Existence of Equilibrium in Tullock Contests with Incomplete Information*

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Abstract

We show that under general assumptions every member of a broad class of generalized Tullock contests with incomplete information has a pure strategy Bayesian Nash equilibrium.

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1 Introduction

The simplest form of a *Tullock contest* – see Tullock (1980) – is a "lottery contest", in which each player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. Tullock (1980) also considers a more general class of contests, in which the probability of success is taken to be the ratio between the individual and the total "productivities" of the efforts, where the productivity of effort is linked to the effort by a power function with a positive exponent[1]. Tullock's framework is a frequent choice in modelling contests with imperfect discrimination, and Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races) which are strategically equivalent to Tullock contests. In addition, Tullock contests arise by design, e.g., in sport competition, internal labor markets – an axiomatic justification for the class of success functions has been offered in a number of studies; see, e.g., Skaperdas (1996) and Clark and Riis (1998).

The existence of (pure strategy) Nash equilibria in Tullock contests with complete information has long been known; such equilibria were studied, e.g., by Perez-Castrillo and Verdier (1992) for symmetric Tullock contests, and by Cornes and Hartly (2005) for asymmetric contests. Szidarovszky and Okuguchi (1997) established existence and uniqueness of equilibrium for contests with a proportional success function more general than that of Tullock (1980), where each player's probability to win is given by the ratio between the productivity of that player's effort and the sum of productivities of all players, and the "production function for lotteries"[2] of each player – determining his effort's productivity – is twice continuously differentiable, strictly increasing, concave, and vanishing at zero.

Although the bulk of research on Tullock contests had the complete information case at its focus, there is a growing number of works dedicated to the study of pure strategy equilibria in Tullock contests with incomplete information. To mention some key contributions, Hurley and Shogren (1998), Malueg and Yates (2004), and Fey (2008) consider two-player Tullock lottery contests in the independent private

[1] The exponent r determines the type of returns to scale of the production function for lotteries: the returns are decreasing when $r < 1$, constant when $r = 1$, and increasing when $r > 1$.

[2] We borrow this term from Szidarovszky and Okuguchi (1997).

value framework. Ryvkin (2010) extends the equilibrium existence results of Fey (2008) into the symmetric multi-player setting, while allowing a general Szidarovszky and Okuguchi (1997) type of the contest success function and a general continuous distribution of the players' cost parameters. Warneryd (2012) also considers mutli-player contests, but in his model the (continuously distributed) value for the prize is common, and the are two types of players who are either completely informed or uninformed about the value. Wasser (2013) proves an equilibrium existence result for general private-value imperfectly discriminating contests, under the assumption that the contest success function is continuous everywhere.[3]

The purpose of this work is to establish an equilibrium existence result for a broad class of Tullock contests with incomplete (and generally asymmetric) information. In our setting, each player's value for the prize, his cost of effort, and the contest success function (specifying the probability distribution that is used to allocate the prize for each profile of efforts) may depend on the state of nature. The set of states of nature need not be finite or countable. Players have a common prior belief, but upon realization of a state of nature, and prior to taking action, each player observes some event that contains the realized state of nature. The information of each player at the moment of taking action is described by a (finite or countable) partition of the set of states of nature. An incomplete information contest is therefore formally described by a set of players, a probability space describing players' uncertainty and their prior belief, a collection of partitions of the state space describing the players' information, a collection of state-dependent functions describing the players' values and costs, and a state-dependent success function. This representation accommodates the Harsanyi types model, in which players' type sets are finite or countable – see Jackson (1993) and Vohra (1999). (In a similar framework, but with finitely many states of nature, Einy *et al* (2001, 2002), Forges and Orzach (2011), and Malueg and Orzach (2009, 2012) study common-value first- and second-price auctions.)

An incomplete information contest will be termed *a generalized Tullock contest* if its success function has the following three properties at each state of nature: (i) when the total effort is positive, each player's probability of winning the prize is continuous

[3] Wasser (2012) also considers modified Tullock lottery contests in which the proportional success function is made continuous when all efforts are zero, by adding a positive fixed "noise" parameter to the numerator and the denominator of the ratio.

with respect to the efforts of all players; (ii) each player's probability of winning is non-decreasing and concave in his own effort (and hence exhibits decreasing returns); (iii) if only one player exerts positive effort, then his probability of winning is 1 – i.e., his effort is perfectly discriminated in this case. We will also assume that all players' cost functions are strictly increasing, convex, continuous, and vanishing at zero.

