# Price Manipulation, Dynamic Informed Trading and Tame Equilibria: Theory and Computation*

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**Abstract.** In this paper, we study a dynamic version of the model proposed in Glosten and Milgrom (1985) with a long-lived informed trader. When the same individual can buy, and in the future sell, the same asset, the trader may profit from price manipulation. We make a fundamental contribution by clarifying the conditions under which a unique equilibrium exists, and in what situations this equilibrium involves price manipulation. We propose a concept that we refer to as a "tame" equilibrium, and derive a bound for the number of trading rounds under (or over) which a unique equilibrium exists (or multiple equilibria exist) for a sufficiently low probability of informed trading. We characterize and numerically compute tame equilibria by showing the monotonicity and continuity of the value functions and bid and ask prices with respect to the market maker's prior belief on the value of the asset. Further, we provide a necessary and sufficient condition under which manipulation arises in equilibrium. We contend that we are able to extend our analysis to a continuous-time setting and thereby provide a reference framework in a discrete-time setting with a unique equilibrium.

**Key Words:** Market microstructure; Glosten–Milgrom; Dynamic trading; Price formation; Sequential trade; Asymmetric information; Bid–ask spreads.

**JEL Classification Numbers:** D82, G12.

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# 1 Introduction

This paper considers dynamic trading in a model proposed by Glosten and Milgrom (1985) with a long-lived informed trader. When the same individual can buy, and in the future sell, the same asset, the trader may profit from this round-trip trade, which we refer to as price manipulation. To analyze dynamic informed trading, we propose the concept of a "tame equilibrium" with desirable properties, such as the continuity of the informed trader's value functions in the market maker's prior belief. We then provide the conditions under which a unique tame equilibrium exists, and under which the equilibrium involves price manipulation. Finally, we provide a method to compute the equilibrium. The possibility then exists to extend our analysis to a continuous-time setting and thereby provide a reference framework in a discrete-time setting with a unique equilibrium and manipulation.

In this paper, we adopt a sequential trade framework with the trading of a risky asset over finitely many periods between the competitive market maker, two types of strategic informed traders, and liquidity traders. At the beginning of the game, nature chooses the liquidation value of a risky asset to be high or low and informs the informed trader, who then trades dynamically. In each period, there is a random determination of whether the informed trader or a liquidity trader trades. To prove the existence of an equilibrium and obtain the conditions for the uniqueness of this equilibrium, we use the Markov equilibrium property and consider the equilibrium where the market maker's belief and the number of remaining trading rounds determine the equilibrium strategy. In this way, we truncate a $T$-period serial problem into that of two-period decision making.

The key element in the analysis is the probability of informed trading. Our approach is to split a unit time interval into subintervals, where the length of each subinterval is a function of the informed trading probability, and to consider the situation where the probability of informed trading is sufficiently small and the number of trading rounds is sufficiently large.

Our results show that when there are relatively few trading rounds, the equilibrium is unique, whereas when there are many trading rounds, there are multiple equilibria. The intuition is simple. As there are two types of informed trader, there are four possible regimes, depending on whether each type manipulates. By a single crossing property of the payoff difference between buy and sell orders, we can prove that there is one equilibrium strategy within each regime when only one type manipulates. However, when there are too many chances to re-trade, both types simultaneously manipulate and this gives more "freedom" for multiple regimes to coexist. This analysis explicitly derives bounds for the number of trading rounds for which these different situations arise.

Our analysis indicates that a tame equilibrium uniquely exists in a very subtle situation, even when the probability of informed trading is sufficiently small. We also demonstrate that when the informed trading probability is sufficiently high, a tame equilibrium may fail to exist. The characterization of a tame equilibrium is given by the following properties.

- The value functions are continuous, piecewise differentiable and monotone with respect to the market maker's belief.
- There are two regions of the market maker's belief, depending on whether the slopes of the value functions are steeper than one. We show that when the informed trader manipulates in equilibrium, the slope needs to be steeper than one, because today he loses one dollar believing that the benefit that he can obtain on the future payoff (namely, the slope of the value function) is more than one dollar. This property specifies a region where manipulation can possibly arise in equilibrium.
- The value functions converge to linear functions when the probability of informed trading goes to $0$.

Finally, in order to see whether a particular equilibrium is unique and compute the manipulation rate, we develop a computational method using linear interpolation. Our computer simulation numerically demonstrates the intuition for the theoretical results.

## 1.1 Related Literature

There is a vast empirical literature on market manipulation. For instance, Aggarwal and Wu (2006) suggest that stock market manipulation may have important impacts on market efficiency. For example, according to the empirical findings in Aggarwal and Wu (2006), while manipulative activities appear to have declined in the main security exchanges, they remain a serious issue in both developed and emerging financial markets, especially in over-the-counter markets.[1]

The theoretical literature begins with market manipulation by uninformed traders. Allen and Gale (1992) provide a model of strategic trading in which some equilibria involve manipulation. Furthermore, Allen and Gorton (1992) consider a model of pure trade-based uninformed manipulation in which asymmetry in buys and sells by liquidity traders creates the possibility of manipulation.

The first paper to consider manipulation by an informed agent within the discrete-time Glosten–Milgrom framework is Chakraborty and Yilmaz (2004). They show that when the market faces uncertainty about the existence of informed traders, and when there are many trading periods, long-lived informed traders will manipulate in every equilibrium. Takayama (2010) furthers this analysis by providing a lower bound for the number of trading periods necessary for the existence of manipulation in equilibrium and shows that if the number of trading periods exceeds this lower bound, every

[1] See Jordan and Jordan (1996) on the cornering of the Treasury note auction market by Solomon Brothers in May 1991, Felixson and Pelli (1999) on closing price manipulation in the Finnish stock market, Mahoney (1999) on stock price manipulation leading up to the US Securities Exchange Act of 1934, Vitale (2000) on manipulation in the foreign exchange market and Merrick *et al.* (2005) on manipulation involving a delivery squeeze on a London-traded bond futures contract. For a useful survey, see Putniņš (2011).

equilibrium involves manipulation. Our results add to this work by showing that there are three possibilities. First, if the number of trading rounds is too small, the equilibrium is unique and manipulation does not arise. Second, if the number of trading rounds is too large, there are multiple equilibria with manipulation. Finally, we show that in the middle case, the equilibrium with manipulation is unique.

Although our main concern in this paper is the dynamic strategic informed trader, Calcagno and Lovo (2006) consider the dynamic informed market maker. In their model, a single market maker receives private information on the value of the asset and repeatedly competes with other uninformed market makers and liquidity traders. Further, in their model, the identity of the informed dealer is commonly known, and thus the uninformed market makers extract information on the value of the asset by observing past quotes posted by the informed market maker. They then show that there is an equilibrium where the informed market maker can possibly manipulate the market where the expected payoff is positive. The literature has investigated conditions based on the relations between prices and trades that rule out manipulation (Jarrow, 1992; Huberman and Stanzl, 2004). This paper also relates to that issue: that is, we start with the simplest possible model and study these relations. We then respond to the questions of when manipulation arises and when the equilibrium with manipulation is unique.

While our model uses a discrete setting, our paper also adds some insights to the literature on continuous-time models. Our analysis relates to two main strands of research. First, De Meyer (2010) studies an $n$-times repeated zero-sum game of incomplete information and shows that the asymptotics of the equilibrium price process converge to a Continuous Martingale of Maximal Variation (hereafter CMMV). One fundamental problem in financial econometrics is to accurately identify the stock price dynamics and analyze how a different market structure affects these dynamics. As De Meyer (2010) points out, this CMMV class could provide natural dynamics that may be useful in financial econometrics, although it remains an open question as to whether the equilibrium dynamics in a non-zero-sum game still belong to the CMMV class. In our analysis, we consider a limit on the probability of informed trading, which could ultimately correspond to the continuous-time setting.

Second, in addition to the Glosten–Milgrom framework, another reference framework is proposed by Kyle (1985). Back (1992) extends the analysis in Kyle (1985) to a continuous-time version. While the uniqueness of the optimal informed trader's strategy either in the original Kyle model or Back (1992) remains unknown, Boulatov *et al.* (2005) and Boulatov and Taub (2013) prove the uniqueness under some technical assumptions. Back and Baruch (2004) study the equivalence of the Glosten–Milgrom model and the Kyle model in a continuous-time setting, and show that the equilibrium in the Glosten–Milgrom model is approximately the same as that in the Kyle model when the trade size is small and uninformed trades occur frequently. Given these two closely related frameworks, our analysis, as a proxy for a continuous-time model, shows the possibility of multiple equilibria and provides insights concerning uniqueness within these frameworks. Finally, Back and Baruch

(2004) conclude that the continuous-time Kyle model is more tractable than the Glosten–Milgrom model, although most markets follow a sequential trade model. This paper is therefore useful for opening the black box lying between these two alternative frameworks, and especially in considering the uniqueness of the informed trader's dynamic strategy and price manipulation in the presence of bid–ask spreads.

Finally, many more studies address the relationship between prices and dynamic trading in a continuous-time setting. For example, Brunnermeier and Pedersen (2005) consider the dynamic strategic behaviour of large traders and show that "overshooting" occurs in equilibrium, while Back and Baruch (2007) analyze different market systems by allowing the informed traders to trade continuously within the Glosten–Milgrom framework. Lastly, within an extended Kyle framework, Collin-Dufresne and Fos (2012) study insider trading where the liquidity provided by noise traders follows a general stochastic process, and show that even though the level of noise trading volatility is observable, in equilibrium the measured price impact is stochastic.

The remainder of the paper is organized as follows. Section 2 details the model and states the main theorems. Section 3 provides the proofs of the theorems and characterizes the tame equilibrium. Section 4 illustrates the results from the numerical simulations and the theoretical findings. The final section concludes.

