Complexity of Payment Network

Hitoshi Hayakawa
University of Tokyo

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Abstract

A graph-theoretic framework is developed to study decentralized settlement in a general payment network. This paper argues settlement efficiency through examining how much settlement fund needs to be provided to settle all given obligations. Observing that required amount of settlement fund depends on in which order those obligations are settled, we focus on a pair of problems that derives its lower-bound and upper-bound, each formalized as a numbering problem on flow network. Our main finding is that twist nature of underlying directed graph (who obliged to whom) is a key factor to form settlement efficiency. The twist nature is captured through our original concepts; *arrow-twisted*, and *vertex-twisted*. Lower-bound of required settlement fund tends to be larger when underlying directed graph is twisted in *arrow-twisted* sense, while upper-bound tends to be smaller when it is twisted in *vertex-twisted* sense.

Keywords: settlement, payment network, interconnected financial system, graph-theoretic model
JEL classification: D53, D85, G20

1 Introduction

The paper proposes a graph-theoretic framework to study decentralized settlement in a general payment network. We formalize and examine a pair of problems that have important applications in interbank settlement systems.

Following simple example hints our framework, also quickly introduces our problems.

The environment for our problems is summarized in the left of Figure 1, which shows a distribution of obligations among economic subjects. Each of the three vertices specifies an economic subject, or a participant in the settlement system. Each of the three arrows traces each relation of obligation between two subjects. Each of the numbers indicates individual amount of each obligation. To summarize the environment, each of the three subjects has one obligation and one claim, each with 10 amount. We suppose each obligation needs to be settled via a transfer of settlement fund in each specified amount. For our analysis, we only allow each unit of obligation to be settled at one time, prohibiting settlement in multiple times.

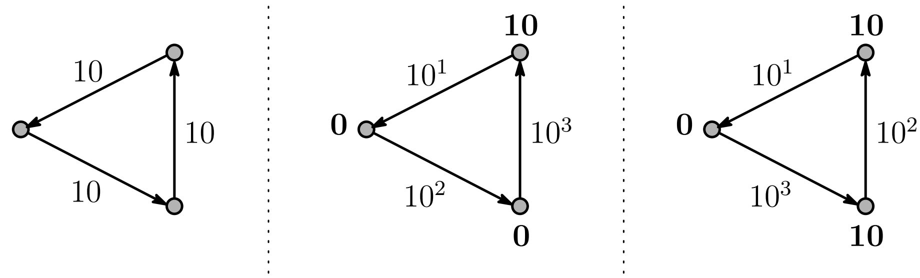

Figure 1:

Under these suppositions, we investigate settlement efficiency, or how much settlement fund needs to be provided to settle all the obligations. We notice that relative order of settlement is crucial. In the middle of Figure 1, each order of settlement is expressed by each number written in the upper-right of each amount of obligation. In this order of settlement, only the subject on the top needs to input settlement fund in the amount of 10, as indicated in boldface. The two other subjects need not input settlement fund because they can recirculate the payments they receive. The figure is interpreted to express a possible settlement procedure for our given distribution of obligations. Under this settlement procedure, 10 amount of settlement are required in total. In contrast, under a different settlement procedure as depicted in the right of Figure 1, total required amount of settlement is now 20 as confirmed similarly.

Supposing any relative order of settlement be possible to realize, the paper specifically examines lower-bound and upper-bound of total required amount of settlement fund. The lower-bound would be attained when order is optimally chosen by a central planner, that is each central bank for the case of settlement in interbank settlement systems. The upper-bound would be possible when order is formed under ill-coordination among subjects. The paper examines these specified problems in a general setting.

Settlement efficiency is one of critical concern in recent interbank settlement systems. Traditionally, interbank settlement systems processed transactions on a net basis; payments are collected and settled only at certain designated time –typically once a day– , and participant banks make net payments; the difference between payments received and payments owed. Realizing that net settlement systems are prone to cascades of defaults, many of interbank settlement systems now adopt real-time gross settlement (RTGS) systems that settle each payment on an individual basis.

Though RTGS systems reduce the risk of cascades of defaults compared to net settlement systems, it tends to require considerable settlement fund. When participants hold insufficient funds for settlement, typically central banks provide settlement fund through intraday lending. For well-functioning of settlement systems, it is crucial to provide sufficient settlement fund. The critical question there is how much settlement fund is required for settlements in each interbank settlement system. This study investigates the question by presenting two formal problems pertaining to each lower-bound and upper-bound of required fund. Accordingly, it is a benchmark for further research into that issue.

One of the main contribution of this paper is to provide a mathematical framework for decentralized settlement. The framework enables to express every possible settlement procedures, which allows us to argue settlement efficiency. Eisenberg and Noe (2001) provided a different framework for decentralized settlement. His framework is not to examine settlement efficiency, by effectively supposing settlement always with infinitely small unit of installment. There, settlement efficiency is assumed highest possible. Our framework is able to examine settlement in arbitrary unit of installment each as a different environment, that means settlement in infinitely small unit is encompassed as one environment. Rotemberg (2011) firstly pointed out in a persuading manner that settlement efficiency gets worse depending on which settlement procedure is realized. The paper focused on certain class of network structure whose underlying directed graph is to be Euler graph, and assumed specific behavioral pattern of subjects. Our research proceeds to a different direction in examining both lower-bound and upper-bound of required settlement fund supposing much wider settlement procedures are possible to realize. In our direction, we show several key network factors contribute to form settlement efficiency.

Our main finding is that twist nature of underlying directed graph is a key factor to form settlement efficiency. The twist nature never appears in the class of network Rotemberg (2011) examined, and it is captured with our original concepts; *arrow-twisted* and *vertex-twisted*.

Here we take up several examples to see how those concepts are defined, how they matter for our problems, and also what they imply in economic sense. Figure 2 presents three example environments for our problems.

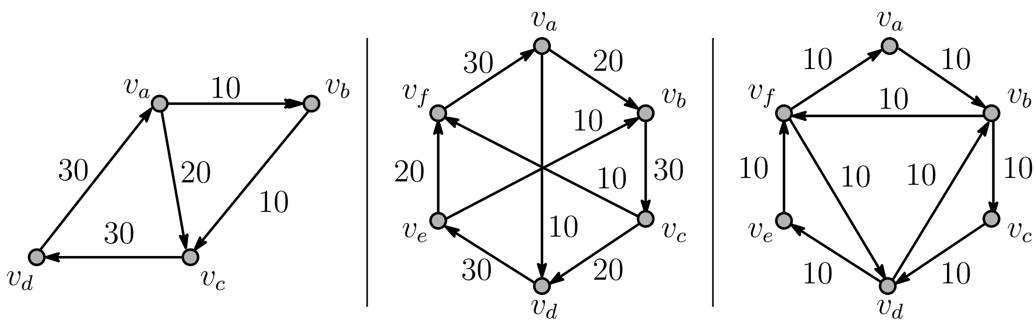

Figure 2: The left of the figure has no relevance neither to *arrow-twisted* nor *vertex-twisted*. The middle is relevant to *arrow-twisted*, while the right is relevant to *vertex-twisted*

We start by an environment expressed in the left of the figure, which has no relevance neither to *arrow-twisted* nor to *vertex-twisted*. Role of *arrow-twisted* and *vertex-twisted* is explained in comparison with this example. There are four subjects v_a, v_b, v_c, and v_d. Five obligations have been formed among those subjects. The lower-bound of required settlement fund is derived as 30, that is realized with a settlement procedure in the left of Figure 3. The upper-bound is 70, that is realized with that in the right of the same figure.

Now hypothetically decompose distribution of obligations into two distributions as shown in Figure 4. In latter analysis, we formally define our decomposition. Here notice that total amount of "decomposed" obligations for each subjects are equal to the original amount. For example, original obligation by v_d to v_a is 30, which is divided into 20

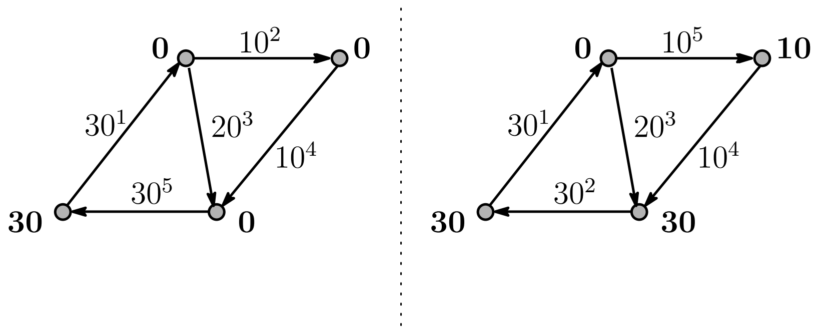

Figure 3: The left settlement procedure attains the lower-bound 30 of required settlement fund for distribution of obligations shown in the left of Figure 2, while the right attains the upper-bound 70.

and 10. Now derive lower-bound and upper-bound for each decomposed distributions of obligations. There we hypothetically suppose each of divided obligations were settled in each unit. The lower-bound for each of the decomposed distributions of obligations is 20, and 10. while the upper-bound is 40, 30 each. For this example, we confirm that mere summation of the lower-bound, upper-bound comes back to each for the original distribution of obligations, as confirmed as $20 + 10 = 30$, while $40 + 30 = 70$.

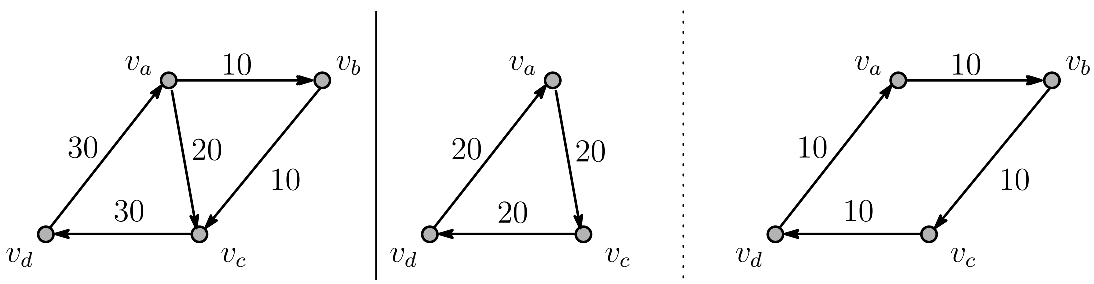

Figure 4: Original distribution of obligations shown in the left of the figure is decomposed into that in the middle, and that in the right.

Let us move to the case expressed with the middle of Figure 2, which is relevant to *arrow-twisted*. The lower-bound is 40 as in Figure 5, and we are going to see how *arrow-twisted* contributes to form that value. Now we have a decomposition as shown in Figure

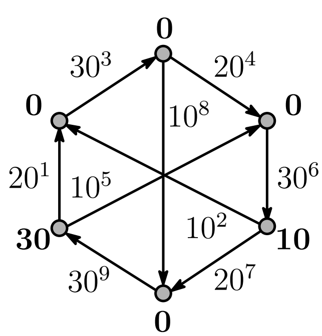

Figure 5: Example settlement procedure that attains lower-bound 40 for the middle of Figure 2.

6. We confirm that mere summation of the lower-bound for each decomposed distribution of obligations is 30, which is less than 40. The reason is that there emerges inconsistency

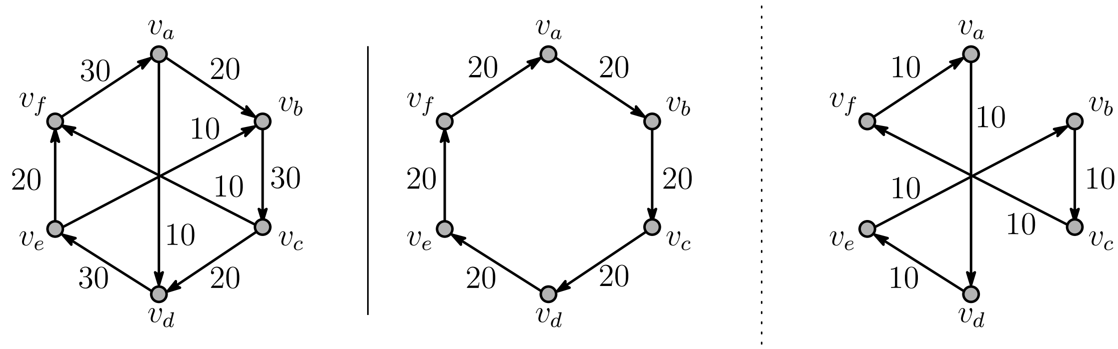

Figure 6: Original distribution of obligations shown in the left of the figure is decomposed into that in the middle, and that in the right.

of synchronization among hypothetically divided obligations. Focus on three obligations, by v_f to v_a, by v_b to v_c, and by v_d to v_e. Observe that for each of the decomposed distributions of obligations, the three obligations are to be settled in an order along with direction indicated by the arrows when each lower-bound is attained. But for the original distribution of obligations, there is no order that is consistent with both of such two orders. This inconsistency is thought to generate negative spillover in the sense that the lower-bound is larger than that without the inconsistency. The inconsistency is captured with our notion of *arrow-twisted* in general.

For our distribution of obligations, suppose obligation by v_f to v_a is to be settled in two units, 20 and 10 as shown in the left of Figure 7. Then, the lower-bound is now 30 as shown in the right of the same figure. We confirm that the inconsistency of synchronization is resolved and the lower-bound gets smaller. Further notice that input of settlement fund by v_f remains zero for those specific example settlement procedures. This indicates that inconsistency of synchronization is interpreted to arise from an externality when we suppose subject who inputs settlement fund incur corresponding financing cost, and also subject who owes obligation can choose unit of settlement.

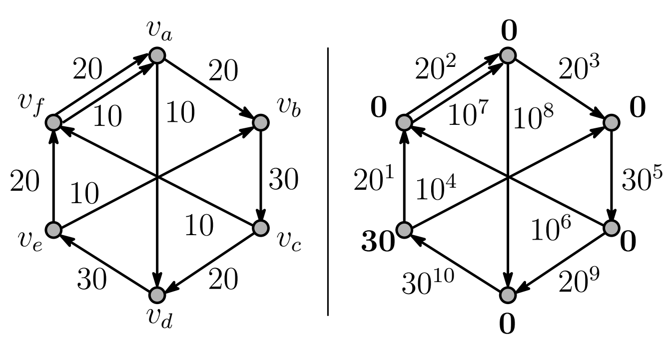

Figure 7: For the left distribution of obligations, the right shows an example settlement procedure that attains its lower-bound 30.

Let us proceed to the case expressed with the right of Figure 2, which is relevant to *vertex-twisted*. The upper-bound is 60 as in Figure 8, and we are going to see how *vertex-twisted* contributes to form that value. We have a decomposition as shown in Figure 9.

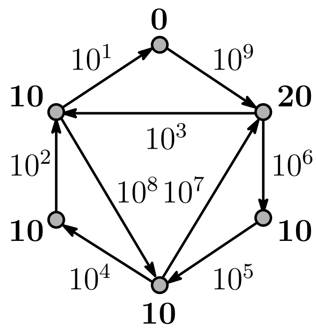

Figure 8: Example settlement procedure that attains upper-bound 60 for the right of Figure 2.

We confirm that mere summation of the upper-bound for each decomposed distribution of obligations is $70 = 50 + 20$, which is larger than 60. The reason is that there emerges inconsistency of synchronization now regarding subjects. For each of the decomposed distribution of obligations, settlement needs to be executed along with the reverse of direction indicated by each arrows. Now for the original distribution of obligations, focus on three subjects v_f, v_b, and v_d each has multiple obligations to make and receive. The obligations cannot be settled in a way that each of the three subjects make all their payments before their receipts as much as possible and also consistent with both of the orders for decomposed distributions of obligations each attains upper-bound of required settlement fund. This inconsistency is thought to generate positive spillover in the sense that the upper-bound is smaller than that without the inconsistency. This type of inconsistency of synchronization is captured with our notion of *vertex-twisted* in general. Confirm that

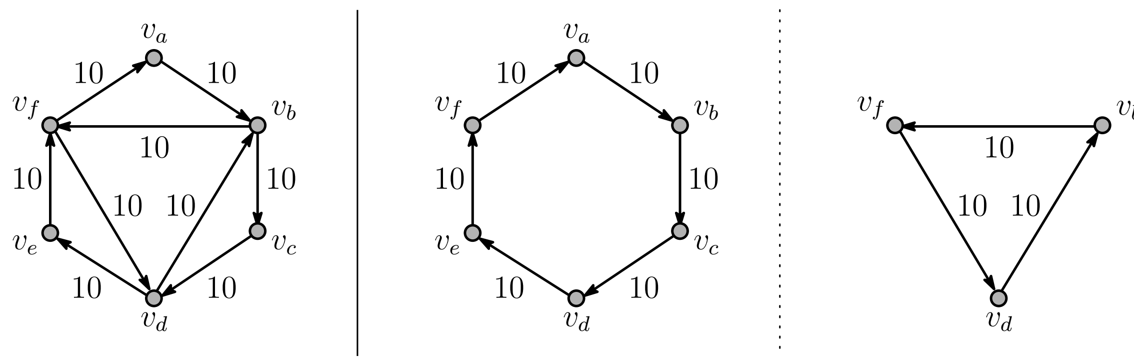

Figure 9: Original distribution of obligations shown in the left of the figure is decomposed into that in the middle, and that in the right.

the inconsistency indicated by *vertex-twisted* is dismissed in Figure 10, where direction of obligation among v_f, v_b, v_d is oppositely formed.

