Iterative Revelation Mechanisms

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Abstract

This paper considers collective decision making processes with quasi-linear dichotomous utilities and complete information. Agents iteratively reveal information about their valuation through an ask-price process. The social planner determines an outcome and monetary transfer on a "pay-as-bid" basis. We show that the efficient allocation is achieved in a subgame perfect Nash equilibrium regardless of details of an ask-price process. Moreover, if an allocation rule is strongly monotone, there exists an equilibrium of threshold strategies, which implements the allocation rule.

The equilibrium strategy is not truthful. We also show that if an iterative revelation mechanism is ex post incentive compatible, it is an ascending-price mechanism. No incentive compatible iterative mechanism exists in typical economic problems such as public goods provision.

Keywords: iterative revelation mechanism, dichotomous preferences, ascending price, single-minded bidders
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1 Introduction

This paper considers a resource allocation problem or collective choice problem with quasi-linear utility. In such problems, we generally focus on direct mechanisms for mechanism design by the Revelation Principle. When we study an equilibrium analysis of particular allocation rules, we often formulate a kind of direct mechanism or bidding games. In many real situations, however, direct mechanism is not practical or applicable, but various type of indirect mechanisms are only available and actually used. In particular, domain of agents' preferences or types is generally very rich and it is practically hard for them to report the full preferences at once. Typical mechanisms employ small massage spaces, so that the social planner can collect only a part of information. In such a case, revelation mechanisms need multiple rounds to collect information enough to find out the efficient outcome.

Consider a public good provision game for example. Suppose the social planner or the government is going to determine whether or not to build a bridge. Although the decision depends on each agent's value for the bridge, an outcome and a corresponding cost sharing are typically determined through simple voting or elections; agents' message space is just binary or at most a handful. A decision may sometimes be made after a sequence of multiple votes.

Motivated by the above example, we consider a situation where the benevolent social planner is not able to ask agents' valuations directly, but asks simple binary questions at once. To achieve the efficient outcome, the planner repeatedly asks questions and gradually specifies their valuations. We formulate dynamic indirect mechanisms that consist of "ask-price processes," called iterative revelation mechanisms. In such a dynamic mechanism, agents observe the progress of questions and responses. Thus, agents have a wide range of strategic behavior contingent on the past history. The main question of this paper is whether the social planner achieves the efficient outcome in equilibrium. We show that the answer is yes for a limited environment in which agents have dichotomous preferences and complete information.

We consider that agents have quasi-linear dichotomous preferences. In a dichotomous preference, outcomes are classified into just two categories: good or bad outcomes. Each agent makes the same value for any outcome in a category. Thus, the type of an agent is represented by a single value with the assumption of quasi-linear utility. Although dichotomous preferences are clearly restrictive, many interesting economic problems are still included. These preferences are studied by Bogomolnaia et al. (2005), Babaioff et al. (2005), and Mishra and Roy (2013) among others.

We show that under complete information, the efficient outcome is achievable in a subgame perfect Nash equilibrium (SPNE) in an arbitrary way of an ask-price process. We consider a multi-round version of Bernheim and Whinston's (1986) menu auction. In each round, the social planner asks an agent a price, and the agent responds whether or not he wants to accept the price for achieving an outcome in his interests. A particular strategy using a simple index is an equilibrium strategy in any iterative revelation mechanism. In addition, with an additional condition of excludability, the equilibrium outcome is in the core.

Our result applies to other allocation rules. We show that if an allocation rule is strongly monotone, the allocation rule is implemented in SPNE of threshold strategies.

In an iterative revelation mechanism, each agent strategically misreports their preferences using the observed information. Therefore, our result critically relies on complete information. When we take care of the case of incomplete information and impose incentive compatibility or strategy-proofness, we show that iterative revelation mechanism must be monotonic price (ascending price). Although ascending price mechanism is necessary for incentives, it is not sufficient to attain incentive compatibility. From the literature of ascending price auctions, it is known that an ascending price mechanism is incentive compatible only in a limited environment.

1.1 Related Literature

This study is related to a growing literature on mechanism design with communication complexity. Single-object auction design under restricted message space is studied by Blumrosen et al. (2007) and Kos (2012). Blumrosen and Feldman (2013) consider implementability of information-theoretically optimal allocation rules in several settings. These studies consider static (simultaneous-move) mechanisms, whereas dynamic mechanisms reduce communication. Mechanism design with dynamic and gradual revelation is studied by Van Zandt (2007). Fadel and Segal (2009) shows that additional communication cost exists for implementing an implementable social choice rule. Mookherjee and Tsumagari (2014) provide a necessary and sufficient condition for Bayesian incentive compatibility.

In multiple-object auctions, preference elicitation is an important issue. In general, bidders have complicated valuations function in which a value is evaluated for each package of goods. Direct revelation is typically unrealistic because there are exponentially many numbers of packages of goods. Conen and Sandholm (2001) provide an algorithm for reducing communication about valuations in auction problem. Nisan and Segal (2006), however, show that the amount of communication for finding the efficient allocation is exponential in number of goods in the worst case.

Ascending auctions are typical mechanisms that reduces the amount of communication compared to direct revelation mechanisms. Ausubel (2004, 2006) provides incentive compatible ascending or dynamic auctions for multiple homogeneous or heterogeneous goods. Ausubel and Milgrom (2002) and Bikhchandani and Ostroy (2002) provide relationships between the Vickrey-Clarke-Groves outcome and core of auction game. The VCG outcome is in the core if goods are substitutes, and they propose ascending auction mechanisms that converge to the VCG outcome. These mechanisms are incentive compatible in ex post equilibrium.

From the viewpoint of implementation theory, it is known that a wide range of allocation rules are implementable in SPNE (Moore and Repullo, 1988). This study is different from the literature of subgame perfect implementation in several ways. First, the goal of this paper is to examine whether a particular class of mechanisms, which mimic collective choice procedures in practice, achieve the efficient allocation in equilibrium, whereas implementation theory tells what class of allocation rules can be implemented. Second, in our iterative revelation mechanism, each agent reports only a part of information about his own valuation. This is very different because agents are often required to report a full state or information regarding the the others' valuations in a typical mechanism for subgame perfect implementation. Instead of thinking of a limited type of mechanisms, we do not consider multiple equilibria very much.

Mishra and Roy (2013) and Babaioff et al. (2005) study implementability under quasi-linear dichotomous preferences. An important example of dichotomous preferences in multiple-object auctions is single-minded bidders, who are interested in particular packages of goods. Auctions with single-minded bidders are studied by Lehmann et al. (2002), Sano (2011), and Milgrom and Segal (2013) among others. A companion paper of mine (Sano, 2015) focuses on ascending price auctions with single-minded bidders but allows flexible pricing rule. That paper also shows that the efficient allocation is achievable in an SPNE, and that the efficiency may not be achievable when a bidder is not single-minded.

In a discrete public good game, the efficient provision of a public good is achievable in a complete information Nash equilibrium (Palfrey and Rosenthal, 1984). Dynamic or sequential contribution mechanism to public goods or a joint project is studied in several works (Admati and Perry, 1991; Varian, 1994). Admati and Perry show that the efficient provision of public good is achieved in an SPNE when monetary transfers are made only if contributions cover the production cost.