# 2 The Model

There is a single risky asset and a numeraire. The terminal value of the risky asset, denoted $\tilde{\theta}$, is a random variable that can take a low or high value, i.e., $L$ or $H$, where $L = 0$ and $H = 1$. We assume that $\Pr(\tilde{\theta} = H) = \delta_0$ for some $\delta_0 \in (0, 1)$. There is a single long-lived informed trader who learns $\tilde{\theta}$ prior to the beginning of trading.

Trade occurs in finitely many periods $t = 1, 2, \ldots, T$. In each period, a single trader comes to the market, the market maker quotes bid and ask prices for the risky asset, and the trader either buys one unit or sells one unit. The agent who goes to the market in period $t$ is a random variable unobserved by the market maker, such that with probability $\mu$ the informed trader is selected. If the informed trader is not selected, the agent selected is a "noise" or "liquidity" trader who (regardless of the quoted prices) buys with probability $\gamma$ and sells with probability $1 - \gamma$. The identities of the selected traders, and the values of the liquidity trader's trades, in the various periods are independent random variables. Prior to period $t$, there is no disclosure of information concerning the values of the random variables in that period.

We focus on equilibria where the market maker's belief and the number of remaining time periods determine an equilibrium strategy. The set of possible actions for the informed trader is denoted $\{B, S\}$, in which $B$ is a buy order and $S$ is a sell order. Here, the market maker's belief $b$ is the probability, from the market maker's point of view, that the state is high, $H$, going into period $t$. The market maker's ask and bid prices are functions of the market maker's belief and given by the functions $\alpha_t : [0,1] \to [0,1]$ and $\beta_t : [0,1] \to [0,1]$, respectively. For each type $\theta$ of informed trader, a trading strategy $\sigma_{\theta t} : [0,1] \to \Delta(\{B,S\})$ for $\theta \in \{H,L\}$ specifies a probability distribution over trades in period $t$ with respect to the bid and ask prices posted in period $t$. In period $t$, the type-$H$ informed trader buys the security with probability $\sigma_{Ht}(b)$ and sells with probability $1-\sigma_{Ht}(b)$, and the type-$L$ trader buys and sells with probabilities $1-\sigma_{Lt}(b)$ and $\sigma_{Lt}(b)$, respectively.

The market maker's posterior belief after observing an order is updated using Bayes' rule on the posterior probability that $\tilde{\theta} = H$. Define the bid and ask functions $A, B : [0,1]^3 \to [0,1]$ by the formulas:

$$A(b,x,y) = \tfrac{[(1-\mu)\gamma+\mu x]b}{(1-\mu)\gamma+\mu bx+\mu(1-b)(1-y)} \quad \text{and} \quad B(b,x,y) = \tfrac{[(1-\mu)(1-\gamma)+\mu(1-x)]b}{(1-\mu)(1-\gamma)+\mu b(1-x)+\mu(1-b)y}.$$

Now, the market maker is really to be thought of as a competitive market of risk-neutral market makers, for instance, a pair of market makers in Bertrand competition or a continuum of identical market makers. The equilibrium condition for the market maker is zero expected profits, which amounts to setting ask and bid prices equal to the posterior expected values of the asset.[2]

Now, we define the Markov equilibrium as follows.

**Definition 1.** *A* Markov equilibrium *is a collection of functions* $\{\alpha_t, \beta_t, \sigma_{Ht}, \sigma_{Lt}\}_{t=1,\cdots,T}$ *with* $\alpha_t, \beta_t : [0,1] \to [0,1]$, $\sigma_{Ht}, \sigma_{Lt} : [0,1] \to \Delta(\{B,S\})$ *and* $J_t, V_t : [0,1] \to I\!R$ *such that for each* $t = 1, \ldots, T$ *and* $b \in [0,1]$,

**(M1)** $\alpha_t(b) = A(b, \sigma_{Ht}(b), \sigma_{Lt}(b))$ *and* $\beta_t(b) = B(b, \sigma_{Ht}(b), \sigma_{Lt}(b))$.

**(M2)**

$$\sigma_{Ht}(b) = \begin{cases} 0, & 1-\alpha_t(b) + J_{t+1}(\alpha_t(b)) < \beta_t(b) - 1 + J_{t+1}(\beta_t(b)), \\ 1, & 1-\alpha_t(b) + J_{t+1}(\alpha_t(b)) > \beta_t(b) - 1 + J_{t+1}(\beta_t(b)), \end{cases}$$

*and*

$$\sigma_{Lt}(b) = \begin{cases} 0, & -\alpha_t(b) + V_{t+1}(\alpha_t(b)) > \beta_t(b) + V_{t+1}(\beta_t(b)), \\ 1, & -\alpha_t(b) + V_{t+1}(\alpha_t(b)) < \beta_t(b) + V_{t+1}(\beta_t(b)). \end{cases}$$

**(M3)**

$$J_t(b) = \mu\left[\sigma_{Ht}(b)(1-\alpha_t(b) + J_{t+1}(\alpha_t(b))) + (1-\sigma_{Ht}(b))(\beta_t(b) - 1 + J_{t+1}(\beta_t(b)))\right]$$

[2]Biais *et al.* (2000) justify this assumption by showing that when there are infinitely many market makers, their expected profit converges to zero. More recently, Calcagno and Lovo (2006) show that a market maker's equilibrium expected payoff is zero if he is uninformed.

$$+(1-\mu)\left[\gamma J_{t+1}(\alpha_t(b))+(1-\gamma)J_{t+1}(\beta_t(b))\right],$$

*and*

$$V_t(b)=\mu\left[(1-\sigma_{Lt}(b))(-\alpha_t(b)+V_{t+1}(\alpha_t(b)))+\sigma_{Lt}(b)(\beta_t(b)+V_{t+1}(\beta_t(b)))\right]$$
$$+(1-\mu)\left[\gamma V(\alpha_t(b))+(1-\gamma)V(\beta_t(b))\right].$$

We see that (M1) states that the ask and bid prices are Bayesian updatings of $b$ conditional on the type of order received; (M2) states that both types of informed seller optimize their order, taking into account the effect on the expected profits from future trades; and (M3) specifies the recursive computation of value functions. Implicitly we are assuming that the functions $J_{T+1}$ and $V_{T+1}$ are identically zero.

Now, we define a tame equilibrium.

**Definition 2.** *We say that a Markov equilibrium is* tame *if the value functions in every period $t$ satisfy the following conditions:*

**(C)** *$J_t$ and $V_t$ are continuous and piecewise differentiable;*

**(M)** *$J_t$ is strictly decreasing and $V_t$ is strictly increasing;*

**(SH)** *there is a $b_H$ such that for any $b_0, b_1 < b_H$, $\frac{J_t(b_1)-J_t(b_0)}{b_1-b_0} < -1$, and for any $b_0, b_1 > b_H$, $\frac{J_t(b_1)-J_t(b_0)}{b_1-b_0} > -1$;*

**(SL)** *there is a $b_L$ such that for any $b_0, b_1 < b_L$, $\frac{V_t(b_1)-V_t(b_0)}{b_1-b_0} < 1$, and for any $b_0, b_1 > b_L$, $\frac{V_t(b_1)-V_t(b_0)}{b_1-b_0} > 1$.*

Properties $(\mathbf{SH})$ and $(\mathbf{SL})$ state that there are indeed two regions for the value functions, and the slope equal to one can divide the entire region into two.

The following property requires that the limit of the value functions with respect to $\mu$ is a linear function. To define this, let $\{\alpha_{t,\mu}, \beta_{t,\mu}, \sigma_{Ht,\mu}, \sigma_{Lt,\mu}\}_\mu$ denote a Markov tame equilibrium in period $t \in \{1, \cdots, T\}$, and $\{J_{t,\mu}\}_\mu$ and $\{V_{t,\mu}\}_\mu$ denote the family of the associated tame equilibrium value functions such that for each $t \in \{1, \cdots, T\}$, a sufficiently small $\varepsilon$, and each $\mu \in (0, \varepsilon)$, $J_{t,\mu}, V_{t,\mu} : [0,1] \to I\!R$. (Similarly, in what follows, we denote a family of functions $f$ by $\{f_\mu\}_{\mu\in(0,\varepsilon)}$.) As the value functions may include kinks, we define $\partial_+ f(b) = \lim_{\epsilon\to 0^+} \frac{f(b+\epsilon)-f(b)}{\epsilon}$.

**Definition 3.** *We say that a family of tame Markov equilibria satisfies* linearity at limit *if the tame equilibrium value functions in every period $t$ satisfy the following condition:*

**(DH)** *for every $\delta_H$, there exists an $\varepsilon^H_\delta \in (0, \varepsilon)$ such that for each $\mu \in (0, \varepsilon^H_\delta)$,*

$$|\partial_+ J_{t,\mu}(b) + \mu(T-t+1)| < \delta_H;$$

**(DL)** *for every $\delta_L$, there exists an $\varepsilon_\delta^L \in (0, \varepsilon)$ such that for each $\mu \in (0, \varepsilon_\delta^L)$,*

$$|\partial_+ V_{t,\mu}(b) - \mu(T - t + 1)| < \delta_L.$$

Now, we state the main theorems in this paper. We consider $T$ to be the number of trades that the informed trader could possibly make, and we segment the time interval $[0, 1]$ into time periods of length $\Delta t$. Let $r > 0$ and $\Delta t = \mu^r$. If $\mu^r$ is too small, we may obtain the situation where both types of informed trader simultaneously manipulate at a particular value of the market maker's belief, and in this situation we obtain multiple equilibria. When $\mu^r$ is too large, manipulation may not arise. The second theorem specifies the intervals of $\mu^r$ for which the unique equilibrium with market manipulation or, instead, multiple equilibria arise.