The remainder of this study is organized as follows. Section 2 introduces our framework and supplies definitions essential for statement of our problems, and Section 3 presents our problems formally. Section 4 offers preliminary analyses, introducing the central

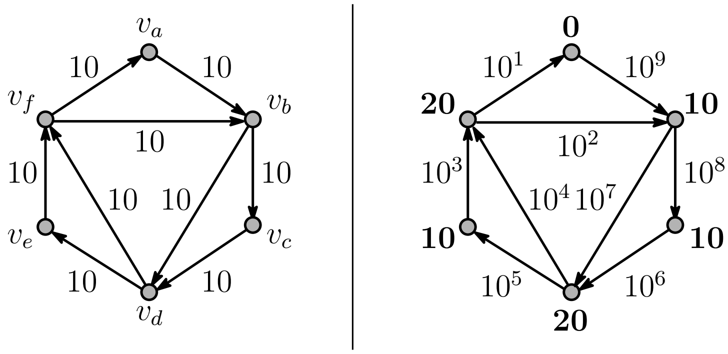

Figure 10: For the left distribution of obligations, the right shows an example settlement procedure that attains its upper-bound 70.

concept of *closed cycle decomposition* alongside several fundamental results. Section 5 displays the first half of our analyses, where we introduce three network properties—*domination*, *arrow-twisted* and *vertex-twisted*—and show how they help characterize the problems. Those properties combined with others are proposed as the key characteristics for our problems. Section 6 extends our analysis in detail, examining several types of transformation of networks in relation to the key characteristics. Section 7 reviews relevant earlier literature, and Section 8 concludes. The Appendix includes proofs for several theorems and additional results relevant to specific literature.

2 Model and Definitions

Our framework consists of five elements, which are expressed with five characters: V, A, f, s, p. The base elements are V and A, where V is a set of *vertices* which expresses economic subjects, while $A = \{(v, w, n)|v, w \in V, n = 1, 2, ..\}$ is a set of *arrows* each of which is an ordered pair of vertices with each index, and expresses payment relation between a pair of subjects. Indices are used to distinguish different payments among the same subjects. If there is no such multiplicity, all the indices are set as 0, and the indices are usually not mentioned in order to avoid notational cumbersome. We do not allow any arrow from and to the same vertex, or exclude payments from and to the same subject. $< V, A >$ constitutes a *directed graph*. An example directed graph is shown in the left of Figure 11.

We are to add additional elements f, s, p to $< V, A >$ to constitute two types of *Networks*; $< V, A, f >$ and $< V, A, f, s, p >$, where $< V, A, f >$ is to express distribution of obligations, and $< V, A, f, s, p >$ is to indicate its relevant settlement procedure. Firstly, $f : A \to R_+$ is called as *flow*, which expresses each amount for each payment. Secondly, $s : A \to \{1, 2, .., |A|\}$ is called as *sequence*, which is one-to-one mapping where $|A|$ denotes the total number of arrows, and economically it expresses relative order of settlement. Lastly, $p : V \to R_{0+}$ is called as *potential*, which expresses amount of settlement fund input by each subject.

We simply term $< V, A, f >$ as *f-Network* and $< V, A, f, s, p >$ as *fsp-Network*. The middle of Figure 11 shows an example of f-Network, and the right of the figure shows an example of fsp-Network constructed by adding s, p to the left f-Network.

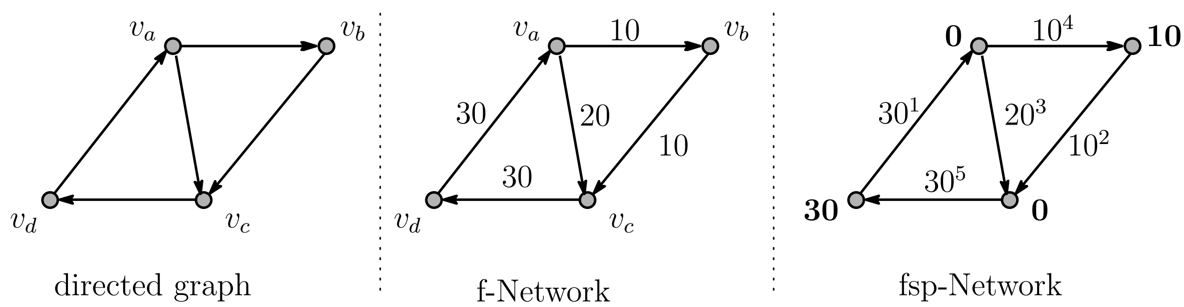

Figure 11: $V = \{v_a, v_b, v_c, v_d\}$, $A = \{(v_a, v_b), (v_a, v_c), (v_b, v_c), (v_c, v_d), (v_d, v_a)\}, f((v_a, v_b)) = f((v_b, v_c)) = 10, f((v_a, v_c)) = 20, f((v_c, v_d)) = f((v_d, v_a)) = 30, s((v_a, v_b)) = 4, s((v_a, v_c)) = 3, s((v_b, v_c)) = 2, s((v_c, v_d)) = 5, s((v_d, v_a)) = 1, p(v_a) = p(v_c) = 0, p(v_b) = 10, p(v_d) = 30$

In order to state our problems, we are to define two properties for the Networks, each of which is to express each economic assumption. One is termed as *closed* property of f-Network, which is to express amounts to make payment and to receive are balanced for each subject, which we call distribution of obligations are balanced. Given f-Network $< V, A, f >$, aggregate amount of payments to receive for $v \in V$ is denoted as $f_v^I \equiv \sum_{v' \in V} f((v', v))$, while aggregate amount of payments to make for $v \in V$ as $f_v^O \equiv \sum_{v' \in V} f((v, v'))$. Now *closed* property is defined as follows.

Definition 1. closed property: balanced distribution of obligations
f-Network $< V, A, f >$ is *closed* if $f_v^I = f_v^O$ for every $v \in V$.

The middle of Figure 11 is an example of *closed* f-Network.

The other property is *e-covered*(exact covered) property for fsp-Network, which is to express settlement procedure is to be proper in a sense that each subject input sufficient amount of settlement fund as well as any of input settlement fund is not to be redundant under attached order of settlement. Given fsp-Network $< V, A, f, s, p >$, suppose periods proceed as $t = 0, 1, .., |A|$ where relative order, or sequence s corresponds to each period t in a way that payments to be executed at the beginning of period t are $arg_a s(a) = t$. Aggregate periodical payments to receive for $v \in V$ at period t is denoted as $f_{v,t}^I = \sum_{v' \in V} 1_{\{s(v',v)=t\}} f((v', v))$, while that to make is denoted as $f_{v,t}^O = \sum_{v' \in V} 1_{\{s(v,v')=t\}} f((v, v'))$. Then periodical holding of money for each subject $v \in V$ at the last of period t is denoted as $p^t(v) = p^{t-1}(v) + (f_{v,t}^I - f_{v,t}^O)$ for $t = 1, 2, .., |A|$ and $p^0 = p(v)$. settlement fund input by each subject is sufficient when every periodical holding is sufficient. Sufficiency condition is defined as *covered* property. $< V, A, f, s, p >$ is *covered* if $p^t(v) \geq 0$ for every $v \in V$ and every $t = 0, 1, .., |A|$. Property of *e-covered* is defined as *covered* property added with property of no redundant settlement fund, as stated below.

Definition 2. e-covered property: proper settlement procedure
fsp-Network $< V, A, f, s, p >$ is *e-covered* (exact covered) if
(no shortage) $< V, A, f, s, p >$ is *covered*, and
(no redundancy) there is no other $p' : V \to R_{0+}$ such that $< V, A, f, s, p' >$ is *covered*, and $p'(v) \leq p(v)$ for every $v \in V$, and $p'(v') < p(v')$ for some $v' \in V$.

The right of Figure 11 shows an example of *e-covered* fsp-Network.

For $< V, A, f, s >$ on closed $< V, A, f >$, e-covered $< V, A, f, s, p >$ is uniquely derived. When $< V, A, f, s, p >$ is e-covered, we term *circulation* for $< V, A, f, s >$ is $\sum_{v \in V} p(v)$.

3 Payment Circulation Problem

We define our problem to derive lower-bound of required settlement fund as *minimum Payment Circulation Problem (min PCP)* which is formally stated as follows;

(min PCP in original form)
Given a closed f-Network $< V, A, f >$*,*
$\min_{s,p} \sum_{v \in V} p(v)$*,*
s.t. fsp-Network $< V, A, f, s, p >$ *is covered.*

Our problem to derive upper-bound of required settlement fund is formalized as *maximum Payment Circulation Problem (max PCP)* as follows;

(max PCP in original form)
Given a closed f-Network $N^f =< V, A, f >$*,*
$\max_{s,p} \sum_{v \in V} p(v)$*,*
s.t. fsp-Network $< V, A, f, s, p >$ *is e-covered.*

We term value derived by each min/max PCP as *min/max circulation.*

Though this paper focuses on these min/max PCP, we can view that the problems belong to a further abstract problem;

Find a set of fsp-Networks which satisfies condition X*.*

In the latter literature section, economic researches in the field of settlement system, emergence of money, and currency area are reviewed in reference to this general form, and our contribution is stated along with the view.

3.1 First thought on the min/max PCP

Let us take up a candidate of approach to derive min/max circulation, which is only partially successful, and whose failure motivates our approach.

Suppose our input is f-Network shown in the left of Figure 12. Then, we have that the middle of the figure shows an fsp-Network which attains the minimum circulation, while the right of the figure is that for the maximum circulation. First for the case of the minimum, we observe that sequence is taken in a way that number is increasing along with direction indicated by the arrows. Actually, start by v_d, move to v_a, then v_c, back to v_d, sequence for corresponding arrows is increasing as 1, 3, 5. For the different route: from v_d, v_a, v_b, v_c, and back to v_d, sequence is also increasing as 1, 2, 4, 5. For the case of the maximum, we observe that sequence is taken in a way that number is increasing along with the opposite direction. Confirm that starting by v_d and move to v_c, v_a back to v_d, sequence is 1, 2, 3. It is similarly confirmed for the other route.

Conversely, we could formulate each of the above ways of taking sequence as an algorithm to derive each min/max circulation combined with some appropriate detailed procedures. These simple algorithms will actually solve min/max PCP for certain class of f-Networks, but not in general. For example, the algorithms do not work for f-Network as shown in the middle of Figure 2.

Our approach departs from constructing algorithms, instead grasp the problems in a topological manner so that relevant economic contexts are to be revealed.

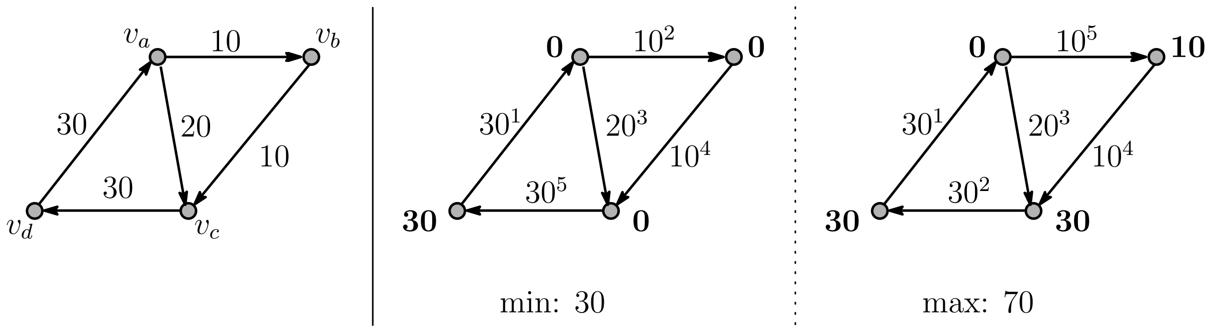

Figure 12: For the left f-Network, the middle fsp-Network attains the minimum circulation, while the right fsp-Network attains the maximum circulation.

4 Preliminary Analysis

This section presents key notions and results for our latter analyses.

We say a f-Network $< V, A, f >$ is not connected when we can divide into $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ such that there is no arrow $(v, v') \in A$ where v and v' belongs to different set with respect to V_1, V_2. In that case, it is apparent if we divide $< V, A, f >$ into two f-Networks along with such V_1, V_2 and associate arrows and flow, we derive min/max circulation for the original f-Network just as summation of that for each divided f-Network. Throughout this article, without loss of generality we focus on f-Networks that are connected.

4.1 Closed Cycle Decomposition

Our approach bases on an observation that closed f-Network can be *decomposed* into several closed f-Networks. *Decomposition* of f-Network is an algebraic notion on f-Networks, which is naturally derived from addition on real number.

Figure 13 shows an example of *decomposition* on f-Network. N^f is *decomposed* into N_1^f and N_2^f, which is denoted as $N^f = N_1^f + N_2^f$.

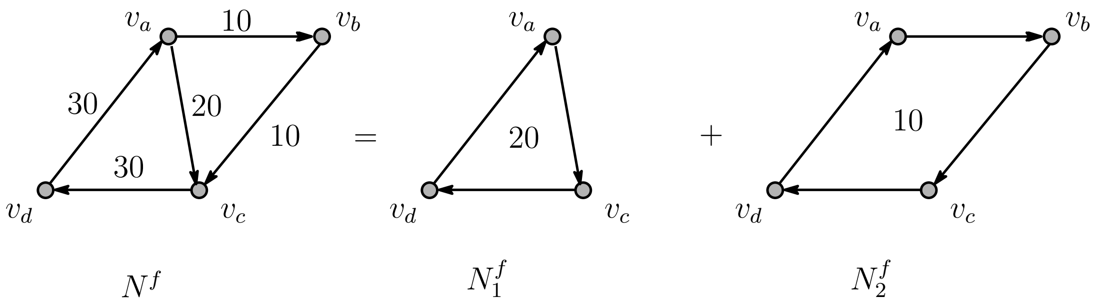

Figure 13: $N^f =< V, A, f >$ is the same as the left of Figure 11. $N_1^f =< V_1, A_1, f_1 >$, where $V_1 = \{v_a, v_c, v_d\}$, $A = \{(v_a, v_c), (v_c, v_d), (v_d, v_a)\}$, $f((v_a, v_c)) = f((v_c, v_d)) = f((v_d, v_a)) = 20$. $N_2^f =< V_2, A_2, f_2 >$, where $V_2 = \{v_a, v_b, v_c, v_d\}$, $A = \{(v_a, v_b), (v_b, v_c)(v_c, v_d), (v_d, v_a)\}$, $f((v_a, v_b)) = f((v_a, v_c)) = f((v_c, v_d)) = f((v_d, v_a)) = 0$. Cycle value for N_1^f is 20, which is expressed in the center of the f-Network, while it is 10 for N_2^f.

Formally, we term that a f-Network $N^f =< V, A, f >$ is *decomposed* into $\left\{N_k^f =< V_k, A_k, f_k >\right\}_{k=1,2,..,K}$ if $V = \cup_{1 \le k \le K} V_k$ and $A = \cup_{1 \le k \le K} A_k$, and $\forall a \in A$, $f(a) = \sum_{k \in K'} f_k(a)$, where $K' = \{k'' | a \in A_{k''}\}$. We denote $N^f = \sum_{k=1}^{K} N_k^f$ for the decomposition.

We find a specific type of *decomposition* is critical for analyses of min/max PCP, which we term as *closed cycle decomposition*. *Decomposition* in Figure 13 is actually a *closed cycle decomposition*, where each of decomposed f-Networks is *closed*, and each consists of one *cycle*. For our formal statement of *closed cycle decomposition*, we define *cycle* and several relevant terminologies.

Given a directed graph $< V, A >$, we denote a set of vertices included in $A' \subseteq A$ as $V_{A'}$, and denote a set of arrows which includes $v \in V$ as A_v. For a directed graph $< V, A >$, $A' \subseteq A$ is a *path* from $v \in V_{A'}$ to $v' \in V_{A'}$ if we can order vertices in $V_{A'}$ such that $(v, v_1, v_2, .., v')$ where each consecutive ordered pair of vertices consists A'. The same arrow is not allowed to appear more than once in a path, but it is allowed for the same vertex. $A' \subseteq A$ is a *cycle* if A' is a path between the same vertex. We say a cycle is *punctured* if it includes the same vertex, and say *non-punctured* if not. For a directed graph G, we denote C_G as the set of cycles included in G, and call it as the cycle set of G.

Our formal definition of *closed cycle decomposition* is as follows.

Definition 3. closed cycle decomposition

A f-Network $N^f =< V, A, f >$ with $G =< V, A >$ is *closed cycle decomposed* into $\left\{N_k^f =< V_k, A_k, f_k >\right\}_{1 \le k \le K}$ if

1) $N^f = \sum_{k=1}^{K} N_k^f$ is a *decomposition*, and

2) $\forall k = 1, 2, ..., K$, each N_k^f consists of mutually different one *cycle* and is also *closed*.

We specifically write $N^f = \sum_{c \in C} < V^c, c, f^c >$ for a closed cycle decomposition with $C \subseteq C_G$, where f^c is referred as cycle value for c. Note that closed cycle decomposition is allowed to include decomposed f-Networks which consist of one *punctured cycle*.

We have following result for *closed cycle decomposition*.

Theorem 1 (Ford and Fulkerson (1962)).

Any closed f-Network can always be closed cycle decomposed.

