuals. Consequentially, each individual may hold heterogeneous beliefs over the strategies employed by the other individuals in the group. The accumulated effects of such heterogeneous belief formation may hamper the performance of group decision-making, which is one of our main concerns in this paper.

At the beginning of each session, as part of the instructions, participants played through two mandatory trial periods. Each session concluded after a short questionnaire. According to the anonymously-recorded questionnaire, 44% of the participants were female. The age distribution was as follows: 31% between the age of 19 and 22, 26% between 23 and 29, 14% between 30 and 39, and 29% between 40 and 67. Heterogeneity in their professions was relatively high: 46% administrative staff, 37% undergraduate students ("Polytechniciens"), 12% Ph.D. students, 1% master students, and 3% researchers. 6% of the participants had previously taken an advanced course in game theory, while 14% had taken an introductory course. 39% said that they had some notions about game theory, while 41% claimed to have no knowledge of game theory.

4 Experimental Results

In this section we present and analyze our experimental results by investigating certain behaviors of the participants at both the individual and group levels.

4.1 Cutoff Strategies

Under our experimental design, a pure strategy of an individual is a function from the realized signal to a binary vote. It is straightforward to show that the best reply of an individual, given any belief on strategies used by the other group members, is a cutoff strategy. There exists a threshold for each individual such that she votes for blue if and only if her signal induces a higher posterior probability of a blue state than the threshold. Since the posterior belief over the two states varies monotonically as a function of the number of blue cards among the 10 revealed ones, a cutoff strategy in our experiment is that each individual votes for blue when the number of observed blue cards is higher than the cutoff value, and for yellow otherwise. Special cases include voting for one of the colors regardless of the signal. The cutoff value is considered as an extreme value (either 0 or 10) for such a behavior.

What we have observed in our data is that the majority of participants used a cutoff strategy with randomization.[17] Theoretically, a cutoff strategy may involve a stochastic

[17] In the post-experiment questionnaire, a few participants expressed reasonings which seemed to have no

behavior only on the exact threshold. In the data, we observed quite a few behaviors in which randomization occurs with two or more realizations of the signal, with the degree of randomization varying monotonically in the right direction (i.e. a higher probability of voting for blue given more blue cards in the signal). We regard such a behavior as a consequence of decision-making with an error or other uncertainties which are not explicitly formalized in the model. Hence, our estimation of the cutoff strategies is derived from a logistic function, in which the cutoff strategies and the error parameter common across individuals are estimated by the maximum likelihood for each phase.

Some robustness checks are in order. During each of the 15 periods in one phase, one signal draw is realized and one voting action is taken by each individual. Therefore, a decent cutoff estimation would require accumulation of a reasonable amount of signal realizations and consequential voting actions over periods. We checked estimations over different intervals of periods and found that estimations based on the data including all 15 periods are robust. Overall, 90.3% of actions are consistent with our estimated cutoff strategies. Inconsistent actions are spread across periods, and the t-statistics of the comparison between the first and last 7 periods are 1.06, 1.32 and 1.39 for the number of inconsistent blue actions, yellow actions and the sum respectively, none statistically significant at $p > 0.10$ level. Also implied is that we do not observe learning over the periods with respect to the threshold strategies.

The average of cutoff strategy estimations is summarized in Table 1.

	500:500	900:200		
Session	$n = 5$	$n = 5$	$n = 9$	$n = 19$
1	4.99	3.92	4.18	4.30
2	4.47	3.89	3.74	3.61
3	4.87	4.09	4.27	4.02
4	5.05	3.77	3.29	3.47
5	5.14	4.40	4.41	4.40
6	4.41	4.21	4.19	4.09
7	4.90	4.02	4.06	3.95
ave.	4.83	4.04	4.02	3.98

Table 1: Phase averages of cutoff estimations in each session.

In Figure 4, histograms of the estimated cutoff values are shown for each phase ($N = 140$). Several remarks are in order. First, we see a clear shift of the distribution from the symmetric payoffs to the asymmetric ones. Most notably, for each of the group sizes of 5, 9 and 19 with asymmetric payoffs, a peak of the frequencies is clearly visible on the intervals $[0, 1)$, representing 7%, 9% and 9% of all cutoff values, respectively. As the cutoff value 0

clear connection with any Bayesian update, such as "I chose yellow when I saw three or more yellow cards aligned in a row, since I thought it was a strong sign that the box is yellow." Such a deviation from rationality is not the type which we aim to analyze here.

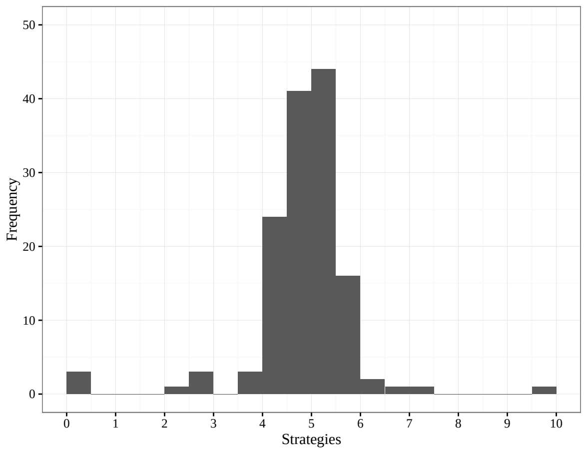
corresponds to the behavior of voting for blue regardless of the obtained signal, the presence of the peaks suggests that a certain amount of participants used the signal-independent voting strategy or at least one close to it. Second, about a half of the estimated cutoff values are included in the interval $[4, 5)$ with asymmetric payoffs. The percentages in this interval for group sizes of 5, 9, and 19 are 51%, 51%, and 66%, respectively. Note that the unbiased strategy is represented by the cutoff value of 5. A cutoff value lower than 5 corresponds to a strategy biased in favor of voting for the ex ante optimal choice, blue. Hence, our estimation implies that about one half to two thirds of participants used a cutoff strategy slightly biased towards the ex ante optimal choice. Third, no single player used a cutoff value higher than 8 with asymmetric payoffs. It is worth underlining that no signal-independent voting behavior to the other extreme direction (i.e. a cutoff value of 10, which corresponds to voting for the ex ante unfavorable alternative, yellow, regardless of the signal) is observed with asymmetric payoffs. Fourth, a non-negligible amount of voting behaviors in favor of yellow are observed, even though they are rather a minority. The frequencies of cutoff values higher than 5 are 15%, 17% and 9%, respectively, in the three phases with asymmetric payoffs.

