# A Dynamic Mechanism Design for Scheduling with Different Use Lengths*

Ryuji Sano†
Institute of Economic Research, Kyoto University

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**Abstract**

This paper considers a dynamic allocation problem. A number of identical perishable goods, such as time slots of a central facility or hotel rooms, are allocated at each period. A number of agents randomly come to a mechanism, and each agent wants to keep winning a good for more than one period to make profits. The seller offers simple long-term contracts that guarantee future allocations to agents. We characterize incentive compatible mechanisms in our domain of mechanisms, and provide a dynamic VCG mechanism that achieves efficient allocations. The seller's revenue is maximized by virtual valuation maximization under a hazard rate condition. Price discounts for long stay agents can be supported as the optimal pricing with certain distributions.

*Keywords*: dynamic mechanism, online mechanism, dynamic population, dynamic VCG mechanism, optimal auction
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†Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan. Telephone: +81-75-7537122. E-mail: sano@kier.kyoto-u.ac.jp

## 1 Introduction

This paper considers a dynamic allocation problem where a seller allocates many perishable objects at each period. A number of agents randomly arrive over time. Agents want to obtain at most one unit of the object at each period. However, they stay for several periods of time, and need to keep obtaining the object during their stay to earn positive profits.

A motivating example is a time-slot allocation problem. Suppose that there are a number of facilities such as city halls, meeting rooms, or shared computer servers, and time slots to use such facilities are allocated over time. Potential users (agents) randomly arrive over time, and slots of time are allocated at a time. Agents often want to use the facilities for several periods of time. For example, an academic conference or an exhibition would be held at a hotel or a convention center for several days or a week. A musician wants to hold a concert at a hall for several days. People would like to stay at a hotel for several nights, or a computer job would need a long time to complete on a server. Agents need the object for different periods to be satisfied, and the necessary period is in general private information of each agent along with the valuation. Agents thus have two dimensional type, value and time period, and earn the value only when they use the facilities for the whole of the time they need it for.

In such a case, the seller often offers long-term allocations to agents and reserves future slots for them in advance. Although a long-term contract might possibly be contingent on future events, it is frequently hard to make such a complex contract in practical situations. Only simple incomplete contracts that are not contingent on future events are available in real situations.

This paper considers a mechanism in which a seller offers simple long-term contracts to agents. Agents randomly arrive over time, having a two dimensional type: valuation and time period. Each agent earns a value only if he gets the object for the length of the period. A seller or a mechanism designer offers an $m_i$-period simple contract to an agent when he arrives. A simple contract specifies a sequence of allocations to an agent regardless of future events.

We focus on dynamic direct mechanisms by the revelation principle. We characterize incentive compatible mechanisms, and provide an efficient mechanism and a revenue-maximizing mechanism in our domain of mechanisms. The standard manner in mechanism design is applied to a dynamic allocation problem of our situation. Given a period type, incentive compatibility requires that the allocation policy for an agent is increasing in his valuation, and that the payment is determined by the envelope theorem formula. In addition, to ensure reporting true period types, the allocation policy needs to satisfy a weak notion of monotonicity on period types. We show that these conditions are also sufficient for incentive compatibility.

We then provide an efficient mechanism and a revenue-maximizing mechanism. The efficient mechanism is straightforwardly provided as an extension of the well-known Vickrey-Clarke-Groves mechanism. Each agent pays the expected externality that he gives to the other current agnets and future agents. The revenue-maximizing or the optimal mechanism is provided as in Myerson (1981). We show that the optimal allocation policy maximizes the discounted virtual valuations under a monotone hazard rate condition. By introducing the notion of residual periods of objects, both the efficient and the optimal allocation policies are described in tractable forms.

In the single unit case, the optimal allocation policy is distorted toward the way that long-stay agents can be more favored than in the efficient allocation policy. In addition, in the optimal pricing, there can exist price discouts in the per period sense for long-stay agents, which is often observed in the real world. However, such a long-stay discount is not robust. The existence of long-stay discout or long-stay premium depends on the type and population distributions.

The contribution of the paper is to formulate a model of a dynamic allocation problem, which has not been covered by preceding studies but is often observed in practical situations. Although several studies such as Bergemann and Valimaki (2010), Pai and Vohra (2011), and Pavan et al. (2009) consider similar environments, they do not consider long-term contracts investigated in this paper. In addition, we characterize incentive compatibility and provide an optimal mechanism in a situation where agents have two dimensional types in a special form. Although the allocation problem is dynamic and complex for the seller, we can deal with the problem in a standard manner. We can also specify the effcient or the optimal mechanisms in several cases, whereas those in preceding studies are complicated and it is typically hard to specify the detail.

## 1.1 Related Literature

In recent years, there have been a number of studies on dynamic auction design.[1] Dynamic auction design in an environment where agents strategically arrive and depart is often called online mechanism design, and has been investigated in the fields of computer sciences and operations research (Lavi and Nisan, 2000).

Parkes and Singh (2003), Mierendorff (2009), and Pai and Vohra (2011) consider an allocation problem of durable goods, such as air tickets sales and hotel room reservations. Buyers arrive over time, stay for several periods, and participate in auctions during the stay. They want to buy an object at most once in their stay. Parkes and Singh (2003) extend the Vickrey-Clarke-Groves mechanism to the case where buyers strategically choose their arrival and departure time. Mierendorff (2009) and Pai and Vohra (2011) characterize incentive compatible and individually rational mechanisms and investigate the optimal mechanism.

Hajiaghayi et al. (2005) and Parkes (2007) consider a perishable goods case such as scheduling of facilities in the presence of strategic arrivals and departures. They consider incentive compatible mechanisms and investigate the efficiency of an allocation policy. Porter (2004) considers a dynamic mechanism in the context of a computer job assignment on a server, and introduce the notion of job length, which corresponds to period type in this paper.

Vulcano et al. (2002), Gershkov and Moldovanu (2009, 2010), Board and Skrzypacz (2010), and Said (2012) consider that agents randomly come to a mechanism. Vulcano et al. (2002) and Gershkov and Moldovanu (2009, 2010) consider the efficient or the revenue-maximizing durable goods sales in which agents are impatient and short-lived. Board and Skrzypacz (2010) also consider a durable goods sale in which agents are patient and stay until the deadline of the sale. Said (2012) considers a perishable goods case. Buyers arrive at random, and stay in the next period with a positive probability common among buyers. He shows that the outcomes in the efficient or the optimal mechanisms can be achieved by repeated ascending auctions in a perfect Bayesian equilibrium.

Most of these papers consider the situation where agents stay in a mechanism for

[1] See Bergemann and Said (2011) for a review. Dynamic mechanism design is also called online mechanism design especially in the fields of operations research and computer sciences. See also Parkes (2007).

a long time and want to win an auction at most once. Hence, agents stay until they win an object, and they are supposed to exit once they win. In other words, agents are assumed to evaluate intertemporal objects as perfect substitutes. On the other hand, our model considers the case where agents evaluate intertemporal objects as perfect complements.

There are also studies on a similar dynamic allocation problem with dynamic information. Bergemann and Valimaki (2010) consider an infinitely repeated allocation problem with a type of agent being drawn at each period by a Markov process. They formulate an incentive compatible efficient mechanism called "dynamic pivot mechanism," which is an extension of the Vickrey-Clarke-Groves mechanism. Athey and Segal (2007) consider a similar situation and provide an efficient budget-balancing mechanism. Pavan et al. (2009) and Kakade et al. (2011) characterize incentive compatibility and provide revenue equivalence in a dynamic information model.

The remainder of the paper is as follows. In section 2, we provide a model of the time-slot scheduling problem. We explain agents' preferences and the domain of dynamic mechanisms. In section 3, we characterize incentive compatible mechanisms, and show a revenue equivalence result. In section 4, we provide the dynamic VCG mechanism. We show that truthful reporting is a dominant strategy equilibrium in the dynamic VCG mechanism. In section 5, we provide the revenue-maximizing mechanism. We show that the revenue-maximizing allocation policy maximizes the virtual welfare under the monotone hazard rate condition. We also investigate the optimal pricing in a special case.

## 2 The Model

We consider an environment with independent and private values in a discrete-time model. $K$ identical objects, such as time slots of a city hall or facilities, are supplied at each period $t = 1, \dots, T$. Suppose $T \leq \infty$; time horizon is either finite or infinite. For simplicity, we consider the infinite horizon. Objects are non-storable and perish at the end of each period. At each period, a finite number of agents enter a mechanism. The set of entrants at $t$ is denoted by $N^t$. Each agent at each period is ex ante homogeneous, and the number of entrants $|N^t|$ is an i.i.d. random variable at each period. Each entrant wants to own at most one unit of the object at a period. The set of agents having entered by $t$ is denoted by $\mathcal{N}^t \equiv \bigcup_{s \leq t} N^s$. An allocation at $t$ is denoted by $a^t = (a_i^t)_{i \in \mathcal{N}^t}$. And, $a_i^t \in [0,1]$ denotes the probability of obtaining the object at $t$. An allocation $a^t$ is said to be *feasible at t* if $\sum_i a_i^t \leq K$ and $a_i^t = 0$ for any $i$ who is not in the mechanism at $t$.