Bernheim and Whinston (1986) formulate a generalized first price auction named menu auction, and show that the efficient (and core) outcome is achieved in a Nash equilibrium. Bergemann and Valimaki (2003), in a framework of common agency, extend to a kind of dynamic environment and a two-stage auction called agenda game. In a first auction, a set of admissible outcomes is chosen by an auction, and the final outcome is auctioned in the second stage from the selected admissible outcomes. Our model is also considered as a dynamic version of the menu auction in a different way from Bergemann and Valimaki (2003).

2 The Model

Let $N \equiv \{0, 1, 2, \ldots, n\}$ be the set of all individuals. The benevolent social planner is denoted by 0, and $I = \{1, \ldots, n\}$ is the set of agents. The social planner chooses an outcome x from a finite set of alternatives X along with monetary transfer. Each agent has a quasi-linear utility function with integer-valued valuation function $u_i : X \to \mathbb{Z}$. Agent i's utility is denoted by $\pi_i = u_i(x) - p_i$, where p_i denotes the monetary transfer to the planner. Each agent has dichotomous preferences: i.e., he has a non-empty set of interests and has the same value for each outcome of his interests. Let $X_i \subset X$ be the set of *interests* of i. Agent i's valuation function takes a form of

$$u_i(x) = \begin{cases} v_i & \text{if } x \in X_i \\ 0 & \text{if } x \notin X_i \end{cases}. \tag{1}$$

Domain of values is bounded and denoted by $V_i \equiv [\underline{v}_i, \bar{v}_i]$.[1] It is possible for agents to have a negative value or cost $v_i < 0$ for outcomes of interest. The social planner's personal utility function is given by $\pi_0 = u_0(x) + \sum_I p_i$, where $u_0 : X \to \mathbb{Z}$ (or $\mathbb{R}$) is the planner's personal valuation function. We do not assume that u_0 is dichotomous, however, u_0 is commonly known to each other.

We assume that each agent's interests X_i and domains of value V_i are all commonly known to each other including the planner. The state of the world is simply the vector of values, $v = (v_1, \ldots, v_n) \in V \equiv \times_{i \in I} V_i \subset \mathbb{Z}^n$. Moreover, we assume that the state v is commonly known to each agent but only the social planner does not know the state.

The objective of the social planner is to achieve an efficient outcome. Let $X^*(v) \subseteq X$ be the set of efficient outcomes with respect to the state of the world v:

$$X^*(v) \equiv \arg\max_{x \in X} \sum_{i \in N} u_i(x).$$

Just for simplicity, the efficient outcome with respect to the true state is uniquely determined by $X^*(v) = \{x^*\}$.

The social welfare function $W(v)$ given a state of the world v is the maximum social welfare in the efficient outcome:

$$W(v) = \max_X \sum_N u_i(x). \tag{2}$$

Similarly, we also consider the social welfare function when an agent i is silent, which is denoted by

$$W_{-i}(v_{-i}) \equiv W(\underline{v}_i, v_{-i}). \tag{3}$$

[1]For simplicity of notation, a set of consecutive integers from a to b is denoted by $[a, b]$.

2.1 Iterative Revelation Mechanism

We consider an environment in which the planner is not able to directly ask each agent his value. The social planner gradually collects information about the state of the world via an "ask-price scheme" to find the efficient outcome. The planner offers prices (or subsidies) for achieving an outcome of their interests at once. When the planner offers p_i to agent i and he is willing to pay p_i (or receive $-p_i$ when $p_i < 0$) for $x \in X_i$, it implies $v_i \geq p_i$. Conversely, when agent i responds that he is not willing to pay p_i, it implies $v_i < p_i$.[2] By asking many times, the planner iteratively identifies v_i.

Formally, an *iterative revelation mechanism* $\Gamma = (\{J^t, p^t\}_t, g, p)$ is defined as follows. For each round $t = 1, 2, \ldots,$ $J^t : H^{t-1} \to 2^I$ determines a set of agents whom the planner offers a price, where $h^t \in H^t$ indicates a history at the end of round t with $H^0 = \{\emptyset\}$. The price the planner offers to agent $i \in J^t(h^{t-1})$ is determined by $p_i^t : H^{t-1} \to \mathbb{Z}$. Each agent $i \in J^t(h^{t-1})$ at t makes a report $a_i^t \in \{yes, no\}$. The mechanism terminates at T when $J^{T+1}(h^T) = \emptyset$. For each entire history $h \in H$, an outcome $g(h) \in X$ and monetary transfers to the planner $p(h) \in \mathbb{Z}^n$ are determined. A decision rule g along with an ask-price scheme $\{J^t, p^t\}$ is sometimes called a *protocol*.

Using the responses by an agent up to t, the planner identifies the set of values $V_i(h^t)$. Any value $\tilde{v}_i \in V_i(h^t)$ is consistent with all the responses from i. Note that $V_i(h^t)$ should have the form of $V_i(h^t) = [\underline{v}_i(h^t), \bar{v}_i(h^t)]$ with $V_i(h^0) = V_i$. Let $V(h^t) \equiv \times_{i \in I} V_i(h^t)$ and $V(h) = V(h^T)$, which indicate set of possible states consistent with current and overall histories, respectively.

An ask-price scheme $\{J^t, p^t\}_t$ is arbitrary, however, the decision function g and payment rule p are specified. We put several requirements for revelation mechanisms.

Assumption 0 (Informative Query) For any t, any h^{t-1}, and any $i \in J^t(h^{t-1})$, $p_i^t \in V_i(h^{t-1}) \setminus \{\underline{v}_i(h^{t-1})\}$.

Informative query is natural and it is without loss of generality.

[2] Revealed preference only implies $v_i \leq p_i$ when an agent rejects the offer. However, we assume rejection is strict in order that the planner certainly identifies the efficient outcome.

Assumption 1 (Efficiency) For any profile of values $\tilde{v} \in V(h)$, the decision function $g(h) \in X^*(\tilde{v})$.

The Efficiency indirectly restricts a protocol $\{J^t, p^t\}_t$ and determines a necessary condition for termination of a process. The social planner needs to continue the ask-price process until the efficient outcome is specified. At the same time, the planner does not have to uniquely determine a state from responses. Typically, a mechanism does not require the full revelation of agents' values, which implies economy of information transmission and preserving privacy. It is relaxed when considering limited communication.

Assumption 2 (Pay-as-bid) The payment rule is determined by

$$p_i(h) = \begin{cases} \underline{v}_i(h) & \text{if } g(h) \in X_i \\ 0 & \text{if } g(h) \notin X_i \end{cases}.$$

If the final outcome is of i's interest, he needs to pay the minimum value in his revealed set. Equivalently, each agent pays the maximum amount that he said "yes" in the mechanism. Thus, this payment rule is a dynamic version of "pay-as-bid" mechanism and can be viewed as an extension of Bernheim and Whinston's (1986) menu auction.

Assumption 2 is restrictive from mechanism design point of view, which considers payment scheme that incentivizes agents. A justification for this assumption is that the social planner has to minimize communication to find the efficient outcome. Fadel and Segal (2009) show that there exists additional communication to calculate an ex post incentive compatible payment rule. If the communication cost may be large, the social planner might give up to collect additional information after the efficient outcome is identified. Then, she might have to employ the simplest payment rule that is individually rational.

Agents observe all the past information. A (pure) strategy $\sigma_i \in \Sigma_i$ of agent i in an iterative revelation mechanism Γ is a profile of actions $a_i(z) \in \{yes, no\}$ at every decision node $z = (h^{t-1}, p_i^t)$ such that $i \in J^t(h^{t-1})$. The equilibrium concept is a subgame perfect Nash equilibrium (SPNE).