The proportional success function underlying the Tullock lottery contest obviously satisfies (i)–(iii), and so do the more general proportional success functions considered in Szidarovszky and Okuguchi (1997); in fact, in order to satisfy (i)–(iii) the production functions for lotteries in Szidarovszky and Okuguchi (1997) set-up need not be differentiable. Furthermore, our assumptions allow for a great deal more interdependence between the players' efforts in determining the winning probabilities compared to what is entailed by the additive separability in aggregating effort productivities in proportional success functions.

Unlike in earlier works, the success function in a generalized Tullock contest may well vary with the state of nature.[4] We do not limit the set of states of nature be finite or countable, although each player can have at most countably many information sets (corresponding to "types" in the Harsanyi framework). In this aspect our work differs from Ryvkin (2010), Warneryd (2012), and Wasser (2013), who do allow types sets that are a one-dimensional continuum. However, as has already been mentioned, in Ryvkin (2010) the players are ex ante symmetric, while the generalized Tullock contests that we consider need not exhibit any symmetry. Further, in Warneryd (2012) the common value for the prize is given by the type of each informed player (as the players can either possess full information about the value or have no information at all besides their prior belief), and in Wasser (2013) the private value for the prize is assumed to be non-decreasing in the player's type[5]. Importantly, our model of generalized Tullock contests (which accommodates both private or common values) imposes no restrictions on how the value depends on the state of nature, or on the players' private information.

[4] We will introduce an assumption (numbered as (vi) in the text) that will limit the variability of the success function with the state of nature, but only when the set of states of nature is uncountable.

[5] Another notable difference between our model and Wasser (2013) is, as has already been mentioned, in his assumption that the contest success function is continuous for *all* effort profiles, which is not the case even in the simplest Tullock lottery contest.

We will show that any generalized Tullock contest has a pure strategy Bayesian Nash equilibrium. Our proof uses the equilibrium existence theorem of Reny (1999), and, to our knowledge, is the first work on Tullock contests to rely on this tool. The main step of the proof is showing that a generalized Tullock contest is a *better-reply-secure* game, which is one of the main premises for Reny's theorem[6]. Better reply security of the expected payoff functions is a weakening of the usual continuity requirement (the latter would have allowed us to use of the standard Nash's equilibrium existence theorem[7]). However, the expected payoff functions are not continuous when all strategies prescribe zero effort at some states of nature, due to the inherent discontinuity of all contest success functions satisfying (i)–(iii) above at the zero effort profile. Thus, it is the possibility to rely on the property of better reply security, weaker than continuity, that allows establishing equilibrium existence, due to the theorem of Reny (1999).

The rest of the paper is organized as follows: Section 2 describes the general setting of contests with incomplete information and introduces generalized Tullock contests, and Section 3 contains our result on the existence a pure strategy Bayesian Nash equilibrium.

2 Generalized Tullock contests

2.1 Contests with Incomplete Information

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by choosing a level of *effort* in $\mathbb{R}_+$. Players' uncertainty about the state of nature is described

[6]The other conditions, which are the compactness of the players' strategy sets and quasi-concavity of the (expected) payoff functions in players' own strategies, hold (either directly or through an equivalent modification) in our model, as the proof will show.

[7]The use of Nash's existence theorem is still a viable alternative when there are finitely many states of nature. In the latter case, Einy *et al* (2013) (the discussion paper upon which the current work is based) provide a proof that first considers "truncated" contests in which players choose efforts from a compact interval with a positive lower bound, on which the expected payoff functions are continuous, thereby allowing the use of the Nash's theorem to deduce the existence of equilibrium. The crux of the proof is to show that a limit point of the sequence of equilibria of truncated contests with a lower bound on players' efforts approaching zero is an equilibrium in the original contest.