4.2 Level-0 Strategy

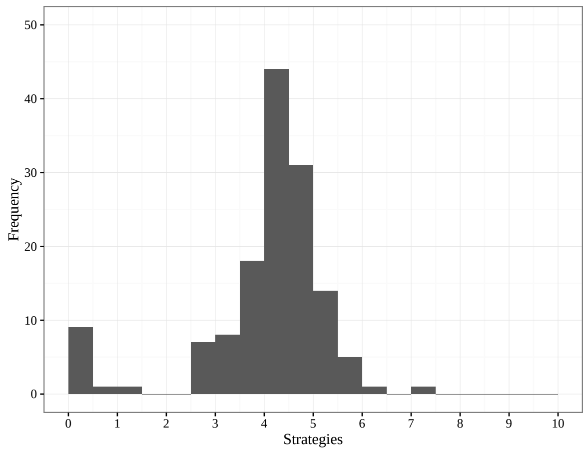
In what follows, we evaluate three cognitive hierarchy models: the standard level-k model (L), the Poisson cognitive hierarchy model (CH), and the endogenous cognitive hierarchy model (ECH), estimating the model parameters which fit best to our experimental data with asymmetric payoffs.

Before proceeding to the estimation, we briefly discuss the choice of the level-0 strategy, which can be supported by the idea of *saliency*. As discussed in Crawford and Iriberri (2007), *inter alia*, some naturally occurring landscapes that are focal across the strategy space may constitute salient non-strategic features of a game and attract naïve assessments. For instance, a strategy space represented by a real interval, say $[m, M]$, may have its minimal point m, its maximal point M, and its midpoint $\frac{m+M}{2}$ as salient locations. Furthermore, a non-strategic, level-0 player would evaluate her choices while disregarding others' strategic incentives. In our game, such a behavior corresponds to a strategy of choosing the ex ante favored choice, always voting for blue. The salient location would then be 0.

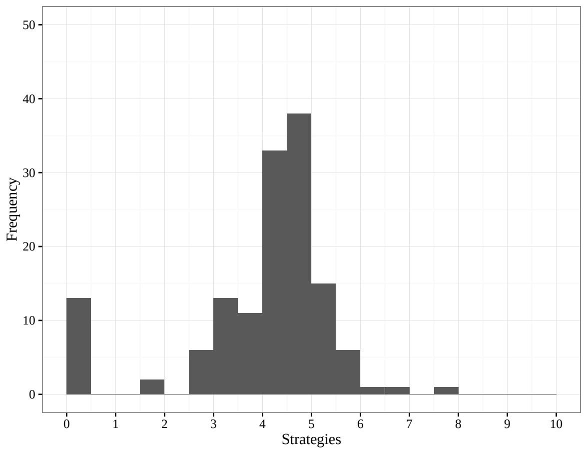
Furthermore, we look for statistical evidence to validate our choice for the level-0 strategy. First, a common choice for the level-0 strategy in the literature (see discussion in Camerer et al. (2004)) is the uniform randomization over all available pure strategies. Second, more specifically in our game, the midpoint strategy of the cutoff value 5 deserves particularly close attention, as it corresponds to the behavior of maximizing the probability of making a right choice, regardless of the winning point (and thus it is not a payoff-maximizing strategy). In our experiment, these two choices for the level-0 strategy make little difference in terms of the level-1 and level-2 strategies, since the payoff function in our game is well-approximated by a linear-quadratic function in which the best reply to a mixed strategy coincides exactly



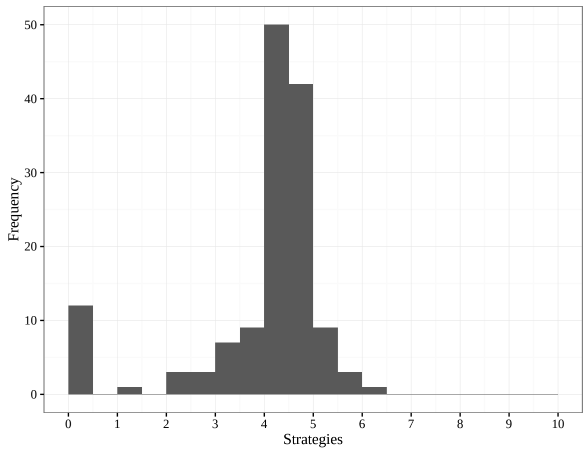
(a) $n = 5$, 500:500



(b) $n = 5$, 900:200



(c) $n = 9$, 900:200



(d) $n = 19$, 900:200

Figure 4: Histogram of the estimated cutoff strategies, 140 observations.

with the best reply to a pure strategy with the expected value of the mixed strategy (see Section 5 for more detail). Lastly, we also consider the opposite extreme behavior: voting for yellow regardless of the obtained signal.

Table 2 provides a comparison based on the maximum likelihood estimation in the ECH model with the group size $n = 5$. Both alternative level-0 choices give a significantly worse fit compared to voting for blue all the time, *i.e.,* level-0 $= 0$. The paired t-test statistic is 2.76 for the comparison of 0 and 5 (one-sided p-value $= 0.031$), while it is 2.40 for the comparison of 0 and 10 (one-sided p-value $= 0.027$). For both cases, the Wilcoxon test gives a p-value of 0.078.[18]

In what follows, we set the level-0 strategy to be signal-independent voting for the ex

[18] All reported tests in this paper are performed in R (see R Core Team (2013)), using the stats package. All paired Wilcoxon signed-rank tests performed in this paper are one-sided.