Agents and the seller discount future payoffs by a common factor $\delta \in (0,1)$. Each agent $i$ has his/her private information $\theta_i \equiv (V_i, l_i) \in [0, \bar{V}] \times \{1, \dots, L\} \equiv \Theta_i$. Agent $i$ of type $\theta_i = (V_i, l_i)$ stays in a mechanism for at least $l_i$ periods. When agent $i$ of type $\theta_i$ enters at $t$, $i$'s utility evaluated at $t$ is given by

$$u_i = \begin{cases} V_i - \sum_{s=t}^{\infty} \delta^{s-t} p_i^s & \text{if } a_i^s = 1 \text{ for } \forall s \in \{t, \dots, t + l_i - 1\}, \\ - \sum_{s=t}^{\infty} \delta^{s-t} p_i^s & \text{otherwise,} \end{cases} \tag{1}$$

where $p_i^s$ denotes $i$'s payment at $s$. Agent $i$ earns a total profit $V_i$ only if he owns the object for $l_i$ periods of time.[2] Otherwise, agent $i$ earns nothing. We call $V_i$, *valuation type* and $l_i$, *period type.* Each agent's type $\theta_i$ is independently drawn from an identical distribution $F$ on $\Theta_i$. Let $F(\cdot|l_i)$ be the cumulative distribution function conditional on $l_i$. Given any $l_i$, $F(\cdot|l_i)$ has a density function $f(\cdot|l_i) \geq 0$ for all $V_i$. In addition, $f(l_i) \equiv \int_0^{\bar{V}} f(V_i, l_i) \mathrm{d}V_i$ denotes the probability that an agent's period type is $l_i$.

The seller offers a long-term contract for $i$. Each agent signs only one contract when he enters a mechanism.[3] There is no renegotication and a contract is never revised or interrupted before expiration. A contract for $i$ at $t$, $z_i^t$, consists of its term, and a sequence of allocations and payments for $i$: $z_i^t \equiv \{a_i^s, p_i^s\}_{s=t}^{t+m_i-1}$. The contract term is denoted by $m_i$, which is assumed to be deterministic. Goods allocation $a_i^s$ is assigned at $s$ with payment $p_i^s$. We limit attention to simple package contracts that depend only on current history. Further, goods allocation is probabilistic only in the first period of the contract. When $a_i^t \in (0,1)$, $i$ obtains an object with probability $a_i^t$ at $t$, and the allocation realized at $t$ is assigned with probability 1 in the later periods until its expiration. In other words, probability is assigned not on periodic allocations but on a package of $m_i$-period allocations $(a_i^t, \dots, a_i^{t+m_i-1}) = (1, \dots, 1)$. Let $\bar{a}_i^t \in \{0,1\}$ be realized allocation for $i$ at $t$.

Note that two contracts $z_i^t$ and $\tilde{z}_i^t$ with identical terms and allocations are indifferent for $i$ whenever $\sum \delta^{s-t} p_i^s = \sum \delta^{s-t} \tilde{p}_i^s$. Hence, abusing notation, a contract can be denoted by $z_i^t = (a_i, m_i, p_i)$, where $a_i$ denotes the (random) allocation at $t$, $m_i$

[2] Note that total profit $V_i$ is evaluated at $t$.

[3] If an agent rejects a contract, then he leaves the mechanism and gets payoff 0.

denotes the contract term, and $p_i \equiv \sum \delta^{s-t} p_i^s$. The set of enforcable contracts for $i$ is denoted by $Z_i \equiv [0,1] \times \{1, \ldots, L\} \times \mathbb{R}$. The set of bundles of contracts at $t$ is denoted by $Z^t \equiv \times_{i \in \mathcal{N}^t} Z_i$. For a contract $z_i^t = (\tilde{a}_i, \tilde{m}_i, \tilde{p}_i)$, we use the following notations for the corresponding components: $a_i(z_i^t) = \tilde{a}_i$, $m_i(z_i^t) = \tilde{m}_i$, and $p_i(z_i^t) = \tilde{p}_i$. In addition, $\bar{a}_i(z_i^t)$ denotes the realized allocation from contract $z_i^t$.

**Remark 1** We restrict the domain of contracts (mechanisms) to incomplete "package contracts," and discuss the efficiency or revenue maximization in the restricted domain. One may consider complete contracts that are contingent on future events. In fact, our model can be included in that of Bergemann and Valimaki (2010) by redefining the notion of agents' types, and fully efficient allocations are implemented by their "dynamic pivot mechanism." However, it is typically intractable to have a specific allocation policy.

## 2.1 Dynamic Direct Mechanisms

We assume that each agent cannot manipulate the arrival time. However, they may manipulate their departures or period types. We also assume that each agent does not observe the past history or the number of the current agents $N^t$ when making a report. This assumption would be natural in many practical situations such as facility scheduling and hotel room assignments. The assumption can also be interpreted as the case where the seller hides the past events so that agents' incentive constraints are the weakest. One might consider the case in which agents observe some information about the past events. In such a case, an incentive constraint is necessary for every history. We will see that the results of the paper remain the same even in such cases.

We can apply the revelation principle, and we limit attention to dynamic direct mechanisms. Each agent reports the type to the seller or the mechanism designer at the arrival time. Then, the seller offers a contract to each agent just once at his arrival time.

At each $t$, each entrant $i \in N^t$ makes a report $\gamma_i^t = (\hat{V}_i, \hat{l}_i) \in \Theta_i$. The profile of types at $t$ is denoted by $\theta^t \equiv (\theta_i)_{i \in N^t}$. The seller makes a contract for each $i \in N^t$ based on the vector of reports $\gamma^t \in \Theta^t \equiv \prod_{i \in N^t} \Theta_i$ and history up to $t$:

$$h_t = (N^1; \gamma^1, z^1, N^2; \ldots; \gamma^{t-1}, z^{t-1}, N^t).$$

Let $\mathcal{H}_t$ be the set of possible history at $t$. A mechanism is denoted by $\{z^t\}_{t=1}^{\infty}$, where

$$z^t : \Theta^t \times \mathcal{H}_t \to Z^t.$$

A mechanism is *feasible* if $z_i^t = \emptyset$ for all $i \notin N^t$ and if $\sum_{i \in N^t} a_i(z_i^t) \leq K_t$, where $K_t$ denotes the number of units that are available for the current entrants.[4]

For a profile of reports $\gamma^t = (\gamma_j^t)_{j \in N^t}$ at $t$, an entrant $i$ at $t$ earns payoff

$$u_i(\gamma^t, \theta_i, h_t) = a_i(z_i^t(\gamma^t, h_t))\mathbb{I}_{\{m_i(z_i^t) \geq l_i\}} V_i - p_i(z_i^t(\gamma^t, h_t)), \tag{2}$$

where $\mathbb{I}$ denotes the indicator function that is 1 if the associated condition holds. Because of verifiability of contracts, we assume that agents can observe the history after the enforcement of their contracts. Let $U_i(\theta^t, h_t) \equiv u_i((\theta_i, \theta_{-i}^t), \theta_i, h_t)$, which indicates $i$'s payoff when every entrant at $t$ reports true information.

A bidder's strategy is a mapping $\gamma_i : \Theta_i \to \Theta_i$. Given that the others report their true types, agent $i$'s interim expected payoff is

$$\pi_i(\gamma_i^t, \theta_i) = \mathbb{E}[u_i(\gamma_i^t, \theta_{-i}^t, \theta_i, h_t)]. \tag{3}$$

Each agent's expected payoff at $t$ is also written as

$$\pi_i(\gamma_i^t, \theta_i) = \alpha_i(\gamma_i^t, l_i) V_i - q_i(\gamma_i^t), \tag{4}$$

where

$$\alpha_i(\gamma_i^t, l_i) = \mathbb{E}\big[a_i(z_i^t(\gamma_i^t, \theta_{-i}^t, h_t))\mathbb{I}_{\{m_i(z_i^t) \geq l_i\}}\big] \tag{5}$$

and

$$q_i(\gamma_i^t) = \mathbb{E}\big[p_i(z_i^t(\gamma_i^t, \theta_{-i}^t, h_t))\big]. \tag{6}$$

Let $\Pi_i(\theta_i) \equiv \pi_i(\theta_i, \theta_i)$, which denotes the expected payoff when $i$ reports his true information. Note that agent $i$'s "winning probability" $\alpha_i$ depends on $i$'s true period type $l_i$ through the indicator function. Abusing notations, let $\alpha_i(V_i, l_i) \equiv \alpha_i((V_i, l_i), l_i)$, which indicates the winning probability when reporting the true period type.

Incentive compatibility and individual rationality are defined in a standard manner.

[4]See Section 4 for a formal definition of $K_t$.