by a probability measure p (representing the players' common prior belief) over a measurable space $(\Omega, \mathcal{F})$ of states of nature. The private information about the state of nature of player $i \in N$ is described by an $\mathcal{F}$-measurable and at most countable partition Π_i of Ω (w.l.o.g. we will assume that $p(\pi_i) > 0$ for each $\pi_i \in \Pi_i$). The value for the prize of each player i is given by an $\mathcal{F}$-measurable and integrable random variable $V_i : \Omega \to \mathbb{R}_{++}$, i.e., if $\omega \in \Omega$ is realized then player i's ("private") value for the prize is $V_i(\omega)$. The cost of effort of each player $i \in N$ is given by a jointly measurable function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, such that:

(i) for every $x \in \mathbb{R}_+$ the random variable $c_i(\cdot, x)$ is integrable,

and

(ii) for any $\omega \in \Omega$ the function $c_i(\omega, \cdot)$ is strictly increasing, continuous, convex, and vanishing at 0.

An incomplete information contest starts by a move of nature that selects a state ω from Ω according to the distribution p. Every player $i \in N$ observes the element $\pi_i(\omega)$ of Π_i which contains ω – the set of states of nature between which i cannot distinguish given ω. Then players simultaneously choose their effort levels, which results in a profile of efforts $(x_1, ..., x_n) \in \mathbb{R}^n_+$. The prize is awarded to the players in a probabilistic fashion, according to a state-dependent *success function* $\rho : \Omega \times \mathbb{R}^n_+ \to \Delta^{n-1}$ that attributes to each $\omega \in \Omega$ and a profile of effort levels $x \in \mathbb{R}^n_+$ a probability distribution $\rho(\omega, x)$ in the $(n-1)$-dimensional simplex $\Delta^{n-1} \subset \mathbb{R}^n_+$, according to which the prize recipient is chosen if ω is realized. Hence, the payoff of player $i \in N$, $u_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$, is given for every $\omega \in \Omega$ and $x \in \mathbb{R}^n_+$ by

$$u_i(\omega, x) = \rho_i(\omega, x) \cdot V_i(\omega) - c_i(\omega, x_i). \tag{1}$$

Thus, an incomplete information contest is described by a collection $(N, (\Omega, \mathcal{F}, p), \{\Pi_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$.

In an incomplete information contest, a pure strategy of player $i \in N$ is a Π_i-measurable function $X_i : \Omega \to \mathbb{R}_+$ (i.e., X_i is constant on every element of Π_i), that represents i's choice of effort in each state of nature following the observation of his private information. We denote by S_i the set of strategies of player i, and by $S = \times_{i=1}^n S_i$ the set of strategy profiles. For any strategy $X_i \in S_i$ and $\pi_i \in \Pi_i$, $X_i(\pi_i)$ will stand for the constant value that $X_i(\cdot)$ obtains on π_i. Also, given a strategy profile $X = (X_1, ..., X_n) \in S$, we will denote by X_{-i} the profile obtained from X by suppressing the strategy of player $i \in N$. Throughout the paper we restrict attention to pure strategies.

Let $X = (X_1, ..., X_n)$ be a strategy profile. We denote by $U_i(X)$ the expected payoff of player i, i.e.,

$$U_i(X) \equiv E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot))].$$

For $\pi_i \in \Pi_i$, we denote by $U_i(X \mid \pi_i)$ the expected payoff of player i conditional on π_i, i.e.,

$$U_i(X \mid \pi_i) \equiv E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot)) \mid \pi_i].$$

An n-tuple of strategies $X^* = (X_1^*, ..., X_n^*)$ is a (Bayesian Nash) equilibrium if

$$U_i(X^*) \geq U_i(X_{-i}^*, X_i) \tag{2}$$

for every player $i \in N$, and every strategy $X_i \in S_i$; or equivalently,

$$U_i(X^* \mid \pi_i) \geq U_i(X_{-i}^*, x_i \mid \pi_i) \tag{3}$$

for every $i \in N$, every $\pi_i \in \Pi_i$, and every effort $x_i \in \mathbb{R}_+$ of player i (viewed here as a strategy in S_i with a constant value x_i on the set π_i).