Session	level-0 = 0	level-0 = 5	level-0 = 10
1	−20.70	−31.88	−32.11
2	−34.16	−42.08	−43.29
3	−25.25	−25.32	−25.34
4	−31.65	−39.41	−39.86
5	−21.63	−20.90	−21.49
6	−22.93	−22.53	−22.56
7	−25.52	−28.72	−28.85

Table 2: Comparison of level-0 specifications by maximum log-likelihood for the ECH model with $n = 5$.

ante favorable alternative, that is, voting for blue regardless of the number of observed blue cards. In terms of the cutoff strategy, it corresponds to the cutoff value of 0.

4.3 Level-k Model

An Lk player (level-k player in the L model) maximizes her payoff holding a belief that all other players follow the $L(k-1)$ strategy. In our game, a best-reply strategy involves an incentive of correcting biases caused by all other players. Therefore, the bias caused by a Lk player with respect to the Nash equilibrium is amplified to the opposite direction as compared to the $L(k-1)$ strategy, and the degree of amplification increases as n increases. For example, for any level-0 strategy which is biased towards one of the alternatives, the $L1$ strategy is to play in an opposite direction with a magnitude increasing in n.[19]

Given the parameter values in our experiments, for all values of $n = 5$, 9, and 19, the cutoff strategy of the $L1$ voter immediately hits 10 as a response to the $L0$ play of 0, that is, to vote for yellow regardless of the signal. The same argument applies to $L2$, which should play 0. Such an oscillation continues in the L model, and a bang-bang solution is obtained perpetually as k increases.

According to the L model, only strategies around the two extremes, 0 and 10, should be observed. Since our experimental data shows a clear inconsistency with such a prediction, we do not further attempt to explain our experimental data with the L model. We underline that Battaglini et al. (2010) are the first to observe a behavioral anomaly of the level-k model in a binary-state binary-decision problem in a committee.

[19] Note that if the level-0 strategy is assumed to be voting for blue regardless of the signal, an $L1$ voter would see that she is never pivotal and would be indifferent. However, since we allow voters to make errors according to the logistic function, the probability of being pivotal is always non-zero.

4.4 CH Model

The Poisson-CH model stipulates that a CHk player (a level-k player in the CH model) maximizes her expected payoff holding a belief that other $n-1$ players have levels up to $k-1$, according to the truncated Poisson distribution. In particular, a $CH1$ player holds the belief that all other players have level 0, which is exactly the same as the belief of an $L1$ player. In our game, the $CH1$ strategy thus has the cutoff value 10 for all $n = 5$, 9 and 19. Our maximum likelihood estimation involves finding the best fit of the $CH2$ strategy which is uniquely determined by the Poisson parameter τ.

	$n=5$		$n=9$		$n=19$	
Session	τ^*	$CH2$	τ^*	$CH2$	τ^*	$CH2$
1	5.0	2.89	3.5	2.58	2.5	2.68
2	4.25	2.07	3.25	2.60	0.25	3.96
3	7.5	2.72	4.0	2.55	3.25	2.75
4	4.5	2.64	2.5	2.79	0.25	3.96
5	6.75	2.74	4.5	2.70	3.0	2.63
6	7.75	2.85	4.0	2.55	3.0	2.77
7	6.25	2.87	4.5	2.70	2.5	2.81

Table 3: Estimated Poisson coefficient τ^* and $CH2$ strategy. Level-0 strategy is 0 and $CH1$ is 10.

Table 3 shows the estimation results of the CH model.[20] An immediate observation is that the best-fitting τ values are decreasing as the group size increases. The decreases from $n = 5$ to $n = 9$ and from $n = 9$ to $n = 19$ are both significant at the $p < 0.01$ level under both the Wilcoxon test and the t-test. Given that τ is the expectation of the level drawn from the Poisson distribution, a smaller τ corresponds to a decrease in the expected cognitive levels, ceteris paribus. Therefore, a decrease in the estimated values of τ may be interpreted as evidence that the average cognitive level decreases as the group size increases, reflecting a larger cognitive load in large groups. This claim is consistent with the findings of Guarnaschelli et al. (2000), in which evidence of decreasing accuracy with larger groups is reported.

However, our interpretation of the decrease of τ under the CH model should come with a caveat. As a function of τ and n, the $CH2$ strategy is decreasing in both variables over the relevant range of our game. This is because of the nature of the $CH1$ strategy, defined as the best reply to the level-0 strategy. Since $CH1$ hits the upper bound of the strategy space, a $CH2$ player faces a large upward bias created by $CH1$ players. Then, her best reply turns in the other direction, toward the lower bound, with the bias increasing in n. Therefore, if the distribution of the observed behaviors shows little change with respect to the group size, the

[20]Level-0 strategy is fixed as 0, and the best-fitting values of τ and ε are estimated by maximum likelihood. We perform a grid search over interval $[0, 10]$ with an increment of 0.25 for both variables.

estimated τ value is expected to be smaller as n increases, implying that a lower expectation of cognitive levels in beliefs does not fully account for the decrease in the estimated values of τ.

Table 3 shows that the difference in the estimated $CH2$ strategies is not statistically significant from $n = 5$ to $n = 9$. The increase from $n = 9$ to $n = 19$ is significant at the $p < 0.10$ level by the t-test, although such an increase is not observed in the histograms of all estimated strategies in Figure 4. The key is an increasing sensitivity of the $CH2$ strategy for large n. Not only is the best-reply function in our game decreasing, but the *slope* of the best-reply function becomes steeper as n increases. Thus, the sensitivity of the best reply to the belief over the other players' strategies also increases as n increases, rendering the reliability of the $CH2$ estimation (and therefore the interpretation of the estimated values of τ) questionable for large n. As we show later in Theorem 1, such an increasing sensitivity leads to different asymptotic properties of the strategy between the CH and ECH model. We provide further discussions in Section 5.