**Definition 1** A dynamic direct mechanism is *(Bayesian) incentive compatible* if for all $i$, all $t$, all $\theta_i$, and all $\gamma_i^t$,

$$\Pi_i(\theta_i) \geq \pi_i(\gamma_i^t, \theta_i).$$

In addition, a mechanism is *individually rational* if for all $i$, all $t$, and all $\theta_i$, $\Pi_i(\theta_i) \geq 0$.

**Definition 2** A dynamic direct mechanism is *dominant strategy incentive compatible* if for all $i$, all $t$, all $h_t$, all $\theta$, and all $\gamma_i^t$,

$$U_i(\theta^t, h_t) \geq u_i((\gamma_i^t, \theta_{-i}^t), \theta_i, h_t).$$

Note that by the ex post availability of the history, the dominant strategy incentive compatibility is a little stronger than the standard definition in the sense that the incentive compatibility is imposed for every history.

**Remark 2** In a dynamic environment, it is hard to construct a dominant strategy incentive compatible mechanism in general. This is because future agents can play a strange strategy contingent on history, which prevents truthtelling from being optimal. Most of the related studies consider ex post incentive compatibility or periodic ex post incentive compatibility (Bergemann and Valimaki, 2010). However, in our model, the payoff of an entrant $i$ at $t$ is determined only by the current entrants and past agents, and future events are independent from $i$'s payoff. Thus, we can establish dominant strategy incentive compatible mechanisms as we do later.

# 3 Characterization of Incentive Compatibility

We characterize incentive compatible mechanisms. In this section, we restrict the domain of mechanisms such that each contract term is set to agent's reported period type.

**Assumption 1** Every dynamic direct mechanism satisfies $m_i(z_i(\theta^t, h_t)) = l_i$ for all $t$, all $h_t$, all $i \in N^t$, and all $\theta^t$.

This restriction would be natural.[5] As we will verify later, it does not reduce the efficiecy in an efficient mechanism design.

Note that since each agent reports his type only once at his unmanipulatable entry period, a dynamic direct mechanism is just static for each agent. Thus, if we suppose that each agent never manipulates his period type, incentive compatibility is characterized in a standard manner (Myerson, 1981).

**Lemma 1 (Myerson, 1981)** *Suppose each agent reports the true period type. Then* $\gamma_i(\theta_i) = \theta_i$ *maximizes i's expected payoff if and only if*

1. $\alpha_i(V_i, l_i)$ *is weakly increasing in* $V_i$ *for every* $l_i$*, and*
2. $\Pi_i(V_i, l_i) = \Pi_i(0, l_i) + \int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu$ *for all* $V_i$ *and* $l_i$*.*

The following theorem is our first main theorem of the paper. An additional condition on the allocation rule is necessary in order to ensure that agents report true period types.

**Theorem 1** *Suppose Assumption 1. A dynamic direct mechanism is incentive compatible if and only if for all i,*

1. $\alpha_i(V_i, l_i)$ *is weakly increasing in* $V_i$ *for every* $l_i$*,*
2. $\int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu$ *is weakly decreasing in* $l_i$ *for every* $V_i$*, and*
3. *there exists a worst-case payoff* $\underline{\Pi}_i$ *independent from* $l_i$*, and*
$$\Pi_i(V_i, l_i) = \underline{\Pi}_i + \int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu \tag{7}$$
*for all* $V_i$ *and all* $l_i$*.*

**Proof.** See Appendix.

Note that $\Pi_i(0, l_i)$ does not depend on $l_i$. Preceding studies (Mierendorff, 2009; Pai and Vohra, 2011) also derive a similar condition with ours in a model of strategic arrival and departure. Pai and Vohra (2011) characterize incentive compatible

[5] In preceding studies such as Parkes and Singh (2003), Hajiaghayi et al. (2005), and Pai and Vohra (2011), there is explicitly no assumption corresponding to Assumption 1. In their model, however, agents determine their arrival and departure times *outside of* a mechanism. Thus, we can interpret a similar assumption is implicitly imposed.

mechanisms in the case where goods are durable and agents want to win only once during their stay. These studies characterize incentive compatibility along with the binding individual rationality; i.e., they *assume* the expected payoff when value is 0 is 0. It is worthy noting that Theorem 1 does not use the individual rationality.

Preceding studies introduce a stronger notion of *monotonicity* of the allocation policy, which requires that the allocation for an agent is monotone in both valuation and length of the stay. We introduce a similar concept in our model. Given Assumption 1 and a mechanism $\{z^t\}$, an *allocation policy* is denoted by $a^t : \Theta^t \times \mathcal{H}_t \to [0,1]^{N^t}$ and

$$a_i^t(\theta^t, h_t) \equiv a_i(z_i^t(\theta^t, h_t)).$$

**Definition 3** An allocation policy is said to be *monotone* if for all $i$, $a_i^t(\theta^t, h_t)$ is weakly increasing in $V_i$ and weakly decreasing in $l_i$.

Note that incentive compatibility does not require that the allocation policy is monotone in period type. Theorem 1 immediately shows that any monotone allocation policy is implementable.

**Corollary 1** *Suppose Assumption 1. If an allocation policy is monotone, then there exists a payment scheme that induces the incentive compatibility.*

## 3.1 Dominant Strategy Incentive Compatibility

Bikhchandani et al. (2006) show that in multi-dimensional model, a deterministic allocation rule is implemented in dominant strategy if and only if it is *weakly monotone*. Indeed, our model can be included by the model of Bikhchandani et al. (2006), and any monotone allocation policy is implemented in dominant strategy when we focus on deterministic allocation policies. The following proposition is provided as a corollary of Bikhchandani et al. (2006), but we provide the proof independently in Appendix.

**Proposition 1** *A deterministic allocation policy is implemented in dominant strategy if it is monotone.*

# 4 An Efficient Mechanism

In this section, we establish an efficient incentive compatible mechanism, which is an extension of the well-known Vickrey-Clarke-Groves mechanism. In order to formulate the social optimization problem, we introduce the *residual periods of $z_i^s$ at $t \geq s$*, which is denoted by $r(t, z_i^s)$ and

$$r(t, z_i^s) \equiv \max\{m_i(z_i^s) + s - t, 0\}.$$

In addition, *the residual periods of objects at $t$* is $K$-dimensional vector $x^t = (x_k^t)_{k=1}^K$, where $x_k^t$ is the $k$-th highest order number of $r(t, z_i^s)$ of all $i$ and all $s < t$ such that $\bar{a}_i(z_i^s) = 1$. Note that $\#\{k | x_k^t \geq 1\}$ units of the object are kept at $t$ by some incumbent agents. The supply at $t$ is denoted by $K_t = K - \#\{k | x_k^t \geq 1\}$. Let $X$ be the set of $x^t$; $X = \{x \in \mathbb{Z}_+^K | \mathbf{0} \leq x \leq (L-1, \ldots, L-1)\}$.

## 4.1 Social Welfare

In what follows, we formulate the socially optimal allocation policy. Because we assume i.i.d. populations and type distributions, the state of the world at $t$ is $(\theta^t, x^t)$. It is easy to verify that the efficient allocation policy $\mathbf{y}^* = \{a^{t*}, m^{t*}\}_{t=0}^{\infty}$ is deterministic and satisfies $m_i^t = l_i$. Hence, we focus on deterministic allocation policies with $m_i^t = l_i$. Since the "allocative state" of the next period, $x^{t+1}$, is determined by the current allocation $y^t$ and the current state $x^t$, let $G$ be the state transition function: $x^{t+1} = G(y^t, x^t)$. The socially optimal welfare at $t$, $W(\theta^t, x^t)$, is written as

$$\begin{aligned} W(\theta^t, x^t) = \max_{y^t} & \sum_{i \in N^t} a_i^t V_i + \delta \mathbb{E} W(\theta^{t+1}, x^{t+1}) \\ \text{s.t.} \qquad & a_i^t \in \{0, 1\}, \\ & m_i^t = l_i, \\ \sum_{i \in N^t} & a_i^t \leq K_t, \\ & x^{t+1} = G(y^t, x^t). \end{aligned} \tag{8}$$

Similarly, we define the social optimization problem without $i \in N^t$. Let $W_{-i}(\theta^t, x^t)$ be the maximized social welfare when $i$ is excluded. The efficient allocation policy at $t$ without $i \in N^t$ is denoted by $\hat{y}_{-i}^t$.

It is easy to verify that the efficient allocation policy $a^*(\theta^t, x^t)$ is monotone.