3 Result

To simplify the analysis, we mainly focus on a special case in which for all t and all h^{t-1}, $|J^t(h^{t-1})| = 1$. It is easy to arrange the result to the case of general number of $|J^t|$, which is described later.

To state the main result, we introduce some notions and an additional assumption on tie-breaking rule. We introduce the notion of *target values*.

Definition 1 The *target value* v_i^t of i at round t is defined by

$$v_i^t \equiv \begin{cases} v_i & \text{if } v_i \in V_i(h^t) \\ \bar{v}_i(h^t) & \text{if } v_i > \bar{v}_i(h^t) \\ \underline{v}_i(h^t) & \text{if } v_i < \underline{v}_i(h^t) \end{cases}. \tag{4}$$

Since agents observe all the past actions, they know the target value of each other. Note that $v_i^t \in V_i(h^t)$.

Because we consider complete information, we need to take care of ties in equilibrium. We impose the following pre-determined tie-breaking for simplicity.

Assumption 3 The personal utility for the planner u_0 is real-valued, and for any x, $x' \in X$, the difference $u_0(x) - u_0(x')$ is not integral; i.e., for any state v, $X^*(v)$ is singleton.

An alternative tie-breaking rule uses target values.

Assumption 4 For each $x \in \cap_{\tilde{v} \in V(h)} X^*(\tilde{v})$, define

$$\tau(x) \equiv \min\{\tau | \forall s \geq \tau, x \in X^*(v^s)\}.$$

Then the decision function satisfies $g(h) \in \arg\min \tau(x)$.

This assumption does not mean that the social planner breaks ties in this manner, but is imposed just for analysis. Assumption 4 is important when the planner's personal utility u_0 takes integer values because a tie case is a general phenomenon and a tie arises in equilibrium. Roughly speaking, Assumption 4 requires that the efficient outcome is favored and chosen in equilibrium tie-breaking. At the same time, the efficient outcome is not favored at the off-equilibrium path tie-breaking.

Definition 2 An agent i's *marginal contribution* under v is

$$M_i(v) \equiv W(v) - W_{-i}(v_{-i}). \tag{5}$$

In addition, we use the notation $M_i^t = M_i(v_i, v_{-i}^t)$ for the marginal contribution under the target values.

The main theorem of this paper is stated as follows. Notably, a specific strategy forms an SPNE for *every* iterative revelation mechanism.

Theorem 1 *Suppose that Assumptions 1, 2, and 3 (or 4) hold. In every iterative revelation mechanism, there exists an SPNE in which each agent i takes the following strategy:*

$$a_i(h^{t-1}, p_i^t) = \begin{cases} yes & \text{if } p_i^t \leq \lceil v_i - M_i^{t-1} \rceil \\ no & \text{otherwise} \end{cases}. \tag{6}$$

The equilibrium outcome is efficient.

All proofs are in the Appendix.

An intuition of Theorem 1 is given as follows. By definition of target values, each agent j is willing to pay at most v_j^{t-1} given the observed actions. Suppose that agents would foresee that the equilibrium outcome is efficient under the current information, $x \in X^*(v^{t-1})$, and that agent i makes an action at round t and $M_i^{t-1} > 0$. A positive marginal contribution indicates the expected outcome $X^*(v^{t-1})$ are of i's interests. By definition of marginal contribution, i can make $X^*(v^t)$ remain the same by revealing $v_i^t \geq v_i - M_i^{t-1}$. Thus, i needs to say yes for any price $p_i^t \leq v_i - M_i^{t-1}$ and say no for any higher price to reduce the payment.

Agents tell lies in equilibrium. They report no for a price over $v_i - M_i^{t-1}$ to reduce their payments. An interesting property is that the equilibrium strategy is independent of the detail of an ask-price process although agents use all the past responses and true values of the other agents. Particularly, the strategy (6) constitutes an SPNE even if agents do not know whom and how much the social planner will ask in the future.

3.1 Core

In addition to the efficiency result of Theorem 1, the equilibrium outcome is located in the core with respect to the true state of the world. To consider the core property of the allocation problem, we additionally impose the excludability on the feasible outcomes X.

Definition 3 A set X of alternatives satisfies the *excludability* if the following holds; For each subset of agents $J \subset I$, there exists a set of feasible alternatives, denoted by $X(J) \subset X$, achievable for $J \cup \{0\}$, and for each $i \notin J$, $X_i \cap X(J) = \emptyset$.

The excludability holds for various environment such as private goods allocation (auction) problems and club goods. The excludability implies that any coalition including the social planner can exclude agents outside the coalition. The coalition value of J is defined as the maximum social welfare generated by J. The coalition value function or characteristic function) $\omega : 2^N \times V \to \mathbb{R}$ is defined in a standard manner by

$$\omega(J; v) = \begin{cases} \max_{X(J_{-0})} \sum_{j \in J} u_j(x) & \text{if } 0 \in J \\ 0 & \text{if } 0 \notin J \end{cases}. \tag{7}$$

In this formulation, we assume that the social planner has the authority to implement an outcome x and thus any coalition by agents only generates zero value. A payoff profile is in the core if it is efficient, individually rational, and not blocked by any coalition. The core given a state of the world is denoted by

$$C(\omega, v) = \{\pi \in \mathbb{R}^{n+1} | (\forall i \in N) \pi_i \geq 0, (\forall J \subseteq N) \sum_{j \in J} \pi_j \geq \omega(J, v)\}.$$

The following theorem states that the payoff profile associated with Theorem 1 is in the core. In the above definition of core, the social planner is also included as a player. This implies that in public goods provision, for example, the government's revenue from agents covers the provision cost. Similarly, in a double auction with single-unit demand and supply, a Walrasian auctioneer has no budget deficit in equilibrium.

Theorem 2 *Suppose that X satisfies the excludability. The equilibrium outcome associated with Theorem 1 is in the core with respect to the true state of the world.*

3.2 When $|J^t| \geq 2$

The above result is easily arranged to the case where the social planner can simultaneously ask prices to many agents. The equilibrium strategy for such cases is basically the same as Theorem 1, but some coordination of responses is necessary to avoid a kind of tragedy, in which many agents simultaneously say no and their contributions fall short to the efficient outcome. Thus, a coordination problem should be solved in each round.

Corollary 1 *Suppose $|J^t|$ can be more than 1. In every iterative revelation mechanism, there exists a following SPNE: for any history h^{t-1} and (J^t, p^t), find a maximal set $J_n^t \subseteq \bar{J}^t \equiv \{i \in J^t | v_i - M_i^{t-1} < p_i^t \leq v_i\}$ satisfying the following conditions:*

1. *Agent i says no if $i \in J_n^t$ or if $p_i^t > v_i$,*
2. *Agent i says yes if $i \in \bar{J}^t \setminus J_n^t$ or if $p_i^t \leq \lceil v_i - M_i^{t-1} \rceil$,*
3. *$X^*(v^{t-1}) \subseteq X^*(v^t)$.*

It is obvious that our result is not affected even when the social planner can ask multiple prices to an agent. For example, we have the same equilibrium when the planner ask a set of prices $\{p_{i1}^t, \ldots, p_{ik}^t\}$ to agent i in a round. Agent i responds with the maximum price that he accepts (or says yes) among the offered prices. Note that a direct revelation of a value is equivalent to asking a set of prices $\{\underline{v}_i, \underline{v}_i + 1, \ldots, \bar{v}_i\}$. Then, Theorem 1 shows that in a direct revelation pay-as-bid mechanism, there exists an efficient Nash equilibrium as a corollary, which is a known result.