4.5 ECH Model

	$n = 5$			$n = 9$			$n = 19$		
Session	τ^*	$ECH1$	$ECH2$	τ^*	$ECH1$	$ECH2$	τ^*	$ECH1$	$ECH2$
1	4.75	4.63	4.50	4.75	4.96	4.71	10.00	5.03	4.85
2	2.00	4.74	4.53	3.25	4.95	4.63	4.25	5.26	4.87
3	10.00	4.48	4.40	8.00	4.85	4.68	10.00	5.00	4.81
4	2.50	4.69	4.49	2.75	5.09	4.74	3.75	5.30	4.86
5	10.00	4.51	4.42	10.00	4.79	4.65	10.00	5.00	4.82
6	10.00	4.48	4.40	6.25	4.82	4.62	9.00	5.03	4.82
7	6.50	4.53	4.41	7.00	4.82	4.64	6.50	5.12	4.84

Table 4: ECH model. Level-0 strategy is 0.

Table 4 provides the best-fitting ECH model estimations for $n = 5$, 9 and 19. First, an immediate observation is that estimated the $ECH1$ and $ECH2$ strategies are both increasing in n. These differences are statistically significant at the $p < 0.01$ level under the Wilcoxon test.[21] Comparing these values with the Nash equilibrium, the increasing $ECH1$ and $ECH2$ are both in line with the increase of the Nash equilibrium with respect to n (see Figure 3). The intuition is that the Nash equilibrium should monotonically converge to the unbiased strategy (i.e. 5), since all individuals equally share the prior bias caused by the asymmetric payoffs in the symmetric Nash equilibrium, and such an individual share converges to zero

[21]The t-statistics for $ECH1$ are 13.9 and 6.94, and for $ECH2$ are 9.65 and 9.74, from $n = 5$ to $n = 9$ and from $n = 9$ to $n = 19$, respectively, implying that all differences are statistically significant at the $p < 0.01$ level.

as n increases. As we discuss later in Section 5, $ECH1$ would converge to a value opposite to the prior bias with respect to the Nash equilibrium, and $ECH2$ would approach to the Nash equilibrium (Theorem 1). Our ECH estimations from the data are consistent with those theoretical predictions.

Second, differences of the estimated τ values across $n = 5$, 9 and 19 are not statistically significant at the $p > 0.10$ level under the Wilcoxon test in the ECH model, unlike in the CH model. We also observe that the estimated τ values hit the upper bound in 7 out of 21 sessions. By definition of the Poisson distribution, such high values of τ correspond to the distribution with the probabilities heavily assigned to higher levels. Since the high ($k \geq 2$) level strategies converge to the Nash equilibrium in the ECH model (cf. Theorem 1), high values of estimated τ mean that the model predicts strategies to be distributed close to the Nash equilibrium. Indeed, all 7 sessions mentioned above coincide exactly with the ones in which only a few (or none) of the level-0 behaviors happened to be realized. As we discuss in greater detail in Section 5, one of the most remarkable properties of the ECH model is that all levels $k \geq 2$ can be classified as "sophisticated" behavior in this class of games. High estimated values of τ in some sessions in our data correspond well to this feature of the ECH model.

Moreover, high values of τ correspond to the responses obtained in our post-experiment questionnaire. In Figure 1(b), we saw that a majority of participants claimed that they assigned a ratio larger than 50% for other participants to use the same strategy, with the peak around 80%. Remember that an ECH2 player assigns a probability $g_2(2) = \frac{\tau^2/2}{1+\tau+\tau^2/2}$ for another player to have the same level. We see that $g_2(2) = 0.8$ (resp. $g_2(2) = 0.5$) corresponds to $\tau \simeq 8.9$ (resp. $\tau \simeq 2.7$). The estimated values of τ in Table 4 are consistent with the ones implied by the responses in the questionnaire.

4.6 Which model fits the best?

	$n = 5$			$n = 9$			$n = 19$		
Session	CH	NE	ECH	CH	NE	ECH	CH	NE	ECH
1	−64.67	−31.84	−20.70	−78.28	−33.47	−22.61	−81.01	−25.47	−18.77
2	−73.73	−42.80	−34.16	−69.53	−40.89	−37.60	−73.65	−42.32	−30.94
3	−60.71	−25.28	−25.25	−74.97	−27.53	−21.89	−68.61	−30.10	−30.55
4	−70.01	−39.38	−31.65	−68.83	−43.85	−34.36	−71.81	−43.17	−35.72
5	−66.21	−21.23	−21.63	−71.33	−21.73	−22.32	−78.94	−24.10	−24.91
6	−60.19	−22.50	−22.93	−69.90	−34.73	−33.41	−73.43	−32.67	−30.86
7	−60.39	−28.70	−25.52	−68.81	−32.38	−30.68	−75.04	−35.44	−29.29

Table 5: Comparison of models by maximum log-likelihood values.

A comparison of the models in terms of the data fit by the maximum log-likelihood is summarized in Table 5. First, note that the ECH model always fits our data better than the CH model in all sessions.[22] This improvement is significant at the $p < 0.01$ level under the Wilcoxon test with $3 \times 7 = 21$ paired observations. The ECH model performs better than the NE in most of the phases, and the p-value of the Wilcoxon test is 0.013.[23] Taking a closer look at the sessions in which the NE estimation performs well, we observe that they correspond exactly to those in which very few (or none) of the level-0 behaviors are realized. It is not surprising that the Nash equilibrium can explain the data well when only a small number of level-0 behaviors are observed.

We conclude that the data provides clear evidence that the endogenous cognitive hierarchy model (ECH) demonstrates a better fit as compared with the standard level-k model (L), the Poisson cognitive hierarchy model (CH), and the full rationality model (NE) in our stylized laboratory experiments.

5 Asymptotic Properties

The main findings from our experimental results in the previous section are that, in an information aggregation problem with asymmetric payoffs, clear deviations from Nash behavior are observed, and that the maximum likelihood estimation shows a better performance of the ECH model fitting the data, as compared with the CH model and the Nash equilibrium.

In this section, our objective is to provide theoretical explanations according to the asymptotic property of the games. We show below that, for a class of games in which the best-reply functions are asymptotically expanding, the distances of the CH and L strategies from the Nash equilibrium diverge away, while that of the ECH strategy is bounded (Theorem 1). On the other hand, for the games in which the best-reply functions are not asymptotically expanding, we show that the strategies in all of these models are bounded (Theorem 2). Our intent here is to provide conditions in as general a form as possible, under which the contrast between the CH and the ECH models arises. We believe that such an analytical explanation would help us better understand the role of the self-awareness, which separates these models.