**Lemma 2** *The efficient allocation policy $a^*$ is monotone.*

**Proof.** See Appendix.

### 4.2 The Dynamic VCG Mechanism

Since the efficient allocation policy is monotone, it is implementable in dominant strategy. The efficient allocation policy is implemented via dynamic Vickrey-Clarke-Groves (VCG) mechanism. As defined in the static VCG mechanism, we introduce the *marginal contribution* of $i$, which is defined by

$$C_i(\theta^t, x^t) \equiv W(\theta^t, x^t) - W_{-i}(\theta^t, x^t).$$

Let $x^{t+1*} \equiv G(y^{t*}, x^t)$, which indicates the allocative state at $t+1$ given an efficient policy at $t$. Similarly, let $\hat{x}^{t+1}_{-i} \equiv G(\hat{y}^t_{-i}, x^t)$, which denotes the allocative state at $t+1$ given the efficient policy $\hat{y}^t_{-i}$ when $i$ is excluded at $t$. Note that agent $i$ makes no report after $t+1$. Hence, we have

$$W_{-i}(\theta^t, x^t) = \sum_{j \in N^t \setminus \{i\}} \hat{a}^t_{-i,j} V_j + \delta \mathbb{E} W(\theta^{t+1}, \hat{x}^{t+1}_{-i}).$$

The payment scheme for the dynamic VCG mechanism is defined so that each agent earns his marginal contribution under the current state. Hence, the (total) monetary transfer $p^*_i(\theta^t, x^t)$ in the dynamic VCG mechanism is defined as follows: for $i \in N^t$,

$$p^*_i(\theta^t, x^t) = \sum_{j \in N^t \setminus \{i\}} (\hat{a}^t_{-i,j} - a^{t*}_j) V_j + \delta(\mathcal{W}(\hat{x}^{t+1}_{-i}) - \mathcal{W}(x^{t+1*})), \tag{9}$$

where $\mathcal{W}(x) \equiv \mathbb{E}W(\theta^{t+1}, x)$.

**Definition 4** A dynamic direct mechanism $\{z^*\} = \{y^*, p^*\}_{t=0}^{\infty}$ is said to be the *dynamic VCG mechanism* if $a^*$ is the efficient allocation policy, $m^*_i = l_i$ for all $i$, and if the payment is determined by (9).

**Theorem 2** *The dynamic VCG mechanism $\{z^*\}$ is dominant strategy incentive compatible and individually rational. The equilibrium payoff of $i \in N^t$ is $U_i(\theta^t, h_t) = C_i(\theta^t, x^t)$.*

**Proof.** Suppose $i \in N^t$. For any $\theta_i$ and reports of the others $\theta^t_{-i}$, $i$'s ex post payoff given a report $\gamma_i$ is

$$\begin{aligned} u_i((\gamma_i, \theta^t_{-i}), \theta_i, h_t) &= a^*_i(\gamma_i, \theta^t_{-i}, x^t)V_i - p^*_i(\gamma_i, \theta^t_{-i}, x^t) \\ &\leq \sum_{N^t} a^*_j(\gamma_i, \theta^t_{-i}, x^t)V_j + \delta W(G(y^*(\gamma_i, \theta^t_{-i}, x^t), x^t)) - W_{-i}(\theta^t, x^t) \\ &\leq \sum_{N^t} a^*_j(\theta^t, x^t)V_j + \delta W(G(y^*(\theta^t, x^t), x^t)) - W_{-i}(\theta^t, x^t) \\ &= U_i(\theta^t, h_t). \end{aligned} \tag{10}$$

Therefore, truth-telling is optimal, and the associated payoff is $W(\theta^t, x^t) - W_{-i}(\theta^t, x^t) = C_i(\theta^t, x^t)$. Since $W(\theta^t, x^t) \geq W_{-i}(\theta^t, x^t)$ by definition, $\{z^*\}$ is individually rational. ■

Bergemann and Valimaki (2010) and Parkes and Singh (2003) provide similar mechanisms that extend the VCG mechanism to dynamic environments. As noted in Remark 1, Bergemann and Valimaki (2010) can include agents' preferences in our model. In their "dynamic pivot mechanism," an allocation at each period is determined by the current profile of types, and the set of available allocations does not change over time. However, in our model, the set of available allocations is constrained by the past contracts as in the case of durable goods.

The availability of the history information for agents does not affect the result. Even if agents can observe some information on the past history, it does not affect the agents' incentive.

### 4.3 Single Unit Case

Our next interest is the specification of the efficient allocation policy. We first consider the case of $K = 1$. The set of the state of the world is $X = \{0, 1, \ldots, L-1\}$. For any $t$ such that $x^t \geq 1$, we have $K_t = 0$ and $a^*_i = 0$ for all $i \in N^t$. Therefore, for $x^t = r \geq 1$,

$$W(\theta^t, r) = \delta^r \mathbb{E} W(\theta^{t+r}, 0) \equiv \delta^r \bar{W}.$$

An allocation problem is considered only when $x^t = 0$. The social optimization problem for $x^t = 0$ is described as

$$W(\theta^t, 0) = \max_{i \in N^t \cup \{0\}} V_i + \delta^{l_i} \bar{W}, \tag{11}$$

where agent 0 is a dummy agent (or the seller), whose type is $\theta_0 = (0, 1)$. The agent who maximizes the value $V_i + \delta^{l_i}\bar{W}$ win the object, or the seller assigns nothing to each agent and waits for the next period.

Suppose agent $i$ wins an object at $t$, and that $j \in N^t \cup \{0\}$ is the second-highest agent who maximizes the social welfare. Then $i$'s payment in the dynamic VCG mechanism is

$$p_i^*(\theta^t) = V_j + (\delta^{l_j} - \delta^{l_i})\bar{W}.$$

To fully characterize the efficient allocation policy, we need to compute the expected social welfare $\bar{W}$. Although it is hard to derive $\bar{W}$ analytically, it is derived by a standard manner.

### 4.4 Binary Period Types

In this section, we consider the case of $K \geq 2$ and $L = 2$: Agents stay for at most 2 periods. The set of possible residual objects is given by

$$X = \{(0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1)\}.$$

Instead of that, we can simply let $x^t$ denote the number of objects reserved for incumbent agents: $x^t \in \{0, \ldots, K\}$. For simplicity of notations, the type of agent $i$ is denoted by $v_i$ for $\theta_i = (v_i, 1)$, and $V_i$ for $\theta_i = (V_i, 2)$. In addition, let $v^{(j)}$ indicate the $j$-th highest order statistics among short-stay agents with $l_i = 1$ at the current period. Let $V^{(j)}$ be the $j$-th highest order statistics among long-stay agents with $l_i = 2$.

It is obvious that only the highest agents among the same period type win the objects in the efficient allocation policy. That is, if agent $i$ with $v_i = v^{(k)}$ wins, then any agent $i'$ with $v_{i'} = v^{(k')}$ where $k' < k$ wins in the efficient allocation. It holds for long-stay agnets as well. In addition, the social welfare improves by allocating a unit to a short-stay agent rather than keeping it unallocated. Hence, all the units are allocated to someone at each period.[6] Let $k^t$ be the number of units allocated to long-stay agents at $t$. Then, we have $x^t = k^{t-1}$ and $K_t = K - k^{t-1}$. The Bellman

[6]We add a sufficient number of dummy agents with $v_i = 0$ when the number of short-stay agents is small.

equation for the efficient allocation policy is

$$W(\theta^t, k^{t-1}) = \max_{k^t \in \{0,\ldots,K_t\}} \sum_{j=1}^{k^t} V^{(j)} + \sum_{j=1}^{K_t-k^t} v^{(j)} + \delta W(k^t). \tag{12}$$

$W(k)$ is decreasing in $k$. In order to have the efficient allocation policy, we first show that $W(k)$ is a concave function. That is, $W(k) - W(k-1)$ is decreasing (increasing in the absolute value) in $k$. The intuition is simple. Since the supply units at the current period is $K_t = K - k$, the difference $W(k-1) - W(k)$ indicates the marginal value for an additional supply of $(K_t+1)$-th unit at the current period. As the number of supply decreases (i.e., $k$ increases), the market at the current period becomes more competitive and an agent with a high value might lose the auction. Thus, the marginal welfare for the additional unit is increasing in $k$. Let $w(\theta^t, k) \equiv -(W(\theta^t, k) - W(\theta^t, k-1))$.

**Proposition 2** *For any $\theta^t$, $W(\theta^t, k)$ is concave in $k$; i.e., $-w(\theta^t, k)$ is decreasing in $k$.*

**Proof.** See Appendix.

Using Proposition 2, we have the efficient allocation policy as follows. Let $w(k) \equiv E[w(\theta^t, k)]$.

**Theorem 3** *Suppose $L = 2$. Then, the efficient allocation policy is given as follows: For the current state $k^{t-1}$, a long-stay agent $i$ with $V_i = V^{(k)}$, where $k \leq K_t = K - k^{t-1}$, wins if and only if*

$$V_i \geq v^{(K_t-k+1)} + \delta w(k). \tag{13}$$

*On the other hand, a short-stay agent $i$ with $v_i = v^{(k)}$ wins if and only if*

$$v_i \geq V^{(K_t-k+1)} - \delta w(K_t - k + 1). \tag{14}$$

**Proof.** See Appendix.

Let $k^*$ be the number of units allocated to long-stay agents. By inspection, the payments of the winning long- and short-stay agents in dynamic VCG mechanism are given by

$$\max\{V^{(k^*+1)}, v^{(K_t-k^*+1)} + \delta w(k^*)\}$$

and

$$\max\{v^{(K_t-k^*+1)}, V^{(k^*+1)} - \delta w(k^*+1)\},$$

respectively.