**Theorem 1.** *Let $r \in (0, +\infty)$. There exists a $\bar{\mu}$ such that for every $\mu < \bar{\mu}$, if $T \leq \lfloor \frac{1}{\mu^r} \rfloor$, a tame Markov equilibrium exists.*

**Theorem 2.** *The following holds:*

*(a) Let $r \in (0, 1]$. Then, there exists a $\mu_0$ such that for every $\mu < \mu_0$, if $T = \lfloor \frac{1}{\mu^r} \rfloor$, the tame equilibrium is the unique equilibrium. Moreover, there is no manipulation in equilibrium.*

*(b) Let $r \in (1, 2)$. Then, there exists a $\mu_1$ such that for every $\mu < \mu_1$, if $T = \lfloor \frac{1}{\mu^r} \rfloor$, the tame equilibrium is the unique equilibrium. Moreover, in equilibrium, manipulation arises such that at most one type of trader manipulates at some belief $b$ in some period $t$.*

*(c) Let $r \in (2, +\infty)$. Then, there exists a $\mu_2$ such that for every $\mu < \mu_2$, if $T = \lfloor \frac{1}{\mu^r} \rfloor$, there are multiple equilibria, including multiple tame equilibria. Moreover, in equilibrium, manipulation arises such that both types of trader simultaneously manipulate at some belief $b$ in some period $t$.*

Our method[3] of proving the existence of equilibrium contrasts with the method proposed in Duffie *et al.* (1994). Duffie *et al.* (1994) develop an approach whereby the existence of an equilibrium for a finite horizon version of a model implies that an "expectations correspondence" possesses certain properties that, in turn, imply the desired existence. In our method, even if there are multiple equilibria, we select a desirable Markov equilibrium where the continuity and monotonicity of the value functions are recursively established. In this way, we recursively show the existence of a tame Markov equilibrium. As pointed out in Duffie *et al.* (1994), their method requires that the agents agree that an irrelevant random variable will determine which of a number of equally valid equilibrium continuations will be followed. Instead, our method finds a property that is needed to carry out backwards induction and shows that there exists an equilibrium such that this property holds.

[3] Although we use continuous value functions in this proof, one can prove the existence of a history-dependent equilibrium by using the fact that bid and ask prices are continuous in belief and strategy. This proof without using the continuity of the value functions is available upon request from the author.

# 3 The Proofs of The Main Theorems and Characterization

## 3.1 Preliminary Results

Our aim in this section is to show the conditions under which the equilibrium strategy is unique and manipulation arises in equilibrium. We start with defining a manipulative strategy. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit.

**Definition 4.** *For* $\theta \in \{H, L\}$ *we say that* the type-$\theta$ trader manipulates at $b$ in period $t$ *if* $\sigma_{\theta t}(b) < 1$.[4]

Our mode of analysis is backwards induction: we assume certain properties of the value functions $J_{t+1}$ and $V_{t+1}$, and from this assumption, derive various properties of the functions $\alpha_t$, $\beta_t$, $\sigma_{Ht}$, $\sigma_{Lt}$, $J_t$ and $V_t$. For $t = T$ the equilibrium conditions have a unique closed-form solution, because $J_{T+1}$ and $V_{T+1}$ are identically zero. The next result is necessary to begin the process of backwards induction. As a direct consequence of optimization, we can prove that there is no manipulation in the last period. By using this, we obtain the following theorem.

**Proposition 1.** *The last-period value functions* $J_T$ *and* $V_T$ *satisfy* $(\mathbf{C}), (\mathbf{M}), (\mathbf{SH}),$ *and* $(\mathbf{SL})$*, and their family satisfies* $(\mathbf{DH})$ *and* $(\mathbf{DL})$.

The proofs are found in Appendix A unless presented immediately after each result. Next, fix $t < T$ and suppose that a family of continuous next-period value functions is given where each $J_{t+1}$ and $V_{t+1}$ satisfies the four properties and the family satisfies $(\mathbf{DH})$ and $(\mathbf{DL})$. For $b \in [0,1]$ let

$$\mathcal{E}(b) = \{(\sigma_{Ht}(b), \sigma_{Lt}(b)) : \text{for the } \alpha_t(b) \text{ and } \beta_t(b) \text{ given by (M1), (M2) holds}\}.$$

Fix $b \in (0,1)$ and $\sigma = (\sigma_H, \sigma_L) \in \mathcal{E}(b)$, and let $\alpha = A(b, \sigma_H, \sigma_L)$ and $\beta = B(b, \sigma_H, \sigma_L)$ be the pair of equilibrium ask and bid prices associated with $b$ and $\sigma$ in period $t$.

**Lemma 1.** $\alpha > b > \beta$ *and* $\sigma_H + \sigma_L > 1$. *In particular,* $\sigma_H, \sigma_L > 0$.

*Proof.* Suppose that $\alpha \le \beta$. Bayes' rule implies that $0 < \alpha, \beta < 1$, so $1 - \alpha > \beta - 1$ and $-\alpha < \beta$, and the monotonicity condition $(\mathbf{M})$ gives

$$\begin{aligned} 1 - \alpha + J_{t+1}(\alpha) &> \beta - 1 + J_{t+1}(\beta); \\ -\alpha + V_{t+1}(\alpha) &< \beta + V_{t+1}(\beta). \end{aligned}$$

Now optimisation implies that $\sigma_H = 1$ and $\sigma_L = 1$, and Bayes' rule gives $\alpha > b > \beta$, a contradiction. In turn, by Bayes' rule, $\alpha > b > \beta$ implies that $\sigma_H + \sigma_L > 1$. $\square$

[4]This is the same definition used by Chakraborty and Yilmaz (2004). Back and Baruch (2004) use the term "bluffing" instead, while Huberman and Stanzl (2004) define price manipulation as a round-trip trade. For additional discussion on how to define price manipulation, see Kyle and Viswanathan (2008).

In equilibrium, the type-$H$ trader does not sell with probability one and the type-$L$ trader does not buy with probability one. This means that the informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case, the informed trader is indifferent between buy and sell orders. This motivates consideration of the slopes of the value functions. By Lemma 1 the bid–ask spread $\alpha - \beta$ is strictly positive. If the type-$H$ trader manipulates, it must be the case that $1 - \alpha + J_{t+1}(\alpha) = \beta - 1 + J_{t+1}(\beta)$, so we have:

$$\frac{J_{t+1}(\alpha) - J_{t+1}(\beta)}{\alpha - \beta} = \frac{\alpha + \beta - 2}{\alpha - \beta} = -1 - \frac{2 - 2\alpha}{\alpha - \beta} < -1. \tag{1}$$

Similarly, if the type-$L$ trader manipulates, we have:

$$\frac{V_{t+1}(\alpha) - V_{t+1}(\beta)}{\alpha - \beta} = \frac{\alpha + \beta}{\alpha - \beta} = 1 + \frac{2\beta}{\alpha - \beta} > 1. \tag{2}$$

Given $\beta < b < \alpha$, and slopes of the value functions between the bid and ask prices larger than one, **(SH)** and **(SL)** imply the following result.

**Lemma 2.** *In equilibrium, the following hold.*

**H.** *If the type-$H$ trader manipulates at* b*, then* $\frac{J_{t+1}(\beta_0) - J_{t+1}(\beta_1)}{\beta_0 - \beta_1} < -1$ *for any* $\beta_0, \beta_1 \leq \beta$.

**L.** *If the type-$L$ trader manipulates at* b*, then* $\frac{V_{t+1}(\alpha_0) - V_{t+1}(\alpha_1)}{\alpha_0 - \alpha_1} > 1$ *for any* $\alpha_0, \alpha_1 \geq \alpha$.

Lemma 2 indicates that manipulation could arise only in a region where the value function is steep. This is intuitive: the informed trader manipulates when the change in the future payoff from manipulating is large.

We now classify equilibria according to the types of trader that sometimes trade against their information. An equilibrium $\sigma$ is in *Regime* $\emptyset$ if $\sigma = (1, 1)$. It is in *Regime* $L$ if $\sigma_L < 1$ and $\sigma_H = 1$; it is in *Regime* $H$ if $\sigma_L = 1$ and $\sigma_H < 1$; and it is in *Regime* $HL$ if $\sigma_L < 1$ and $\sigma_H < 1$. We say that a regime *arises* at a belief $b$ if $\mathcal{E}(b)$ contains an equilibrium in that regime.

Our first objective is to show that the equilibrium strategy is unique within Regime $H$ and Regime $L$. To do so, we first prove that the difference in payoffs between trading for and against the information is monotone across the relevant region of prior beliefs. Define

$$\begin{aligned} D_H(b, x, y) &= -A(b, x, y) + J_{t+1}(A(b, x, y)) - B(b, x, y) - J_{t+1}(B(b, x, y)) + 2; \\ D_L(b, x, y) &= B(b, x, y) + V_{t+1}(B(b, x, y)) + A(b, x, y) - V_{t+1}(A(b, x, y)). \end{aligned} \tag{3}$$

Then $D_\theta(b, x, y)$ is the difference in payoffs between trading for and against the information given prices $A(b, x, y)$ and $B(b, x, y)$ for each type $\theta \in \{H, L\}$. Notice that for any $\sigma \in [0, 1]$ and prior $b \in [0, 1]$, by Bayes' rule, both the bid and ask prices are equal to $b$. Thus, we have

$$D_L(b, \sigma, 1 - \sigma) > 0 \quad \text{and} \quad D_H(b, \sigma, 1 - \sigma) > 0. \tag{4}$$

For each $\theta \in \{H, L\}$, define $x^\theta$ and $y^\theta$ by:

$$x^\theta := \begin{cases} \min\{x : D_\theta(b, x, 1) = 0\} & \text{if} \quad \{x : D_\theta(b, x, 1) = 0\} \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$y^\theta := \begin{cases} \min\{y : D_\theta(b, 1, y) = 0\} & \text{if} \quad \{y : D_\theta(b, 1, y) = 0\} \neq \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

The following lemma shows that when only one type manipulates, the payoff difference is monotonically decreasing with respect to each type's strategy in $[x^\theta, 1]$ or $[y^\theta, 1]$ for each $\theta \in \{H, L\}$.