Corollary 2 (Bernheim and Whinston, 1986) *In a one-shot (static) menu auction, there exists an efficient Nash equilibrium, in which each agent submits a bid of $b_i = \lceil v_i - M_i(v_i, b_{-i}) \rceil$.*

3.3 Limited Communication

We have considered the efficient allocation rule and assumed that the social planner can ask as many questions as possible until the efficient outcome is specified. However, the social planner often has communication constraints that prevent from so many questions. For example, the planner may be able to ask only a limited number of questions to each agent or in total. Alternatively, it may take a cost to ask an agent a question. In the presence of communication constraints, the social planner may not or cannot specify the efficient outcome but carefully designs a protocol that finds an approximately efficient outcome. In a single-object auction problem, Blumrosen et al. (2007) and Kos (2012) examine an informationally efficient protocol that maximizes the social welfare given a limited number of actions.[3] However, it is quite challenging to answer the question how we can maximize social welfare in a more general allocation problem with limited communication.

In the current study we do not consider any specific form of communication constraints or algorithmic methods that increase social welfare. Instead, we drop Assumption 1 and extend the previous result to a wide range of allocation rules consistent with limited communication.

For any protocol $(\{J^t, p^t\}, g)$, let $\phi : V \to H$ be a mapping from a state to the associated history, assuming sincere reporting by agents. That is, ϕ is an inverse mapping of $V(h)$. Then, a direct allocation rule $f : V \to X$ associated with $(\{J^t, p^t\}, g)$ is formulated as $f = g \circ \phi$. The efficient allocation rule is denoted by f^* and

$$f^*(v) \in \arg\max_X \sum_{i \in N} u_i(x).$$

An allocation rule f is said to be *monotone* if for all $i \in I$, all $v \in V$, and all $\tilde{v}_i > v_i$,

$$f(v) \in X_i \Rightarrow f(\tilde{v}_i, v_{-i}) \in X_i.$$

Moreover, an allocation rule f is said to be *strongly monotone* if for all $i \in I$, all $v \in V$, and all $\tilde{v}_i > v_i$

$$f(v) \in X_i \Rightarrow f(\tilde{v}_i, v_{-i}) = f(v).$$

An iterative revelation mechanism Γ is said to be *(strongly) monotone* if the associated direct allocation rule f is (strongly) monotone. Given a strongly monotone

[3]See Blumrosen and Feldman (2013) as well.

allocation rule f, the *critical value* $c_i^f(v_{-i})$ is defined by

$$c_i^f(v_{-i}) = \begin{cases} \min\{\tilde{v}_i \in V_i | f(\tilde{v}_i, v_{-i}) \in X_i\} & \text{if exists} \\ \bar{v}_i + 1 & \text{otherwise} \end{cases}. \tag{8}$$

The following theorem states that if an iterative revelation mechanism Γ is strongly monotone, there exists an SPNE in which each agent takes a threshold strategy, and the allocation rule f is implemented in the equilibrium.

Theorem 3 *Suppose that Assumptions 2 and 3 (or 4) hold. If an iterative revelation mechanism Γ is strongly monotone, then there exists an SPNE in which each agent i takes the following strategy:*

$$a_i(h^{t-1}, p_i^t) = \begin{cases} yes & \textit{if } p_i^t \leq c_i^f(v_{-i}^{t-1}) \\ no & \textit{otherwise} \end{cases}. \tag{9}$$

The SPNE implements f.

Notice that the efficient allocation rule f^* is strongly monotone. The critical value for the efficient allocation rule is given by $c_i^f(v_{-i}) = \lceil v_i - M_i(v) \rceil$, thus that Theorem 3 is a generalization of Theorem 1.

4 Incentive Compatibility

Theorem 1 shows that any iterative revelation mechanism achieves efficiency in an SPNE but each agent may strategically misreport his preferences. Moreover, each agent uses the others' true values, so the result critically relies on the complete information. We are naturally interested in incentive compatible mechanism. We here consider ex post incentive compatibility.

Definition 4 An iterative revelation mechanism is *ex post incentive compatible* if for any state v and any history, sincere reporting is an ex post equilibrium:

$$a_i(h^{t-1}, p_i^t) = \begin{cases} yes & \text{if } p_i^t \leq v_i \\ no & \text{otherwise} \end{cases}.$$

In addition to the ex post incentive compatibility, we also impose tightness for a mechanism.

Definition 5 An iterative revelation mechanism associated with f is *tight* if for all t and all h^t, it holds that $i \notin J^{t+1}(h^t)$ if $\forall \tilde{v} \in V(h^t)$, $f(\tilde{v}) \notin X_i$ or if $\forall \tilde{v} \in V(h^t), f(\tilde{v}) \in X_i$. An iterative revelation mechanism associated with f is *weakly tight* if for all t and all h^t, it holds that $i \notin J^{t+1}(h^t)$ if $\forall \tilde{v} \in V(h^t)$, $f(\tilde{v}) \notin X_i$.

Notice that if $\forall \tilde{v} \in V(h^t)$, $X^*(\tilde{v}) \cap X_i = \emptyset$ or $X^*(\tilde{v}) \subseteq X_i$, the social planner does not have to specify agent i's value any further since i's value is no longer critical for determining the efficient outcome. A smart social planner would only ask questions relevant to determining the efficient outcome.

Theorem 4 *If an iterative revelation mechanism is monotone, ex post incentive compatible, and weakly tight, and satisfies Assumptions 0 and 2, then it is an ascending-price mechanism: $p_i^s < p_i^t$ for all i and all $s < t$.*

As a corollary of Theorem 4, the English auction is a unique iterative mechanism that is efficient and ex post incentive compatible in a single-object auction.

Corollary 3 *In a single-object auction environment, the English auction is a unique iterative revelation mechanism that is efficient, ex post incentive compatible, and tight, and has the pay-as-bid monetary transfer.*

Theorem 4 provides several findings along with a knowledge from ascending price auctions theory. When we focus on "ascending-price auctions," the following result is known.

Proposition 1 (Ausubel and Milgrom, 2002; etc.) *There exists an efficient and incentive compatible ascending-price mechanism if for any state v, coalition value function is submodular: for any $J' \subseteq J$,*

$$\omega(J' \cup i; v) - \omega(J'; v) \geq \omega(J \cup i; v) - \omega(J; v).$$

It is easily verified that in many environments of dichotomous preferences, the coalition value function is not submodular. Roughly speaking, the submodular condition corresponds to substitutes condition, whereas dichotomous preferences typically exhibit complementarity. In a public good problem, for example, each agent's marginal contribution to the economy $\omega(J) - \omega(J_{-i})$ is non-decreasing in the size of the economy J. Thus, there exists no incentive compatible iterative revelation mechanism in general.

The negative result depends on both the pay-as-bid pricing rule and tightness. Fadel and Segal (2009) show that even when we drop Assumption 2, an efficient protocol may not be implemented in ex post equilibrium. This is because an ex post incentive compatible payment rule is equivalent to that of the Vickrey-Clarke-Groves mechanism, whereas an efficient protocol may not collect information enough to calculate the Vickrey-Clarke-Groves payments. However, Fadel and Segal also show that every efficient allocation rule (protocol) is implemented in Bayesian Nash equilibrium using a payment rule similar to the "AGV mechanism." In the multi-object auction model, Mishra and Parkes (2007) propose an ex post incentive compatible ascending auction for general valuations, which does not satisfy pay-as-bid rule or tightness.