## 5 Revenue Maximization

In this section, we provide a revenue maximizing mechanism in a similar manner with Myerson (1981). We introduce the *virtual valuation*, which is denoted by $\phi(\theta_i)$, and

$$\phi(\theta_i) \equiv V_i - \frac{1 - F(V_i|l_i)}{f(V_i|l_i)}. \tag{15}$$

From Theorem 1 and some calculations, we have the expected revenue from agent $i \in N^t$, $\mathbb{E}[q_i(\theta_i)]$ as follows:

$$\mathbb{E}[q_i(\theta_i)] = -\underline{\Pi}_i + \int_{\theta_i} \alpha_i(\theta_i)\phi(\theta_i)f(\theta_i)\mathrm{d}\theta_i.$$

Hence, the expected revenue raised at $t$ is given by

$$\mathbb{E}\big[\sum_{i\in N^t} q_i(\theta_i)\big] = \mathbb{E}\big[-\sum_{i\in N^t}\underline{\Pi}_i + \int_{\theta^t}\sum_{i\in N^t}[a_i(\theta^t, h_t)\phi(\theta_i)]f^t(\theta^t)\mathrm{d}\theta^t\big],$$

where $f^t(\theta^t) = \prod_{i\in N^t} f(\theta_i)$ and the expectation is taken over $h_t$. From Theorem 1, the revenue maximizing problem is rewritten as follows.

**Proposition 3** *Suppose Assumption 1. The optimal allocation policy is a solution of the following problem:*

$$\begin{aligned}
\max_{\{a^s\}_{s=t}^{\infty}} \mathbb{E}\Big[&\sum_{s=t}^{\infty}\delta^{s-t}\big[\sum_{i\in N^s}\big(-\underline{\Pi}_i\big) + \sum_{i\in N^s} a_i^s\phi(\theta_i)\big]\Big] \\
&s.t.\ \alpha_i(\theta_i) \textit{ is weakly increasing in } V_i, \\
&\quad \int_0^V \alpha_i(\nu, l_i)\mathrm{d}\nu \textit{ is weakly decreasing in } l_i, \\
&\quad \underline{\Pi}_i \geq 0, \\
&\quad a_i^s \in [0,1], \\
&\quad \sum_{i\in N^s} a_i^s \leq K_s.
\end{aligned} \tag{16}$$

Obviously, $\underline{\Pi}_i = 0$ for all $i$ in the optimal mechanism. Then, let us consider a relaxed problem (in a recursive form) below:

$$
\begin{aligned}
R(\theta^t, x^t) = \max_{a^t} & \sum_{i \in N^t} a_i^t \phi(\theta_i) + \delta \mathbb{E} R(\theta^{t+1}, x^{t+1}) \\
\text{s.t. } & a_i^t \in \{0, 1\}, \\
& \sum_{i \in N^t} a_i^t \leq K_t, \\
& x^{t+1} = G((a^t, l^t), x^t),
\end{aligned}
\tag{17}
$$

where $R(\theta^t, x^t)$ denotes the optimal "virtual welfare function."[7] This problem is the same as the social optimization problem (8), except that the valuation types are replaced with the virtual valuations. Consider a solution of the virtual social optimization problem (17). From Lemma 2, the solution $a^{**}$ is weakly increasing in $\phi$.

In order that a solution of the relaxed problem (17) also solves the original problem (16), we need a regularity condition. A sufficient condition is that the virtual valuation $\phi$ is incerasing in $V_i$ and weakly decreasing in $l_i$. We impose the following monotone hazard rate condition.

**Assumption 2** The conditional hazard rate $\lambda_{l_i}(V_i) \equiv \frac{f(V_i|l_i)}{1-F(V_i|l_i)}$ is weakly increasing in $V_i$ and weakly decreasing in $l_i$.[8]

Roughly speaking, Assumption 2 requires that the longer the period type, the higher valuation for an agent. Indeed, when the assumption holds, the distribution $F(\cdot|l_i)$ stochastically dominates $F(\cdot|l_i')$ for $l_i' < l_i$. This would likely be the cases in real situations. From Lemma 2 and Assumption 2, in a solution of the relaxed problem, $\alpha_i$ is decreasing in $l_i$ because when $l_i$ gets shorter with $V_i$ constant, then the virtual value $\phi$ weakly increases. Since the solution for (17) is monotone in terms of $\phi$, it must be monotone.

**Theorem 4** *Under Assumptions 1 and 2, the allocation policy derived by a relaxed problem (17) is monotone and maximizes the expected revenue for the seller.*

[7]Similarly ot the social welfare maximization, the optimal allocation policy must be deterministic, and $x^t$ is adopted for a state variable.

[8]Pai and Vohra (2011) impose a similar hazard rate condition. On the monotonicity of $V_i$, it is sufficient to assume $\phi(\theta_i)$ is increasing in $V_i$, as considered in Myerson (1981).

The optimal allocation policy $a^{**}$ is in general very different from the efficient allocation policy $a^*$ even if type distribution is identical. In the standard static optimal auction design, the optimal allocation policy is constrained efficient in the sense that the agent who is awarded an object has the highest value. However, in our model, the virtual value $\phi$ depends on both valuation type and period type, and it generates asymmetry between different period types. In addition, the social welfare function and the optimal virtual welfare function are different, so that the optimal policies $a^*$ and $a^{**}$ are different.

Under the assumptions, the availability of the history information for agents again does not affect the result, similarly to the efficient mechanism design.

## 5.1 Single Unit Case

The optimal allocation policy is derived in the same manner with the efficient allocation policy. Consider the case of $K = 1$. An allocation problem is considered only when $x^t = 0$. Since

$$R(\theta^t, r) = \delta^r \mathbb{E} R(\theta^{t+r}, 0) \equiv \delta^r \bar{R},$$

the Bellman equation is

$$R(\theta^t, 0) = \max_{i \in N^t \cup \{0\}} \phi(\theta_i) + \delta^{l_i} \bar{R}, \tag{18}$$

where 0 denotes a dummy agent, who has the *reservation type* $\theta_0 = (\hat{V}, 1)$. The reservation value $\hat{V} \in (0, \bar{V})$ is determined by $\phi(\hat{V}, 1) = 0$.

In an incentive compatible mechanism, winning agents pay the threshold value in order to win given the others' types. Suppose agent $i$ wins at $t$ and agent $j$ is the second highest: $j \in \arg\max_{j' \in N^t_{-i} \cup \{0\}} \phi(\theta_{j'}) + \delta^{l_{j'}} \bar{R}$. Let $\phi^{-1}(\cdot | l_i)$ be the inverse function of $\phi(\cdot | l_i)$ given $l_i$ fixed. Then agent $i$ pays the total amount of

$$p_i^{**}(\theta^t) = \phi^{-1}\big(\phi(\theta_j) + (\delta^{l_j} - \delta^{l_i})\bar{R} | l_i\big).$$

A interesting question is how the efficient and the optimal allocation policies are different. To investigate this, let us consider the value indifferent from $\bar{\theta} \equiv (\bar{v}, 1)$ for the seller. In the optimal allocation policy, an agent with $\theta_i = (V_i, l)$ is evaluated equally to $\bar{\theta}$ when

$$V_i - \frac{1}{\lambda_l(V_i)} + \delta^l \bar{R} = \bar{v} - \frac{1}{\lambda_1(\bar{v})} + \delta \bar{R}.$$

Hence, we define $V^o(l;\bar{\theta})$ as

$$V^o(l;\bar{\theta}) \equiv \bar{v} + (\delta - \delta^l)\bar{R} + \frac{1}{\lambda_l(V^o)} - \frac{1}{\lambda_1(\bar{v})}. \tag{19}$$

Similarly, the agent with the period type $l$ is evaluated equally to $\bar{\theta}$ in the efficient mechanism when the valuation type is

$$V^e(l;\bar{v}) \equiv \bar{v} + (\delta - \delta^l)\bar{W}. \tag{20}$$

Note that the expected revenue for the seller $\bar{R}$ is strictly less than the expected social welfare $\bar{W}$; $\bar{R} < \bar{W}$. We can conclude that $V^o(l) \leq V^e(l)$ if

$$\frac{1}{\lambda_l(V^o(l;\bar{v}))} - \frac{1}{\lambda_1(\bar{v})} \leq (\delta - \delta^l)(\bar{W} - \bar{R}). \tag{21}$$

Basically, it is unclear whether (21) is satisfied. Generally, distributions of values are asymmetric among different period types even with the symmetric type distribution, so that it is difficult to obtain a clear conclusion about the direction of distortion in the optimal allocation policy. Hence, let us consider a special case in which the total value $V_i$ and the period type $l_i$ are independently distributed. That is, we have $\lambda_l(\cdot) = \lambda(\cdot)$ for all $l$. Since $V^o(l) > \bar{v}$ for $l \geq 2$, we have $\lambda(V^o) \geq \lambda(\bar{v})$ when we assume non-decreasing hazard rate. Thus, (21) is satisfied.