**Lemma 3.** *For each $\theta \in \{H, L\}$,*

- *the payoff difference $D_\theta(b, x, 1)$ is monotonically decreasing as $x$ increases for all $x \geq x^\theta$;*
- *the payoff difference $D_\theta(b, 1, y)$ is monotonically decreasing as $y$ increases for all $y \geq y^\theta$.*

Further, for each $\theta \in \{H, L\}$, define $\tilde{y}_\theta : [0, 1] \to [0, 1]$ by $D_\theta(b, x, \tilde{y}_\theta(x)) = 0$ if $D_\theta(b, 1, 1) \leq 0$. Then, we obtain the following result:

**Lemma 4.** *For each $\theta \in \{H, L\}$, $\tilde{y}_\theta$ is continuous and is strictly decreasing in $x$.*

By Lemma 3 and Lemma 4, we obtain the following result.

**Proposition 2.** *The following holds:*

*(a) if Regime $HL$ arises at $b$, then $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) < 0$;*

*(b) if $D_L(b, 1, 1) \geq 0$ and $D_H(b, 1, 1) \geq 0$, then $\mathcal{E}(b) = \{(1, 1)\}$, so that only regime $\emptyset$ arises at $b$;*

*(c) if $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$, then $\mathcal{E}(b)$ is a singleton whose unique element is in Regime $L$;*

*(d) if $D_H(b, 1, 1) < 0$ and $D_L(b, 1, 1) \geq 0$, then $\mathcal{E}(b)$ is a singleton whose unique element is in Regime $H$;*

*(e) if $D_H(b, 1, 1) < 0$ and $D_L(b, 1, 1) < 0$, then at most one element within Regime $H$ is in $\mathcal{E}(b)$ and at most one element within Regime $L$ is in $\mathcal{E}(b)$.*

It may be easy to see that the monotonicity of $\tilde{y}_\theta$ by Lemma 4 yields a "single crossing property." By using this property, we obtain the following result.

**Lemma 5.** *At prior belief $b$ in period $t$,*

**Case I.** *if $x^L < x^H < 1 \quad and \quad 1 > y^L > y^H$, then only Regime $HL$ arises;*

**Case II.** *if $1 > x^L > x^H \quad and \quad y^L < y^H < 1$, then Regime $H$, Regime $L$ and Regime $HL$ arise.*

*Proof.* First, we show that Regime $HL$ arises in both cases. By symmetry, assume

$$x^L < x^H < 1 \quad \text{and} \quad 1 > y^L > y^H. \tag{5}$$

Lemma 3 and (5) indicate that $D_H(b,1,1) < 0$ and $D_L(b,1,1) < 0$. By Lemma 3, $x^H < 1$. Because $\tilde{y}_L(x^L) = 1$, $x^L < x^H < 1$ and $\tilde{y}_L(1) = y^L$, $\tilde{y}_L(x^H) \in (y^L, \tilde{y}_L(x^L))$ by Lemma 4. Thus, we obtain $\tilde{y}_L(x^H) < 1 = \tilde{y}_L(x^L)$. Thus, by (5) we obtain that $\tilde{y}_L(x^H) < \tilde{y}_H(x^H) = 1$ and $y^L = \tilde{y}_L(1) > \tilde{y}_H(1) = y^H$. By applying the intermediate value theorem to $\tilde{y}_H$ and $\tilde{y}_L$, there exists an $x \in (x^H, 1)$ to satisfy $\tilde{y}_H(x) = \tilde{y}_L(x)$. Thus, by the definition of $\tilde{y}_H$ and $\tilde{y}_L$, we obtain $x$ and $y = \tilde{y}_H(x)$ to satisfy both of the indifference conditions. Thus, Regime $HL$ arises.

Now, if $y^L < y^H$ and $x^L > x^H$, Regimes $H$, $L$ and $HL$ arise by Lemma 5, because Lemma 3 indicates that $D_H(b,1,y^L) > 0$ and $D_L(b,x^H,1) > 0$. We can prove Case II symmetrically. □

Our second objective is to show the next proposition, which states that when $\mu$ converges to zero, the speed of convergence in the bid–ask spread is faster than the evolution of a number of trading rounds if $r < 1$.

**Proposition 3.** *Take $x \in (0,1]$ and $y \in (0,1]$ and suppose that $x > 1 - y$. For $r < 1$, as $\mu$ goes to $0$, $\frac{A_\mu(b,x,y) - B_\mu(b,x,y)}{\mu^r}$ goes to $0$. For $r > 1$, as $\mu$ goes to $0$, $\frac{A_\mu(b,x,y) - B_\mu(b,x,y)}{\mu^r}$ goes to $+\infty$.*

*Proof of Proposition 3.* For future reference, we compute that

$$\begin{aligned} A_\mu(b,x,y) - b &= \tfrac{\mu b(1-b)(x-1+y)}{\mu[bx+(1-b)(1-y)]+(1-\mu)\gamma}; \\ b - B_\mu(b,x,y) &= \tfrac{\mu b(1-b)(x-1+y)}{\mu[b(1-x)+(1-b)y]+(1-\mu)(1-\gamma)}. \end{aligned}$$

As $x > 1 - y$, for $r < 1$,

$$\begin{aligned} \lim_{\mu \to 0} \frac{(A_\mu(b,x,y) - B_\mu(b,x,y))}{\mu^r} &= \lim_{\mu \to 0} \left( \frac{b(1-b)(x-1+y)}{\mu[bx+(1-b)(1-y)]+(1-\mu)\gamma} + \frac{b(1-b)(x-1+y)}{[\mu b(1-x)+(1-b)y]+(1-\mu)(1-\gamma)} \right) \cdot \mu^{1-r} \\ &= 0. \end{aligned}$$

The second statement is obtained symmetrically. □

Proposition 2 and Proposition 3 yield the following result.

**Proposition 4.** *Let $T = \lfloor \frac{1}{\mu^r} \rfloor$ for some $r > 0$ and a sufficiently small $\mu$. Then,*

*(a) if $r \in (0,1]$, Regime $\emptyset$ arises and manipulation does not arise at any belief in period $t$;*

*(b) if $r \in (1,+\infty)$, Regime $H$ arises at some belief $b$ that is sufficiently close to $1$, and Regime $L$ arises at some belief $b$ that is sufficiently close to $0$ in period $t$;*

*(c) if $r \in (1,2)$, Regime $HL$ never arises at any belief in period $t$;*

*(d) if $r \in (2,+\infty)$, Regime $H$, Regime $L$ and Regime $HL$ arise at some belief $b$ in period $t$.*

*Proof.* We first show that when $r \leq 1$, Regime $\emptyset$ arises for a sufficiently small $\mu$. Proposition 2's (a) indicates that if Regime $\emptyset$ does not arise, then an honest strategy is not optimal for at least one type. For notational simplicity, we write

$$\bar{A} := A(b,1,1) \quad \text{and} \quad \bar{B} := B(b,1,1).$$

Aiming to obtain a contradiction, by $(\mathbf{DL})$ suppose that there exists an arbitrarily small $\epsilon_A$ and $\epsilon_B$ for which the following holds:

$$\Big(\mu\lfloor\tfrac{1}{\mu^r}\rfloor - \mu(t-1)\Big)\left(\bar{A}-\bar{B}\right) + \epsilon_A^L\bar{A} - \epsilon_B^L\bar{B} \;>\; \left(\bar{A}+\bar{B}\right). \tag{6}$$

As $\mu\lfloor\frac{1}{\mu^r}\rfloor - \mu(t-1) \leq \mu\lfloor\frac{1}{\mu^r}\rfloor$,

$$\mu\lfloor\tfrac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right) + \epsilon_A^L\bar{A} - \epsilon_B^L\bar{B} \;>\; \left(\bar{A}+\bar{B}\right). \tag{7}$$

When $r \leq 1$, (7) does not hold as the left-hand side (LHS) is arbitrarily close to $0$ by Proposition 3 and the right-hand side (RHS) is strictly greater than $0$ for $b \in (0,1)$. This is a contradiction. By symmetry, we can also prove that $D_H(b,1,1) \geq 0$, and (a) in Proposition 2 completes the first claim.

Second, let $r \in (1,2)$. Then, let $b = \mu$. As $\mu$ is sufficiently small, similarly to (7), Regime $L$ arises if, for an arbitrarily small $\epsilon_{\bar{A}}$, $\epsilon_{\bar{B}}$, $\epsilon_A^L$ and $\epsilon_B^L$,

$$\mu\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right) + \epsilon_A^L\mu(1+\epsilon_{\bar{A}}) - \epsilon_B^L\mu(1+\epsilon_{\bar{B}}) > \mu\cdot\left(2+\epsilon_{\bar{A}}+\epsilon_{\bar{B}}\right),$$

which indicates

$$\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right) > 2+\epsilon_{\bar{A}}+\epsilon_{\bar{B}} - \epsilon_A^L(1+\epsilon_{\bar{A}}) + \epsilon_B^L(1+\epsilon_{\bar{B}}). \tag{8}$$

By Proposition 3, for $r > 1$, the LHS is sufficiently large and the RHS is sufficiently close to 2. Therefore, the above holds. By the same argument, we can see that Regime $H$ does not arise at $b = \mu$ by (c) of Proposition 2 because Proposition 3 indicates that $\mu\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right)$ is sufficiently close to $0$ for $r > 1$ and so

$$\mu\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right) + \epsilon_B^H\mu(1+\epsilon_{\bar{B}}) - \epsilon_A^H\mu(1+\epsilon_{\bar{A}}) < 2 - \mu\cdot\left(2+\epsilon_{\bar{A}}+\epsilon_{\bar{B}}\right).$$

On the other hand, symmetrically we can prove that at $b = 1-\mu$, Regime $H$ arises and Regime $L$ does not arise.