For the sake of tractability, in this section we focus on linear quadratic games (Currarini and Feri (2015)), which have desirable features for our analysis. First, they are fully aggregative games (Cornes and Hartley (2012)), in which the action profile of the players affects the payoff of each player through the aggregate of the strategies of all players and her own strategy. This fits well with our current objective, as our goal here is to understand analytically how the optimal strategy of a player would be affected by the belief over the type of the other players. The fact that the strategies of the other players are explicitly visible in an aggregative form allows us to obtain straightforward insights on the relationship between the shape of the best-reply functions and the players' belief over the strategies of

[22]As mentioned in Subsection 4.3, the L model predicts a bang-bang solution, and thus the predicted values are out of comparison. Hence, we removed the L model from the table.

[23]The Nash equilibrium estimation is computed using the estimated logistic error term ε for each session.

the other players. Second, in a more technical convenience, linear quadratic games have a property such that, when a player holds a stochastic belief over the strategies of the other players, the maximizer of her expected payoff coincides with the best reply against the pure strategy which takes the expected value of the aggregate. This is because the order of the partial derivative and the expectation can be switched, as the former is linear. Then, facing heterogeneous beliefs over the other players' strategies, our analysis can simply focus on the best reply against the expectation of the beliefs, which provides us with a high tractability of the models.[24]

Consider n individuals, each of whom takes an action $x_i \in \mathbb{R}$. The payoff of player i in a linear quadratic game is a function of her own action x_i and the aggregate of the other players' actions $X_{-i} = \sum_{j \neq i} x_j$ in the following form:

$$u_i\left(x_i, X_{-i}\right) = \lambda^t x + x^t \Gamma x \tag{1}$$

where $x = \left(x_i \quad X_{-i}\right)^t$, $\lambda = \left(\lambda_x \quad \lambda_X\right)^t$ and

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}.$$

There are several games of interest which fall in the class of linear quadratic games.

Example **1** *(A simple quadratic game) Suppose* $u_i\left(x_i, x_{-i}\right) = -\left(\sum_j x_j\right)^2$. *Then,* $\lambda^t = \left(0 \quad 0\right)$ *and*

$$\Gamma = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Example **2** *(Cournot competition) Consider a Cournot competition. Suppose that the inverse demand function is linear* $P\left(Q\right) = a - bQ$*, and each firm has a constant marginal cost* c_i*. Let* q_i *be the quantity produced by firm* i *and* $Q_{-i} := \sum_{j \neq i} q_j$*. The profit of firm* i *is:*

$$\Pi_i = q_i\left(a - b\left(q_i + Q_{-i}\right) - c_i\right).$$

Then, $\lambda^t = \left(a - c_i \quad 0\right)$ *and*

$$\Gamma = \begin{pmatrix} -b & -\frac{b}{2} \\ -\frac{b}{2} & 0 \end{pmatrix}.$$

Example **3** *(Keynesian beauty contest games) Suppose that each of* n *players chooses a number* x_i *simultaneously, and each player's payoff is quadratic with respect to the distance*

[24] Obtained insights could be extended to a game with more general payoff functions, to the extent that the second-degree Taylor expansion of the payoff function with respect to the aggregate strategy provides an approximation.

between her own choice and the average of all players' choices multiplied by a constant $p \in (0,1)$. *Then,*

$$u_i(x_i, X_{-i}) = -\left(x_i - p\left(\frac{x_i + X_{-i}}{n}\right)\right)^2.$$

Then, $\lambda^t = (0 \quad 0)$ *and*

$$\Gamma = \begin{pmatrix} -\left(1-\frac{p}{n}\right)^2 & \left(1-\frac{p}{n}\right)\frac{p}{n} \\ \left(1-\frac{p}{n}\right)\frac{p}{n} & -\left(\frac{p}{n}\right)^2 \end{pmatrix}.$$

Example 4 *(Public good provision game) Suppose that each agent contributes* x_i *to a public good and the cost is quadratic:*

$$u_i(x_i, X_{-i}) = \theta_i(x_i + X_{-i}) - c_i x_i^2.$$

Then, $\lambda^t = (\theta_i \quad \theta_i)$ *and*

$$\Gamma = \begin{pmatrix} -c_i & 0 \\ 0 & 0 \end{pmatrix}.$$

We impose some regularity conditions on the linear quadratic game in the form (1). First, we assume $\gamma_{11} < 0$. This implies that u_i has a unique maximizer for any X_{-i} and thus the best-reply function is well-defined. It is straightforward to show that the game defined by (1) has a unique symmetric Nash equilibrium:

$$x^* := -\frac{\lambda_x}{2(\gamma_{11} + (n-1)\gamma_{12})}.$$

We assume that the denominator is non-zero so that the symmetric Nash equilibrium is well-defined. By applying a parallel transformation $y_i := x_i - x^*$, (1) becomes:

$$u_i = \lambda^t x + x^t \Gamma x = \lambda_y^t y + y^t \Gamma y + c$$

where $y = (y_i \quad Y_{-i})^t$, $\lambda_y = (0 \quad \lambda_Y)^t$, and λ_Y and c are independent of y. As the terms $\lambda_y^t y = \lambda_Y Y_{-i}$ and c have no strategic consequence on player i's behavior (i.e. the best-reply function of player i is unaffected), we can assume $\lambda_Y = 0$ and $c = 0$ without loss of generality. Therefore, in the following, we focus our attention on the games with the payoff function:

$$u_i = y^t \Gamma y, \tag{2}$$

with $\gamma_{11} < 0$ and $\gamma_{11} + (n-1)\gamma_{12} \neq 0$ (as in Angeletos and Pavan (2007)). Notice that there is a unique symmetric Nash equilibrium $y_i^* = 0$ for all i.