**Proposition 4** *Suppose that $K = 1$ and that the valuation type and the period type are independently distributed. Then, agents with a long period type are more favored in the optimal mechanism than in the efficient mechanism.*

## 5.2 Long-Stay Discount vs. Premium

In what follows, we investigate the seller's optimal pricing. We consider the single unit case and assume that $|N^t| \leq 1$ for all $t$. At most one agent enters the mechanism in a period, and the arrival rate is given by $\eta \in (0,1]$. Both the incentive compatible efficient and the optimal mechanisms are posted prices: the seller sets a proper price for each period length.

In the real world situations such as hotel rooms, the seller often offer a discounted price for long-stay people. However, assigning slots to a long-stay agent makes the seller give up the potential revenue from the future agents. Hence, it is uncertain whether the long-stay discount is supported by the optimal pricing.

First consider the efficient mechanism as a benchmark. It is easy to verify that the efficient total price for each period length, $P^*(l)$, is given by $P^*(l) = (\delta - \delta^l)\bar{W}$.[9] Thus, the average price $p^*(l)$ for $l$-period stay is given by

$$p^*(l) = \frac{\delta - \delta^l}{1 - \delta^l}\bar{w},$$

where $\bar{w} = (1-\delta)\bar{W}$ indicates the average welfare. Since $\frac{\delta-\delta^l}{1-\delta^l}$ is increasing in $l$, the per period price $p^*(l)$ is increasing in period length in the efficient mechanism. Thus, the long-stay discount does not exist but the long-stay premium necessarily exists.

Now consider the optimal mechanism. We show that long-stay discount is optimal under a certain situation. Suppose again that the total value and the period type are independently distributed: $\lambda_l(\cdot) = \lambda(\cdot)$ for all $l$. As we have observed in the previous section, long-stay agents are more favored in the optimal mechanism. The total payment $P^{**}(l)$ for $l$-period stay in the optimal mechanism is given by $\phi(P^{**}(l)) = (\delta - \delta^l)\bar{R}$. Hence, the average price $p^{**}(l)$ satisfies

$$p^{**}(l) = \frac{\delta - \delta^l}{1 - \delta^l}\bar{r} + \frac{1-\delta}{(1-\delta^l)} \cdot \frac{1}{\lambda(P^{**}(l))}, \tag{22}$$

where $\bar{r} \equiv (1-\delta)\bar{R}$ is the average revenue.

From Assumption 2 and $P^{**}(l) < P^{**}(l+1)$, we have $\lambda(P^{**}(l)) \leq \lambda(P^{**}(l+1))$. Hence,

$$\begin{aligned} p^{**}(l+1) - p^{**}(l) &= \frac{\delta^l(1-\delta)^2}{(1-\delta^l)(1-\delta^{l+1})}\bar{r} + \frac{1-\delta}{(1-\delta^l)(1-\delta^{l+1})}\Big(\frac{1-\delta^l}{\lambda(P^{**}(l+1))} - \frac{1-\delta^{l+1}}{\lambda(P^{**}(l))}\Big) \\ &\leq \frac{\delta^l(1-\delta)^2}{(1-\delta^l)(1-\delta^{l+1})}\Big(\bar{r} - \frac{1}{\lambda(P^{**}(l))}\Big). \end{aligned} \tag{23}$$

Therefore, we have $p^{**}(l+1) < p^{**}(l)$ when

$$\lambda(P^{**}(l))\bar{r} \leq 1. \tag{24}$$

We have to remark that the average revenue $\bar{r}$ is endogenously determined by the dynamic optimization problem (18) and depends on the type distribution. However, $\bar{r}$ has a degree of freedom by the parameter of the population dynamics $\eta$. Hence, whether (24) holds or not depends on $\eta$.

[9] In this section, we use capital character $P$ for the total payment and small character $p$ for the average (per period) payment.

Note that $p^{**}(1) = P^{**}(1)$ and $\lambda(P^{**}(1))p^{**}(1) = 1$. The seller earns the revenue $p^{**}(1)$ when an agent $i$ with $V_i \geq p^{**}(1)$ and $l_i = 1$ enters, and $l_i = 1$ is the best period type. Therefore, it is obvious that $\bar{r} < p^{**}(1)$ regardless of the arrival rate $\eta$.

**Proposition 5** *Suppose that $K = 1$, $|N^t| \leq 1$, and that the valuation type and the period type are independently distributed. The long-stay discount, $p^{**}(l) < p^{**}(l+1)$, holds if $\lambda(P^{**}(l))\bar{r} \leq 1$. In particular, it holds that $p^{**}(2) < p^{**}(1)$.*

Although the long-stay discount exists in the above specification, it does not in another specification. Consider next the case where the per period value $v_i$ and the period type $l_i$ are independently drawn. Let us assume that the value per period $v_i$ is drawn from a density function $f > 0$ on $[0, 1]$. The distribution of $v_i$ satisfies the monotone hazard rate condition, and then Assumption 2 is satisfied. Then, we have the virtual valuation by a simple calculation

$$\phi(\theta_i) = \frac{1 - \delta^{l_i}}{1 - \delta}\tilde{\phi}(v_i),$$

where $\tilde{\phi}(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$. Hence, the per period price is given by

$$p^{**}(l) = \tilde{\phi}^{-1}\Big(\frac{\delta - \delta^l}{1 - \delta^l}\bar{r}\Big), \tag{25}$$

which indicates the long-stay premium.

In the presence of the long-stay premium, a long-stay agent would not like to accept a long term contract but make a short-term contract at each period, which is not considered in our model. The result would change if agents are allowed to make a new contract after the expiration of a contract. It is an open question how we can characterize the incentive compatibility and what the optimal mechanism is in such a case.[10] Nevertheless, long-stay premium might exist even if agents can repeat short-term contracts because long-stay agents accepting a short-term contract face risk to lose the object in a future auction.

# 6 Conclusion

We formulate a model of the dynamic allocation problem in which agents want to obtain an object for periods of time. We characterize incentive compatible mechanisms, and construct the efficient mechanism and the optimal mechanism in a domain

[10] If current outcome is determined by all the history in a mechanism, this kind of deviation can easily be overcome.

where the seller offers simple long-term contracts. The dynamic VCG mechanism achieves efficiency in a dominant strategy equilibrium. With a monotone hazard rate condition, the optimal allocation policy maximizes the virtual welfare, similarly to Myerson (1981). It is worthy noting that it is possible to specify the efficient and the optimal allocation policies in several cases in our model. It is typically quite complicated or intractable in models of preceding studies such as Bergemann and Valimaki (2010), Mierendorff (2009), and Pai and Vohra (2011).

There are several avenues for future research. First, we need to investigate the efficient or optimal allocation policy further in detail. Although we focus on simple incomplete constracts, it is difficult to specify the allocation policy when there are multiple units. Second, characterization of incentive compatibility is open question when agents can repeat short-term contracts. Third, it would be interesting and important to consider the case in which agents often make new contracts after expiration, as considered by Bergemann and Valimaki (2010), Pavan et al. (2009), and Kakade et al. (2011). We would need to consider a dynamic structure to both the seller and agents in order consider various practical situations.

# A Proofs

## A.1 Proof of Theorem 1

(Only if part.) Suppose a mechanism is incentive compatible. Then, we have

$$\alpha_i(V_i, l_i)V_i - q_i(V_i, l_i) \geq \alpha_i(V_i', l_i)V_i - q_i(V_i', l_i),$$

hence,

$$(\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i \geq q_i(V_i, l_i) - q_i(V_i', l_i).$$

Similarly, we have

$$(\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i' \leq q_i(V_i, l_i) - q_i(V_i', l_i).$$

Therefore,

$$(\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i' \leq (\alpha_i(V_i, l_i) - \alpha_i(V_i', l_i))V_i.$$

Therefore, $V_i' < V_i$ implies $\alpha_i(V_i', l_i) \leq \alpha_i(V_i, l_i)$.

From the standard argument of the envelope theorem (Milgrom and Segal, 2002), if $V_i \in \arg\max_{\nu\in[0,\bar{V}]} \alpha_i(\nu, l_i)V_i - q_i(\nu, l_i)$, then

$$\frac{\partial \Pi_i(V_i, l_i)}{\partial V_i} = \alpha_i(V_i, l_i)$$

almost everywhere, and

$$\Pi_i(V_i, l_i) - \Pi_i(V_i', l_i) = \int_{V_i'}^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu. \tag{26}$$

Suppose $l_i' > l_i$. Then, $\alpha_i((V_i', l_i'), l_i) = \mathbb{E}[a_i(V_i', l_i', \theta_{-i}^t, h_t)\mathbb{I}_{\{m_i(z_i^t)=l_i'\geq l_i\}}] = \alpha_i(V_i', l_i')$. Then incentive compatibility implies for all $V_i$,

$$\begin{aligned}\Pi_i(V_i, l_i) &\geq \alpha_i((V_i, l_i'), l_i)V_i - q_i(V_i, l_i') \\ &= \alpha_i(V_i, l_i')V_i - q_i(V_i, l_i') \\ &= \Pi_i(V_i, l_i').\end{aligned} \tag{27}$$

From the envelope formula (26), (27) yields

$$\Pi_i(0, l_i) + \int_0^{V_i} \alpha_i(\nu, l_i)\mathrm{d}\nu \geq \Pi_i(0, l_i') + \int_0^{V_i} \alpha_i(\nu, l_i')\mathrm{d}\nu. \tag{28}$$

For $V_i = 0$, it also holds that $\Pi_i(0, l_i) = -q_i(0, l_i)$. Incentive compatibility requires $-q_i(0, l_i) \geq -q_i(0, l_i')$ for any $l_i$ and $l_i'$, thus that $-q_i(0, l_i)$ does not depend on $l_i$. Hence, $\Pi_i(0, l_i) = \underline{\Pi}_i$ for all $l_i$.