The theorem ensures that our observation of closed cycle decomposition in Figure 13 is not just luck. However, it is shown that closed cycle decomposition in Figure 13 is a rather special case regarding *uniqueness* of closed cycle decomposition. We say f-Network is uniquely closed cycle decomposed if there is no other closed cycle decomposition for the same f-Network such that cycle sets are different[1]. Uniqueness of closed cycle decomposition is characterized with the notion of *disjoint*. Intuitively, two sets of cycles are *disjoint* if we cannot create the other cycle using the arrows included in the two sets. Formally, given a set of cycles C in some directed graph, we say $C', C'' \subseteq C$ are *disjoint*(regarding C) if $\forall K \in C$ and $K \subset A_{C' \cup C''}$, $K \notin C \setminus (C' \cup C'')$. We say C is a *disjoint set* if $\forall C', C'' \subseteq C$, C' and C'' are disjoint.

Corollary 1 (Uniqueness of closed cycle decomposition).

Given a closed f-Network $N^f =< V, A, f >$ with $G =< V, A >$, N^f is uniquely closed cycle decomposed if and only if C_G is a disjoint set.

We turn to *decomposition* on fsp-Network, which is similarly defined with that on f-Network. Figure 14 is an example of *decomposition* on fsp-Network.

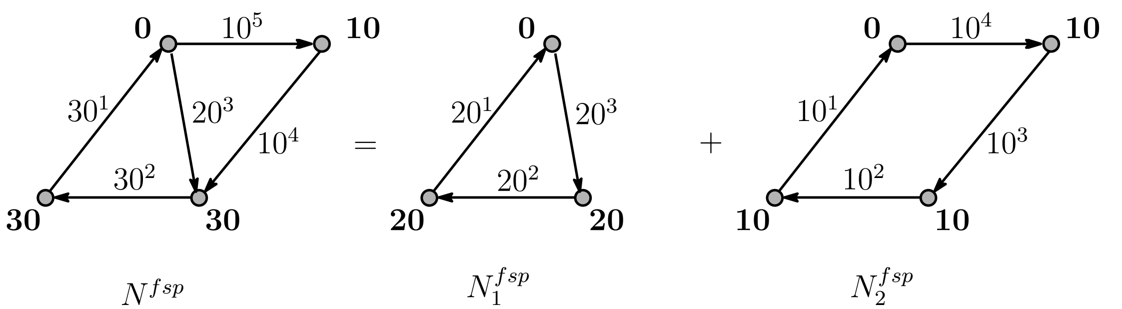

Figure 14: example for decomposition on fsp-Network

Formally, we say fsp-Network $N^{fsp} =< V, A, f, s, p >$ is *decomposed* into fsp-Networks $\left\{ N_k^{fsp} =< V_k, A_k, f_k, s_k, p_k > \right\}_{k=1,2,..,K}$ if

(1) $< V, A, f >$ is decomposed into $\{< V_k, A_k, f_k >\}_{k=1,2,..,K}$,

(2) each sequence s_k is consistent with s in the sense that the ordering is preserved, and

(3) $\sum_k p_k(v) = p(v)$ for every $v \in V$.

When a decomposition on fsp-Network $N^{fsp} =< V, A, f, s, p >$ is also a closed cycle decomposition on corresponding f-Network, we write as: $< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$.

Notice that in Figure **??** and 14, e-covered fsp-Network is decomposed into fsp-Networks which are all e-covered.

[1]For uniqueness of closed cycle decomposition, we ignore trivial differences such that difference is only on sets of vertices, such that whether isolated vertices are included or not

Theorem 2. *decomposition on e-covered fsp-Network*

Given a closed f-Network $< V, A, f >$*, for any e-covered fsp-Network* $N^{fsp} =< V, A, f, s, p >$*, there exists decomposition* $N^{fsp} = \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$ *such that* $< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$*.*

Proof. See Appendix A.2. □

5 Key Characteristics for the min/max PCP

Purpose of this section is to reveal key characteristics for the min/max PCP. Rearranging the original min/max PCP utilizing *closed cycle decomposition* leads us to find them.

Notice that in Figure **??**, fsp-Network that attains min circulation is expressed with *decomposed* fsp-Networks. Figure 14 shows that for the case of maximum. Stepping further, the next two theorems show that each of the min/max PCP is rewritten in a form to choose fsp-Networks for *closed cycle decomposed* f-Networks of given f-Network, while the original form is just to choose fsp-Network for given f-Network.

Theorem 3. *min PCP in decomposed form*

Given a closed f-Network $N^f =< V, A, f >$*, the following problem gives the same value with the min PCP on* N^f*;*

$\min_{s, C \in C_{N^f}, \{f^c\}_{c \in C}} \sum_{c \in C} \sum_{v \in V^c} p^c(v)$.
s.t. $N^f = \sum_{c \in C} < V^c, c, f^c >$ *is a closed cycle decomposition,*
$< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$*, and*
$< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$

Proof. see the appendix A.3. □

Theorem 4. *max PCP in decomposed form*

Given a closed f-Network $N^f =< V, A, f >$*, the following problem gives the same value with the max PCP on* N^f*;*

$\max_{s, C \in C_{N^f}, \{f^c\}_{c \in C}} \sum_{c \in C} \sum_{v \in V^c} p^c(v)$.
s.t. $N^f = \sum_{c \in C} < V^c, c, f^c >$ *is a closed cycle decomposition,*
$< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$*, and*
$< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$ *is e-covered.*

Proof. See Appendix A.4. □

The above decomposed forms of the min/max PCP need to be rearranged so as to reveal their own worth. Each of the decomposed form problems is to be separated into *decomposition choice part* and *sequence choice part.* We first define a sub-problem for each min/max PCP, each of which corresponds to the *sequence choice part.*

(sub-problem for min PCP)

Given a closed f-Network N^f *and its closed cycle decomposition that is characterized with* $C \in C_{N^f}$ *and* $\{f^c\}_{c \in C}$*,*

$\min_{\{s^c,p^c\}_{c\in C}} \sum_{c\in C}(\sum_{v\in V^c} p^c(v) - f^c)$
s.t. $< V^c, c, f^c, s^c, p^c >$ *is exact covered for every* $c \in C$*, and*
$< V, A, f, s, p >= \sum_{c\in C} < V^c, c, f^c, s^c, p^c >$.

(sub-problem for max PCP)

Given a closed f-Network N^f *and its closed cycle decomposition that is characterized with* $C \in C_{N^f}$ *and* $\{f^c\}_{c\in C}$,
$\min_{\{s^c,p^c\}_{c\in C}} \sum_{c\in C}((|c|-1)f^c - \sum_{v\in V^c} p^c(v))$,
s.t. $< V^c, c, f^c, s^c, p^c >$ *is e-covered for every* $c \in C$*, and*
$< V, A, f, s, p >= \sum_{c\in C} < V^c, c, f^c, s^c, p^c >$ *is e-covered.*

, where $|c|$ denotes the number of arrows which constitute cycle c.

Denote each value as $R^{min/max}(N^f, C, \{f^c\}_{c\in C})$.

Next lemma ensures $R^{max}(.)$ has some value for any closed cycle decomposition.

Lemma 1.

Given a closed f-Network $< V, A, f >$ *and its closed cycle decomposition* $< V, A, f >= \sum_{c\in C} < V^c, c, f^c >$ *Then we can always take fsp-Network* $< V, A, f, s, p >$ *and associated e-covered fsp-Networks* $\{< V^c, c, f^c, s^c, p^c >\}_{c\in C}$ *such that* $< V, A, f, s, p >= \sum_{c\in C} < V^c, c, f^c, s^c, p^c >$ *is e-covered.*

Proof. Take arbitrary v-number $s_v : V \to \{1, 2, .., |V|\}$. Denote each set of vertices $V_k = arg_{v\in V} s_v(v) = k$ for $k = 1, 2, .., |V|$. Take sequence on arrows $a_k \in A$ that starts from $v \in V_k$ so that $\sum_1^{k-1} |V_{k-1} < s(a_k) < \sum_1^k V_k$. Such sequence s let us take p^c that each decomposed fsp-Networks is e-covered. Now for each vertex $v \in V$, take any two out-arrows $a' = (v, v'), a'' = (v, v'') \in A$. Then, there is no in-arrow $a''' = (v''', v) \in A$ such that $s(a') < s(a''') < s(a'')$. It is true for any two out-arrows. When we take $p(v) = \sum_{c\in C} p^c(v)$ for each $v \in V$, it says that the combined fsp-Network $< V, A, f, s, p >$ is also e-covered. □

Now we rewrite the min/max PCP in each separated form.

(min PCP in separated form)

Given a closed f-Network $N^f =< V, A, f >$,
$\min_{C\in C_{N^f},\{f^c\}_{c\in C}} \sum_{c\in C} f^c + R^{min}(N^f, C, \{f^c\}_{c\in C})$
s.t. $N^f = \sum_{c\in C} < V^c, c, f^c >$ *is a closed cycle decomposition.*

(max PCP in separated form)

Given a closed f-Network $N^f =< V, A, f >$,
$\max_{C\in C_{N^f},\{f^c\}_{c\in C}} \sum_{c\in C}(|c|-1)f^c - R^{max}(N^f, C, \{f^c\}_{c\in C})$
s.t.$N^f = \sum_{c\in C} < V^c, c, f^c >$ *is a closed cycle decomposition*

For a given closed f-Network N^f, let $x^{min/max}(N^f)$ denote each min/max circulation.

In the rest of this sections, we are to reveal certain network properties help characterize each of the decomposition choice part and the sequence choice part of min/max PCP. We show *arrow-twist* property is key to the sequence choice part of min PCP, while *vertex-twist* property to the same part of max PCP, and *domination* property is to the decomposition choice part both for min/max PCP.

5.1 Property of *arrow-twist* and min PCP

Let f-Network shown in the left of Figure 15 be our input for the min PCP. We know that fsp-Network shown in the right of the figure realizes its min circulation 40. The min circulation is derived with a closed cycle decomposition shown in Figure 16. We confirm that the residual is solved as 10. In search of sources of the non-zero residual, we focus on two of the decomposed cycles. For each of the cycles, take sequence in a way that it is increasing along with direction indicated by its arrows. Suppose we start by the arrow (v_f, v_a). Then, focusing on two of the other arrows (v_b, v_c) and (v_d, v_e), in the left cycle sequence for (v_b, v_c) needs to be smaller than (v_d, v_e) while it is opposite for the right cycle. That is the source of non-zero value for residual, which is captured with the notion of *arrow-twisted*. Below we formally define *arrow-twisted* and related notions.

For Networks which include $G =< V, A >$ and sequence s on A, let cycle c consists of arrows $(a_1, a_2, ..a_n, a_{n+1} = a_1)$ where $a_k = (v_k, v_{k+1})$ for $k = 1, 2, .., n$, then the arrow-reverse number is defined as $r^{atwi}(c, s) = \sum_{k=1}^{n} 1_{\{s(a_k)>s(a_{k+1})\}}$. When there exist multiple ways to index arrows for a cycle and accordingly multiple values of *arrow-reverse number* (which is possible when it is punctured), set arrow-reverse number as the minimum among those. We say cycles in $C \subseteq C_G$ are in *arrow-twisted* relation, or just say they are *arrow-twisted* if we cannot take any sequence s such that $r^{atwi}(c, s) = 1$ for every $c \in C$. We say cycles in $C \subseteq C_G$ are *minimum arrow-twisted* when there exists no *arrow-twisted* cycles $C' \subset C$. Going back to Figure 16, the two of the decomposed cycles are *arrow-twisted* and *minimum arrow-twisted*. Note that *minimum arrow-twisted* cycles are not always a pair, as confirmed in Figure 17.

Property of *arrow-twist* for given f-Network among its cycle sets refers to whole relations of *arrow-twisted* among its sets of cycles as well as their *arrow-reverse numbers*.

For *arrow-twist* property, following lemma is fundamental for our analyses.

Lemma 2. *arrow-twisted and* $R^{min}(.)$

Given a closed f-Network N^f *and its closed cycle decomposition that is characterized with* $C \in C_{N^f}$ *and* $\{f^c\}_{c\in C}$*,*

$R^{min}(N^f, C, \{f^c\}_{c\in C}) = 0$ *if and only if* C *is not arrow-twisted.*

Proof. When C is not arrow-twisted, then we can always take sequence so that arrow-reverse number for every $c \in C$ is one. It lets us take $\sum_{v\in V^c} p^c(v) = f^c$ for every $c \in C$. Conversely, when $R^{min}(.) = 0$, we can always take arrow-reverse number is one for all $c \in C$ under any sequence that realises $R^{min}(.) = 0$. □

We have a basic result for the case of *disjoint*.

Lemma 3.

For given closed f-Network $< V, A, f >$ *with* $G =< V, A >$ *and its cycles* $C \subseteq C_G$*, we have*

C *is not disjoint if* C *is arrow-twisted.*

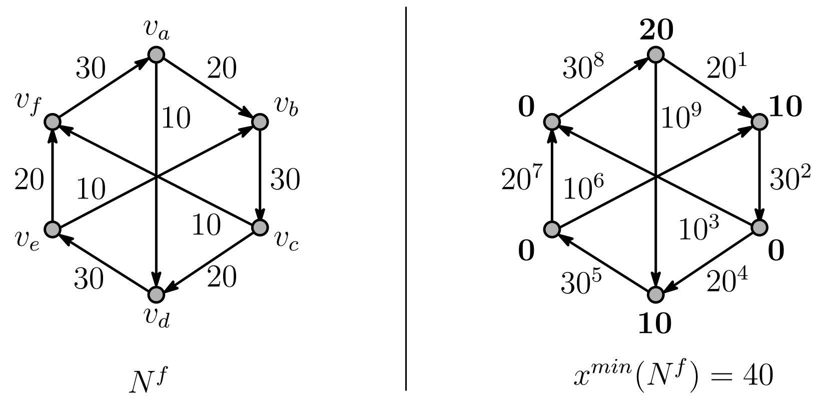

Figure 15:

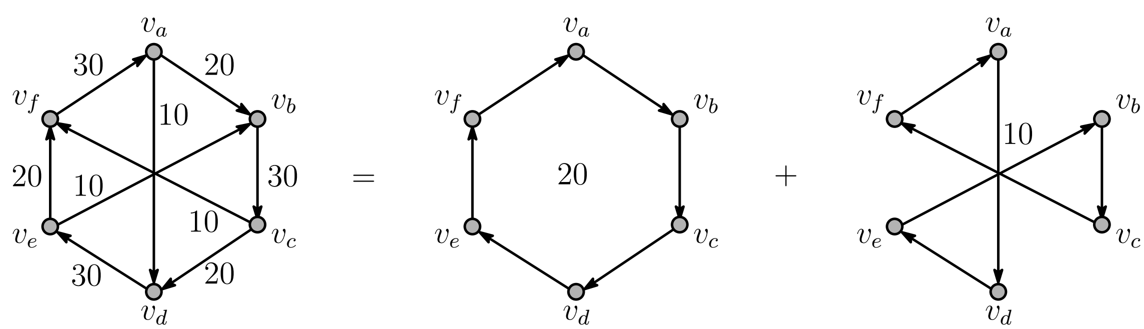

Figure 16:

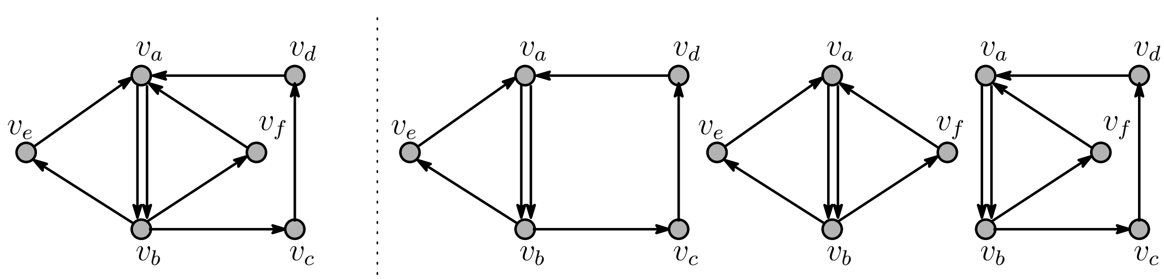

Figure 17: Example for arrow-twisted

For the left directed graph, the rest three cycles are minimum arrow-twisted.

Proof. For a closed f-Network, suppose some of its cycles C is *arrow-twisted*. Denote A as the set of arrows which constitutes C. Since C is *arrow-twisted*, we can take at least two cycles using arrows in A such that the two cycles have at least two common arrows which are not successive. Otherwise it is straight that we take sequence in a way that arrow-reverse number is all one for every possible cycles in A, which let C be not arrow-twisted.

If two cycles have two common arrows which are not successive, we can immediately take another cycle using part of arrows both from the two cycles. It says C is not disjoint, which ends our proof. □

The next theorem shows that min circulation is derived in a straight way for the case of *disjoint*.

Theorem 5. *min PCP for f-Network whose cycle set is disjoint*

For a closed f-Network $N^f =< V, A, f >$ with $G =< V, A >$, if C_G is disjoint, then

$x^{min}(N^f) = \sum_{c \in C_G} f^c$*, and*

with its closed cycle decomposition $N^f = \sum_{c \in C_G} < V^c, c, f^c >$.

Proof. From Lemma 2 and 3, we need not consider into sequence choice part for the min PCP in separated form. Further, Corollary 1 states that there exists only one *closed cycle decomposition* for the case of *disjoint*, which ends our proof. □

We show our additional results after introducing *domination* property in latter part.