5 Conclusion

Iterative revelation mechanism is a class of indirect mechanisms in which the social planner iteratively asks a binary choice question and identifies the state of the world after a sequence of questions and responses. With complete information and dichotomous preferences, the efficient outcome is achievable in an SPNE regardless of the way of asking prices. However, agents strategically misreport their values contingent on the others' past responses. From the perspective of incentive compatibility, price asking scheme must be monotonic. However, an incentive compatible iterative revelation mechanism often fails to exist in general.

We have assumed that the planner has knowledge about agents' interests. It is an open question when the planner does not know agents' interests and needs to ask their interests too. Another extension is relaxing the assumption of dichotomous preferences. Knowledge from combinatorial auctions tells us that there exists an incentive compatible ascending combinatorial auctions when goods are substitutes. Unfortunately, however, Sano (2015) shows that in the presence of complementarities, an ascending combinatorial auction does not achieve the efficiency in SPNE.

A Proofs

In the proofs, we use the following fact that is easily verified: if $X^*(v) \cap X_i \neq \emptyset$, then $X^*(\tilde{v}_i, v_{-i}) \subseteq X_i$ for all $\tilde{v}_i > v_i$. In addition, the set of agents whose interests include x is denoted by $N(x) \equiv \{i \in I | x \in X_i\}$.

A.1 Proof of Theorem 1

We give the proof under Assumption 4 here. Proof under Assumption 3 is similar and omitted. Under Assumption 4, u_0 is integer, and $M_i(v)$ is integer for all v too.

For each subgame starting from a decision node $z = (h^{t-1}, p_i^t)$, let $I(z)$ be the set of all agents who make actions in the subgame. In addition, let $\tau^{\leq t}(x) = \min\{\tau | (t \geq \forall s \geq \tau), x \in X^*(v^s)\}$, and define $X^*(h^t) \equiv \arg\min_{x \in X^*(v^t)} \tau^{\leq t}(x) \subseteq X^*(v^t)$. An outcome in $X^*(h^t)$ is denoted by $x^t \in X^*(h^t)$. We prove by backward induction.

Step 1. $|I(z)| = 1$; That is, agent i is a unique agent who makes actions in the subgame from z.

Step 1.1. Suppose that $x^{t-1} \notin X_i$ for all $x^{t-1} \in X^*(h^{t-1})$. It implies $W(v^{t-1}) = W_{-i}(v^{t-1})$ and $M_i^{t-1} = 0$.

Suppose $v_i < \underline{v}_i(h^{t-1}) = v_i^{t-1}$. Further suppose that there is a path achieving an outcome $\tilde{x} \in X_i$. (If not, any action profile is optimal in the subgame and we have done.) Then, agent i's payment must be $p_i = \underline{v}_i(h^T) > \underline{v}_i(h^{t-1}) > v_i$. Thus, the resulting payoff is $v_i - p_i < 0$. When agent i plays the proposed strategy, he reports as if he has a value $v_i^{t-1} = \underline{v}_i(h^{t-1})$. Hence, $\underline{v}_i(h^{t-1}) \in V_i(h^s)$ for all $s \geq t$ and $v^t = v^{t-1} = (\underline{v}_i(h^{t-1}), v_{-i}^{t-1}) \in V(h)$. By the efficiency of the mechanism, the resulting outcome is $g(h) = x \notin X_i$ and i's payoff is 0.

Suppose $v_i \in V_i(h^{t-1})$ and $v_i = v_i^{t-1}$. Further suppose that there is a path achieving an outcome $\tilde{x} \in X_i$. Let $\tilde{V}_i(h) = [\underline{\tilde{v}}_i(h), \bar{\tilde{v}}_i]$ be the revealed set at the termination of such a path. Since $V_j(h) = V_j(h^{t-1})$ for all $j \neq i$, $\tilde{x} \in X^*(\underline{\tilde{v}}_i(h), v_{-i}^{t-1})$. By $\tilde{x} \notin X^*(v_i, v_{-i}^{t-1})$, we have $v_i < \underline{\tilde{v}}_i(h) = p_i(h)$, and agent i results in a negative payoff. When agent i plays the proposed strategy, he results in zero payoff by the same argument in the previous paragraph.

Suppose $v_i > \bar{v}_i(h^{t-1}) = v_i^{t-1}$. Since $x^{t-1} \notin X_i$ and $V_i(h) \subseteq V_i(h^{t-1})$, the final outcome must be x^{t-1} regardless of i's actions, thus we have done.

Step 1.2. Suppose that there exists $x^{t-1} \in X_i$ and $M_i^{t-1} \geq 0$.

Suppose $v_i < \underline{v}_i(h^{t-1}) = v_i^{t-1}$. Since $x^{t-1} \in X^*(\underline{v}_i(h^{t-1}), v_{-i}^{t-1})$, we have $x^{t-1} \in X^*(\tilde{v}_i, v_{-i}^{t-1})$ for all $\tilde{v}_i \in V_i(h)$ regardless of i's actions. For payment minimization, it is optimal to report as if he has a value of $\underline{v}_i(h^{t-1})$. This is consistent with the proposed strategy.

Suppose $v_i \geq \underline{v}_i(h^{t-1})$ and $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$. Consider the terminal revealed set $\tilde{V}_i(h)$ of i satisfying $g(h) \in X_i$. For all $\tilde{v}_i \in \tilde{V}_i(h)$, $(\tilde{v}_i, v_{-i}^{t-1}) \in V(h)$ and

$$\tilde{v}_i + \sum_{j \in N(g(h))} v_j^{t-1} + u_0(g(h)) \geq W_{-i}(v_{-i}^{t-1}). \tag{10}$$

Inequality (10) is equivalent to $\tilde{v}_i \geq v_i - M_i^{t-1}$. Hence, $\underline{\tilde{v}}_i(h) \geq v_i - M_i^{t-1}$ is necessary and sufficient for $g(h) \in X_i$. When $\underline{v}_i(h^{t-1}) \geq v_i - M_i^{t-1}$, we have $g(h) \in X_i$ regardless of i's actions. Hence, the proposed strategy is optimal by the payment minimization. When $\underline{v}_i(h^{t-1}) < v_i - M_i^{t-1}$, the final outcome is $x^{t-1} \in X_i$ or some $\tilde{x} \notin X_i$. The resulting payoff for $\tilde{x}$ is 0. When agent i plays the proposed strategy, $v_i - M_i^{t-1} \in V_i(h)$ and thus $g(h) = x^{t-1}$. In addition, agent i says no at $s \geq t$ for any $p_i^s > v_i - M_i^{t-1}$ and thus $p_i(h) \leq v_i - M_i^{t-1}$. Hence, i's payoff is $v_i - p_i(h) \geq M_i^{t-1} \geq 0$. (On the other hand, if under the proposed strategy $p_i(h) < v_i^{t-1} - M_i^{t-1}$, then it holds that $\underline{v}_i(h^T) + \sum_{N(g(h))} v_j^{t-1} + u_0(g(h)) < W_{-i}(v_{-i}^{t-1})$, which is a contradiction. Thus $p_i(h) = v_i - M_i^{t-1}$.)