(If part.) Since $\alpha_i$ is monotone, each agent's expected payoff $\pi_i((\nu, l_i), (V_i, l_i))$ satisfies the single crossing condition on $(V_i, \nu)$. Given $l_i$, a standard argument and Lemma 1 implies for all $V_i'$,

$$\alpha_i(V_i, l_i)V_i - q_i(V_i, l_t) \geq \alpha_i(V_i', l_i)V_i - q_i(V_i', l_i).$$

Suppose $l_i' > l_i$. Note that $\alpha_i((V_i, l_i'), l_i) = \alpha_i(V_i, l_i')$. Hence, for any $V_i'$,

$$\begin{aligned}\alpha_i((V_i', l_i'), l_i)V_i - q_i(V_i', l_i') &= \alpha_i(V_i', l_i')V_i - q_i(V_i', l_i') \\ &\leq \Pi_i(V_i, l_i') \\ &\leq \Pi_i(V_i, l_i).\end{aligned}$$

The last inequality holds from the conditions 2 and 3.

The envelope condition (7) implies

$$\begin{aligned}-q_i(V_i, l_i) &= \underline{\Pi}_i + \int_0^{V_i} [\alpha_i(\nu, l_i) - \alpha_i(V_i, l_i)]\mathrm{d}\nu \\ &\leq \Pi_i(0, l_i),\end{aligned}$$

where inequality comes from the monotonicity of $\alpha_i$. Then, for all $V_i'$ and all $l_i' < l_i$,

$$\begin{aligned}\alpha_i((V_i', l_i'), l_i)V_i - q_i(V_i', l_i') &= -q_i(V_i', l_i') \\ &\leq \Pi_i(0, l_i') \\ &= \Pi_i(0, l_i) \\ &\leq \Pi_i(V_i, l_i).\end{aligned}$$

The second equality comes from condition 3, and the last inqeuality is from the monotonicity and (7). ■

## A.2 Proof of Proposition 1

Given a monotone policy $\{a^t\}$ with Assumption 1 and any period type $l_i$, define a threshold type $\hat{\theta}_i(\theta^t, h_t)$ as $\hat{\theta}_{i2} = l_i$ and

$$\hat{\theta}_{i1}(\theta^t, h_t) \equiv \hat{V}_i(l_i, \theta_{-i}^t, h_t) = \inf\{\tilde{V}_i | a_i((\tilde{V}_i, l_i), \theta_{-i}^t, h_t) = 1\}. \tag{29}$$

Let $\hat{V}_i(\theta_{-i}^t, h_t, l_i) = \infty$ if such $\tilde{V}_i$ does not exist. Fix any $\theta_{-i}^t$ and $h_t$. Abusing notaions, we drop $(\theta_{-i}^t, h_t)$ and use the notation $\hat{V}_i(l_i)$.

Note that $\hat{V}_i(l_i)$ is weakly increasing in $l_i$. If not, $\hat{V}_i(l_i) < \hat{V}_i(l_i')$ for some $l_i > l_i'$. Suppose $\theta_{i\epsilon} = (\hat{V}_i(l_i) + \epsilon, l_i)$ and $\theta_{i\epsilon}' = (\hat{V}_i(l_i) + \epsilon, l_i')$. Then, by definition of $\hat{V}_i$, $a_i(\theta_{i\epsilon}) = 1$ for all $\epsilon > 0$ and $a_i(\theta_{i\epsilon}') = 0$ for $\epsilon \in (0, \hat{V}_i(l_i') - \hat{V}_i(l_i))$, which is contradiction.

Now, consider a payment policy, which is defined as

$$p_i(\theta^t, h_t) = \begin{cases} \hat{V}_i(l_i, \theta_{-i}^t, h_t) & \text{if } a_i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

**Case 1**: $V_i < \hat{V}_i(l_i)$.

If $i$ reports the truth, he loses and the payoff is 0. When $i$ reports $\theta_i'$ such that $l_i' \geq l_i$ and wins, then his payoff is

$$V_i - \hat{V}_i(l_i') \leq V_i - \hat{V}_i(l_i) < 0.$$

Hence, it is not profitable. When $i$ reports $\theta_i'$ such that $l_i' < l_i$ and wins, then his payoff is $-\hat{V}_i(l_i') \leq 0$. Therefore, truth-telling is optimal.

**Case 2**: $V_i \geq \hat{V}_i(l_i)$.

When $i$ reports truthfully, then he wins with a payoff $V_i - \hat{V}_i(l_i) \geq 0$. When $i$ reports $\theta_i'$ such that $l_i' \geq l_i$ and wins, then his payoff is

$$V_i - \hat{V}_i(l_i') \leq V_i - \hat{V}_i(l_i).$$

When $i$ reports $\theta_i'$ such that $l_i' < l_i$ and wins, then his payoff is $-\hat{V}_i(l_i') \leq 0$. Therefore, truth-telling is optimal. ■

## A.3 Proof of Lemma 2

Suppose that $a_i^*(\theta^t, x^t) = 1$ for some $i \in N^t$ of $\theta^t = (V_i, l_i)$ under $(\theta^t, x^t)$. First, consider $\theta_i' = (V_i', l_i)$ where $V_i' > V_i$. Note that the maximum social welfare without $i$ is independent from $\theta_i$; $W_{-i}(\theta_i', \theta_{-i}^t, x^t) = W_{-i}(\theta^t, x^t)$. On the other hand, the maximized social welfare is at least the same as the value in the case where $a^*(\theta^t, x^t)$ is assigned at $t$. Hence,

$$\begin{aligned}
W(\theta_i', \theta_{-i}^t, x^t) &\geq V_i' + \sum_{j \in N^t \backslash \{i\}} a_i^*(\theta^t, x^t) V_j + \delta \mathbb{E} W(\theta^{t+1}, G(y^*(\theta^t, x^t), x^t)) \\
&> W(\theta^t, x^t) \\
&\geq W_{-i}(\theta^t, x^t) = W_{-i}(\theta_i', \theta_{-i}^t, x^t).
\end{aligned}$$

Therefore, we have $a_i^*(\theta_i', \theta_{-i}^*, x^t) = 1$.

Now consider $l_i \geq 2$ and $\theta_i' = (V_i, l_i')$ where $l_i' < l_i$. Suppose that the allocation at $t$ is determined by $a^*(\theta^t, x^t)$ and that the mechanism designer limits the supply of the objects to $K_s - 1$ from period $t + l_i'$ to $t + l_i - 1$. Then, the social welfare must be the same as in the case of $(\theta^t, x^t)$. Hence,

$$\begin{aligned}
W(\theta_i', \theta_{-i}^t, x^t) &> \max_{\{y^s\}_{s=t+1}^{t+l_i-1}} \{ \sum_{j \in N^t} a_j^*(\theta^t, x^t) V_j + \mathbb{E}\Big[ \sum_{s=t+1}^{t+l_i-1} \delta^{s-t-1} \sum_{j \in N^s} a_j^s V_j + \delta^{t+l_i} \mathbb{E} W(\theta^{t+l_i}, x^{t+l_i}) \Big] | \Omega \} \\
&= W(\theta^t, x^t) \\
&\geq W_{-i}(\theta^t, x^t) = W_{-i}(\theta_i', \theta_{-i}^t, x^t),
\end{aligned}$$

where $\Omega$ denotes the condition on the maximization. And,

$$\begin{aligned}
\Omega = \{ &(t+1 \leq \forall s \leq t + l_i' - 1) \sum_{j \in N^s} a_j^s \leq K_s, \\
&(t + l_i' \leq \forall s \leq t + l_i - 1) \sum_{j \in N^s} a_j^s \leq K_s - 1, \\
&x^{t+l_i} = G(y^{t+l_i-1}, x^{t+l_i-1}) \}.
\end{aligned}$$

The strict inequality must hold because there is a positive probability of arrival of an agent $j$ at $s \in [t + l'_i, t + l_i - 1]$ having a type $\theta_j = (V_j, 1)$. Therefore, we have $a^*_i(\theta'_i, \theta^t_{-i}, x^t) = 1$. ■

## A.4 Proof of Proposition 2

For the current state $(\theta^t, k)$, let $k^*(\theta^t, k)$ be the efficient allocation policy that determines the number of units allocated to long-stay agents. To show the proposition, we first show the following lemma.