5.2 Property of *vertex-twist* and max PCP

Let f-Network shown in the left of Figure 18 be our input for the max PCP. We know that fsp-Network shown in the right of the figure realizes its max circulation 110. The max circulation is derived with a closed cycle decomposition shown in Figure 19. We confirm that the residual is solved as -10. Let us focus on two of the decomposed cycles. For each of the cycle, take sequence in a way that it is increasing along with direction indicated by its arrows. Suppose we start by the arrow (v_f, v_a). Now examine in which order each vertex appears under supposed sequence. Focusing on the three of the vertices v_f, v_b, v_d, in the left cycle sequence for v_b needs to come before v_d while it is opposite for the right cycle. That is the reason for non-zero value for residual, and it is captured with the notion of *vertex-twisted*. Below we define *vertex-twisted* and related notions.

We prepare a different type of sequence for our model. For $< V, A >$, define *vertex-sequence* (sequence for vertex) $s_v : V \to \{1, 2, .., |V|\}$ as one-to-one mapping.

Let cycle c consists of $v_1 v_2 .. v_n v_{n+1}$ with $v_{n+1} = v_1$, then *vertex-reverse number* is defined as $r^{vtwi}(c, s_v) = \sum_{k=1}^{n} 1_{\{s_v(v_k) > s_v(v_{k+1})\}}$. When there exist multiple ways to index vertices for a cycle and accordingly multiple values of *vertex-reverse number* (which is possible when it is punctured), set *vertex-reverse number* for the cycle as the minimum among those. We say cycles in $C \subseteq C_G$ are in *vertex-twisted* relation, or just say they are *vertex-twisted* if we cannot take any vertex-sequence s^v such that $r^{vtwi}(c, s) = 1$ for every $c \in C$. We say cycles in $C \subseteq C_G$ are *minimum vertex-twisted* when there exists no vertex-twisted cycles $C' \subset C$.

Note that although any punctured cycle as in Figure 20 is *vertex-twisted* by itself, notion of *vertex-twisted* is not trivial in the sense that cycles which are not punctured can be also *vertex-twisted* as already shown in Figure 19, and as in Figure 21.

Property of *vertex-twist* for given f-Network refers to whole relations of *vertex-twisted* among its sets of cycles as well as their *vertex-reverse numbers*.

Notice that if cycles in C are *arrow-twisted*, then they are also *vertex-twisted* as stated in the following Lemma 4. The reverse is not always true as easily confirmed.

Lemma 4. *arrow-twisted and vertex-twisted*
Given $G =< V, A >$ and C_G, then for any $C \subseteq C_G$,
C are vertex-twisted if C are arrow-twisted.

Proof. Suppose C is not *vertex-twisted*. Then the definition says that we can take *vertex-sequence* s_v on vertices in C such that *vertex-reverse number* is one for all $c \in C$. Take *sequence* s^c on arrows for each $c \in C$ such that $s^c((v, v')) = s_v(v)$. We have $r^{atwi}(c, s^c) = 1$ for every $c \in C$. Since we can always take a *sequence* for arrows on C such that it is consistent with all the s^c, we know C is not *arrow-twisted*.

□

For *vertex-twist* property, the following result is fundamental for our analyses.

Lemma 5. *vertex-twisted and $R^{max}(.)$*
Given a closed f-Network N^f and its closed cycle decomposition that is characterized with $C \in C_{N^f}$ and $\{f^c\}_{c \in C}$,
$R^{max}(N^f, C, \{f^c\}_{c \in C}) = 0$ if and only if C is not vertex-twisted.

Proof. When C is not *vertex-twisted*, then from its definition, we can always take *vertex-sequence* s_v on vertices in C such that *vertex-reverse number* is $|c| - 1$ for all $c \in C$. Denote each set of vertices $V_k = arg_{v \in V} s_v(v) = k$ for $k = 1, 2, .., |V|$. take sequence on arrows $a_k \in A$ that start from $v \in V_k$ so that $\sum_1^{k-1} |V_{k-1} < s(a_k) < \sum_1^k V_k$. Since there exist no vertex-twisted, such sequence s let us take p^c that each decomposed fsp-Networks is e-covered and $\sum_{v \in V^c} p^c(v) = (|c| - 1)f^c$. What needs to be shown is that combined fsp-Network with the decomposed fsp-Network is e-covered. For each vertex $v \in V$, take any two out-arrows $a' = (v, v'), a'' = (v, v'') \in A$. Then, there is no in-arrow $a''' = (v''', v) \in A$ such that $s(a') < s(a''') < s(a'')$. It is true for any two out-arrows. It says that the combined fsp-Network is e-covered.

For the converse direction, take a sequence s that realizes $R^{max}(N^f, C, \{f^c\}_{c \in C}) = 0$. Under the sequence s, for each cycle $c \in C$ with its set of vertices V^c, take a vertex $v^c \in V^c$ such that $s((v, v^c)) = argmin_{a \in c} s(a)$ and call it *head-vertex* for c. Then, for every vertices $v' \in V^c \setminus v^c$ with its arrow $(v', v) \in c$, there is no arrow $a = (v'', v') \in C$ such that $s(a) < s((v', v))$ since otherwise it immediately leads to $R^{max}(N^f, C, \{f^c\}_{c \in C}) > 0$. It is true for every cycles $c \in C$. Then, we can naturally define partial order $<$ on $v \in V^c$ from sequence s in a way that each *head-vertex* is largest and gets smaller along with the direction opposite to that indicated by the arrows. We can always take vertex-sequence consistent with the order $<$, and under such vertex-sequence, vertex-reverse number is 1 for every cycles $c \in C$. It says C is not vertex-twisted.

□

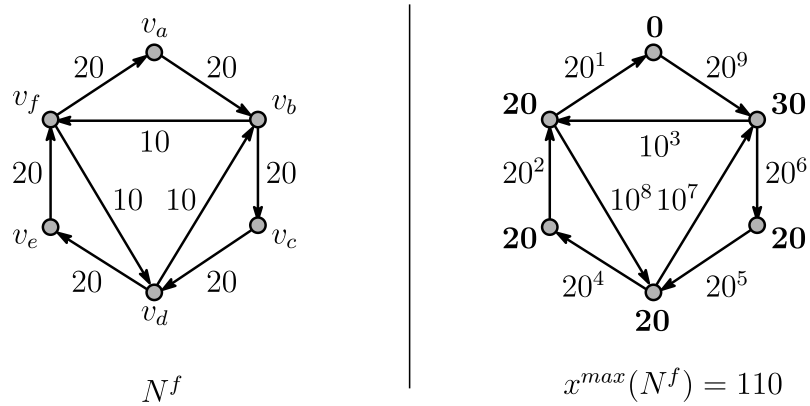

Figure 18:

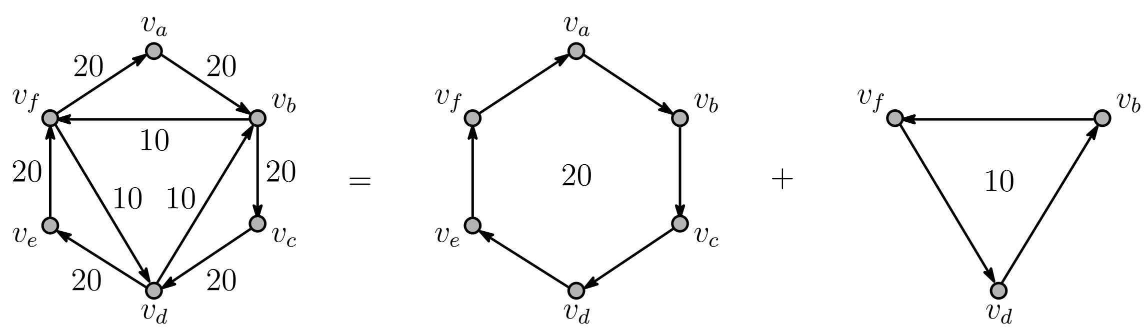

Figure 19:

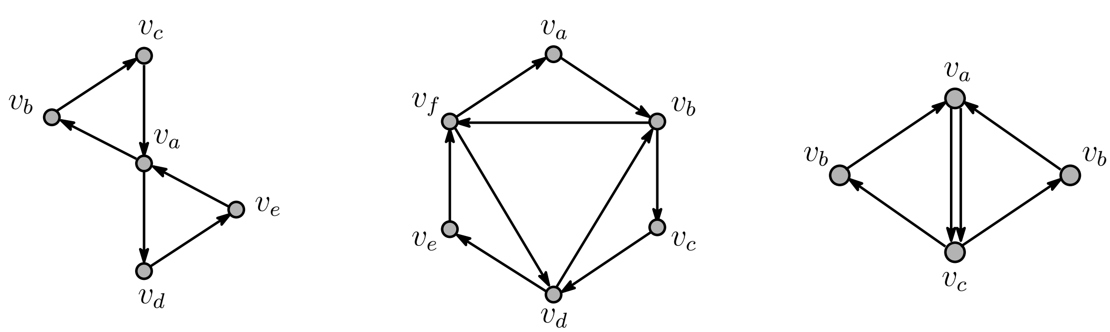

Figure 20: example for vertex-twisted cycle : Each of the above directed graph constitutes a punctured cycle.

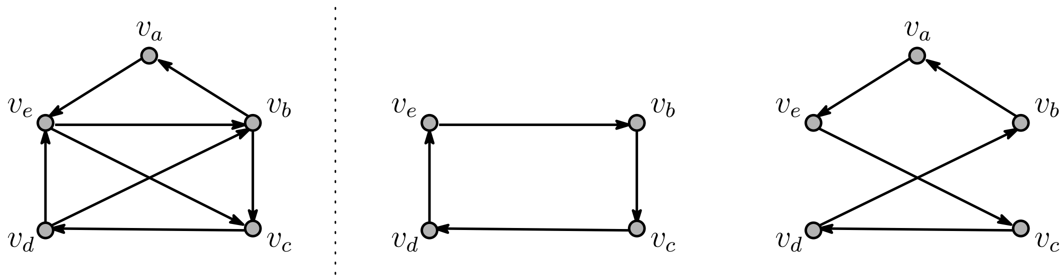

Figure 21: example for vertex-twisted cycles : for directed graph at the left, two cycles in the right are *vertex-twisted*.

We have a basic result for the case of *disjoint*.

Lemma 6.

For given closed f-Network $< V, A, f >$ *with* $G =< V, A >$ *and its cycles* $C \subseteq C_G$*, we have*

C *is not disjoint if* C *is vertex-twisted.*

Proof. For a closed f-Network, suppose some of its cycles C is *vertex-twisted*. When there exists any punctured cycle, it is immediate C is not disjoint. Suppose not. Denote A as the set of arrows which constitutes C. Since C is *vertex-twisted*, we can take at least two cycles using arrows in A such that the two cycles have at least two common vertices which are not included in successive common arrows of the two cycles. Otherwise it is straight that we take sequence in a way that vertex-reverse number is all one for every possible cycles in A, which let C be not vertex-twisted.

If two cycles have such two common vertices, we can immediately take another cycle using part of arrows both from the two cycles. It says C is not disjoint, which ends our proof. □

The next theorem shows that max PCP are derived in a straight way for f-Networks which are disjoint.

Theorem 6. *max PCP for f-Network whose cycle set is disjoint*

For a closed f-Network $N^f =< V, A, f >$ *with* $G =< V, A >$*, if* C_G *is disjoint, then*

$x^{max}(N^f) = \sum_{c \in C_G} (|c| - 1) f^c$

with its closed cycle decomposition $N^f = \sum_{c \in C_G} < V^c, c, f^c >$.

Proof. From Lemma 6, when C is *disjoint*, then C is not vertex-twisted. Further, Corollary 1 states that there exists only one *closed cycle decomposition* with cycles C_G for the case of *disjoint*. From Lemma 5 and with the same procedure of taking sequence shown in its proof, we always take sequence and potential so that associated fsp-Network is e-covered and has $\sum_{c \in C_G} (|c| - 1) f^c$.

□

We show our additional results after introducing *domination* property.

5.3 Property of *domination* and min/max PCP

For a closed f-Network shown in the left of Figure 22, min circulation is derived as 30, which for example is realized with a fsp-Network in the right of the figure.

In Figure 23, observe that the same f-Network is closed cycle decomposed into different number of closed f-Networks. The minimum circulation is derived with the decomposition that has smaller number of cycles, that is captured with notion of *domination* formally defined below.

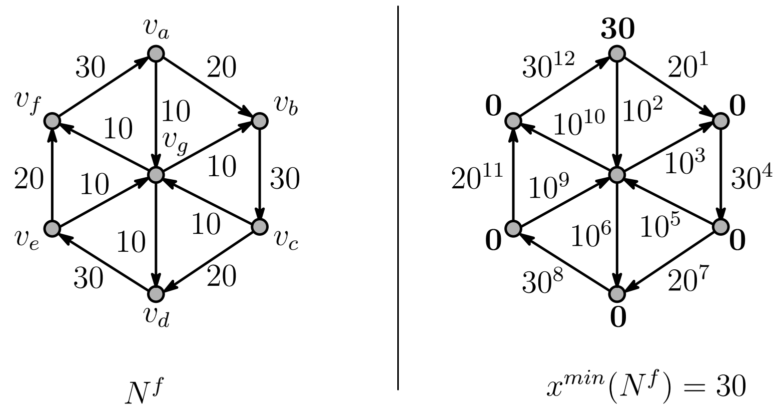

Figure 22:

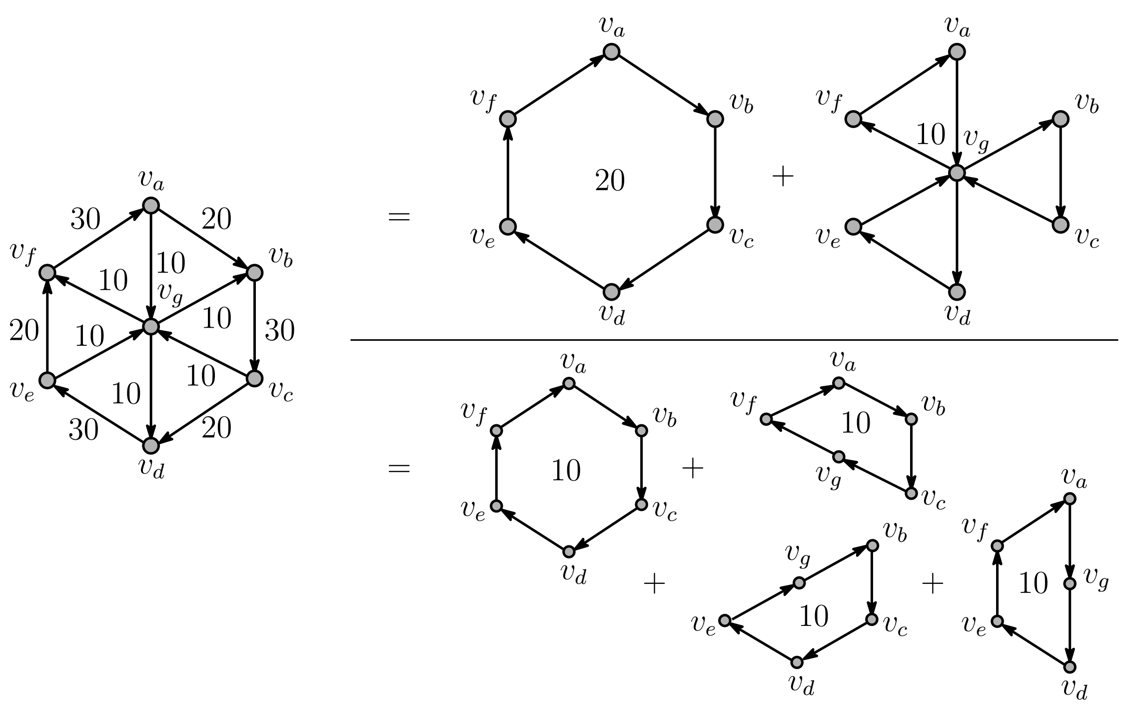

Figure 23:

Given a directed graph G and its cycle set C_G, a set of cycles $C \subseteq C_G$ is termed as *dominated* by another set of cycles $C' \subseteq C_G$ when there exist the same f-Network N^f

and closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >= \sum_{c' \in C'} < V^{c'}, c', f^{c'} >$ with $\sum_{c \in C} f^c < \sum_{c' \in C'} f^{c'}$. Note that *dominated* is well-defined since closed cycle decomposition with the same cycle set C leads to unique value of $\sum_{c \in C} f^c$ for a given closed cycle decomposition, though there exist room for the choice of flow for each cycle. We especially say $c \in C_G$ *singular dominates* $C' \subseteq C_G$ if c dominates C'. We say $C \subseteq C_G$ is *undominated* in C_G if there is no $C' \subseteq C_G$ which dominates C. We say a set of cycle C has *no domination* if there exists no $C', C'' \subseteq C$ such that C' dominates C''. A set of cycles C is *undominated* in C_G if C is not *domintated* by any $C' \subseteq C_G$.

Note that any punctured cycle *dominates* the set of its component non-punctured cycles. For example in Figure 23, a punctured cycle $v_a v_g v_b v_c v_g v_d v_e v_g v_f v_a$ *dominates* $\{v_a v_g v_f v_a, v_c v_g v_b v_c, v_e v_g v_d v_e\}$. Though number of decomposed cycles seemingly a determinant of *dominated* relation, Figure 24 shows it is not always true.