The first-order condition of player i is:

$$\frac{\partial u_i}{\partial y_i} = 2\gamma_{11} y_i + 2\gamma_{12} Y_{-i}.$$

When player i holds a stochastic belief over the strategies of the other players, the aggregate of the other players' strategies is a random variable $\tilde{Y}_{-i}$. Since the first-order condition is linear in Y_{-i} in quadratic games, the best reply against a mixed-strategy profile coincides with the best reply against the aggregate strategy which takes deterministically the expected value of the random variable:

$$BR_i\left(\tilde{Y}_{-i}\right) = -\frac{\gamma_{12}}{\gamma_{11}}\mathbb{E}\left[\tilde{Y}_{-i}\right]. \tag{3}$$

In order to describe asymptotic properties, we consider a sequence of linear quadratic games in which the number of players increases. More precisely, let $G(n) = \langle n, \mathbb{R}, (u_i^n)_{i=1}^n \rangle$ be a normal-form game with n players where the set of pure strategies is fixed as the set of real numbers $\mathbb{R}$,[25] and u_i^n is the payoff function of player i which satisfies (2). We analyze asymptotic properties of the strategies under the sequence of games $\{G(n)\}_{n=2}^{\infty}$.

Remind that the three models under our scrutiny here, L, CH and ECH, differ only in the assumption imposed on players' beliefs on the types of the other players. For each model, the strategy in each level is defined in the same way as in Definition 1. The only difference is that the frequency $g_k(h)$, assigned in the belief of a level-k player to the event in which each of the other players should be level-h, is specified by the equation (ECH) in Section 2 in the ECH model, but it is replaced by (L) (resp. (CH)) in the L (resp. CH) model. It is worth emphasizing that our results in this section do not hinge on the Poisson assumption concerning the underlying distribution f_k. We consider a sequence of level-symmetric strategies $\sigma = (\sigma_k)_{k\geq 0}$ where for each $k \geq 1$, σ_k maximizes the expected payoff under the belief $g_k(h)$.

For each game $G(n)$ and each of the three models, the level-0 strategy σ_0 is exogenously given, allowing the possibility for a mixed strategy. In order to make the comparison explicit across the models for $k \geq 1$, we add a superscript which represents the model, such as σ_k^L, σ_k^{CH}, and σ_k^{ECH}. Note that, by (3), $\sigma_k^M(n)$ are all pure strategies for $k \geq 1$ for each model $M \in \{L, CH, ECH\}$.

We assume that the following limit exists, allowing infinity:

$$A := \lim_{n\to\infty}\left|\frac{\gamma_{12}}{\gamma_{11}}n\right| \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Remember that $-\frac{\gamma_{12}}{\gamma_{11}}$ is the slope of the best-reply function (3). Since $\tilde{Y}_{-i}$ is the sum of the strategies of the other players, A is the limit of the slope of player i's best-reply function, as a function of the *average* of the other players.

[25]The assumption of the one-dimensional, unbounded strategy space allows us to obtain clear insights on the convergence and/or divergence of the strategies. In the games with a compact, one-dimensional strategy space, these insights could be inherited with some adjustments, e.g. divergence corresponds to a bang-bang corner solution.

First, we consider the case $A = \infty$. In such games, we say that the sequence of the games is asymptotically expanding, denoting the property that the sensitivity of one's strategy to the average strategy of the other players increases without a bound. We show that the strategies diverge from the Nash equilibrium in the L and the CH models, while it is bounded in the ECH model.

Theorem 1 *Consider a sequence of games $\{G(n)\}_{n=2}^{\infty}$ in which the payoff functions satisfy (2) for each n. Consider any σ_0 and let $\mu := \mathbb{E}[\sigma_0]$. Suppose $A = \infty$. For any $\mu \neq 0$, $\lim_{n\to\infty} \left|\sigma_k^L(n)\right| = \infty$ and $\lim_{n\to\infty} \left|\sigma_k^{CH}(n)\right| = \infty$, while $\lim_{n\to\infty} \left|\sigma_1^{ECH}(n)\right| < \infty$ and $\lim_{n\to\infty} \left|\sigma_k^{ECH}(n)\right| = 0$ for all $k \geq 2$.*

Among the examples described above, the sequence $\{G(n)\}_{n=2}^{\infty}$ is asymptotically expanding ($A = \infty$) in the simple quadratic game (Example 1) and in the linear Cournot competition (Example 2). A common feature of these games is that the aggregate of all players' strategies enters in each player's payoff in a way that the aggregate term does not dissipate for large n. When $A = \infty$, we show that the behaviors in the ECH model show a stark contrast with those in the L or in the CH model. The presence of the self-awareness condition thus leads to an intrinsic difference in the prediction. Moreover, we show that the ECH strategy converges to the Nash equilibrium for any level $k \geq 2$.

We interpret these properties of the ECH model as accurately capturing the phenomenon which is frequently observed in the behavioral data. In addition to a naïve strategy (level-0), we often observe sophisticated behaviors, with possibly heterogeneous degrees of sophistication. What is implied by Theorem 1 for asymptotically expanding games is that there are fundamentally three degrees of strategic sophistication: naïve (level-0), partially sophisticated (level-1), and highly sophisticated (level-2 or more). Since the strategies of level-2 or higher all converge to the Nash equilibrium, behaviors in this class of games fall in one of the following three classes asymptotically: (i) naïve strategy which does not maximize the expected payoff, (ii) level-1 strategy which maximizes the payoff but under an inconsistent belief, and (iii) fully sophisticated strategy, which maximizes the payoff under the consistent belief.

Now, consider a sequence of games which satisfies the same conditions assumed in Theorem 1, except for that in A.

Theorem 2 *Suppose $A < \infty$. For any μ, $\left|\sigma_k^L(n)\right|$, $\left|\sigma_k^{CH}(n)\right|$ and $\left|\sigma_k^{ECH}(n)\right|$ are all bounded as $n \to \infty$, for all $k \geq 1$.*

In the standard Keynesian beauty contest games (Example 3), we have:

$$A = \lim_{n\to\infty} \left| \frac{\left(1 - \frac{p}{n}\right) \frac{p}{n} n}{-\left(1 - \frac{p}{n}\right)^2} \right| = p < \infty.$$

In the games with $A < \infty$, the slope of the best-reply function is bounded as n goes to infinity. Hence, even in a game with a large number of players, the optimal strategy of a player does not diverge away. In the beauty contest games, we see that the aggregate term is relevant in each player's payoff to the degree of the *average* of all players.