Third, let $r < 2$. Seeking a contradiction, suppose that in period $t$, there exist $\bar{\alpha}$ and $\bar{\beta}$ satisfying the two indifference conditions. Then, by substituting $(\mathbf{DH})$ and $(\mathbf{DL})$ into the indifference conditions, for sufficiently small $\varepsilon_L$'s and $\varepsilon_H$'s,

$$\begin{aligned} \left(\mu\lfloor \tfrac{1}{\mu^r} \rfloor - \mu(t-1)\right)(\bar{\alpha}-\bar{\beta}) + \varepsilon_L^{\bar{\alpha}} - \varepsilon_L^{\bar{\beta}} &= \bar{\alpha}+\bar{\beta}; \\ -\left(\mu\lfloor \tfrac{1}{\mu^r} \rfloor - \mu(t-1)\right)(\bar{\alpha}-\bar{\beta}) + \varepsilon_H^{\bar{\alpha}} - \varepsilon_H^{\bar{\beta}} &= \bar{\alpha}+\bar{\beta}-2. \end{aligned} \tag{9}$$

Then, notice that we must have $\bar{\alpha}+\bar{\beta} \approx 1$, because for a sufficiently small $\mu$, combining the two in (9) yields $0 \approx 2(\bar{\alpha}+\bar{\beta}) - 2$. Thus, we obtain:

$$\left(\mu\lfloor \frac{1}{\mu^r} \rfloor - \mu(t-1)\right)(\bar{\alpha}-\bar{\beta}) \approx 1. \tag{10}$$

Note that as $t \in \{1, \cdots, \lfloor \frac{1}{\mu^r} \rfloor - 1\}$,

$$\mu\lfloor \frac{1}{\mu^r} \rfloor(\bar{\alpha}-\bar{\beta}) > \left(\mu\lfloor \frac{1}{\mu^r} \rfloor - \mu(t-1)\right)(\bar{\alpha}-\bar{\beta}) \geq (\bar{\alpha}-\bar{\beta}). \tag{11}$$

By applying the squeeze theorem to (11) and Proposition 3, the LHS of (10) is sufficiently close to $0$, which contradicts (10).

Finally, it suffices to show that when $r > 2$, Regime $H$ and Regime $L$ simultaneously arise at some belief $b$, because by Lemma 5, Regime $HL$ also arises. First, we show that $D_H(b,1,1) < 0$ and $D_L(b,1,1) < 0$ hold simultaneously at some belief $b$. Note that when Regime $HL$ arises, $\bar{\alpha}+\bar{\beta} \approx 1$ holds, as indicated in the proof of the previous claim, and thus $\bar{A}+\bar{B} \approx 1$ holds because for any $x$ and $y$, $A_\mu(b,x,y)$ and $B_\mu(b,x,y)$ both converge to $b$.

Property $(\mathbf{DL})$ indicates that:

$$\frac{V_{t+1}(\bar{A}) - V_{t+1}(\bar{B})}{\bar{A}-\bar{B}} \approx \mu\lfloor \frac{1}{\mu^r} \rfloor - \mu(t-1). \tag{12}$$

For a sufficiently small $\mu$, as $\bar{A}+\bar{B} \approx 1$,

$$\frac{\bar{A}+\bar{B}}{\bar{A}-\bar{B}} \approx \frac{1}{\bar{A}-\bar{B}}. \tag{13}$$

Proposition 3 implies that $\mu\lfloor \frac{1}{\mu^r} \rfloor \cdot (\bar{A}-\bar{B})$ is sufficiently large and must be larger than $1$. Thus, comparing (12) and (13) yields:

$$\frac{V_{t+1}(\bar{A}) - V_{t+1}(\bar{B})}{\bar{A}-\bar{B}} > \frac{\bar{A}+\bar{B}}{\bar{A}-\bar{B}}. \tag{14}$$

Therefore, we can conclude that $D_L(b,1,1) < 0$ holds, and by symmetry we can also prove that $D_H(b,1,1) < 0$ at the same time. Then, by (4) and Lemma 3 there exists a $y_L$ such that $D_L(b,1,y^L) =$

0. Let $\bar{\alpha}_0 = A(b, 1, y^L)$ and $\bar{\beta}_0 = B(b, 1, y^L)$. Then, we consider the type-$H$ trader. As $\bar{\alpha}_0$ and $\bar{\beta}_0$ are sufficiently close to $b$, and $(\bar{\alpha}_0 - \bar{\beta}_0)\mu(T - t + 1)$ is sufficiently large due to Proposition 3,

$$\bar{\beta}_0 - 1 + \bar{\beta}_0\mu(T - t + 1) < 1 - \bar{\alpha}_0 + \bar{\alpha}_0\mu(T - t + 1), \tag{15}$$

and for any sufficiently small $\varepsilon_H$'s,

$$\bar{\beta}_0 - 1 + \bar{\beta}_0\mu(T - t + 1) + \varepsilon_H^\beta < 1 - \bar{\alpha}_0 + \bar{\alpha}_0\mu(T - t + 1) + \varepsilon_H^\alpha. \tag{16}$$

Given that (16) and $(\mathbf{DH})$ indicate that $D_H(b, 1, y_L) > 0$, Regime $L$ arises. Symmetrically, we can prove that Regime $H$ arises. By Lemma 5, Regime $HL$ also arises. Therefore, these three different regimes coexist at belief $b$ and time $t$. This completes the proof. □

One implication of Proposition 4 is straightforward. When the number of trading periods grows more rapidly than the informed trading probability, Regime $HL$ arises. On the other hand, if there are not enough trading periods, manipulation itself does not arise.

Proposition 4 also indicates that when the value functions are close to a linear function, Regime $L$ arises around $b = \mu$, which is sufficiently small. Proposition D4 in Appendix D computes the slopes of the value functions at $b = 0$ and $b = 1$, which also shows how the slopes at the edges grow over time. Manipulation arises when the market maker is almost correct or very wrong. For a sufficiently small $\mu$, Proposition 4 implies that manipulation arises when the market maker is almost correct. The important factor is $\bar{A} + \bar{B}$ or $2 - (\bar{A} + \bar{B})$. This can be thought of as the difference of costs that the type-$L$ or the type-$H$ trader has to incur in order to manipulate. When the value functions are almost linear, the effect of manipulation, which can be measured by the slopes of the value functions, is almost constant everywhere in the region. When this constant effect is not so large, the informed trader would only manipulate when the cost of manipulating is small. As such, Regime $HL$ does not arise as the two regions where the cost of manipulation is small for each type do not overlap.

## 3.2 The Proof of Theorem 1

*Proof of Theorem 1.* Proposition 2 implies that even when there are multiple regimes, the equilibrium strategy is unique within Regime $H$ or $L$. Therefore, when there are multiple equilibria, we select an equilibrium in Regime $H$ (or $L$). We provide a series of lemmas to prove each property in Appendix A. Here, we explain how we establish each property. Continuity and piecewise differentiability are consequences of the fact that Bayes' rule and the next-period value function also hold these properties, and the implicit function theorem yields the result (see Lemma A2). Then, we show that a family of value functions satisfies $(\mathbf{DH})$ and $(\mathbf{DL})$ if all the constituent functions satisfy $(\mathbf{C})$. Note that when $\mu$ converges to 0, both bid and ask prices converge to each belief and these functions become linear. As the summation of these functions, we also obtain linearity at the limit for the current-period value functions (see Lemma A3). We obtain $(\mathbf{SH})$ and $(\mathbf{SL})$ from $(\mathbf{DH})$ and $(\mathbf{DL})$ (see Lemma A4). To obtain monotonicity of the current-period value functions, we require the monotonicity of the bid and ask prices. This is easy to show in Regime $\emptyset$ using Bayes' rule. In Regime $H$ or $L$, the indifference condition and Lemma 2 give us the results (see Lemma A6). By mathematical induction and Proposition 1, we conclude that the equilibrium is unique for a sufficiently small $\mu$. Finally, take a supremum of such a $\mu$ and call it $\bar{\mu}$. This completes the proof. $\square$

### 3.3 The Proof of Theorem 2

To prove the second theorem, we use Proposition 2 recursively by applying backwards induction. An intuition behind the second theorem is that when $r \leq 1$, the equilibrium is still unique but there is no manipulation as there are too few trading rounds for manipulation to arise in equilibrium. Conversely, when $r > 2$, there would be too many chances to trade and Regime $HL$ may arise. Indeed, $\mu^r$ for $r \in (1, 2)$ is the interval of trading periods for which the equilibrium is unique and Regime $HL$ does not arise.

*Proof of Theorem 2.* To prove (a), by Proposition 4, in equilibrium manipulation does not arise when $r \leq 1$ for a sufficiently small $\mu$. Take a supremum of such a $\mu$ and call it $\mu_0$. Similarly, we can prove (b) and (c). $\square$

One may wonder if the result holds for $r = 2$. When $r = 2$, as $\mu$ goes to $0$, following the proof of Proposition 3, we obtain:

$$\lim_{\mu \to 0} \frac{(\alpha_{t,\mu}(b) - \beta_{t,\mu}(b))}{\mu^{r-1}} = b(1-b)(\sigma_{H,\mu} - 1 + \sigma_{L,\mu}) \left( \frac{1}{\gamma} + \frac{1}{1-\gamma} \right).$$

By substituting the above into (11), we can see that whether a pair of bid and ask prices to satisfy (10) exists depends on $\gamma$. Still, the following result holds.