2.2 Generalizing Tullock Contests in the Incomplete Information Setup

The subclass of incomplete information contests that we now introduce is characterized by some simple properties of the contest success function. For $x \in \mathbb{R}_+^n$ we denote by $x_{-i} \in \mathbb{R}_+^{n-1}$ the profile of efforts obtained from x by suppressing the effort of player i, and by $\mathbf{0} \in \mathbb{R}_+^n$ the zero vector (i.e., the profile of zero efforts in our context). A *generalized Tullock contest* is an incomplete information contest in which the success function ρ has the following properties at every $\omega \in \Omega$:

(iii) $\rho(\omega, \cdot)$ is continuous on $\mathbb{R}_+^n \diagdown \{\mathbf{0}\}$;

(iv) for every $i \in N$ and $x_{-i} \in \mathbb{R}_+^{n-1}$, $\rho_i(\omega, x_{-i}, x_i)$ is non-decreasing and concave in the effort x_i of player i;

and

(v) for every $i \in N$ and $x_i > 0$, $\rho_i(\omega, \mathbf{0}_{-i}, x_i) = 1$, i.e., if all players but i make zero effort at ω, any positive effort by i guarantees that he gets the prize with probability 1 at that state of nature.[8]

If the set of states of nature Ω is finite or countable, no restriction will be put on the dependence of the success function ρ on the state of nature. Only if Ω is uncountable, we will assume that each player's chance of success, $\rho_i(\cdot, x)$, is constant given his information set π_i; this will hold, in particular, when the chances of success are common knowledge of the players at every state of nature:

(vi) If Ω is uncountable, then for every $i \in N$ and $x \in \mathbb{R}^n_+$, $\rho_i(\cdot, x)$ is measurable with respect to Π_i.

A *Tullock lottery contest* is a particular case of a generalized Tullock contest, in which the state-independent success function ρ^T is given for each $x \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ and $i \in N$ by

$$\rho_i^T(x) = \frac{x_i}{\sum_{j=1}^n x_j}. \tag{4}$$

It is easy to see that ρ^T satisfies conditions (iii), (iv) and (v). More generally, conditions (iii)–(v) are satisfied by any success function ρ that is given for any $\omega \in \Omega$, $x \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ and $i \in N$ by

$$\rho_i(\omega, x) = \frac{g_i(\omega, x_i)}{\sum_{j=1}^n g_j(\omega, x_j)}, \tag{5}$$

where for every $\omega \in \Omega$ and $j \in N$ the state-dependent *production function for lotteries* $g_j(\omega, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, describing the productivity of j's efforts, is strictly increasing, continuous, concave, and vanishes at 0, and $g_j(\cdot, x_j)$ is Π_i-measurable for every $x_j \in \mathbb{R}_+$. Thus, incomplete information contests with success functions given by (5) are also generalized Tullock contests.[9] In particular, the functional form in (5) can accommodate the commonly assumed contest success functions with $g_i(\omega, x_i) = x_i^r$, if the "impact parameter" r belongs to $(0, 1]$.

[8] Notice that (v) implies (iv) for $x_{-i} = \mathbf{0}_{-i}$ (due to the assumption that $\rho_i(\omega, \mathbf{0}) \leq 1$), and hence it would have sufficed to state property (iv) only for $x_{-i} \in \mathbb{R}^{n-1}_+ \diagdown \{\mathbf{0}_{-i}\}$.

[9] Existence of equilibrium in contests with success functions belonging to this class was established for the complete information case by Szidarovszky and Okuguchi (1997), under an additional assumption that each $g_j(\omega, \cdot)$ is twice continuously differentiable.

3 Existence of Equilibrium

This section contains our equilibrium existence result.

Theorem. *Every generalized Tullock contest has a (pure strategy) Bayesian Nash equilibrium.*

Proof. Let $C = (N, (\Omega, \mathcal{F}, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$ be a generalized Tullock contest.

Step 1. This step is needed only if Ω is uncountable. In this case, we will construct an equivalent generalized Tullock contest with *at most countable* set of states of nature, each of which occurs with positive probability.