**Lemma 3** *For any $\theta^t$, $k^*(\theta^t, k)$ is non-increasing in $k$.*

**Proof.** Suppose for contradiction that there exists $k$ and for some $\theta^t$, $k^*(\theta^t, k) > k^*(\theta^t, k-1)$. Let $k^* \equiv k^*(\theta^t, k)$ and $k^{**} \equiv k^*(\theta^t, k-1)$. Since $k^{**} < K-k$, we have[11]

$$
\begin{aligned}
W(\theta^t, k) &= \sum_{j=1}^{k^*} V^{(j)} + \sum_{j=1}^{K-k-k^*} v^{(j)} + \delta W(k^*) \\
&> \sum_{j=1}^{k^{**}} V^{(j)} + \sum_{j=1}^{K-k-k^{**}} v^{(j)} + \delta W(k^{**}),
\end{aligned}
$$

and hence,

$$
\sum_{j=k^{**}+1}^{k^*} V^{(j)} - \sum_{j=K-k-k^*+1}^{K-k-k^{**}} v^{(j)} + \delta(W(k^*) - W(k^{**})) > 0. \tag{30}
$$

Therefore, we have

$$
\begin{aligned}
W(\theta^t, k-1) - W(\theta^t, k) &= \sum_{j=1}^{k^{**}} V^{(j)} + \sum_{j=1}^{K-k-k^{**}} v^{(j)} + \delta W(k^{**}) - \sum_{j=1}^{k^*} V^{(j)} - \sum_{j=1}^{K-k-k^*} v^{(j)} - \delta W(k^*) \\
&= - \sum_{j=k^{**}+1}^{k^*} V^{(j)} + \sum_{j=K-k-k^*+1}^{K-k-k^{**}} v^{(j)} - \delta(W(k^*) - W(k^{**})) + v^{(K-k-k^{**}+1)} \\
&< v^{(K-k-k^{**}+1)}.
\end{aligned}
\tag{31}
$$

However, it is obvious that

$$
\begin{aligned}
W(\theta^t, k-1) - W(\theta^t, k) &\geq v^{(K-k-k^*+1)} \\
&\geq v^{(K-k-k^{**}+1)},
\end{aligned}
$$

[11] If equality holds, $k^{**}$ is also an efficient allocation given $(\theta^t, k)$, and we can construct a non-increasing allocation policy.

which is a contradiction to (31). □

Thus, $k^* \leq k^{**}$. Next, we show $k^{**} \in \{k^*, k^*+1\}$.

**Lemma 4** *If $W(k)$ is concave, $k^{**} \in \{k^*, k^*+1\}$.*

**Proof.** Suppose $W(k)$ is concave. Suppose for contradiction that for some $k$ and $\theta^t$, $k^*+2 \leq k^{**}$.

Since $k^*+1 \leq k^{**}-1 \leq K-k$,

$$\begin{aligned} W(\theta^t, k) &= \sum_{j=1}^{k^*} V^{(j)} + \sum_{j=1}^{K-k-k^*} v^{(j)} + \delta W(k^*) \\ &> \sum_{j=1}^{k^*+1} V^{(j)} + \sum_{j=1}^{K-k-k^*-1} v^{(j)} + \delta W(k^*+1), \end{aligned}$$

and hence,

$$-V^{(k^*+1)} + v^{(K-k-k^*)} + \delta w(k^*+1) > 0. \tag{32}$$

Since $W(k)$ is concave, we have

$$\begin{aligned} 0 &< -V^{(k^*+1)} + v^{(K-k-k^*)} + \delta w(k^*+1) \\ &\leq -V^{(k^*+2)} + v^{(K-k-k^*-1)} + \delta w(k^*+2) \\ &\leq \dots . \end{aligned} \tag{33}$$

Therefore, we have

$$\begin{aligned} & W(\theta^t, k-1) - \big[\sum_{j=1}^{k^*+1} V^{(j)} + \sum_{j=1}^{K-k-k^*} v^{(j)} + \delta W(k^*+1)\big] \\ &= \sum_{j=k^*+2}^{k^{**}} V^{(j)} - \sum_{j=K-k-k^{**}+2}^{K-k-k^*} v^{(j)} - \delta(W(k^*+1) - W(k^{**})) \\ &\leq \sum_{j=k^*+1}^{k^{**}-1} V^{(j)} - \sum_{j=K-k-k^{**}+2}^{K-k-k^*} v^{(j)} - \delta(w(k^*+2) + w(k^*+3) + \dots + w(k^{**})) \\ &\leq \sum_{j=k^*+1}^{k^{**}-1} V^{(j)} - \sum_{j=K-k-k^{**}+2}^{K-k-k^*} v^{(j)} - \delta(w(k^*+1) + w(k^*+2) + \dots + w(k^{**}-1)) \\ &< 0. \end{aligned}$$

The second inequality comes from the concavity of $W(k)$. This contradicts the optimality of $W(\theta^t, k-1)$. □

Finally, we show the proposition. Since the Bellman operator is a contraction, it suffice to show that if any continuation value $\hat{W}(\theta^t, k)$ is concave in $k$, then the mapping is also concave. Let $k^{***} \equiv k^*(\theta^t, k-2)$.

**Case 1:** $k^{**} = k^*$ or $k^{***} = k^{**}$.

Note that

$$w(\theta^t, k) \geq v^{(K-k-k^*+1)}.$$

Since $k^* \leq K - k$ and $k^{***} \leq k^* + 1 \leq K - k + 1$,

$$\begin{aligned}
& w(\theta^t, k-1) \\
& \leq \sum_{j=1}^{k^{***}} V^{(j)} + \sum_{j=1}^{K-k-k^{***}+2} v^{(j)} + \delta W(k^{***}) - \sum_{j=1}^{k^{***}} V^{(j)} - \sum_{j=1}^{K-k-k^{***}+1} v^{(j)} - \delta W(k^{***}) \\
& = v^{(K-k-k^{***}+2)} \\
& \leq v^{(K-k-k^*+1)} \\
& \leq w(\theta^t, k).
\end{aligned}$$

**Case 2:** $k^{**} = k^* + 1$ and $k^{***} = k^{**} + 1 = k^* + 2$.

Note that

$$W(\theta^t, k-1) - W(\theta^t, k) = V^{(k^*+1)} + \delta(\hat{W}(k^*+1) - \hat{W}(k^*))$$

and

$$W(\theta^t, k-2) - W(\theta^t, k-1) = V^{(k^*+2)} + \delta(\hat{W}(k^*+2) - \hat{W}(k^*+1)).$$

Hence,

$$W(\theta^t, k-1) - W(\theta^t, k) \geq W(\theta^t, k-2) - W(\theta^t, k-1).$$

■

## A.5 Proof of Theorem 3

Suppose agent $i \in N^t$ is a long-stay agent with a type $V_i = V^{(k)}$ and satisfies

$$V_i \geq v^{(K_t+1-k)} + \delta w(k).$$

Then, we have

$$\begin{aligned}
\cdots \geq V^{(k-1)} \geq V^{(k)} & \geq v^{(K_t+1-k)} + \delta w(k) \\
& \geq v^{(K_t+2-k)} + \delta w(k-1) \\
& \geq \cdots.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{j=1}^{k} V^{(j)} + \sum_{j=1}^{K_t-k} v^{(j)} + \delta W(k) \\
&\geq \sum_{j=1}^{k-1} V^{(j)} + \sum_{j=1}^{K_t+1-k} v^{(j)} + \delta W(k-1) \\
&= \sum_{j=1}^{k-2} V^{(j)} + \sum_{j=1}^{K_t+2-k} v^{(j)} + \delta W(k-2) + \left[V^{(k-1)} - v^{(K_t+2-k)} - \delta w(k-1)\right] \\
&\geq \sum_{j=1}^{k-2} V^{(j)} + \sum_{j=1}^{K_t+2-k} v^{(j)} + \delta W(k-2) \\
&\geq \dots .
\end{aligned}$$

Therefore, if we ignore the case of ties, the optimal solution does not lie in $\{0, \dots, k-1\}$ but lies in $\{k, \dots, K_t\}$. Agent $i$ wins the object.

Conversely, suppose

$$V_i \leq v^{(K_t+1-k)} + \delta w(k).$$

Then, we have

$$\begin{aligned}
\dots \leq V^{(k+1)} \leq V^{(k)} &\leq v^{(K_t+1-k)} + \delta(W(k-1) - W(k)) \\
&\leq v^{(K_t-k)} + \delta(W(k) - W(k+1)) \\
&\leq \dots .
\end{aligned}$$

Similar calculations as above yield

$$\sum_{j=1}^{k-1} V^{(j)} + \sum_{j=1}^{K_t+1-k} v^{(j)} + \delta W(k-1) \geq \sum_{j=1}^{k'} V^{(j)} + \sum_{j=1}^{K_t-k'} v^{(j)} + \delta W(k')$$

for all $k' \geq k$. Hence, the optimal solution lies in $\{0, \dots, k-1\}$. ■

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