Implied by Theorem 2 is that the self-awareness condition has little relevance in the games with $A < \infty$.[26] Therefore, in these games, the ECH model does not add much to the existing models of strategic thinking, such as the standard level-k model or the Poisson cognitive hierarchy model. Above all, the ECH model does not undermine the experimental success of the existing models in these games.

As far as our knowledge goes, most of the remarkable results in the literature of strategic thinking have treated the games with $A < \infty$, such as the beauty contest games. Other examples include finite games, such as market entry games, coordination games, or centipede games, to name a few. Even for the finite games, as long as the game is dominance solvable, we can consider in a large sense that the game falls into the class of $A < \infty$, since the infinite iteration of applying the best-reply function leads to a convergence to the Nash equilibrium.

It is rather straightforward to see why dominance-solvable games have been the most frequently chosen object of research in this literature. When the iterative application of the best-reply function leads to the unique Nash equilibrium, it corresponds to the high-level strategies converging to the Nash equilibrium. Such a property is often considered to fit well to the idea that the deviation from the Nash equilibrium caused by the limited cognitive ability of the low-level players dissipates as the cognitive limit goes to infinity.

Theorem 1 and Theorem 2 together do not show the superiority of the ECH model per se. What is implied by Theorem 1 is simply that the presence of the self-awareness condition *does* matter in describing asymptotic behaviors in the games with $A = \infty$. The only way to statistically test the hypothesis of self-awareness is to refer to the actual data. As we have seen in Section 4, in the case of the Condorcet Jury Theorem, our experimental data provided a clear answer in favor of the ECH model.[27] Our results do not contradict with the findings by Camerer et al. (2004), in which the estimations with the self-awareness condition do not provide better fits to the data. Since the games considered there fall in the category of $A < \infty$, our Theorem 2 implies that the presence of the self-awareness condition do not necessarily improve the fit. What we would like to point out here is that the self-awareness

[26] We do not claim that there is no difference in the predictive power among the diverse models. Papers in the literature provide comparative studies over different models, e.g. Breitmoser (2012). All we claim here is that the self-awareness condition does not make an intrinsic difference in the games with $A < \infty$ as we saw in the games with $A = \infty$.

[27] In our experiment, the results in Theorem 1 cannot be applied immediately, since the games are not exactly linear quadratic. However, it is straightforward to show that the slope of the best-reply functions diverges, inheriting the insights obtained by the case $A = \infty$ in Theorem 1. Indeed, the second-order Taylor expansion of the payoff function provides a highly suitable approximation.

condition is relevant in the games with $A = \infty$ and hence the ECH model may provide a significantly better fit to the data.

6 Conclusion

We have introduced an endogenous cognitive hierarchy model which allows self-awareness in the belief over cognitive levels of other players in an information aggregation problem of the Condorcet Jury Theorem. We found that asymptotic properties of the group decision-making, especially asymptotic efficiency, exhibit a stark contrast depending on whether the self-awareness condition is admitted in the model or not. Results from our laboratory experiment provide evidence for (i) systematic deviations from Nash equilibrium behavior, and (ii) a better fit to the data for the model with the presence of self-awareness.

Our theoretical analysis implies that the asymptotic property of the slope of the best-reply function is the key ingredient to determining whether the asymptotic properties differ between the models with and without the self-awareness condition. The increasing sensitivity of the best-reply function to the beliefs leads to a divergence of the strategies from Nash behavior in cognitive hierarchy models without self-awareness, in classes of games with asymptotically expanding best-reply functions. Since the same property is shared by the best-reply function of the information aggregation problem of the Condorcet Jury Theorem studied in this paper, the presence of the self-awareness prevents strategies from diverging away from the symmetric Nash equilibrium, and hence provides asymptotic efficiency as the group size increases.

As far as our knowledge goes, most of the games studied with cognitive hierarchy models share the property that the best-reply function is asymptotically non-expansive (i.e. $A < \infty$). In such games, we show that the presence of self-awareness has little relevance, at least asymptotically. A major example is the classical Keynesian beauty contest game. Since the presence of self-awareness makes little difference, results obtained in the existing cognitive hierarchy models without the self-awareness condition do not lose their validity, even though its explanatory power may vary as a function of model settings and parameters. Similar insights are inherited in the games with best-reply functions that are 'contractive' in a broader sense, such as dominance-solvable games, coordination games, and market entry games, among others. We hence do not expect any intrinsic improvement of the predictive power of the cognitive hierarchy model with the presence of the self-awareness condition in this class of games.

The main message of this paper is that there are games in the other class in which the presence of self-awareness matters. We think there are a lot of interesting games in this class that are worth pursuing further analysis.

Our interests go beyond the analytical results obtained in this paper. A crucial difference induced by the presence of self-awareness is the existence of 'sophisticated' players. A highest-level player in the ECH model best replies holding the correct belief concerning the distribution of the levels of other players. This is not the case in the cognitive hierarchy models without self-awareness. Players are supposed to maintain Savage rationality, but full consistency of their beliefs is not postulated even for the highest level. In that sense, simply the existence of fully-sophisticated players may suffice to convey our message. Our model consists of players who are naïve (level-0), best-replying but with inconsistent beliefs (level-1), and sophisticated (level-2). The beauty of the cognitive hierarchy models lies, we believe, in the heterogeneous degrees of belief inconsistency that can be explicitly accommodated. We would like to further understand the role of heterogeneous degrees of inconsistent beliefs under the existence of fully sophisticated players. We leave this for our future research.

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A Proofs

A.1 Proof of Theorem 1

Proof: Let

$$\alpha_n := -\frac{\gamma_{12}}{\gamma_{11}}(n-1).$$

By (3), α_n is the slope of the best-reply function with respect to the *average* of the other players' strategies. Using α_n, we can explicitly write the level-k strategy under each of the three models, L, CH, and ECH. Note that these models differ only in the belief held by each player, specified in equations (L), (CH) and (ECH) in Section 2.[28] By definition, $\lim_{n\to\infty}|\alpha_n| = A$.