Let Ω' be the set of all positive probability elements of Π, where $\Pi = \vee_{i \in N} \Pi_i$ is the coarsest partition of Ω that refines each Π_i. Note that, as each Π_i is at most countable, so is Ω'. Define the probability distribution p' on Ω' by $p'(\{\omega'\}) = p(\omega')$ for every $\omega' \in \Omega'$, and, for every $i \in N$, consider the partition Π_i' of Ω' that consists of the sets $\pi_i' = \{\omega' \in \Omega' \mid \omega' \subset \pi_i\}$ for every $\pi_i \in \Pi_i$. Furthermore, for every $\omega' \in \Omega'$, $x \in \mathbb{R}_+^n$, and $i \in N$, define

$$V_i'(\omega') \equiv E[V_i(\cdot) \mid \omega'] \text{ and } c_i'(\omega', x_i) \equiv E[c_i(\cdot, x_i) \mid \omega'],$$

and also note that

$$\rho_i'(\omega', x) \equiv \rho_i(\omega, x) \text{ if } \omega \in \omega'$$

is well-defined as ρ_i is Π_i-measurable by condition (vi). It is easy to see that the functions $\{V_i'\}_{i \in N}$ are integrable on Ω', and that $\{c_i'\}_{i \in N}$ and ρ' satisfy conditions (i)–(v) with Ω' as the new set of states of nature (specifically, the continuity of $c_i'(\omega', \cdot)$ in condition (ii) is an implication of the dominated convergence theorem and the assumptions on c_i). Thus, $C' = (N, (\Omega', 2^{\Omega'}, p'), \{\Pi_i'\}_{i \in N}, \{V_i'\}_{i \in N}, \{c_i'\}_{i \in N}, \rho')$ also constitutes a generalized Tullock contest. Denote by $u_i'(\cdot, \cdot)$ the state-dependent payoff function of player i in C', and by U_i' his expected payoff function.

Since every strategy X_i of player $i \in N$ obtains the constant value $X_i(\pi_i)$ on each $\pi_i \in \Pi_i$, it is identifiable with his strategy X_i' in C' that obtains the value $X_i(\pi_i)$ on each $\pi_i' \in \Pi_i'$ (where $\pi_i = \cup \pi_i'$ up to a zero-probability set); the map $X_i \longrightarrow X_i'$ is a bijection. Given a strategy profile $X = (X_1, ..., X_n)$, observe that for every $\omega' \in \Omega'$

$$\begin{aligned} E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot)) \quad &| \quad \omega'] \\ &= \rho_i'(\omega', x) \cdot E[V_i(\cdot) \mid \omega'] - E[c_i(\cdot, x_i) \mid \omega'] \\ &= \rho_i'(\omega', x) \cdot V_i'(\omega') - c_i'(\omega', x_i) \\ &= u_i'(\omega', (X_1'(\omega'), ..., X_n'(\omega')). \end{aligned}$$

It follows that

$$\begin{aligned} U_i(X) &= E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot))] \\ &= \sum_{\omega' \in \Omega'} E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot)) \mid \omega'] \cdot p(\omega') \\ &= \sum_{\omega' \in \Omega'} u_i'(\omega', (X_1'(\omega'), ..., X_n'(\omega')) \cdot p'(\{\omega'\}) \\ &= U_i'(X'). \end{aligned}$$

The contests C and C' are therefore equivalent (under the above identification of strategies).

We conclude that it entails no loss of generality to assume that the set of states of nature Ω in the given contest C is *at most countable*, and that each state in Ω occurs with positive probability.[10] These assumptions on C will be maintained henceforth.

Step 2. We will now construct a "bounded" variant of the given contest C, in which the strategy sets are compact.

Since the cost function of each player is strictly increasing and convex in the player's effort, $\lim_{x_i \to \infty} c_i(\cdot, x_i) = \infty$, and hence $\lim_{x_i \to \infty} E[c_i(\cdot, x_i) \mid \pi_i] = \infty$ by Fatou's lemma for every $i \in N$ and $\pi_i \in \Pi_i$. It follows that every $i \in N$ and $\pi_i \in \Pi_i$ there exists $Q^i_{\pi_i} > 0$ such that $E[V_i(\cdot) \mid \pi_i] < E[c_i(\cdot, Q^i_{\pi_i}) \mid \pi_i]$. Since $E[c_i(\cdot, 0) \mid \pi_i] = 0$, and $E[c_i(\cdot, x_i) \mid \pi_i]$ is continuous in x_i on the interval $\left[0, Q^i_{\pi_i}\right]$ (the latter property follows from the dominated convergence theorem and the monotonicity of $c_i(\cdot, x_i)$ in x_i), there exists $0 < \overline{Q}^i_{\pi_i} < Q^i_{\pi_i}$ such that

$$E[V_i(\cdot) \mid \pi_i] < E[c_i(\cdot, \overline{Q}^i_{\pi_i}) \mid \pi_i] < E[V_i(\cdot) \mid \pi_i] + 1. \tag{6}$$

[10] If Ω is countable in the original contest C, simply strike out all zero-probability states of nature to obtain an equivalent contest.