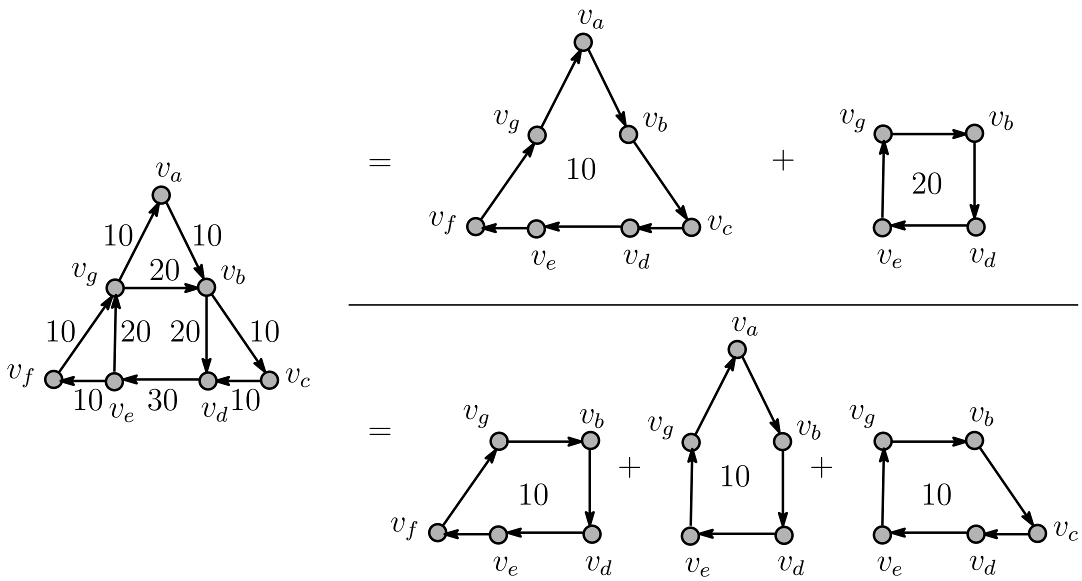

Figure 24:

Theorem 7. *min circulation under no arrow-twisted*

Given a closed f-Network $N^f =< V, A, f >$ on $G =< V, A >$, if there exist no arrow-twisted cycles in C_G, then,

$x^{min}(N^f) = \sum_{c \in C} f^c$ *with any closed cycle decomposition* $N^f = \sum_{c \in C} < V^c, c, f^c >$ *such that $C \subseteq C_G$ is undominated in C_G.*

Proof. Lemma 2 ensures $R^{min}(.)$ is always zero. Definition of *undominated* ensures our choice of closed cycle decomposition.

□

Theorem 8. *max circulation under no vertex-twisted*

Given a closed f-Network $N^f =< V, A, f >$ on $G =< V, A >$, if there exist no vertex-twisted cycle (punctured cycle) nor vertex-twisted cycles in C_G, then,

$x^{max}_{N^f} = \sum_{c \in C} (|c| - 1) f^c$

with any closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$ *such that* $C \subseteq C_G$ *is undominated in* C_G.

Proof. Take a closed cycle decomposition $C \in C_{N^f}$, $\{f^c\}_{c \in C}$ that is undominated in C_G. By taking sequence for given f-Network shown in the proof of Lemma 5. We can take e-covered fsp-Networks for decomposed f-Networks with p^c so that $\sum_{v \in V^c} p^c(v) = \sum_{c \in C} (|c| - 1) f^c$ for every $c \in C$ and combined fsp-Network is e-covered. □

Note that f-Networks with punctured cycles are excluded from the above theorem, since any punctured cycle is *vertex-twisted* by itself. The next theorem allows punctured cycles but not for the other vertex-twisted cycles. For $G =< V, A >$, denote $C^p_G \subset C_G$ as the set of punctured cycles for G, and denote $C^{np}_G \subseteq C_G \setminus C^p_G$ as the set of non-punctured cycles for G.

Theorem 9. *max circulation under no vertex-twisted except for punctured cycles*

Given a closed f-Network $N^f =< V, A, f >$ *on* $G =< V, A >$*, if there exists no vertex-twisted cycles, then,*

$x^{max}_{N^f} = \sum_{c \in C} (|c| - 1) f^c$

with any closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$ *such that* $C \subseteq C_G$ *is undominated in* C^{np}_G.

Proof. When there exits no *vertex-twisted* cycles in C^{np}_G, take closed cycle decomposition for C^{np}_G with undominated cycles. Suppose we can make a punctured cycle c from two of the cycles c', c''. Consider *closed cycle decomposition* that has c instead of c' and c'' for the amount of flow z. In separated form, it always increase the former part by z since $|c| = |c'| + |c''|$. However, since c itself is *vertex-twisted*, it always $R^{max}(.)$ part at least by z. We confirm that taking into account a punctured cycle generated by the two cycles never increase circulation of that without it. Any punctured cycle is able to be constituted by iterating combination of two cycles, which always leads to the same result.

Among *closed cycle decomposition* within C^{np}_G, the largest circulation is realized with cycles which are *undominated* from its definition. It ends our proof. □

Property of *domination* for given f-Network refers to whole relations of *dominated* among its sets of cycles.

5.4 The Key Characteristics

We have seen properties of *arrow-twist*, *vertex-twist*, and *domination* help characterize the min/max PCP. In addition, notice that number of vertices is also an important property for the max PCP, which we capture with a notion of *weighted*. For a closed f-Network $N^f =< V, A, f >$, given a closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$, we call each cycle $c \in C$ is *weighted* by $|V^c - 1|$. Property of *weight* for given f-Network refers to *weighted* amounts for decomposed cycles for every possible *closed cycle decompositions*.

For our following analyses, we call *arrow-twist*, *vertex-twist*, *domination*, and *weight* are the key characteristics of the min/max PCP.

6 Effects of Network Transformations

The previous section introduced key characteristics and showed relevant basic results. This section examines in more detail how those characteristics work. For that purpose, we specifically examine how min/max circulation are affected when our input, f-Network is transformed into some other f-Network in various manner.

6.1 Definitions of Local Operations, Semi-Global Operations

We take up five types of transformations as shown in Figure 25. Suppose our original input for min/max PCP is that shown in the left of the figure, which consists one cycle $v_a v_b v_c v_d v_e v_f$ with flow 10. For the f-Network, shown in the right part of the figure are transformed f-Networks, each of which is derived through following *operations*: *arrow separation* on an arrow (v_a, v_b) into (v_a, v_g) and (v_g, v_b) with added vertex v_g for the upper-left, *arrow slicing* on $(v_a, v_b, 0)$ into $(v_a, v_b, 0)$ and $(v_a, v_b, 1)$ for the upper-middle, *vertex contraction* on v_a, v_d to v_a for the upper-right, *cycle addition* $v_a v_d v_e v_b v_c v_f v_a$ with flow 20 for the lower-right, *cycle separation* on cycle $v_a v_b v_c v_d v_e v_f v_a$ in the amount of 10 for the lower-right. We call first three of the operations; *arrow separation, arrow slicing, vertex contraction* as *local operations*, while *semi-global operations* for the latter two.

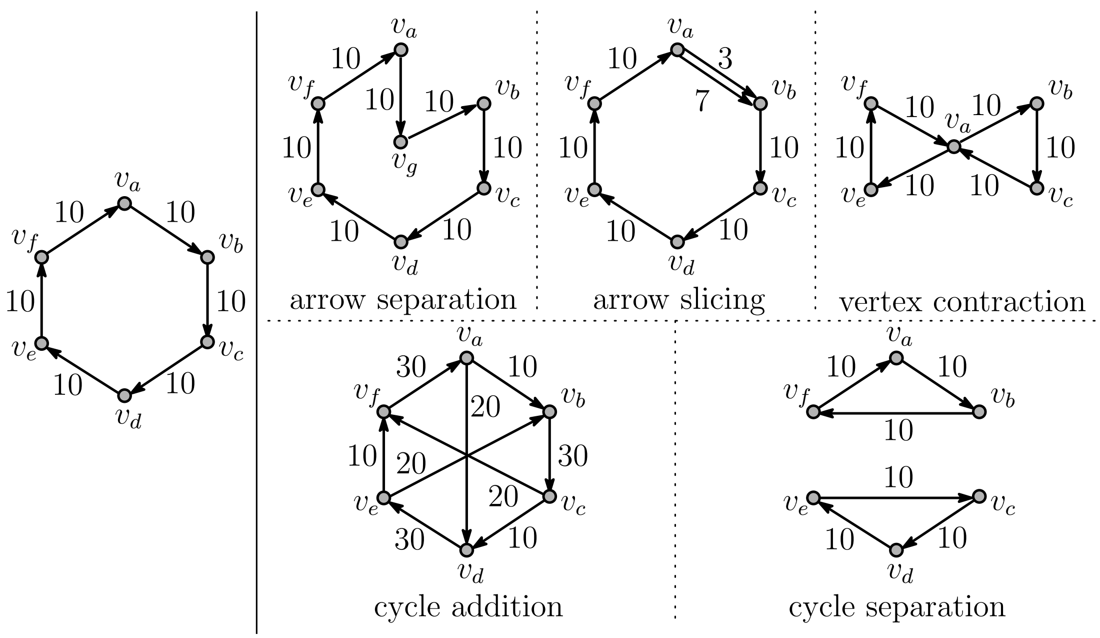

Figure 25:

Formally, given a f-Network $< V, A, f >$, each operation is defined as follows. We say *arrow separation* on $a = (v, v') \in A$ into a' and a'' with v'' to have $< V, A', f' >$ when $A' = A \cup a' \cup a'' \setminus a$ and $a' = (v, v''), a'' = (v'', v')$ with $f'(a') = f'(a'') = f(a)$, while $f'(a''') = f(a''')$ for every $a''' \in A \setminus a$. We say *arrow slicing* on $a \in A$ into a and a' to have $< V, A', f' >$ when arrows a, a' are between the same vertices, $A' = A \cup a'$, $f(a) = f'(a) + f'(a')$, and $f'(a'') = f(a'')$ for every $a'' \in A \setminus a$. We say *vertex contraction* for $N^f =< V, A, f >$ on $v, v' \in V$ to v to have $< V', A', f' >$ when $V' = V \setminus v'$, and all the arrows from or to v' in A are replaced by arrows from or to v in A', and f' is determined accordingly. We exclude vertices v, v' such that both $(v, v'), (v', v)$ exists in A. We say *cycle addition* $c \subseteq A$ with its flow f^c on $N^f =< V, A, f >$ to $N^{f'} =< V', A', f' >$ when $V' = V \cup V_c$, $A' = A \cup A_c$, and $f'(a) = f(a) + f^c$ for every $a \in c$ and $f'(a) = f(a)$ otherwise. Note that *flow increase* is a special case of *cycle addition*. We say a closed graph $G =< V, A >$ is *separated cycle* graph if any two cycles $c, c' \in C_G$ have no common vertex. We say *cycle separation* for $N^f =< V, A, f >$ on cycle $c \subseteq C_{<V,A>}$ in the amount of $f^k (\leq \min_{a \in c} f(a))$ to have $N^{f'} =< V, A', f' >$ when for some $a_1 = (v_1, v_1'), a_2 = (v_2, v_2') \in c$ such that $v_1' \neq v_2$ and $v_2' \neq v_1$, we take $a_1' = (v_1, v_2'), a_2' = (v_2, v_1')$, and $f'(a_1) = f(a_1) - f^k$, $f'(a_2) = f(a_2) - f^k$, $f'(a_1') = f(a_1') + f^k$, $f'(a_2') = f(a_2') + f^k$.

Note that those five operations are sufficient to examine network transformations in the following sense. For any given two closed f-Networks, we can always attain some closed f-Network from each of the two f-Networks through combinations of *arrow slicing, vertex contraction, cycle addition*. Notice that though *arrow separation* and *cycle separation* are redundant there, each has its own worth. *Arrow separation* reveals simpler cases within *vertex contraction*, while *cycle separation* reveals effects which are not directly captured by each of the other operations alone.

For latter reference, Table 6.1 summarizes our results in this section. It shows how each of the operations affects min/max circulation in total, as well as how each of the key characteristics contributes to the effect. For example, we read that *arrow separation* has no effect regarding min circulation, while it tends to increase max circulation through affecting *weight* property.

		min. circulation			max. circulation			
		dom.	a-twi.	total	wei.	dom.	v-twi.	total
Local	arrow sep.	$-$	$-$	$-_{10}$	$\uparrow$	$-$	$-$	$\uparrow_{11}$
	arrow slice.	$\downarrow$	$\downarrow$	$\downarrow_{12}$	$-$	$\downarrow$	$\uparrow$	$-_{13}$
	vertex cont.	$\downarrow$	$\downarrow$	$\downarrow_{14}$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow_{15}$
semi-Global	cycle add.	$\Downarrow_{16}$	$\Uparrow_{18}$	$\Downarrow\Uparrow$	$-$	$\Uparrow_{17}$	$\Downarrow_{17,19}$	$\Downarrow$
	cycle sep.	$\uparrow_{22}$	$\downarrow_{23}$	$\downarrow\uparrow$	$-$	$\downarrow_{24}$	$\uparrow_{25}$	$\downarrow\uparrow$

Table 1: Effects of Operations in relation to the Key Characteristics
"$\uparrow$"("$\downarrow$") and "$\Uparrow$"("$\Downarrow$") show that the corresponding *operation* has weakly increasing(decreasing) effect through the corresponding property on either problem, while "$-$" means no effect. "$\Uparrow$","$\Downarrow$" especially express possibility of "multiplier effect". Numbers in the cells show those for related theorems.

6.2 Effects of Local Operations

arrow separation

For the min PCP, arrow separation has no effect.

Theorem 10. *no effect of arrow separation on min circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any arrow separation on $a \in A$ to have $N^{f'}$, we have
$x^{min}(N^{f'}) = x^{min}(N^f)$.

Proof. For any sequence on N^f, we can take a sequence on $N^{f'}$ such that the relative orders are all the same and separated arrows have successive numbers. Under the sequence, circulation for $N^{f'}$ is the same as that on N^f. Conversely, for any sequence on $N^{f'}$, we can take sequence on N^f such that the relative orders are the same when we correspond either of the two of separated arrows in $N^{f'}$ to arrow a in N^f. Under that sequence, circulation for N^f is equal to or smaller than that for $N^{f'}$. □

Observe that *arrow separation* has no effect on either property of *domination* or *arrow-twist*, which leads to no effect on min circulation.

Theorem 11. *increasing effect of arrow separation on max circulation*
Given a closed f-Network $N^f =< V, A, f >$, for any arrow separation on $a \in A$ to have $N^{f'}$, we have
$x^{max}(N^{f'}) = x^{max}(N^f) + f(a)$.

Proof. Denote arrow separation on $a = (v, v')$ into $a' = (v, v'')$ and $a'' = (v'', v''')$. Given a sequence which realizes the maximum circulation for N^f, it is always possible to take sequence s for $N^{f'}$ such that$s(a') > s(a'')$ while the other orderings are unchanged. Circulation under the sequence s for $N^{f'}$ is $x^{max}(N^f) + f(a)$. Conversely, suppose there exists sequence for $N^{f'}$ such that its circulation is larger than $x^{max}(N^f) + f(a)$. Then, we can always take sequence for N^f such that orderings are the same when we correspond either of a' or a'' to a. It decreases circulation by at most $f(a)$, which contradicts max circulation for N^f is $x^{max}(N^f)$. □

Observe that *arrow separation* has no effect on either property of *domination* or *vertex-twist* but has effect on *weight* property, which is the source of increase of max circulation.

arrow slicing

Theorem 12. *decreasing effect of arrow slicing on min circulation*
Given a closed f-Network $N^f =< V, A, f >$, for any arrow slicing on $a \in A$ to have $N^{f'}$, we have
$x^{min}(N^{f'}) \leq x^{min}(N^f)$.

Proof. Given a sequence which realizes the minimum circulation for the original f-Network, take sequence for arrow-sliced f-Network so that sliced arrows have successive number, and maintain the ordering for the other arrows. It never increase the circulation. □

Arrow slicing has decreasing effect on min circulation both through affecting property of *domination* and *arrow-twist*. Figure 26 shows effect through *domination*, and Figure 27 shows effect through *arrow-twist*.

We observe that decreasing effect through *domination* property can be partly cancelled out through *arrow-twist* property by generating new *arrow-twisted* cycles, as shown in Figure 28.

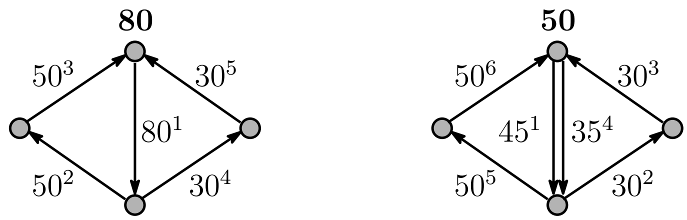

Figure 26: arrow slicing (decrease of min circulation through *domination*)

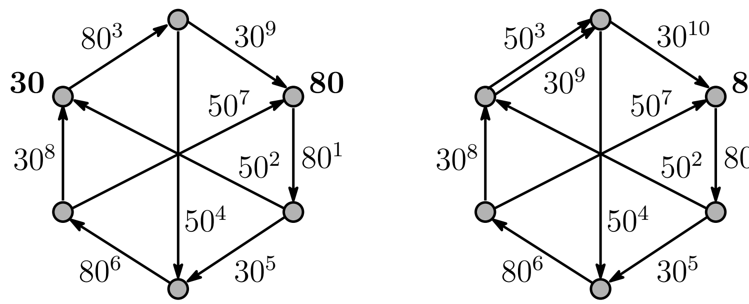

Figure 27: arrow slicing (decrease of min circulation through *arrow-twist*)

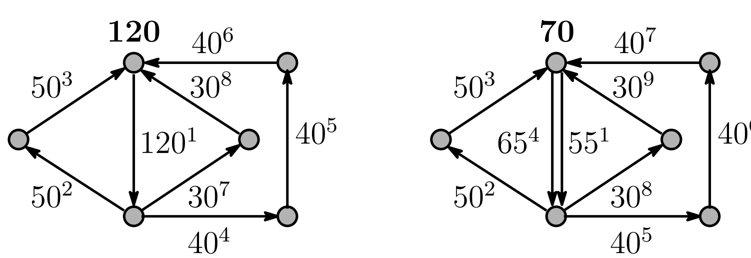

Figure 28: arrow slicing (partly cancel-out effect on min circulation through *arrow-twist*): Observe that there is no *arrow-twisted* cycles in the left fsp-Network. In the right shows an arrow-sliced fsp-Network, whose directed graph is shown in Figure 17, where there emerges a cycle which *dominates* cycles in the left fsp-Network as well as *arrow-twisted* cycles. It realizes the minimum circulation 70, which is larger than the maximum flow 65. The difference amounts to cancel-out effect by arrow-twist.