Therefore, the proposed strategy is optimal. Under the strategy, it holds that $x^{t-1} \in X^*(v^t)$ for all t and $g(h) \in X^*(h^{t-1})$ in equilibrium by the tie-breaking rule.

Step 2. $|I(z)| = m \geq 2$. In addition, the proposed strategy consists an SPNE for all $m' \leq m-1$ and $|I(z)| = m'$. In addition, after each action of agent i at round t, $|I(h^t, p_j^{t+1})| = m - 1$.

Step 2.1. Suppose that $x^{t-1} \notin X_i$ for all $x^{t-1} \in X^*(h^{t-1})$. It implies $W(v^{t-1}) = W_{-i}(v^{t-1})$ and $M_i^{t-1} = 0$.

Suppose that after agent i's action at round t, $x^t \in X_i$. By the tie-breaking assumption, it implies for all $x \in X^*(v^{t-1})$, $x \notin X^*(v^t)$. Then, induction hypothesis implies that $g(h) = x^t$. There are two cases for such an event arises:

1. $v_i^t = \underline{v}_i(h^t) > v_i^{t-1} = \underline{v}_i(h^{t-1}) > v_i$.

2. $v_i^t = \underline{v}_i(h^t) > v_i^{t-1} = v_i$.

In each case, $p_i^t = v_i^t = \underline{v}_i(h^t) > v_i$. By induction hypothesis, agent i earns $v_i - p_i(h) \leq v_i - \underline{v}_i(h^t) < 0$. On the other hand, if he plays the proposed strategy and reports no for $p_i^t > v_i$, it holds $v_i^t = v_i^{t-1}$ and $X^*(v^t) = X^*(v^{t-1})$, thus that i earns zero payoff by induction hypothesis. Therefore, the proposed strategy is optimal and $X^*(v^t) = X^*(v^{t-1})$.

Step 2.2. Suppose that there exists $x^{t-1} \in X_i$.

Step 2.2.1. Suppose $p_i^t \leq v_i - M_i^{t-1}$. Under the proposed strategy, agent i reports yes. This implies $v_i^t = v_i^{t-1}$. Note that $v_j^t = v_j^{t-1}$ for all $j \neq i$, so

$$W(v^t) = W(v^{t-1}) \geq W_{-i}(v_{-i}^{t-1}) = W_{-i}(v_{-i}^t). \tag{11}$$

Hence, $x^{t-1} \in X^*(v^t)$ and by induction hypothesis x^{t-1} is the final outcome in equilibrium. Agent i earns a payoff of $v_i - p_i(h) \geq v_i - p_i^t \geq 0$.

If agent i reports no, then it implies $\bar{v}_i(h^t) = p_i^t - 1$, (and by the induction hypothesis, $\bar{v}_i(h) = p_i^t - 1$). Since

$$\begin{aligned} p_i^t - 1 + \sum_{j \in N(x^{t-1}) \setminus \{i\}} v_j^t + u_0(x^{t-1}) &< v_i - M_i^{t-1} + \sum_{j \in N(x^{t-1}) \setminus \{i\}} v_j^t + u_0(x^{t-1}) \\ &= W_{-i}(v^t), \end{aligned} \tag{12}$$

we have

$$x^{t-1} \notin X^*(v^t). \tag{13}$$

By induction hypothesis, the final outcome in equilibrium is in $X^*(v^t)$, thus that agent i earns zero payoff. Therefore, the proposed strategy is optimal, and it holds $X^*(v^{t-1}) \subseteq X^*(v^t)$

Step 2.2.2. Suppose $p_i^t > v_i - M_i^{t-1}$. In the proposed strategy, agent i reports no. It is obviously optimal to report no when $p_i^t > v_i$. Suppose $v_i - M_i^{t-1} < p_i^t \leq v_i^{t-1}$, where $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$. When agent i reports yes, we have $v^t = v^{t-1}$ and $x^{t-1} \in X_i$ is achieved by induction hypothesis. Agent i's payoff is $v_i - \underline{v}_i(h) \leq v_i - p_i^t$. When agent i reports no as in the proposed strategy, $v_i^t = \bar{v}_i(h^t) = p_i^t - 1$ and $v_{-i}^t = v_{-i}^{t-1}$.

Then for $x^{t-1} \in X_i$,

$$\begin{aligned}\sum_{j\in N(x^{t-1})} v_j^t + u_0(x^{t-1}) &= p_i^t - 1 + \sum_{j\in N(x^{t-1})\setminus\{i\}} v_j^{t-1} + u_0(x^{t-1}) \\ &\geq v_i - M_i^{t-1} + \sum_{j\in N(x^{t-1})\setminus\{i\}} v_j^{t-1} + u_0(x^{t-1}) \qquad (14) \\ &= W(v_{-i}^{t-1}) = W(v_{-i}^t).\end{aligned}$$

Hence, $x^{t-1} \in X^*(v^t)$ and by the induction hypothesis it is achieved in equilibrium. Then, agent i's payoff is $v_i - \underline{v}_i(h) \geq v_i - \underline{v}_i(h^{t-1}) > v_i - p_i^t$. Therefore, the proposed strategy is optimal and $X^*(v^{t-1}) \subseteq X^*(v^t)$.

Step 3. $|I(z)| = m \geq 2$. In addition, the proposed strategy constitutes an SPNE for all $m' \leq m-1$ and $|I(z)| = m'$. Suppose that in round $t+\tau$ after z, $|I(h^{t+\tau-1}, p_j^{t+\tau})| = m$ and $|I(h^{t+\tau}, p_{j'}^{t+\tau+1})| \leq m-1$. Take the largest τ for any subgame starting from z. We show that for each $m \geq 2$ and each $\tau \geq 0$, the proposed strategy is optimal by induction.

We have shown that the proposed strategy is optimal for each m and $\tau = 0$ at Step 2. Suppose $\tau \geq 1$ and the proposed strategy constitutes an SPNE for all $\tau' \leq \tau - 1$.

Step 3.1. Suppose that $x^{t-1} \notin X_i$. It implies $W(v^{t-1}) = W_{-i}(v^{t-1})$ and $M_i^{t-1} = 0$.

Suppose that after agent i's action at round t, $x^t \in X_i$. By the tie-breaking assumption, it implies for all $x \in X^*(v^{t-1})$, $x \notin X^*(v^t)$. Then, induction hypothesis implies that $g(h) = x^t$. There are two cases for such an event arises:

1. $v_i^t = \underline{v}_i(h^t) > v_i^{t-1} = \underline{v}_i(h^{t-1}) > v_i$.
2. $v_i^t = \underline{v}_i(h^t) > v_i^{t-1} = v_i$.

In each case, $p_i^t = v_i^t = \underline{v}_i(h^t) > v_i$. By induction hypothesis, agent i earns $v_i - p_i(h) \leq v_i - \underline{v}_i(h^t) < 0$. On the other hand, if he plays the proposed strategy and reports no for $p_i^t > v_i$, it holds that $v_i^t = v_i^{t-1}$ and $X^*(v^t) = X^*(v^{t-1})$, so that i earns zero payoff. Therefore, the proposed strategy is optimal and $X^*(v^t) = X^*(v^{t-1})$.

Step 3.2. Suppose that there exists $x^{t-1} \in X_i$.