Consider a variant $\overline{C}$ of the given contest C, in which the effort set of each player i is restricted to be the bounded interval $\left[0, \overline{Q}^i_{\pi_i}\right]$ given his information set π_i. In $\overline{C}$, the set of strategies of player i, $\overline{S}_i$, is identifiable with the *compact* and metrizable product set $\times_{\pi_i \in \Pi_i} \left[0, \overline{Q}^i_{\pi_i}\right]$ via the the bijection $X_i \longleftrightarrow (X_i(\pi_i))_{\pi_i \in \Pi_i}$, and player i's expected payoff function U_i is *concave* in i's own strategy (as the state-dependent payoff function $u_i(\cdot, x)$ is concave in the variable x_i, which follows from conditions (ii) and (iv)).

For each $i \in N$, the expected payoff function U_i is *not* continuous on $\overline{S} = \times_{i=1}^n \overline{S}_i$, but we will show that it is continuous on $\overline{S}_+$, where $\overline{S}_+ \subset \overline{S}$ is the set that consists of strategy-profiles X such that $X(\cdot) \neq \mathbf{0}$ on Ω. Indeed, consider a sequence $\left(X^k\right)_{k=1}^\infty \subset \overline{S}_+$ of strategy profiles that converge (pointwise) to a profile $X \in \overline{S}_+$. Then

$$\lim_{k\to\infty} E[\rho_i\left(\cdot, X^k(\cdot)\right) \cdot V_i(\cdot)] = E[\rho_i(\cdot, X(\cdot)) \cdot V_i(\cdot)]$$

by the dominated convergence theorem and the fact that ρ is continuous on $\mathbb{R}^n_+ \diagdown \{\mathbf{0}\}$ by condition (iv), and

$$\lim_{k\to\infty} E[c_i\left(\cdot, X^k_i(\cdot)\right)] = E[c_i(\cdot, X_i(\cdot))]$$

by the dominated convergence theorem[11] and the continuity of the cost function which is ensured by condition (ii). It now follows from (1) that $\lim_{k\to\infty} U_i\left(X^k\right) = U_i(X)$.

Each function U_i is, moreover, *lower semi-continuous* in the variable $X_i \in \overline{S}_i$; i.e., for a fixed $X_{-i} \in \overline{S}_{-i} \equiv \times_{j\neq i}\overline{S}_j$ and every sequence $\left(X^k_i\right)_{k=1}^\infty \subset \overline{S}_i$ that converges (pointwise) to X_i, $\liminf_{k\to\infty} U_i\left(X_{-i}, X^k_i\right) \geq U_i(X_{-i}, X_i)$. Indeed, since the i^{th} component of the success function, ρ_i, is lower semi-continuous in $x_i \in \mathbb{R}_+$ as follows from conditions (iii) and (v), and the cost function is continuous in $x_i \in \mathbb{R}_+$, using (1) we obtain $\liminf_{k\to\infty} u_i\left(\cdot, X_{-i}(\cdot), X^k_i(\cdot)\right) \geq u_i(\cdot, X_{-i}(\cdot), X_i(\cdot))$. It follows from Fatou's lemma that $\liminf_{k\to\infty} U_i\left(X_{-i}, X^k_i\right) \geq U_i(X_{-i}, X_i)$.