Theorem 13. *no effect of arrow slicing on max circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any arrow slicing on $a \in A$ to have $N^{f'} =< V, A', f' >$, we have

$x^{max}(N^{f'}) = x^{max}(N^f)$.

Proof. See Appendix A.5. □

We observe that *arrow separation* affect *vertex-twist* and *domination* in opposite direction, and we can interpret the effects are canceled out regarding max circulation. More clearly, when we confine us to non-punctured cycles, we observe that *arrow slicing* never affect *v-twit* nor *domination*, which amounts to no effect on max circulation in total.

vertex contraction

Theorem 14. *decreasing effect of vertex contraction on min circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any vertex contraction for N^f on $v, v' \in V$ to have $N^{f'}$, we have

$x^{min}(N^{f'}) \leq x^{min}(N^f)$.

Proof. The original sequence for the generated f-Network let the associated fsp-Network still covered. □

Vertex contraction has decreasing effect both through affecting *domination* and *arrow-twist*. Figure 29 shows effect through *domination*, where *vertex contraction* generates a cycle which dominates existent cycles. Figure 30 shows effect through *arrow-twist*, where *vertex contraction* let *arrow-reverse number* for two *arrow-twisted* cycles be less.

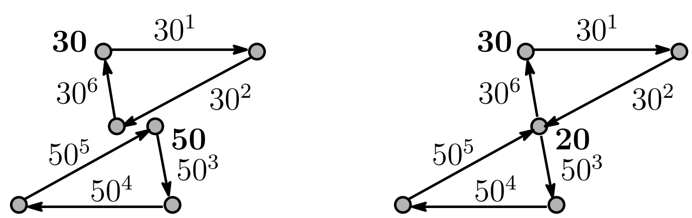

Figure 29: Vertex Contraction (decrease of min circulation through *domination*)

Theorem 15. *decreasing effect of vertex contraction on max circulation*

Given a closed f-Network $N^f =< V, A, f >$, for any vertex contraction on $v, v' \in V$ to have $N^{f'} =< V, A', f' >$, we have

$x^{max}(N^{f'}) \leq x^{max}(N^f)$.

Proof. We say $s : A \to \{1, 2, .., |A|\}$ and $s' : A' \to \{1, 2, .., |A'|\}$ are the same sequence when $s(a) = s'(a)$ for every $a \in A$ supposing each of $v, v' \in V$ is equal with $v \in V'$. For any same sequence s, s', take associated exact covered fsp-Network $< V, A, f, s, p >$ and $< V, A', f', s', p' >$, then we have $\sum_{v'' \in V \setminus (v,v')} p(v'') = \sum_{v'' \in V \setminus v} p'(v'')$, and $p(v) + p(v') \geq p'(v)$. □

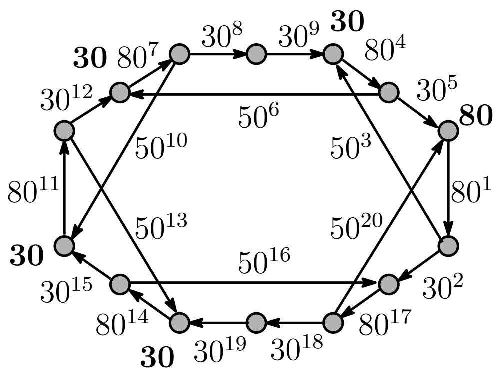

Figure 30: Vertex Contraction (decrease of min circulation through *arrow-twist*)

Vertex contraction affects *vertex-twist*, which itself has decreasing effect of max circulation as show in Figure 31, while its decreasing effect can be canceled out through affecting *domination* as shown in Figure 32. Notice that cycle set is unaffected in the former example.

When we confine us to non-punctured cycles, we interpret that *vertex contraction* never affect *domination*.

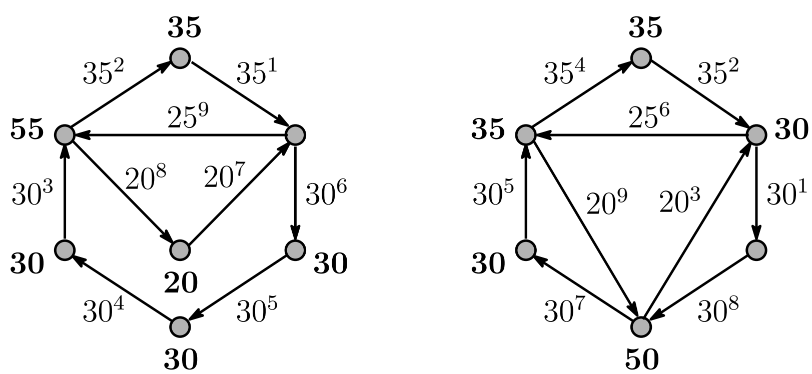

Figure 31: vertex contraction (decrease of max circulation through *vertex-twist*)

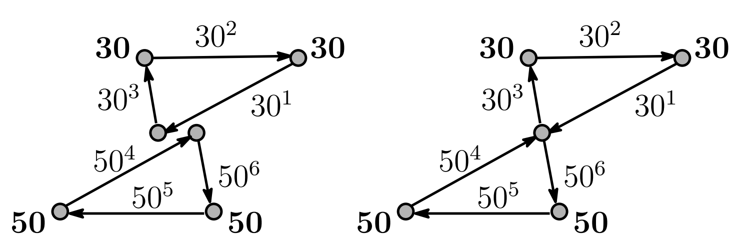

Figure 32: vertex contraction (no effect on max circulation)

6.3 Effects of Semi-Global Operations

cycle addition

We especially take up two special cases for the cycle addition which clarify heterogeneity of effects through *domination* and *arrow-twist*.

We say a directed graph $< V, A >$ is a *separated-cycles graph* when it consists of cycles which have no common vertex each other.

Theorem 16. *decreasing multiplier effect of cycle addition on min circulation*

Given a closed f-Network $N^f =< V, A, f >$ *where* $< V, A >$ *is a separated-cycles graph, make cycle addition* c *with* f^c *to* $N^{f'} =< V', A', f' >$ *so that* c *has at least one common vertex with* n *cycles but has no common arrows with any of the cycles. Then,*

$x^{min}(N^{f'}) - x^{min}(N^f) - f^c = -f^c * n + \sum_{c' \in C} \max\{f^c - f(c'), 0\}$.

Proof. It is straight since the cycle addition always add a punctured cycle which consists of $n + 1$ cycles. □

We say multiplier for the above N^f and $N^{f'}$ is $m = (x^{min}(N^{f'}) - x^{min}(N^f) - f^c)/f^c$.

We know that the multiplier is as large as n if f^c is sufficiently small, but it decreases as f^c gets larger, which is $\frac{\sum_{c' \in C} f^{c'}}{f^c}$ if $f^c > \max_{c' \in C} f^{c'}$.

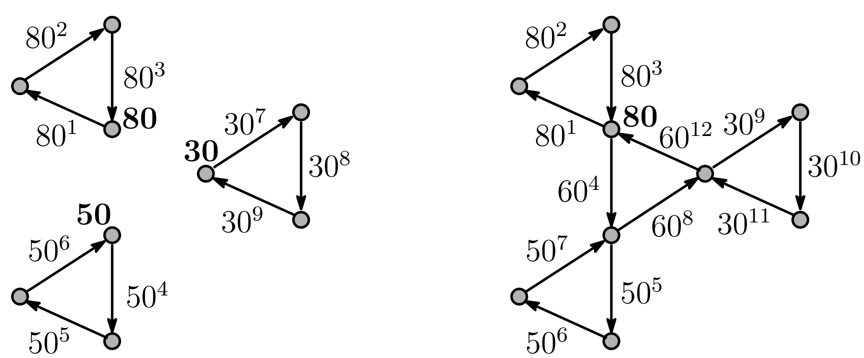

Figure 33: cycle addition (decreasing multiplier effect on min circulation through *domination*): min circulation decreases by $60 * 3 - \max\{60 - 80, 0\} - \max\{60 - 50, 0\} - \max\{60 - 30, 0\} - 60 = 80$.

The following theorem contrasts effect of *cycle addition* on max circulation to that on min circulation.

Theorem 17. *no effect of cycle addition on max circulation: separated non-punctured graph*

Given a closed f-Network $N^f =< V, A, f >$ *where* $< V, A >$ *is a separated non-punctured cycle graph, make cycle addition non-punctured cycle* c *with* f^c *to* $N^{f'} =< V', A', f' >$ *so that* c *has* $k \leq 2$ *common vertices with* n *cycles but has no common arrows with any of the cycles. Then,*

$x^{max}(N^{f'}) = x^{max}(N^f) + (|c| - 1)f^c$.

Proof. Denote $G =< V, A >$, $G' =< V', A' >$. When $k \leq 2$, there never appear vertex-twisted cycles other than punctured cycles including c. Theorem 9 ensures that the maximum circulation for $N^{f'}$ is mere summation of that of N^f and $(|c| - 1)f^c$. □

Next theorem shows effects through *arrow-twist* on min circulation, which has opposite effect of *domination*. The theorem also shows that *arrow-twist* is actually a determining property on min circulation for certain cases.

Theorem 18. *increasing multiplier effect of cycle addition on min circulation*

Given a closed f-Network which consists of only one non-punctured cycle $N^f =< V, c, f^c >$*. make cycle addition non-punctured cycle* c' *with* $f^{c'}$ *to have* $N^{f'} =< V', A', f' >$ *so that it has n arrow-reverse number with c and there exist no punctured cycle. Then,*

$x^{min}(N^{f'}) - x^{min}(N^f) - f^c = n * min\left\{f^c, f^{c'}\right\}$

Proof. We say two arrow-twisted cycles c, c' are *base n arrow-twisted* cycles if each cycle consists $2*(n+2)$ arrows, and they have $(n+2)$ common vertices so that they have n arrow-twists. The left graph of Figure 15 includes base 1 arrow-twisted cycles; $\{v_a v_b v_c v_d v_e v_f v_a, v_a v_d v_e v_b v_c v_f v_a\}$.

Suppose cycle addition is executed so that the added cycle is base n arrow-twisted with the other cycle. Then, closed cycle decomposition for $N^{f'}$ is realized either with $\{c, c'\}$, or with cycles which consists of 4 arrows; two arrows from common arrows of c, c' and each one arrow from cycles c, c', and either or both of c, c'. It is straight the minimum circulation is derived as above. For cycles which are n arrow-twisted but not base n arrow-twisted, our previous results for *Local Operations* ensures that constructed f-Network can always be transformed into that with two base n arrow-twisted cycles while the minimum circulation is unchanged. □

Note that the multiplier is as large as n if $f^{c'}$ is small enough, but decreases as $f^{c'}$ is larger, which is $\frac{n*f^c}{f^{c'}}$ if $f^{c'} > f^c$.

The theorem indicates how sequence needs to be taken to attain min circulation under existence of *arrow-twisted* cycles. As confirmed in the right of Figure 34, cycle with larger flow is endowed priority to the other in the sense that sequence is increasing along with the former cycle while not for the latter.

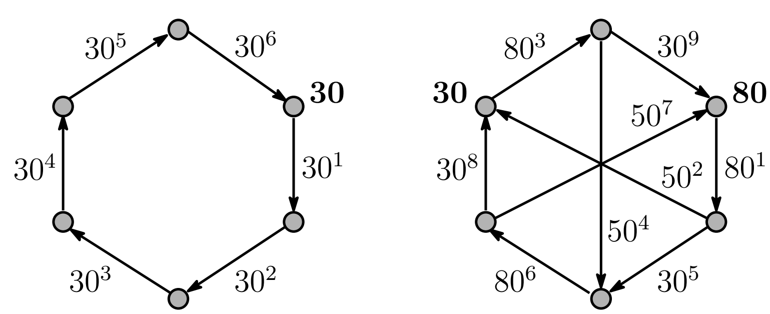

Figure 34: cycle addition (increasing multiplier effect on min circulation through *arrow-twist*): cycle addition with its flow 50 lets minimum circulation increase by $1 * \min\{30, 50\} + 50 = 80$.

There exists more complicated case where both properties of *domination* and *arrow-twist* take part, as shown in Figure 35.

Next we see effects of *vertex-twist* on max circulation. We take up a special class of f-Networks. Given $G =< V, A >$, we say two cycles $c, c' \in C_G$ are n *opposite cycles* for $n =$

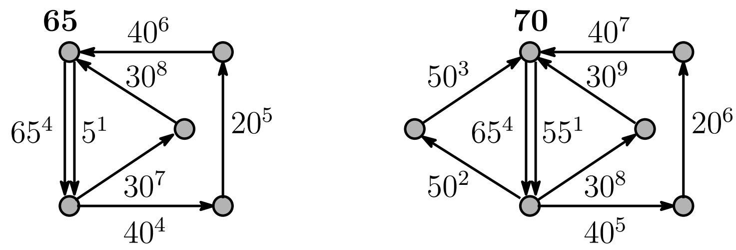

Figure 35: cycle addition (effect on min circulation through both *domination* and *arrow-twist*): cycle addition with its flow 50 lets minimum circulation decrease by 50(minus the added cycle value) through domination part, but arrow-twist partly cancels out the decrease by 5.

$1, 2, ..$ if c, c' have $n+2$ common vertices and no common arrow, and the common vertices appear exactly the opposite order. For example, in the left of Figure 18, $v_a v_b v_c v_d v_f v_a$ and $v_b v_f v_d$ are 1 *opposite cycles*.

Theorem 19. *decreasing multiplier effect of cycle addition on max circulation: n opposite cycles*

Given a closed f-Network which consists of only one non-punctured cycle $N^f =< V, c, f^c >$, *make cycle addition non-punctured cycle* c' *with* $f^{c'}$ *to have* $N^{f'}$ *so that it has either* c, c' *are* n *opposite cycles. Then,*

$x^{max}(N^{f'}) - x^{max}(N^f) - (|c'| - 1)f^{c'} = -n * \min\left\{f^c, f^{c'}\right\}$

Proof. When the cycles are n opposite cycles, Theorem 9 says that we only need to examine a closed cycle decomposition where all cycles are non-punctured. We have such a closed cycle decomposition which consists of $n+2$ cycles with flow $\min\left\{f^c, f^{c'}\right\}$, and either of the cycle c, c' with flow $|f^c - f^{c'}|$. Without loss of generality, suppose $f^c \leq f^{c'}$. For $n+2$ cycles with f^c, sum of maximum number of reverse equals to the number of cycles $n+2$. We have $x^{max}_{N^f} = f^c*(|c|+|c'|-n-2)+(f^{c'}-f^c)(|c'|-1) = (|c|-1)f^c+(|c'|-1)f^{c'}-n*f^c$. $\square$

Figure 36 is an example for the decreasing multiplier effect. Note that the multiplier is as large as n if $f^{c'}$ is small enough, but decreases as $f^{c'}$ is larger, which is $\frac{n*f^c}{f^{c'}}$ if $f^{c'} > f(c)$.

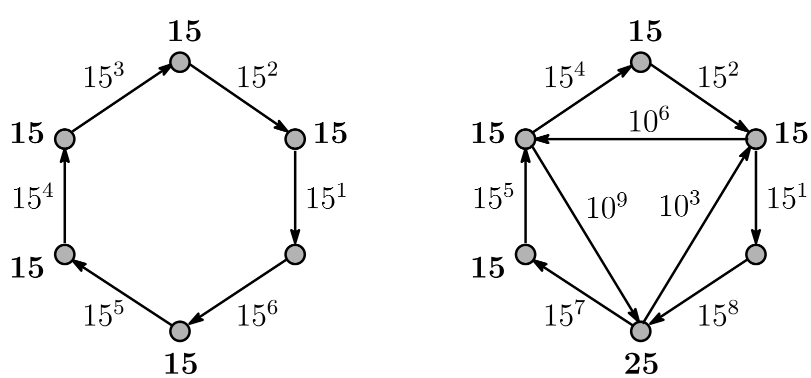

Figure 36: cycle addition (decreasing multiplier effect on max circulation through *vertex-twist*: cycle addition with its flow 10 lets the max circulation for the combined f-Network turn to be $15 * 5 + 10 * 2 - 1 * \min\{10, 20\} = 85$, which is less than $15 * 5 + 10 * 2$, the summation of max circulation for two combined f-Networks.

As a special case of *cycle addition*, we define *flow increase*. For a closed f-Network $N^f =< V, A, f >$, we say *flow increase* on N^f to $N^{f'} =< V, A, f' >$ when $f'(a) \geq f(a)$ for every $a \in A$ and there exists $a' \in A$ such that$f'(a') > f(a')$, and $N^{f'}$ is still closed.