Step 3.2.1. Suppose $p_i^t \leq v_i - M_i^{t-1}$. Under the proposed strategy, agent i reports yes. This implies $v_i^t = v_i^{t-1}$. Note that $v_j^t = v_j^{t-1}$ for all $j \neq i$, so

$$W(v^t) = W(v^{t-1}) \geq W_{-i}(v_{-i}^{t-1}) = W_{-i}(v_{-i}^t). \tag{15}$$

Hence, $x^{t-1} \in X^*(v^t)$ and by induction hypothesis x^{t-1} is the final outcome in equilibrium. Agent i earns a payoff of $v_i - p_i(h) \geq v_i - p_i^t \geq 0$.

If agent i reports no, then it implies $\bar{v}_i(h^t) = p_i^t - 1$. Since

$$\begin{aligned} p_i^t - 1 + \sum_{j \in N(x^{t-1}) \setminus \{i\}} v_j^t + u_0(x^{t-1}) &< v_i - M_i^{t-1} + \sum_{j \in N(x^{t-1}) \setminus \{i\}} v_j^t + u_0(x^{t-1}) \\ &= W_{-i}(v^t), \end{aligned} \tag{16}$$

we have

$$x^{t-1} \notin X^*(v^t). \tag{17}$$

By induction hypothesis, the final outcome in equilibrium is in $X^*(v^t)$, thus that agent i earns zero payoff. Therefore, the proposed strategy is optimal and it holds that $X^*(v^{t-1}) \subseteq X^*(v^t)$.

Step 3.2.2. Suppose $p_i^t > v_i - M_i^{t-1}$. In the proposed strategy, agent i reports no. It is obviously optimal to report no when $p_i^t > v_i$. Suppose $v_i - M_i^{t-1} < p_i^t \leq v_i^t$. When agent i reports yes, we have $v^t = v^{t-1}$ and $x^{t-1} \in X_i$ is achieved. Agent i's payoff is $v_i - \underline{v}_i(h) \leq v_i - p_i^t$. When agent i reports no as in the proposed strategy, $v_i^t = \bar{v}_i(h^t) = p_i^t - 1$ and $v_{-i}^t = v_{-i}^{t-1}$. Then for x^{t-1},

$$\begin{aligned} \sum_{j \in N(x^{t-1})} v_j^t + u_0(x^{t-1}) &= p_i^t - 1 + \sum_{j \in N(x^{t-1}) \setminus \{i\}} v_j^{t-1} + u_0(x^{t-1}) \\ &\geq v_i - M_i^{t-1} + \sum_{j \in N(x^{t-1}) \setminus \{i\}} v_j^{t-1} + u_0(x^{t-1}) \\ &= W(v_{-i}^{t-1}) = W(v_{-i}^t). \end{aligned} \tag{18}$$

Hence, $x^{t-1} \in X^*(v^t)$ and by the induction hypothesis it is achieved in equilibrium. Then, agent i's payoff is $v_i - \underline{v}_i(h) > v_i - p_i^t$ since $\bar{v}_i(h) < p_i^t$. Therefore, the proposed strategy is optimal and $X^*(v^{t-1}) \subseteq X^*(v^t)$. ■

A.2 Proof of Theorem 2

Because the equilibrium allocation x^* associate with Theorem 1 is efficient, for each feasible allocation $x \in X$, we have

$$u_0(x^*) + \sum_{N(x^*)} \underline{v}_j(h) \geq u_0(x) + \sum_{N(x)\cap N(x^*)} \underline{v}_j(h) + \sum_{N(x)\setminus N(x^*)} \bar{v}_j(h). \tag{19}$$

Consider any coalition J including the social planner. Obviously, it suffices to consider the case of $J = N(x) \cup \{0\}$ for any feasible outcome $x \in X$.

$$\begin{aligned}
\sum_J \pi_j &= u_0(x^*) + \sum_{N(x^*)} \underline{v}_j(h) + \sum_{N(x)} \pi_j \\
&\geq u_0(x) + \sum_{N(x^*)\cap N(x)} \underline{v}_j(h) + \sum_{N(x)\setminus N(x^*)} \bar{v}_j(h) + \sum_{N(x)} \pi_j \\
&= u_0(x) + \sum_{N(x^*)\cap N(x)} v_j + \sum_{N(x)\setminus N(x^*)} \bar{v}_j(h) \\
&\geq u_0(x) + \sum_{N(x^*)\cap N(x)} v_j + \sum_{N(x)\setminus N(x^*)} v_j \\
&= \omega(J; v).
\end{aligned} \tag{20}$$

The third line comes from the fact that $\pi_j = v_j - \underline{v}_j(h)$ for every $j \in N(x^*)$ and $\pi_j = 0$ for $j \notin N(x^*)$. Note that in the SPNE of Theorem 1, $M_i^t = 0$ for all $i \notin N(x^*)$ and all t. Hence, $\bar{v}_j(h) \geq v_j$ for each $i \notin N(x^*)$, which induces the fourth line. ■

A.3 Proof of Theorem 3

A decision node is denoted by $z = (h^{t-1}, p_i^t)$. Let $I(z)$ be the set of agents who make an action in the subgame starting from z.

Step 1. Suppose $|I(z)| = 1$. Agent i is a unique agent making actions at z and all the subsequent nodes.

Step 1.1. Suppose $f(v^{t-1}) \notin X_i$. If $\forall \tilde{v}_i \in V_i(h^{t-1})$, $f(\tilde{v}_i, v_{-i}^{t-1}) \notin X_i$, then agent i earns zero payoff regardless of his responses, and any strategy is indifferent and optimal. Hence, suppose $\exists \tilde{v}_i \in V_i(h^{t-1})$, $f(\tilde{v}_i, v_{-i}^{t-1}) \in X_i$. Because $f(v^{t-1}) \notin X_i$, we have $v_i^{t-1} < c_i^f(v_{-i}^{t-1})$ and $c_i^f(v_{-i}^{t-1}) \in V_i(h^{t-1})$. Because $v_i^{t-1} \neq \bar{v}_i(h^{t-1})$, we have $v_i^{t-1} = \max\{v_i, \underline{v}_i(h^{t-1})\} \geq v_i$. Under the proposed strategy, agent i responses sincerely regarding v_i (equivalently v_i^{t-1}), so that the final allocation must be $f(v^{t-1})$ and agent i earns zero payoff. If agent i deviates and takes another strategy achieving some $x \in X_i$, then the social planner knows that i's revealed value is $\tilde{v}_i \geq c_i^f(v_{-i}^{t-1})$. This indicates that agent i says yes for a price $p_i^s \geq c_i^f(v_{-i}^{t-1})$ at a round $s \geq t$, and that i's payoff is negative: $v_i - p_i(h) \leq v_i - p_i^s < 0$. Hence, the proposed strategy is optimal.

Step 1.2. Suppose $f(v^{t-1}) \in X_i$. If $v_i^{t-1} = \underline{v}_i(h^{t-1}) \geq v_i$, it implies for any $\tilde{v}_i \in V_i(h^{t-1})$, $f(\tilde{v}_i, v_{-i}^{t-1}) \in X_i$. For payment minimization, it is optimal to say no for all prices, which is consistent with the proposed strategy because $c_i^f(v_{-i}^{t-1}) \leq \underline{v}_i(h^{t-1})$.