Given the compactness of $\overline{S}_i$ and the concavity of U_i in the variable $X_i \in \overline{S}_i$, for each $i \in N$, existence of equilibrium in $\overline{C}$ is guaranteed by Theorem 3.1 of Reny (1999), provided $\overline{C}$ is in addition *better-reply-secure*: if (a) $\left(X^k\right)_{k=1}^\infty \subset \overline{S}$ is a sequence such that the (pointwise) limit $X \equiv \lim_{k\to\infty} X^k$ exists and X is not a Bayesian Nash

[11] The cost of i is bounded from above by the function that is equal to $c_i(\cdot, \overline{Q}^i_{\pi_i})$ on each π_i, which is integrable by the second inequality in (6).

equilibrium in $\overline{C}$; and (b) $w_i \equiv \lim_{k\to\infty} U_i(X^k)$ exists for every $i \in N$, then there must be some player i that can *secure* a payoff greater than w_i at X, i.e., there exist $Y_i \in \overline{S}_i$, $z_i > w_i$, and an open neighborhood $W \subset \overline{S}_{-i}$ of X_{-i} such that $U_i(X'_{-i}, Y_i) \geq z_i$ for every $X'_{-i} \in W$.

Step 3. We will show that $\overline{C}$ is, indeed, better-reply-secure.

Let $\left(X^k\right)_{k=1}^{\infty}$, X, and $(w_i)_{i\in N}$ be as above. If $X \in \overline{S}_+$, then the functions $(U_i)_{i\in N}$ are continuous at X and hence $w_i = U_i(X)$ for every $i \in N$. Since X is not an equilibrium by assumption, there exist $i \in N$ and $Y_i \in \overline{S}_i$ such that

$$U_i(X_{-i}, Y_i) > w_i + \varepsilon \tag{7}$$

for some $\varepsilon > 0$. It can be assumed w.l.o.g. that Y_i is strictly positive, as U_i is lower semi-continuous in the i^{th} variable. By the continuity of U_i at $(X_{-i}, Y_i) \in \overline{S}_+$, $U_i(X'_{-i}, Y_i) \geq z_i \equiv w_i + \frac{\varepsilon}{2}$ for every X'_{-i} in some open neighborhood W of X_{-i}, and thus i can secure at X a payoff greater than w_i.

Assume now that $X \in \overline{S}\backslash\overline{S}_+$; thus, $X(\omega^*) = \mathbf{0}$ for some $\omega^* \in \Omega$. Since Ω is at most countable, which can be assumed w.l.o.g. following step 1 of the proof, the set $\left(\Delta^{n-1}\right)^{\Omega}$ (where, recall, Δ^{n-1} denotes the $n-1$-simplex in $\mathbb{R}^n$) is metrizable and hence sequentially compact in the product topology. We can therefore consider an accumulation point $(\widetilde{p}(\omega))_{\omega\in\Omega}$ of the sequence $\{\left(\rho\left(\omega, X^k(\omega)\right)\right)_{\omega\in\Omega}\}_{k=1}^{\infty}$. Assume w.l.o.g. (passing to a subsequence if necessary) that $\lim_{k\to\infty}\left(\rho\left(\omega, X^k(\omega)\right)\right)_{\omega\in\Omega} = (\widetilde{p}(\omega))_{\omega\in\Omega}$. Define, for every $\omega \in \Omega$ and $i \in N$,

$$\widetilde{w}_i(\omega) \equiv \widetilde{p}_i(\omega) \cdot V_i(\omega) - c_i(\omega, X_i(\omega)).$$

By the continuity of the cost function and the dominated convergence theorem, $w_i = E(\widetilde{w}_i(\cdot))$.

Since $\widetilde{p}(\omega^*)$ is a probability vector, there exists $i \in N$ for whom

$$\widetilde{p}_i(\omega^*) < 1. \tag{8}$$

For any $0 < \varepsilon < Q$, consider a strategy $Y_i^{\varepsilon} \in \overline{S}_i$ given by $Y_i^{\varepsilon}(\cdot) \equiv \max\{X_i(\cdot), \varepsilon\}$. (In particular, $Y_i^{\varepsilon}(\pi_i(\omega^*)) = \varepsilon$.) Then for any $\omega \in \Omega$ with $X(\omega) \neq \mathbf{0}$,

$$\lim_{\varepsilon\to 0+} u_i(\omega, X_{-i}(\omega), Y_i^{\varepsilon}(\omega)) = \lim_{\varepsilon\to 0+}\left[\rho_i(\omega, X_{-i}(\omega), Y_i^{\varepsilon}(\omega)) \cdot V_i(\omega) - c_i(\omega, Y_i^{\varepsilon}(\omega))\right] = \widetilde{w}_i(\omega), \tag{9}$$