In the L model, the strategy of the level-$(k+1)$ player is defined as the best reply to the level-k player. By (3) and (L),

$$\sigma_{k+1}^{L}(n) = \alpha_n \sigma_k^{L}(n) \quad \text{for } k \geq 0.$$

Hence,

$$\sigma_k^{L}(n) = (\alpha_n)^k \mu \quad \text{for } k \geq 1.$$

Therefore, for any $\mu \neq 0$ and any $k \geq 1$, we have $\lim_{n\to\infty}\left|\sigma_k^{L}(n)\right| = \infty$ if $A = \infty$, and bounded if $A < \infty$.

In the CH model, by (3) and (CH),

$$\sigma_k^{CH}(n) = \alpha_n \left(\sum_{h=0}^{k-1} g_k^{CH}(h)\, \sigma_h^{CH}(n)\right). \tag{4}$$

Especially, $\sigma_1^{CH}(n) = \alpha_n \mu$. For the sake of induction, assume that $\sigma_h^{CH}(n)$ is a polynomial of degree h with respect to α_n for $h \leq k-1$. Then, by (4), $\sigma_k^{CH}(n)$ is a polynomial of degree k with respect to α_n. Therefore, we have:

$$\sigma_k^{CH}(n) = \varphi_k(\alpha_n)\, \mu$$

where $\varphi_k(\cdot)$ is a polynomial of degree k. Therefore, for any $\mu \neq 0$ and any $k \geq 1$, we have $\lim_{n\to\infty}\left|\sigma_k^{CH}(n)\right| = \infty$ if $A = \infty$, and bounded if $A < \infty$.

In the ECH model, by (3) and (ECH),

$$\sigma_k^{ECH}(n) = \alpha_n \left(\sum_{h=0}^{k} g_k^{ECH}(h)\, \sigma_h^{ECH}(n)\right).$$

[28] In the proof, we write the (possibly mixed) strategy of a level-0 player as $\sigma_0 = \mu$, identifying it with its expected value, since expectation is the only relevant term which determines the best reply in the linear quadratic games.

Hence,

$$\sigma_k^{ECH}(n) = \frac{\alpha_n \sum_{h=0}^{k-1} g_k^{ECH}(h)\, \sigma_h^{ECH}(n)}{1 - \alpha_n g_k^{ECH}(k)}. \tag{5}$$

Now, suppose $A = \infty$. For $k = 1$,

$$\sigma_1^{ECH}(n) = \frac{\alpha_n g_1^{ECH}(0)\, \sigma_0}{1 - \alpha_n g_1^{ECH}(1)}.$$

As $\lim_{n\to\infty} |\alpha_n| = \infty$, we have $\lim_{n\to\infty} \sigma_1^{ECH} = -\frac{g_1^{ECH}(0)}{g_1^{ECH}(1)}\mu = -\frac{f_0}{f_1}\mu$.[29]

For $k = 2$, by (5),

$$\sigma_2^{ECH}(n) = \frac{\alpha_n \left(g_2^{ECH}(0)\, \sigma_0 + g_2^{ECH}(1)\, \sigma_1^{ECH}(n)\right)}{1 - \alpha_n g_2^{ECH}(2)}.$$

As $\lim_{n\to\infty} |\alpha_n| = \infty$, we have:

$$\lim_{n\to\infty} \sigma_2^{ECH}(n) = -\frac{g_2^{ECH}(0)\, \mu + g_2^{ECH}(1) \left(-\frac{f_0}{f_1}\mu\right)}{g_2^{ECH}(2)}.$$

Since $\frac{g_2^{ECH}(0)}{g_2^{ECH}(1)} = \frac{f_0}{f_1}$, we have $\lim_{n\to\infty} \sigma_2^{ECH}(n) = 0$. For $k > 2$,

$$\begin{aligned} & \lim_{n\to\infty} \sigma_k^{ECH}(n) \\ = & \lim_{n\to\infty} \left(\frac{\sum_{h=0}^{k-1} g_k^{ECH}(h)\, \sigma_h^{ECH}(n)}{\frac{1}{\alpha_n} - g_k^{ECH}(k)} \right) \\ = & -\frac{1}{g_k^{ECH}(k)} \left(g_k^{ECH}(0)\, \mu + g_k^{ECH}(1) \left(-\frac{f_0}{f_1}\mu\right) + \sum_{h=2}^{k-1} g_k^{ECH}(h) \lim_{n\to\infty} \sigma_h^{ECH}(n) \right). \end{aligned}$$

The first two terms in the bracket cancel out, since $\frac{g_k^{ECH}(0)}{g_k^{ECH}(1)} = \frac{f_0}{f_1}$. For the sake of induction, assume $\lim_{n\to\infty} \sigma_h^{ECH}(n) = 0$ for $2 \le h \le k-1$. Then, $\lim_{n\to\infty} \sigma_k^{ECH}(n) = 0$. ■

A.2 Proof of Theorem 2

Proof: Suppose $A < \infty$. Then by (5), for $k \ge 1$,

$$\lim_{n\to\infty} \sigma_k^{ECH}(n) = \frac{A \sum_{h=0}^{k-1} g_k^{ECH}(h)\, \sigma_h^{ECH}(n)}{1 - A g_k^{ECH}(k)}.$$

Especially, for $k = 1$,

$$\lim_{n\to\infty} \sigma_1^{ECH}(n) = \frac{A g_1^{ECH}(0)\, \mu}{1 - A g_1^{ECH}(1)} < \infty.$$

[29]Remember that g_k is the truncated distribution induced by f, the underlying distribution over levels defined in Definition 1.

For the sake of induction, assume $\lim_{n\to\infty}\left|\sigma_h^{ECH}(n)\right| =: s_h < \infty$ for $1 \leq h \leq k-1$. Then, for $k \geq 2$,

$$\lim_{n\to\infty}\left|\sigma_k^{ECH}(n)\right| \leq \left|\frac{A\sum_{h=0}^{k-1} g_k^{ECH}(h)\, s_h}{1 - A g_k^{ECH}(k)}\right| < \infty.$$

■