Suppose $v_i > \underline{v}_i(h^{t-1})$ and $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$. If $c_i^f(v_{-i}^{t-1}) \leq \underline{v}_i(h^{t-1})$, then the proposed strategy is clearly optimal as in the previous paragraph. Hence, suppose $c_i^f(v_{-i}^{t-1}) > \underline{v}_i(h^{t-1})$. By $f(v^{t-1}) \in X_i$, it holds that $c_i^f(v_{-i}^{t-1}) \leq v_i^{t-1} \leq v_i$. If agent i reports as if his value is $\tilde{v}_i < c_i^f(v_{-i}^{t-1})$, then his resulting payoff is zero. If agent i takes the proposed strategy, it holds that $c_i^f(v_{-i}^{t-1}) \in V_i(h)$ regardless of the ask-price process. Hence, we have

$$g(h) = g\big(\phi(c_i^f(v_{-i}^{t-1}), v_{-i}^{t-1})\big) = f\big(c_i^f(v_{-i}^{t-1}), v_{-i}^{t-1}\big) = f(v^{t-1}) \in X_i. \tag{21}$$

The first equality is from $v_j^{t-1} \in V_j(h)$ for all $j \neq i$. The third equality is from strong monotonicity. In addition, agent i says no for any $p_i^s > c_i^f(v_{-i}^{t-1})$ for all $s \geq t$, his payoff is at least $v_i - c_i^f(v_{-i}^{t-1}) \geq 0$. It is clearly suboptimal for i to say yes for $p_i^s > c_i^f(v_{-i}^{t-1})$ for some $s \geq t$.

Therefore, we have shown that the proposed strategy constitutes an SPNE and $g(h) = f(v^{t-1})$ when $|I(z)| = 1$.

Step 2. Now we consider any decision node z of round t and the corresponding mover i. We impose the following induction hypothesis; For every subsequent node z' of round $t' > t$ after z, the proposed strategy is an SPNE and $f(v^{t'}) = f(v^{t'-1})$ for all $t' > t$. Hence, $f(v^t)$ is chosen in the SPNE by the hypothesis.

Step 2.1. Suppose $f(v^{t-1}) \notin X_i$ and $p_i \in V_i(h^{t-1}) \setminus \{\underline{v}_i(h^{t-1})\}$. If $c_i^f(v_{-i}^{t-1}) > \bar{v}_i(h^{t-1})$, we have $f(v^t) \notin X_i$ regardless of i's action.[4] By the induction hypothesis,

[4]It is not guaranteed that $f(v^t) = f(v^{t-1})$.

i's resulting payoff is zero, and the proposed strategy is optimal.

Suppose $c_i^f(v_{-i}^{t-1}) \leq \bar{v}_i(h^{t-1})$. Because $v_i^{t-1} < c_i^f(v_{-i}^{t-1}) < \bar{v}_i(h^{t-1})$, we have $v_i^{t-1} = \max\{v_i, \underline{v}_i(h^{t-1})\}$. If $p_i^t < c_i^f(v_{-i}^{t-1})$, then $v_i^t \in \{v_i^{t-1}, p_i^t, p_i^t - 1\}$ after any response of agent i. Hence, $f(v^t) \notin X_i$ regardless of i's response. Hence, the proposed strategy is optimal, and under the strategy we have $v_i^t = v_i^{t-1}$, which induces $f(v^t) = f(v^{t-1})$.

Suppose $p_i^t \geq c_i^f(v_{-i}^{t-1})$. If, following the proposed strategy, agent i says no, then $v_i^t = v_i^{t-1}$ and $f(v^t) = f(v^{t-1})$. If he deviates and says yes, then $v_i^t = p_i^t$ and $f(v^t) = f(p_i^t, v_{-i}^{t-1}) \in X_i$. By induction hypothesis, the final allocation is $f(v^t)$. Because i pays at least p_i^t, his payoff is negative.

Step 2.2. Suppose $f(v^{t-1}) \in X_i$ and $p_i \in V_i(h^{t-1}) \backslash \{\underline{v}_i(h^{t-1})\}$. If $c_i^f(v_{-i}^{t-1}) < \underline{v}_i(h^{t-1})$, we have $f(v^t) = f(v^{t-1}) \in X_i$ regardless of i's action by strong monotonicity. Hence, for payment minimization, the proposed strategy in which agent i says no for all $p_i^t \in V_i(h^{t-1})$ is optimal.

Suppose $c_i^f(v_{-i}^{t-1}) > \underline{v}_i(h^{t-1})$. Because $v_i^{t-1} \geq c_i^f(v_{-i}^{t-1})$, we have $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$. If $p_i^t \leq c_i^f(v_{-i}^{t-1})$, agent i says yes under the proposed strategy, and we have $v_i^t = v_i^{t-1}$ (because $p_i^t \leq v_i^{t-1}$). Hence, $f(v^t) = f(v^{t-1}) \in X_i$ and by induction hypothesis, $g(h) = f(v^t)$. Because agent i says no for any price $p_i^s > v_i$ for any subsequent node, his payoff must be nonnegative $v_i - p_i(h) \geq 0$. If agent i deviates and says yes, then $v_i^t = p_i^t - 1 < c_i^f(v_{-i}^{t-1})$ and $f(v^t) \notin X_i$. By induction hypothesis, i's payoff is zero, so that the proposed strategy is optimal and $f(v^t) = f(v^{t-1})$ holds.

If $p_i^t > c_i^f(v_{-i}^{t-1})$, agent i says no under the proposed strategy, and we have $v_i^t = \min\{v_i, p_i^t - 1\}$.[5] Because $p_i^t - 1 \geq c_i^f(v_{-i}^{t-1})$, we have $f(v^t) = f(v^{t-1}) \in X_i$ by strong monotonicity. By induction hypothesis, $g(h) = f(v^t)$. Agent i's payment is at most $p_i^t - 1$, so that his resulting payoff is at least $v_i - p_i^t + 1$. If agent i deviates and says yes, then $v_i^t = \max\{v_i^{t-1}, p_i^t\} \geq v_i^{t-1}$ and $f(v^t) = f(v^{t-1}) \in X_i$. By induction hypothesis, $g(h) = f(v^t)$. Agent i's payment is at least p_i^t, so that his resulting payoff is at most $v_i - p_i^t$. Therefore, the proposed strategy is optimal and $f(v^t) = f(v^{t-1})$ holds. ■

[5] Remember that $v_i^{t-1} = \min\{v_i, \bar{v}_i(h^{t-1})\}$ and $p_i^t \leq \bar{v}_i(h^{t-1})$.

A.4 Proof of Theorem 4

Suppose for contradiction there exists agent i and $\exists s < t$, $p_i^s > p_i^t$. By informative query, $a_i^s = no$. Consider the revealed information at round $t-1$. By weak tightness, there exists $v \in V(h^{t-1})$ and $f(v) \in X_i$. In addition, $v \in V(h^{s-1})$ since $V_j(h^{t-1}) \subseteq V_j(h^{s-1})$ for each agent. Suppose that the true state is $(\hat{v}_i, v_{-i})$ with $\hat{v}_i \in V_i(h^{s-1})$ and $\hat{v}_i > p_i^s$. By monotonicity, $f(\hat{v}_i, v_{-i}) \in X_i$. Under sincere reporting, agent i reports yes at round s with p_i^s, and he earns a payoff $\hat{v}_i - p_i \leq \hat{v}_i - p_i^s$. If agent i reports no at p_i^s and pretends to have v_i, then the final outcome is still in X_i and he earns a payoff $\hat{v}_i - p_i > \hat{v}_i - p_i^s$, which is a contradiction. ■

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