since ρ_i is continuous at $X(\omega) \neq \mathbf{0}$ and therefore $\lim_{\varepsilon \to 0+} \rho_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) = \lim_{k\to\infty} \rho_i\left(\omega, X^k(\omega)\right) = \widetilde{p}_i(\omega)$. And for any $\omega \in \Omega$ with $X(\omega) = \mathbf{0}$,

$$\lim_{\varepsilon \to 0+} u_i(\omega, X_{-i}(\omega), Y_i^\varepsilon(\omega)) = \lim_{\varepsilon \to 0+} \left[\rho_i(\omega, 0_{-i}, \varepsilon) \cdot V_i(\omega) - c_i(\omega, \varepsilon)\right] = V_i(\omega) \geq \widetilde{w}_i(\omega) \tag{10}$$

by property (v) of ρ, with a *strict inequality* for $\omega = \omega^*$ as follows from (8) and the assumption that every $V_i(\omega)$ is strictly positive. It is then implied by (9) and (10) and the dominated convergence theorem that[12]

$$\lim_{\varepsilon \to 0+} U_i(X_{-i}, Y_i^\varepsilon) > E\left(\widetilde{w}_i(\cdot)\right) = w_i. \tag{11}$$

Now fix some $\varepsilon > 0$ for which $U_i(X_{-i}, Y_i^\varepsilon) > w_i + \varepsilon$ (it exists by (11)), and denote $Y_i \equiv Y_i^\varepsilon$. By definition, (X_{-i}, Y_i) satisfies (7), and repeating the arguments following (7) shows that i can secure a payoff greater than w_i. Thus $\overline{C}$ is better-reply-secure.

We conclude that $\overline{C}$ possesses some Bayesian Nash equilibrium X^*. In particular, X^* satisfies (3) for every $i \in N$, $\pi_i \in \Pi_i$, and $x_i \in \left[0, \overline{Q}^i_{\pi_i}\right]$. But note that every $x_i > \overline{Q}^i_{\pi_i}$ leads to a negative expected payoff to player i conditional on $\pi_i \in \Pi_i$ (this follows from the first inequality in (6)), which can be improved upon by lowering the effort on π_i to zero. Thus, in contemplating a unilateral deviation from $X_i^*(\pi_i)$ conditional on π_i, player i is never worse off by limiting himself to efforts $0 \leq x_i \leq \overline{Q}^i_{\pi_i}$. But this means that X^* satisfies (3) for every $x_i \in \mathbb{R}_+$. Since this is the case for every $i \in N$ and every $\pi_i \in \Pi_i$, X^* is a Bayesian Nash equilibrium of the original contest C. ■

Our theorem makes no assumptions about players' private information, and applies regardless of whether players have private or common values, or whether their costs of effort are the same or different. It also implies existence of a Bayesian Nash equilibrium in a generalized Tullock contest in Harsanyi's types model, where each player's uncertain type represents his private information, and players have a common prior distribution over all possible realizations of types, provided each player's type set is at most countable. (In the discrete case, these two models of incomplete information games are equivalent – see Jackson (1993) and Vohra (1999).)

[12] Recall that the cost of i is bounded from above by an integrable function that is equal to $c_i(\cdot, \overline{Q}^i_{\pi_i})$ on each π_i, and that w.l.o.g. (following step 1 of the proof) $p(\{\omega^*\}) > 0$.

The remark below, which follows from Theorem 1 of Ewerhart and Quartieri (2013), states sufficient conditions under which an equilibrium in a generalized Tullock contest is unique.

Remark. *Suppose that in a generalized Tullock contest: (a) Ω is finite, (b) the state-dependend production functions $(g_i(\omega,\cdot))_{i\in N,\omega\in\Omega}$ in (5) are twice differentiable and $\rho_i(\omega,0)<1$ for every $i\in N$ and $\omega\in\Omega$, (c) players' state-dependent cost functions $(c_i(\omega,\cdot))_{i\in N,\omega\in\Omega}$ are twice differentiable, and (d) each player i's value function v_i has the form $v_i(\omega)=v(\omega)\cdot k_i(\omega)$, where $v:\Omega\to R_{++}$, and $k_i:\Omega\to R_{++}$ is Π_i-measurable. Then the contest has a* unique *(pure strategy) Bayesian Nash equilibrium.*

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