**Proposition 5.** *Let $r = 2$, $\gamma = \frac{1}{2}$. Then, there exists a $\mu'$ such that for every $\mu < \mu'$, if $T = \lfloor \frac{1}{\mu^r} \rfloor$, the unique equilibrium is tame. Moreover, in equilibrium, at most one type of trader manipulates at some belief $b$ in period $t$.*

*Proof of Proposition 5.* When $r = 2$ and $\gamma = \frac{1}{2}$, as $\mu$ goes to $0$, following the proof of Proposition 3, we obtain:

$$\lim_{\mu \to 0} \frac{(\alpha_{t,\mu}(b) - \beta_{t,\mu}(b))}{\mu^{r-1}} = b(1-b)(\sigma_{H,\mu} - 1 + \sigma_{L,\mu}) \cdot 4.$$

When $\mu$ is sufficiently small, $\bar{A} \approx b$ and $\bar{B} \approx b$. Applying a similar idea in (9) of the proof of Proposition 4, we show that the indifference conditions would not hold simultaneously for both types.

Proposition 2's (a) indicates that if Regime $HL$ arises, both of the following hold:

$$\begin{aligned} \lim_{\mu\to 0}\tfrac{(\alpha_{t,\mu}(b)-\beta_{t,\mu}(b))}{\mu^{r-1}} &= b(1-b)(\sigma_{H,\mu}-1+\sigma_{L,\mu})\cdot 4 &>& \quad 2b \\ -\lim_{\mu\to 0}\tfrac{(\alpha_{t,\mu}(b)-\beta_{t,\mu}(b))}{\mu^{r-1}} &= -b(1-b)(\sigma_{H,\mu}-1+\sigma_{L,\mu})\cdot 4 &<& \quad -2(1-b). \end{aligned}$$

Adding these together, we obtain $4(\sigma_{H,\mu}-1+\sigma_{L,\mu})>4$, which is impossible. Thus, Regime $HL$ does not arise. By Proposition 4, manipulation arises in equilibrium as $r\in(1,+\infty)$. This completes the proof. □

## 3.4 Characterization of a Tame Markov Equilibrium

Here we start with an illustration of a possible complication in the period preceding that in which Regime $HL$ arises through backwards induction. When Regime $HL$ arises, even when there is only one equilibrium strategy that satisfies the indifference conditions, there could be one pair of bid and ask prices to satisfy these conditions. Thus, it may be the case that in some interval of prior beliefs, bid and ask prices become constant in this equilibrium. Then, a current-period value function would become constant for this interval, and hence the value functions may not satisfy $(\mathbf{M})$. Indeed, when $\mu$ is sufficiently high, this situation may arise.

Consider the case where $\mu$ is sufficiently close to one (instead of zero). We can see that in the region of beliefs sufficiently close to 0 or 1, the value functions would have a spike, because if the market maker was very wrong the informed trader would earn non-negligible profits, whereas in the rest of the region, as the bid–ask spread is quite large, the informed trader's profit would be very close to zero. This intuition is clearly observed later in Figure 2 of Section 4. The value functions do not converge to a linear function as described in $(\mathbf{DH})$ and $(\mathbf{DL})$. Even if we replace $(\mathbf{DH})$ and $(\mathbf{DL})$ with the property that the value functions have a spike, one can prove that Lemma 5 indicates that Case I occurs and thus only Regime $HL$ arises at some belief.[5]

By (c) in Theorem 2 and Proposition 2, for a sufficiently small $\mu$, even when there are multiple equilibria, Regime $H$ and Regime $L$ also arise with Regime $HL$. By selecting either Regime $H$ or $L$ in period $t$, we can move on in backwards induction, and all the five properties are held, as proved in Section 3.3. Therefore, we can obtain the following theorem by Theorem 2.

**Theorem 3.** *Let $r\in(0,+\infty)$. Then, for every $\mu<\bar{\mu}$, if $T=\lfloor\frac{1}{\mu^r}\rfloor$, in a tame Markov equilibrium,*

- *ask and bid prices $\alpha_t$ and $\beta_t$ are continuous and monotonically increasing with respect to the market maker's belief in every period $t$;*
- *manipulation arises if and only if $D_\theta(b,1,1)<0$ for at least one $\theta\in\{H,L\}$;*

[5]A more rigorous analysis is available upon request.

- *only one type manipulates even when manipulation arises.*

*Proof.* When we restrict our attention to an equilibrium such that whenever Regime $HL$ arises together with Regime $H$ or Regime $L$, we select Regime $H$ or Regime $L$, then (b) of Lemma A2 and (b) of Lemma A6 prove the second statement. Third, suppose that Regime $H$ arises. Then,

$$D_H(b, x^H, 1) = 0 \quad \text{and} \quad D_L(b, x^H, 1) \geq 0.$$

By Lemma 3, we obtain $D_H(b, 1, 1) < 0$. Similarly, if Regime $L$ arises, $D_L(b, 1, 1) < 0$. As we only consider the equilibrium where Regime $\emptyset$, $H$ or $L$ arises and $HL$ does not, this completes the proof for the "if" part of the first statement. The "only if" part is proved by Proposition 2. □

# 4 Calibration of Equilibrium

This section explains our calibration results for the equilibrium. A detailed procedure is available in the Appendix C that contrasts our method from that used in Back and Baruch (2004). The Glosten–Milgrom framework in Back and Baruch (2004) is a continuous-time stationary case and their program attempts to find the value functions as a fixed point. To do this, they use an extrapolation method that requires calculating the slopes of the value functions. Because of this problem, Back and Baruch (2004) wrote that even though all the equilibrium conditions hold with a high degree of accuracy, the strategies were not estimated very accurately when manipulation arises. To avoid this problem, we use a linear interpolation method.

We approximate a continuous value function by linear segments and then solve the equilibrium. Given that no type of trader manipulates in the last period of the game, we can calculate the value functions in the last period along with the bid and ask prices. We then split the whole interval $[0, 1]$ into $n$ segments and linearly interpolate the value function for each type of trader in each interval. We then attempt to find whether a pair of ask and bid prices exists such that that each type of informed trader becomes indifferent between buy and sell orders in each interval of the market maker's belief. Using the bid and ask prices we obtain using this procedure, we calculate the current-period value functions and repeat the procedure in the following periods. Finally, we consider the last case, where both types manipulate.

## Characteristics of Equilibrium

We first consider $\gamma = \frac{1}{2}$, so that the equilibrium is symmetric[6]. Figure 1 exhibits the equilibrium bid and ask prices with respect to the market maker's prior belief for the periods from $201$ to $400$. The solid curves that present the highest and lowest points represent the ask and bid prices for the case

[6] The proof of a symmetric equilibrium is in Appendix B.

where there is no manipulation. In the bid and ask price figures, there is a region of beliefs in which the bid or ask prices differ between the periods. It is in this region of beliefs that manipulation arises in equilibrium. As the informed trader's strategy differs between periods because the manipulation rate is time dependent, the bid and ask prices also differ between periods. The thick curves in the middle are indeed a stack of $200$ lines, and as manipulation arises, these curves do not coincide with the single line for the no-manipulation case. Although it is obvious from Bayes' rule, from this figure we can also see that manipulation indeed decreases the bid–ask spread in a given interval of prior beliefs.

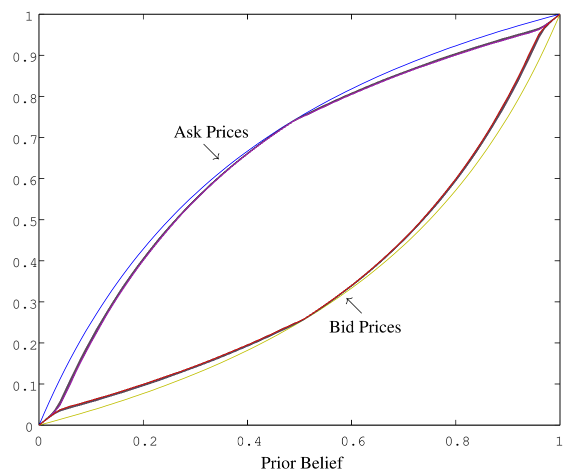


Figure 1: Bid and Ask Prices when $\mu = 0.5, \gamma = 0.5, t \in \{201, 400\}$

As shown in Calcagno and Lovo (2006), along with some empirical and experimental evidence (see Koski and Michaely (2000), Krinsky and Lee (1996) and Venkatesh and Chiang (1986)), our simulation also shows that the bid–ask spread is largest in the last period. In our analysis, this is a direct consequence of the fact that there is no manipulation in the last period. In the Calcagno and Lovo (2006) analysis, we observe this because the winner's curse increases when the terminal period comes near. Although our mechanisms differ, we observe a similar result here as well.

## Manipulation

The results of the simulation also show that the type-$H$ trader manipulates in a region of beliefs close to $0$ and the type-$L$ trader manipulates in a region of beliefs close to $1$. This result is somewhat counterintuitive because, for example, if the type-$H$ trader manipulates in a region of beliefs close to $0$, the bid price will be very low and the trader can only obtain a little money. However, to affect the future payoffs through the updating of the market maker's beliefs, they will manipulate when the bid–ask spread is small and the slope of the next-period value function is steep. This is consistent with our result in Theorem 3.