Theorem 20. *Regime Change for min circulation*

For a closed f-Network $N^f =< V, A, f >$, suppose $< V, A, f, s, p >$ realizes the min circulation. We consider flow increase on N^f to $N^{f'} =< V, A, f' >$. Then,

there exists f' such that the minimum circulation is not attained with the original sequence s if and only if there exists arrow-twisted cycles.

Proof. When there exists no arrow-twisted cycles, the same sequence always gives the minimum circulation for any flow increase since flow increase never change domination. Next, suppose there exist arrow-twisted cycles for $< V, A, f >$, then for each sequence which realizes the minimum circulation we always find a cycle c where there exists more than one reverse. Take such sequence s and some other sequence s' which lets reverse for the cycle c be one. Circulation for s is less than that for s' by certain amount with the original flow f. When we increase flow for the cycle c, difference of circulations between under s and s' gets smaller in proportion to the increase. There exits some point where the original amount of difference is canceled-out and more increase lets circulation for s be larger than that for s', which completes our proof. $\square$

Regime Change for min circulation is only through *arrow-twist* since *domination* is unaffected by *flow increase*. Figure 37 shows an example.

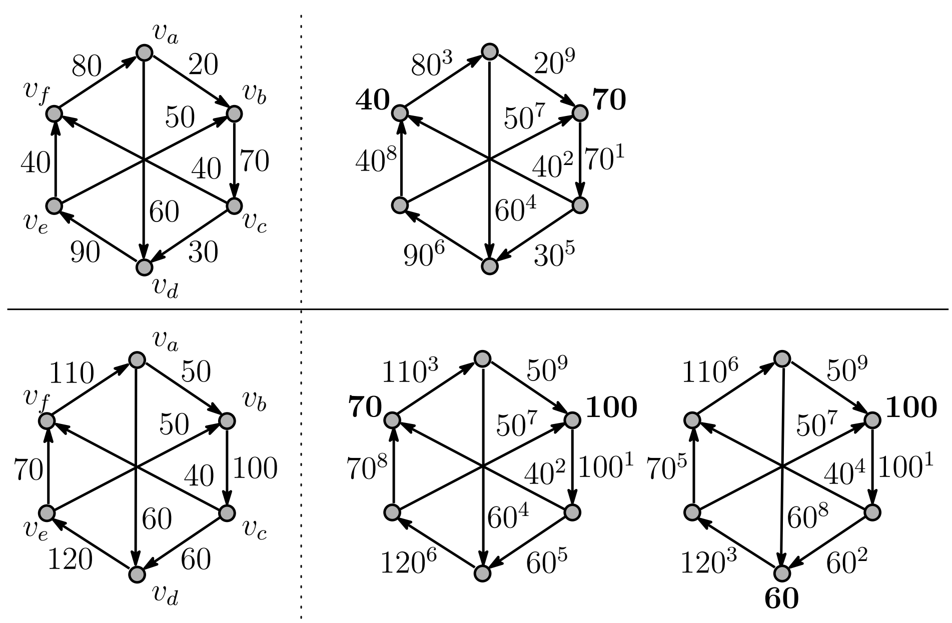

Figure 37: Regime Change; flows are increased for $v_av_bv_cv_dv_ev_fv_a$ by 30 (from the upper left to the lower left f-Network). The upper right realizes min circulation for the original f-Network. The lower middle is an exact covered fsp-Network for the new f-Network with the same sequence. The lower right realizes the minimum circulation for the new f-Network, where the sum of potentials is actually smaller than that of the lower middle by 10.

Theorem 21. *Regime Change for max PCP*

For a closed f-Network $N^f = < V, A, f >$ with $G = < V, A >$, suppose $< V, A, f, s, p >$ realizes max circulation. We consider flow increase on N^f to $N^{f'} = < V, A, f' >$. Then,

1). For any f', exact covered fsp-Network $< V, A, f', s, p' >$ always realizes the maximum circulation for $N^{f'}$ if there exists no vertex-twisted cycles.

2). There exists f' such that the maximum circulation is not attained with the original sequence s if there exists arrow-twisted cycles in C_G.

Proof. 1) is straight from the definition of vertex-twisted. 2) is the same as the counterpart theorem for min PCP. □

Figure 38 is an example for the Regime Change when vertex-twisted but not arrow-twisted cycles exist. Note that Regime Change may not occur depending on sequence for this case.

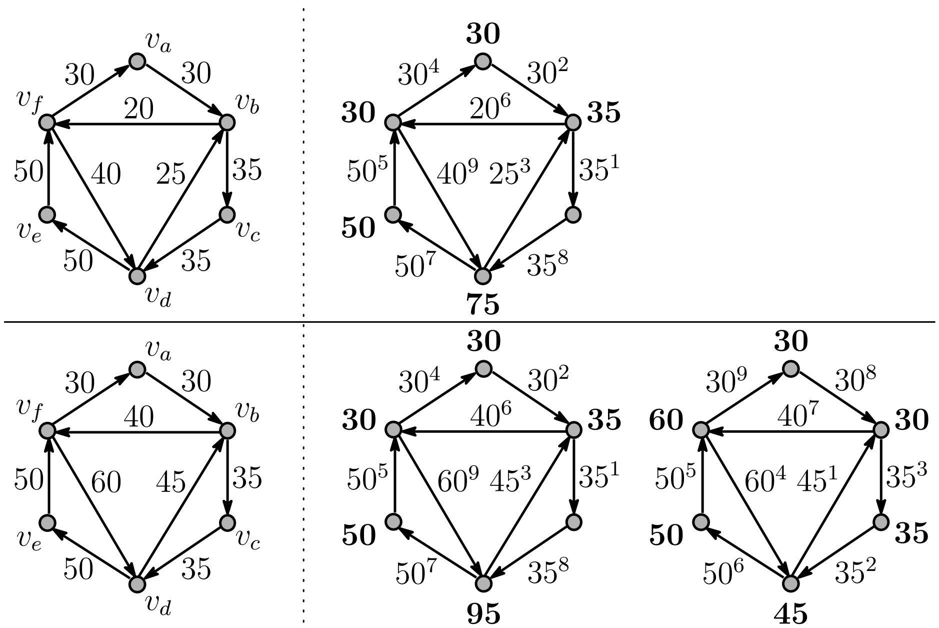

Figure 38: Regime Change; flows are increased for $v_b v_f v_d v_b$ by 20 (from the upper left to the lower left f-Network). The upper right realizes the maximum circulation for the original f-Network. The lower middle is an exact covered fsp-Network for the new f-Network with the same sequence. The lower right realizes the maximum circulation for the new f-Network, where the sum of potentials is confirmed as larger than that of the lower middle by 10

cycle separation

Theorem 22. *increasing effect of cycle separation on min circulation: one cycle*

Given a closed f-Network N^f which consists of a cycle c with its flow f^c. Make cycle separation on c in the amount of f^k to have a closed f-Network $N^{f'}$. Then,

$x^{min}(N^{f'}) - x^{min}(N^f) = f^k$.

Proof. Cycle separation add two more cycles, and the all three cycles are disjoint and not arrow-twisted. It is straight minimum circulation increases by f^k. □

We prepare terminologies for the next theorem. For n arrow-twisted cycles $c = v_1..v_kv_1, c'$ and its common arrows $A = c \cap c'$, replace $a = (v_{k'}, v_{k'+1}) \in A$ by $a' = (v, v') \notin A$ for c so that we have $c'' = v_1..v_{k'-1}vv'v_{k'+2}..v_kv_1$ is a cycle. We say $a \in A$ *contributes* its arrow-twistedness when c', c'' is no more n arrow-twisted. We say c, c' are *quasi-base* n arrow-twisted cycles if every common arrows contribute its arrow-twistedness.

Theorem 23. *decreasing effect of cycle separation on min circulation: quasi-base arrow-twisted non-punctured cycles*

Take a closed f-Network N^f such that $N^f =< V^c, c, f^c > + < V^{c'}, c', f^{c'} >$, where c, c' are quasi-base arrow-twisted non-punctured cycles. Suppose $f^c < f^{c'}$. Make cycle separation on c in the amount of $f^k (\leq f^c)$ so that separation is not on the common arrows, and each separated cycle has at least 2 common arrows with c'. Denote the generated f-Network as $N^{f'}$. Then,

$x^{min}(N^{f'}) - x^{min}(N^f) = -f^k$

Proof. Before the cycle separation, since the cycles are n arrow-twisted, the minimum circulation is $f(c') + (n+1) * f(c)$. c has $n+1$ reverse number under the minimum circulation. After the separation, sum of reverse numbers for the separated cycles is n under sequences which realizes the new minimum circulation. It is not the case when either separated cycle has at most 1 common arrow with c', since reverse number for each cycle cannot be less than 1 under any sequence. □

Decreasing effect of cycle separation is through decreasing the arrow-reverse number. Effects of cycle separations are shown in Figure 39, 40.

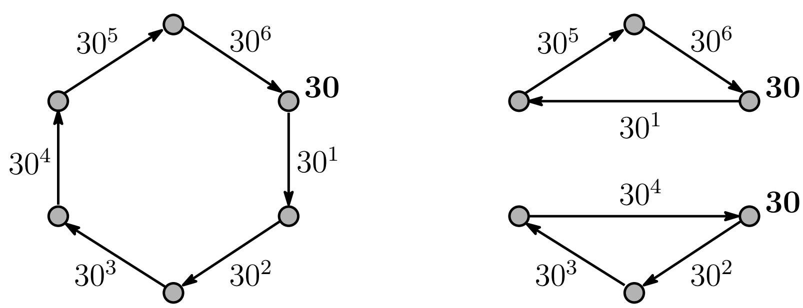

Figure 39: cycle separation (increasing effect on min circulation through *domination*)

Theorem 24. *decreasing effect of cycle separation on max circulation: one cycle*

Given a closed f-Network N^f which consists of a cycle c with its flow f^c. Make cycle separation on c in the amount of $f^k (\leq f^c)$ to have a closed f-Network $N^{f'}$. Then,

$x^{max}(N^{f'}) - x^{max}(N^f) = -f^k$.

Proof. It is straight and omitted. □

Theorem 25. *increasing effect of cycle separation on max circulation: opposite non-punctured cycles, quasi-base arrow-twisted non-punctured cycles*

Take a closed f-Network N^f such that $N^f =< V^c, c, f^c > + < V^{c'}, c', f^{c'} >$, where c, c' are either opposite non-punctured cycles or quasi-base arrow-twisted non-punctured cycles. Suppose $f^c < f^{c'}$.

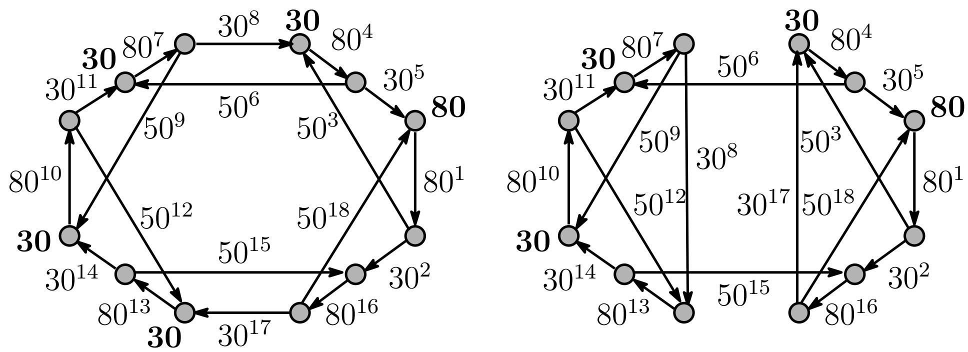

Figure 40: cycle separation (decreasing effect on min circulation through *arrow-twist*)

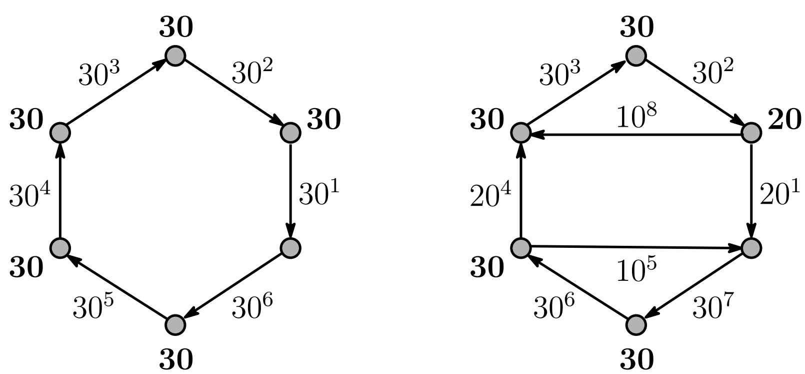

Figure 41: cycle separation (decreasing effect on max circulation through *weight*)

Make cycle separation on c in the amount of $f^k (\leq f^c)$ so that separation is not on the common arrows, and each separated cycle has at least 2 common arrows with c' when cycles are quasi-base arrow-twisted, while each separated cycle has at least 3 common vertices with c' for the case of opposite cycles. Denote the generated f-Network as $N^{f'}$. Then,

$$x^{max}(N^{f'}) - x^{max}(N^f) = f^k$$

Proof. We can prove exactly the same as the case for the min PCP for the case of arrow-twisted. For the case of opposite cycles, we have $x^{max}(N^f) = (|c'|-1)f^{c'} + (|c|-n)f^c$ from Theorem 9. By the separation, since separated cycles has at least 3 common vertices, $x^{max}_{N^{f'}} = (|c'|-1)f^{c'} + (|c|-n)(f^c - f^k) + (|c|-n+1)f^k$. It leads to $x^{max}(N^{f'}) = (|c'|-1)f^{c'} + (|c|-n)f^c + f^k$. □

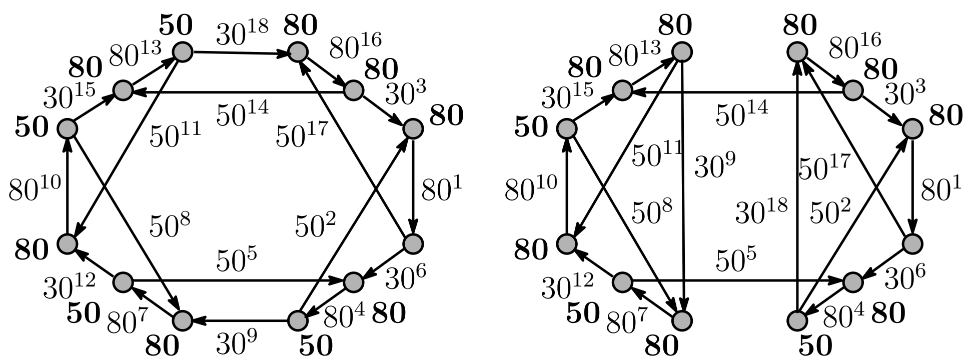

Figure 42: cycle separation (increasing effect on max circulation through *arrow-twist*)

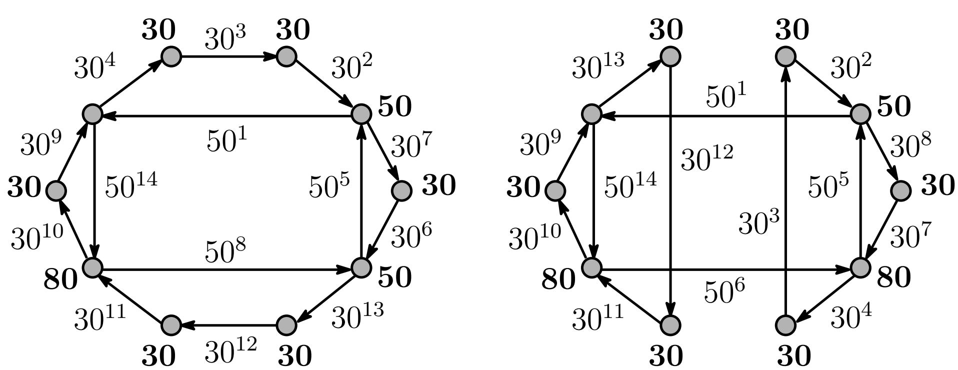

Figure 43: cycle separation (increasing effect on max circulation through *vertex-twist*)

7 Related Literature

Our contribution lies on the cross-point of financial economics and graph theory.

Quesnay (1758) provided the basis for graph-theoretic analysis on payment networks[2]. Mainly for the analysis of the reproduction of goods, he analyzed a simpler class of payment networks, which is embedded in our general model as a special case where payments

[2]For network analysis in the other field of economics, Jackson (2008) provides a wide survey.

are among three representative subjects (Proprietary, Productive, Sterile). Our notion of *closed* for payment flows is a generalized expression for one of the assumptions in the Tables.

The following main contributions to the graph-theoretic analysis on payment networks were accomplished in three fields: settlement systems, emergence of money, and currency area. As already stated in our introductory section, in the field of settlement systems, Eisenberg and Noe (2001) provided a mathematical framework to examine payment network but without element of order of settlement. Rotemberg (2011) firstly pointed out in a persuading manner that order of settlement possibly matters for liquidity problem in the field of settlement systems, taking up a specific class within Euler graph, examining order of settlement under certain exogenous way of decentralized decision pattern. Along with this literature, this paper provides a framework that treats a general class of payment network, and enables to examine every possible order of settlements for given distribution of obligations.