To take a more careful look at a manipulative strategy, Figure 3 shows the equilibrium strategy for the type-$H$ and the type-$L$ trader in the $[0, 1]$ interval of prior beliefs. Interestingly, we can see that when the prior belief is close to $1$, the type-$H$ trader also manipulates; similarly, the type-$L$ trader manipulates when the belief is close to $0$. Indeed, this result is consistent with our calculation in Proposition D4 of Appendix D. When the belief is very close to $0$ or $1$, the market maker almost knows the value of the asset. As shown in Proposition D4, the slopes of the value functions geometrically increase. As a result, the type-$H$ or the type-$L$ trader starts to manipulate as the number of remaining trading periods increases.

Table 1 describes how manipulation starts to arise. As discussed in Proposition 4, manipulation starts to arise when the market maker is almost correct or very wrong. In a sense, there are two types of manipulation. In Proposition 4, manipulation for $r \in (1, 2)$ corresponds to that which arises when the market maker is almost correct. Our simulation shows that as the slope becomes steeper, manipulation arises only when the market maker is very wrong. In other words, the other type of

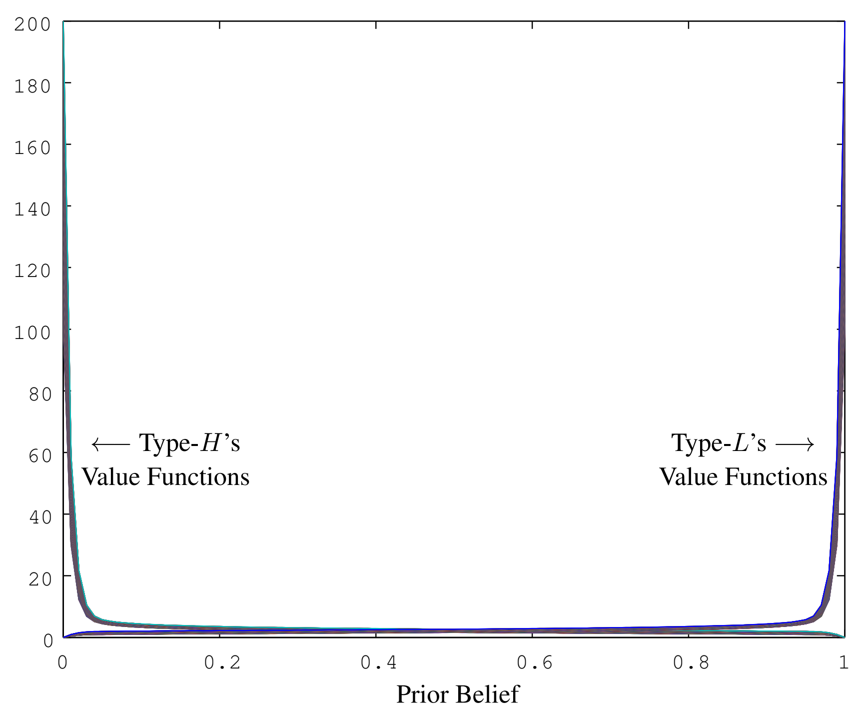


Figure 2: Bid and Ask Prices when $\mu = 0.5, \gamma = 0.5, t \in \{201, 400\}$

manipulation disappears and the remaining type of manipulation expands. This conveys the intuition that the slopes of the value functions are indeed an incentive for the informed trader, and as they increase, manipulation begins to take place over a wider range.

| | $b = 0.01$ | $b = 0.02$ | $\cdots$ | $b = 0.98$ | $b = 0.99$ |
|---|---|---|---|---|---|
| $t = 11$ | $L$ | $\emptyset$ | $\cdots$ | $\emptyset$ | $H$ |
| $t = 12$ | $L$ | $H$ | $\cdots$ | $L$ | $H$ |
| $t = 13$ | $H$ | $H$ | $\cdots$ | $L$ | $L$ |

Table 1: Regimes and Beliefs for $t = 11, 12, 13$

In this way, Regime $HL$ starts to arise in our simulation as each region that each type of trader manipulates overlaps with the other. In our simulation, in period $t = 52$, Regime $HL$ starts to arise at belief $0.5$, and it similarly starts to arise at beliefs $b = 0.01$ or $0.99$ in period $t = 271$.[7] As the market maker is completely wrong at $b = 0$ or $1$, the informed trader does not manipulate because the prices are favourable for the type-$H$ ($b = 0$) or the type-$L$ ($b = 1$) trader. On the other hand, in the region close to $b = 0$ or $1$, as the value function is steep, the type-$H$ trader manipulates around $b = 0$ or the type-$L$ trader manipulates around $b = 1$. Therefore, there is a spike in the rate of informed trading near the edges. As we can see from Figure 3, there appears to be a discontinuity in the informed strategy when the market maker is very wrong. The important idea in our theory of a tame equilibrium is to make manipulation arise near the other edge (that is, when the market maker is almost correct) and not near this edge (when the market maker is very wrong).

As we can see in Figure 2, the value functions are not globally convex. Near $b = 0$ or $1$, some parts appear to be concave. Thus, manipulation arises as the slope near $b = 0$ or $1$ becomes sufficiently steep. As shown in the proof of (b) in Theorem 2, this manipulation remains in the unique equilibrium when $\mu$ becomes sufficiently small and $T$ is sufficiently large. In other words, when the value functions become sufficiently "linear," the moment when the market maker begins to make a mistake is the only opportunity to manipulate. Theorem 2's (b) presents this intuition.

## Effects of an Asymmetric Liquidity Distribution

To this point, we have considered the symmetric case in the sense that the liquidity for a buy is equally likely as the liquidity for a sell. Here we consider how an asymmetric liquidity distribution affects bid and ask prices and who manipulates in equilibrium. The following four figures in Figure 4 and Figure

[7] From Figure 1, it may be difficult to see that at $b = 0.5$, the bid and ask prices in Regime $\emptyset$ differ from the simulated prices for $t = 201, \cdots, 400$, because the manipulation rates at $b = 0.5$ are quite small. Indeed, the bid price without manipulation is $0.25$ and the ask price without manipulation is $0.75$, while the simulated equilibrium bid prices for these periods range from $0.2511$ to $0.2528$ and the ask prices for these periods range from $0.7489$ to $0.7472$.

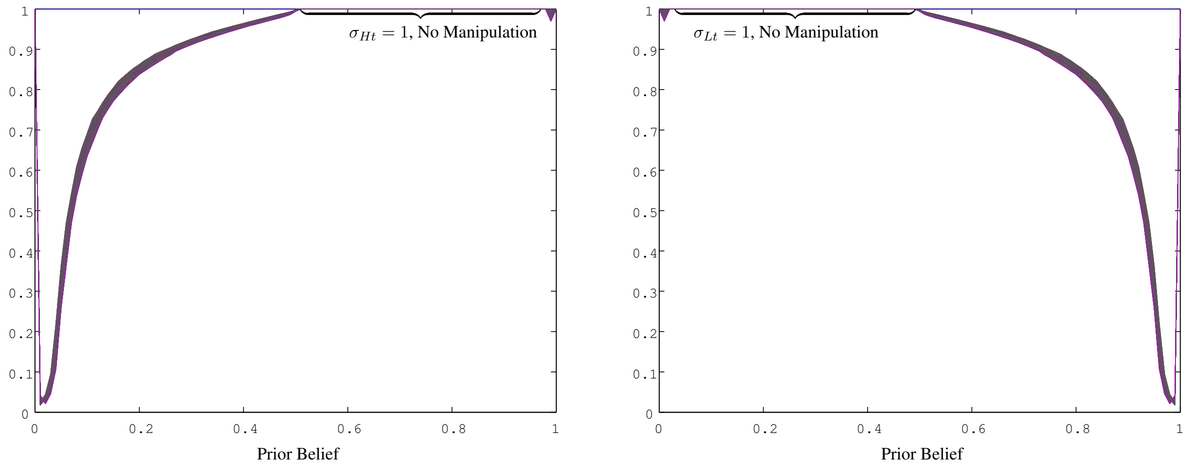


Figure 3: Manipulation Rates $\sigma_{Ht}$ and $\sigma_{Lt}$ when $\mu = 0.5, \gamma = 0.5, t \in \{201, 400\}$

5 show the bid and ask prices, as well as value functions, when $\gamma = 0.2$ and $\gamma = 0.8$. Comparing Figure 4 and 5, we can see that the type-$H$ trader at belief $b$ is a mirror image of the type-$L$ trader at belief $1 - b$. This result is consistent with Proposition B1 in Appendix A. We also observe that Theorem 3 continues to hold in the sense that the informed trader manipulates when the slope is steep. As we can predict from Bayes' rule, as $\gamma$ decreases, the type-$H$ trader's value functions are positioned closer to 0. As such, the region where its value functions are steep becomes smaller and the region in which the type-$H$ trader manipulates become more restricted. The reverse holds for the type-$L$ trader.