In the literature which covers the emergence of money, focus has been rather in showing the reason of existence of monetary substance, not much in examining a general network structure. Kiyotaki and Wright (1989) examined the circulation of the "medium of exchange" on"Wicksellian Triangle"[3], whose network structure is within the simplest cases in our model. Many of the studies that followed (Kiyotaki and Wright (1993),Trejos and Wright (1995),Lagos and Wright (2005)) adopted networks with one cycle. Yasutomi (2000) in section 4 focused on diversity of network and the related "strength" of money to the diversity. This was a major step in examining a class of graphs which contains multiple cycles, though still without *twisted* relations in the terminology of this paper. His study is interpreted to relate multiplicity of *closed cycle decomposition* to the emergence of "strong" or "weak" money. From the view of the *min PCP*, one of its important results is rephrased as "strong" money emerges only when the maximum flow is equal to the *minimum circulation*. In its historical analysis on the multiplicity of currency, Kuroda (2003) compared two types of graphs: the pyramid type and the horizontal type. His study is also related to the notion of *closed cycle decomposition*. He examined situations where each different type of money circulates within each set of *decomposed* cycles.

There are researches which focus on specific properties of payment network. In the analysis of financial contagion, "connectedness" or "connectivity" of network is shown to be a useful notion, as in the case of Allen and Gale (2000), Freixas, Parigi, and Rochet (2000), Lagunoff and Schreft (2001),Cifuentes, Shin, and Ferrucci (2005), Nier, Yang, Yorulmazer, and Alentorn (2007), Caballero and Simsek (2009),Gai and Kapadia (2010), Castiglionesi and Navarro (2008), and Allen, Babus, and Carletti (2010). Among them, "density" of network is proposed as an analytical tool in Zawadowski (2011). In the field of settlement system, the existence of a cycle itself is known as a potential source of gridlock(Beck and Sorämaki (2001)). Our original concepts of *arrow-twisted*, *vertex-twisted* properties provide a different aspect of payment network from those papers.

Related to properties of network in a broad sense, several concepts of properties have been proposed in the literature of social network[4]. Our analysis implies that properties relevant to "payment network" in the field of economics are not necessarily the same as those proposed for "social network", but possibly requires its own concepts.

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[3] a cycle with 3 subjects and 3 arrows among them

[4] See Jackson (2008) for analysis on social network.

From the graph-theoretic view, the *min/max PCP* can be thought of as a variant of "numbering" problem which contains aspects of "flow" problem[5]. The closest "numbering" problem would be the Bandwidth problem[6], where the objective being minimized is determined only with its sequence, while in the *min/max PCP*, the objective is more indirect, in the way that the amount of each flow takes part in. The "flow" aspect of the *min/max PCP* lets us utilize the method of *decomposition*, which has been shown as useful for some "flow" problem[7]. But its "numbering" aspect leads us to a distinct approach– examination of relation among *decomposed* cycles.

8 Concluding Remarks

We set up a graph-theoretic framework to express circulation of settlement fund, or money under network structures. From the view of the equation of exchange[8], its distinguishing characteristic is to capture relative[9] velocity of money through its element of *sequence*. This allows us to examine relation between velocity and quantity for money circulation. The framework has potential for wide applications in financial economics: in the field of settlement system, emergence of money, or currency area.

Under the framework, we have presented a pair of graph-theoretic problems that are fundamental for analysis on gross settlement systems. The problems have unique characteristics from the view of network problems in that the problems have both "numbering" and "flow" aspect. Utilizing a cycle decomposition approach, we have specified several key network properties for the problems. We have examined the effects of network transformations to show how each network property affects minimum and maximum circulation each.

Because we have concentrated on determining the general properties of the problems, it is not discussed which classes of payment networks well captures networks in our real world. Our future task is to specify appropriate classes of payment networks to execute more detailed analysis.

Turning to the graph-theoretic view, we conducted a qualitative characterization of the problems. One of the remaining tasks is to probe whether the problem is NP hard or not. It is known that although many types of "flow" problems are not NP hard, many "numbering" problems, including Bandwidth Problem, are NP hard. In our view, though the min/max PCP would be as "easy" as many "flow" problems regarding *domination* property, they will still contain "difficulty" in relation to *arrow-twist* and *vertex-twist* property. The latter two properties would be the key for the probing.

[5] Diaz, Petit, and Serna (2002) provides a survey for "numbering" problems in the view of graph layout on some dimension. For "flow" problems, see Ahuja, Magnanti, and Orlin (1993)

[6] See Chinn, Chvatalova, Dewdney, and Gibbs (1982) for the Bandwidth Problem.

[7] Goldberg, Plotkin, and Tardos (1991)

[8] Fisher (1911)

[9] The framework is not to capture absolute velocity of money. The framework clarifies the notional difference between *absolute* and *relative* velocity of money itself.

A Appendix

A.1 Local Minimum/Maximum

Given a covered fsp-Network $N^{fsp} =< V, A, f, s, p >$, we say s attains *local minimum* for $A' \subseteq A$ when we cannot take s' such that 1). s' gives the same ordering with s for $A \setminus A'$, and 2). for exact covered fsp-Network $< V, A, f, s', p' >$, $\sum_{v \in V} p'(v) < \sum_{v \in V} p(v)$. We similarly define *local maximum*.

The following theorem is useful.

Theorem 26. *local minimum/maximum*

Given a closed f-Network $N^f =< V, A, f >$, if an exact covered fsp-Network $N^{fsp} =< V, A, f, s, p >$ realizes the minimum/maximum circulation for N^f, then, s attains local minimum/maximum for every $A' \subset A$.

The theorem is straight from the definition of min/max PCP.

A.2 Proof of Theorem 2.

For vertices which have multiple inflows and/or outflows, we execute "unbundling" (see Figure 44). First "unbundle" some vertex to several "hypothetical" vertices such that each "hypothetical" vertex has one inflow and one outflow, and has exact amount of potential for each corresponding sequences. We can always execute such "unbundling", and the derived "unbundled" fsp-Network is also exact covered. We can continue this "unbundling" until each fsp-Network has no vertex which has multiple inflows and/or outflows, and any derived fsp-Network consists of exact covered fsp-Networks, each with one cycle.

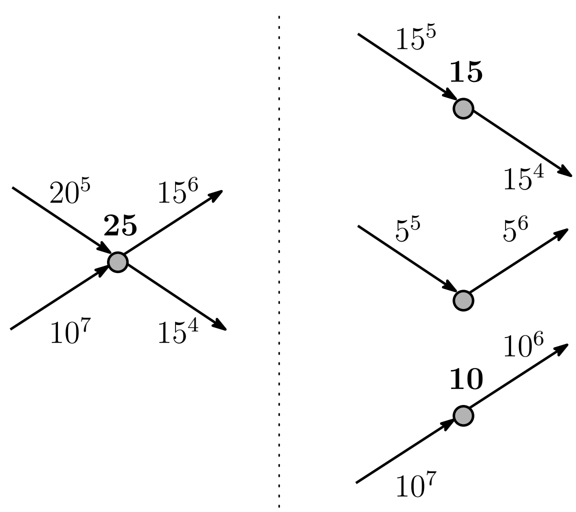

Figure 44: Example of "unbundling" procedure. "Unbundle" vertex in the left to hypothetical vertices in the right

A.3 Proof of Theorem 3.

Theorem 2 ensures that our search for fsp-Networks on the basis of *closed cycle decomposed* f-Networks always include right fsp-Networks in the sense they realize min circulation. What remains to be shown is that we correctly choose right fsp-Networks by minimizing circulation for *closed cycle decomposed* f-Networks. Next lemma ensures that part.

Lemma 7.
Given a closed f-Network $N^f =< V, A, f >$*,*
For any closed cycle decomposition $N^f = \sum_{c \in C} < V^c, c, f^c >$*, if* $< V^c, c, f^c, s^c, p^c >$ *is exact covered for every* $c \in C$*, and we can take* $s : V \to \{1, 2, .., |A|\}$ *which is consistent of* $\{s^c\}_{c \in C}$*, then*
$< V, A, f, s, p >= \sum_{c \in C} < V^c, c, f^c, s^c, p^c >$ *is covered .*

Proof. As long as $\{s^c\}_{c \in C}$ is consistent with s, it is straight that combining covered fsp-Networks always emerge a covered fsp-Network. □

The lemma states that our search on the basis of *closed cycle decomposed* f-Networks never let us find smaller circulation than "true" min circulation. Combining Theorem 2 and Lemma 7, we complete our proof.

A.4 Proof of Theorem 4.

Our proof is similar to that for Theorem 3. Theorem 2 ensures that our search for fsp-Networks on the basis of *closed cycle decomposed* f-Networks always include right fsp-Networks in the sense they realize max circulation. What remains to be shown is that we correctly choose right fsp-Networks by maximizing circulation for *closed cycle decomposed* f-Networks. Since we confine us to search cases where combined fsp-Networks become e-covered, Lemma 7 ensures our search on the basis of *closed cycle decomposed* f-Networks never let us reach larger circulation than "true" max circulation.

A.5 Proof of Theorem 13

We show that the maximum circulation for N^f is always attained for $N^{f'}$, and also maximum circulation for $N^{f'}$ is always attained for N^f. For the former part, given N^{fsp} which attains the maximum circulation for N^f, endow successive sequence for the two sliced arrow with the ordering of the arrows are unchanged with N^{fsp}, which ensures the same amount of circulation. For the latter part, we examine local maximum for sliced vertices $a, a' \in A'$ based on Theorem 26.

Suppose there exits sequence s for $N^{f'}$ such that it attains max circulation for $N^{f'}$. Take a new sequence s' for $N^{f'}$ such that a, a' has successive order in a way that $s(a) = s'(a)$ and $s'(a') = s'(a)+1$, and orderings among $A \setminus a'$ is the same between two sequences s and s'. Then, circulation realized with s' and $N^{f'}$ is equal to or smaller than that with s and N^f. When circulation gets smaller, take another sequence s'' such that $s(a') = s''(a')$ and $s''(a) = s''(a') + 1$ and orderings among $A \setminus a$ is the same between two sequence s and s''. It never changes circulation. It is clarified by dividing the above step to take a new sequence in two steps.

Consider we take a new sequence s' on $N^{f'}$ through following steps. The first step is to remove arrow a from fs-Network $N^{f'}$ with s. Take a temporal fs-Network which maintains all the orderings among $A \setminus a$. The second step is to add arrow a with a s'.

We have following lemma.

Lemma 8.

When circulation gets smaller for fs-Network with s' and $N^{f'}$ that that with s and N^f,

1) circulation of the temporal fs-Network needs to get less than that with s and $N^{f'}$, and

2) circulation of fs-Network with s' and $N^{f'}$ is equal to or larger than that of the temporal fs-Network, though the difference is less than that between circulation of the temporal fs-Network and that of s and $N^{f'}$.

Proof. Firstly, if circulation for the temporal fs-Network become larger, the sequence s contradicts to our assumption that it leads to max circulation. It is confirmed that we can take another sequence where a has the last order while maintaining the other orderings, which attains the same circulation by the temporal fs-Network.

Secondly, circulation for the temporal fs-Network is not larger than that for fs-Network with N^f and s'. It is because in that case when we further remove arrow a' from the temporal fs-Network, circulation gets larger, which leads to a contradiction as above. □

Now when we take the dividing steps on a', circulation of temporal fs-Network needs to the same as that with s and $N^{f'}$. It comes from the latter part of Lemma 8, which amounts to state that increasing flow for a' alone did not increase its circulation under s immediately. It tells that removing a' never decreases circulation, and we already confirmed it never increases.

Further, combining with the former part of the same lemma, we know that circulation of temporal fs-Network needs to be the same as that with s'' and $N^{f'}$. The lemma amounts to state that increasing flow only on a never decrease circulation, and we confirm that increase of circulation led to a contradiction of our assumption that s and $N^{f'}$ attains max circulation.

When two sliced arrows have successive ordering, we have no reason to distinguish sliced arrows from the original arrow regarding circulation, which completes our proof.

A.6 Results relevant to Rotemberg (2011)

We maintain notations in Rotemberg (2011) regarding its target class of network.

The next corollary shows that Rotemberg (2011) treated one of the simplest classes of f-Network with no arrow-twisted cycles.

Corollary 2. *case for Rotemberg (2011)*

For a closed f-Network $N^f =< V, A, f >$ which is in a class of C_N^K with flow z [10] *, we have*

$x_{N^f}^{min} = z.$

[10] We obey the definition of Rotemberg (2011) on C_N^K. N subjects indexed by $i \in [0, 1, .., N-1]$ are arrayed in a circle so that $N-1$ is followed by firm 0. Each subject i has payment in the amount of z to subjects $i+j$ with $j \leq K$, where the addition is taken modulo N. $2K \leq N-1$ is assumed. C_N^K is an associated f-Network.

Proof. Since C_N^K is based on an Euler graph, we can take a cycle c which consists of all the arrows. Further, since flow for each arrow is equal in the amount of z, we can take a closed cycle decomposition with unique undominated cycle c with flow $f(c) = z$. Since Euler graph has no arrow-twisted cycles, the minimum circulation is realized with c, and the derived value is z.

□

The next corollary shows that Rotemberg (2011) treated one of the simplest classes of f-Network with no vertex-twisted cycles.

Corollary 3. *case for Rotemberg (2011)*

For a closed f-Network $N^f = < V, A, f >$ which is in a class of C_N^K with flow z, suppose $N/(K!)$ is integer. Then, we have

$x_{N^f}^{max} = z \sum_{k=1}^{K} k * (\frac{N}{k} - 1)$

Proof. When $N/(K!)$ is integer, there exists no vertex-twisted cycles within C_G^{np} for any associated graph G. Further, we have $C_{N^f}^{nd,np} = C_{N^f}^{ud,np}$. We can take $C \in C^{nd,np}$ such that $|C| = \sum_{k=1}^{K} k$ and each $c \in C$ consists of $\frac{N}{k}$ vertices for $k = 1, 2, ..K$. Considering into weight $\frac{N}{k} - 1$ for each cycle with k vertices, we have the value as stated in the theorem. □

Note that if $N/(K!)$ is not integer, then there exists vertex-twisted cycles within C_G^{np}. We proceed to examine effects of several network transformation on each maximum circulation.

References

Ahuja, R. K., T. L. Magnanti, and J. B. Orlin (1993): *Network Flows*, New Jersey, U.S.: Prentice Hall.

Allen, F., A. Babus, and E. Carletti (2010): "Financial Connections and Systemic Risk," *NBER Working Paper No.16177.*

Allen, F. and D. Gale (2000): "Financial Contagion," *Journal of Political Economy*, 108, 1–33.

Beck, M. L. and K. Sorämaki (2001): "Gridlock resolution in Interbank Payment System," *Bank of Finland*, Discussion Paper 9・2001.

Caballero, R. J. and A. Simsek (2009): "Complexity and Financial Panics," *NBER Working Paper No. 14997.*

Castiglionesi, F. and N. Navarro (2008): "Optimal Fragile Financial Networks," *Second Singapore International Conference on Finance 2008 EFA 2008 Athens Meetings Paper.*

Chinn, P. Z., J. Chvatalova, A. K. Dewdney, and N. E. Gibbs (1982): "The Bandwidth Problem for Graphs and Matrices A Survey," *Journal of Graph Theory*, 6, 223–254.

Cifuentes, R., H. S. Shin, and G. Ferrucci (2005): "Liquidity Risk and Contagion," *Journal of the European Economic Association*, 3, 556–566.

Diaz, J., J. Petit, and M. Serna (2002): "A survey of graph layout problems," *ACM COMPUTING SURVEYS*, 34, 313–356.

Eisenberg, L. and T. H. Noe (2001): "Systemic Risk in Financial Systems," *Management Science*, 47, 236–249.

Fisher, I. (1911): *The Purchasing Power of Money*, New York: The Macmillan Company.

Ford, L. R. and D. R. Fulkerson (1962): *Flows in Networks*, Princeton, NJ: Princeton University Press.

Freixas, X., B. M. Parigi, and J.-C. Rochet (2000): "Systemic Risk, Interbank Relations, and Liquidity Provision by the Central Bank," *Journal of Money, Credit, and Banking*, 32, 611–638.

Gai, P. and S. Kapadia (2010): "Contagion in Financial Networks," *Working Paper No.383, Bank of England.*

Goldberg, A. V., S. A. Plotkin, and E. Tardos (1991): "Combinatorial Algorithms for the Generalized Circulation Problem," *Mathematics of Operations Research*, 16, 351–381.

Jackson, M. O. (2008): *Social and Economic Networks*, Princeton University Press.

Kiyotaki, N. and R. Wright (1989): "On Money as a Medium of Exchange," *Journal of Political Economy*, 97, 927–954.

——— (1993): "A Search-Theoretic Approach to Monetary Economics," *American Economic Review*, 83, 63–77.

Kuroda, A. (2003): *KaheiSisutemu-no-Sekaishi(World History of Currency System)*, Tokyo, Japan: Iwanami Shoten.

Lagos, R. and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113, 463–484.

Lagunoff, R. and S. L. Schreft (2001): "A Model of Financial Fragility," *Journal of Economic Theory*, 99, 220–264.

Nier, E., J. Yang, T. Yorulmazer, and A. Alentorn (2007): "Network Models and Financial Stability," *Journal of Economic Dynamics and Control*, 31, 2033–2060.

Quesnay, F. (1758): *Tableau économique.*

Rotemberg, J. J. (2011): "Minimal Settlement Assets in Economies with Interconnected Financial Obligations," *Journal of Money, Credit, and Banking*, 43, 81–108.

Trejos, A. and R. Wright (1995): "Search, Bargaining, Money, and Prices," *Journal of Political Economy*, 103, 118–141.

Yasutomi, A. (2000): *Kahei-no-Hukuzatusei(Complexity of Money; Theory of emergence and collapse)*, Tokyo, Japan: Sobunsya.

Zawadowski, A. (2011): "Entangled Financial Systems," *Working Paper*.