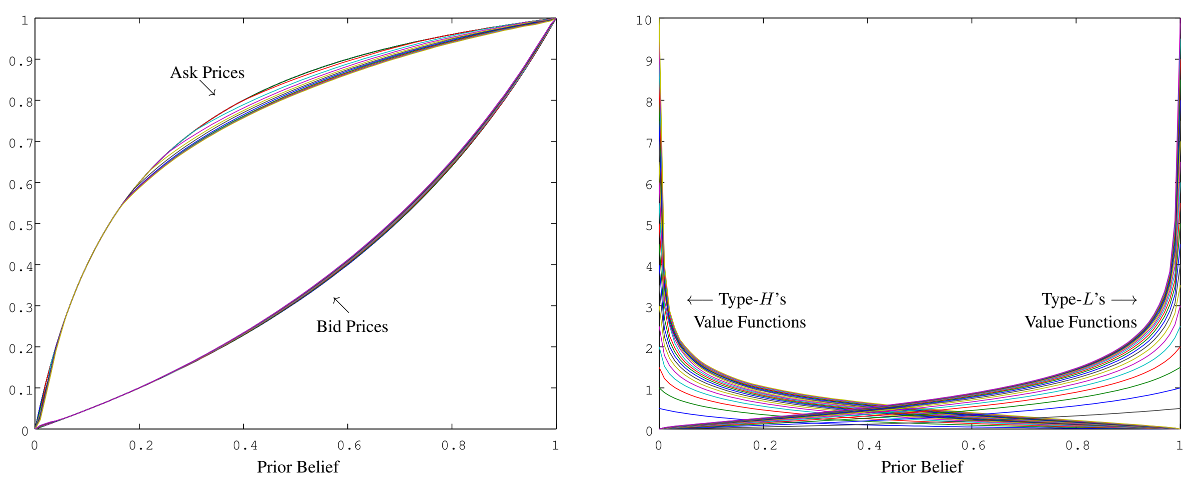


Figure 4: Bid and Ask Prices and Value Functions when $\mu = 0.5, \gamma = 0.2, t \in \{1, 20\}$

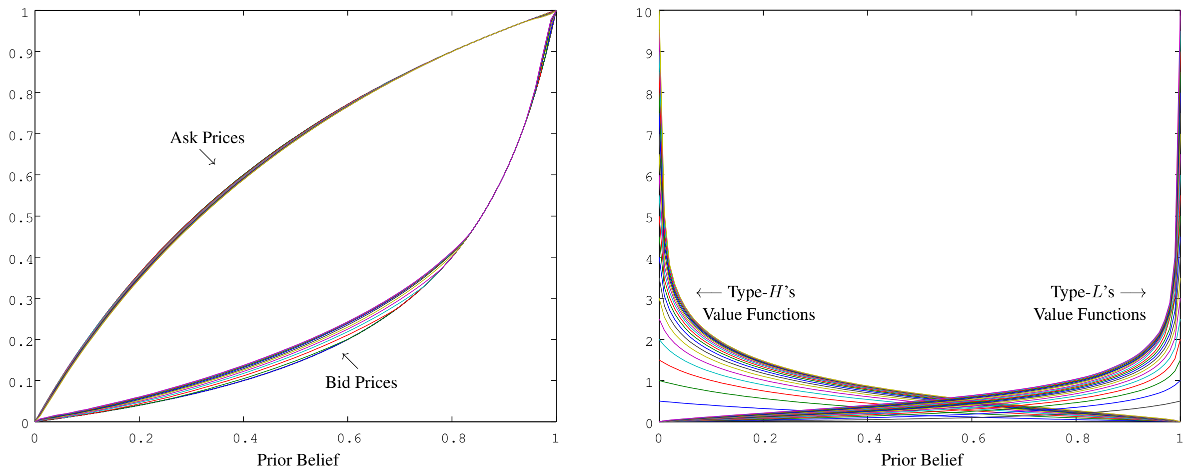


Figure 5: Bid and Ask Prices and Value Functions when $\mu = 0.5, \gamma = 0.8, t \in \{1, 20\}$

# 5 Concluding Remarks

In this paper, we developed a model of dynamic informed trading from a canonical framework in the market microstructure literature. We make a fundamental contribution to the literature by showing the existence of multiple equilibria involving price manipulation. We provided a theorem describing conditions under which multiple equilibria or a unique equilibrium arises. We also provided a computational method to approximate the equilibrium. We can readily extend our findings in the discrete-time setting to a continuous-time setting, and in this sense, this paper provides a different approach to proving the uniqueness of an equilibrium in the continuous-time model.

From our analysis, several important research questions arise. First, as discussed in the introduction, given the association with De Meyer (2010), our paper provides a fundamental framework for a non-zero-sum trading game. Adding a time discount factor to the informed trader's profit to bring our analysis into a continuous-time setting is an obvious extension. Second, the existence of a unique equilibrium in the Kyle model remains an open question in the literature. As shown in Back and Baruch (2004), the equilibrium in the Glosten–Milgrom model converges to that in the Kyle model. We show that there is a possibility of multiple equilibria; however, when $\mu$ is sufficiently small and $T$ satisfies a certain condition, there exists a unique equilibrium in the dynamic Glosten–Milgrom setting. Conceptually the model with a very small $\mu$ is analogous to a continuous-time setting, because the possibility of informed trading is very small in the infinitesimal time intervals of continuous-time models. In this sense, we could use our analysis to understand how a unique equilibrium in the dynamic Glosten–Milgrom model converges to the equilibrium in the Kyle model.

Third, one may question whether the market maker's belief concerning the risky asset's payoff converges to the truth as the number of trading periods tends to infinity. Recently, there has been renewed interest in private information and learning. Examples include Golosov *et al.* (2009) and Loertscher and McLennan (2013). Although their settings are quite different from ours, both question whether uninformed agents learn private information. In the market microstructure literature, Glosten and Milgrom (1985) originally show that such convergence is obtained almost surely if the only available trade size is the unit trade size and an informed trader can trade only once. Ozsoylev and Takayama (2010) show a similar result where the informed trader can trade only once but in multiple sizes. We expect that this result will also hold in our framework, as an intuition similar to the Martingale convergence theorem holds when the equilibrium is unique. However, as there is a possibility that multiple equilibria will arise, and especially that both types of trader will manipulate at the same time, it would be interesting to see how this type of manipulation affects the market's learning.

# A Lemmas and Proofs

*Proof of Proposition 1.* Note that in the last period, Regime $\emptyset$ arises in the whole interval $[0, 1]$ as there is no chance to re-trade. That $\sigma_{HT}$ and $\sigma_{LT}$ are identically one is an immediate consequence of optimization, and the equations are derived by substituting and simplifying.

In any Markov equilibrium, $\sigma_{HT}$ and $\sigma_{LT}$ are identically one, and:

$$\begin{aligned} \alpha_T(b) &= \tfrac{b(\mu+(1-\mu)\gamma)}{\mu b+(1-\mu)\gamma}; \\ \beta_T(b) &= \tfrac{b(1-\mu)(1-\gamma)}{\mu(1-b)+(1-\mu)(1-\gamma)}; \\ J_T(b) &= \mu(1-\alpha_t(b)) = \tfrac{\mu(1-b)(1-\mu)\gamma}{\mu b+(1-\mu)\gamma}; \\ V_T(b) &= \mu\beta_t(b) = \tfrac{\mu b(1-\mu)(1-\gamma)}{\mu(1-b)+(1-\mu)(1-\gamma)}. \end{aligned}$$

*Property* $(\mathbf{C})$. In Regime $\emptyset$, bid and ask prices are continuous by Bayes' rule. Therefore, $J_T(b) = \mu(1-\alpha_T(b))$ and $V_T(b) = \mu\beta_T(b)$ are both continuous in $b$ for any $\mu$. Moreover, notice that the last-period value functions $J_T$ and $V_T$ are continuously differentiable on $[0, 1]$ by Bayes' rule.

*Property* $(\mathbf{M})$. By Bayes' rule, bid and ask prices are monotonically increasing in prior belief. Therefore, $J_T(b) = \mu(1-\alpha_T(b))$ is monotonically decreasing and $V_T(b) = \mu\beta_T(b)$ is monotonically increasing in $b$ for any $\mu$.

*Property* $(\mathbf{SH})$ *and* $(\mathbf{SL})$. The last-period ask price, $\alpha_T(b)$, is strictly concave in $b$, and the last-period bid price, $\beta_T(b)$, is strictly convex in $b$. Therefore, the result follows.

*Property* $(\mathbf{DH})$ *and* $(\mathbf{DL})$. Done by Bayes' rule. $\square$

**Proposition A1.** *The set-valued mapping $\mathcal{E}$ has a closed graph.*

*Proof of Proposition A1.* The result follows by the continuity of the next-period value functions and Bayes' rule. $\square$

**Proposition A2.** *If $J_{t+1}$ and $V_{t+1}$ are continuous, then $\mathcal{E}(b)$ is nonempty for each $b \in [0, 1]$.*

*Proof of Proposition A2.* For $(\sigma_H, \sigma_L) \in [0, 1]^2$ let $B(\sigma_H, \sigma_L)$ be the pair of posterior beliefs given by (M1). Evidently $B : [0, 1]^2 \to [0, 1]^2$ is a continuous function. For $(\alpha, \beta) \in [0, 1]^2$ let $BR^b(\alpha, \beta)$ be the set of pairs $(\sigma_H, \sigma_L)$ satisfying (M2). Given that $J_{t+1}$ and $V_{t+1}$ are continuous, $BR^b$ is an upper semicontinuous correspondence. Its value is always a Cartesian product of two elements of the set $\{\{0\}, [0, 1], \{1\}\}$, so it is convex valued. The composition $BR^b \circ B$ is thus an upper semicontinuous convex-valued correspondence, so Kakutani's fixed point theorem implies that it has a fixed point. $\square$

**Lemma A1.** *If $0 < \bar{x}, \bar{y} \leq 1$, $\bar{x} + \bar{y} > 1$, $\theta \in \{H, L\}$ and $D_\theta(b, \bar{x}, \bar{y}) = 0$, then*

- *the payoff difference $D_\theta(b, x, \bar{y})$ is strictly decreasing as $x$ increases for all $x \geq \bar{x}$;*
- *the payoff difference $D_\theta(b, \bar{x}, y)$ is strictly decreasing as $y$ increases for all $y \geq \bar{y}$.*

*Proof of Lemma A1.* By symmetry it suffices to prove that $D_L(b, x, \bar{y})$ and $D_H(b, x, \bar{y})$ are decreasing in $x$. First, consider $D_L(b, x, \bar{y})$. Now,

$$V_{t+1}(A(b, \bar{x}, \bar{y})) - V_{t+1}(B(b, \bar{x}, \bar{y})) = A(b, \bar{x}, \bar{y}) + B(b, \bar{x}, \bar{y}). \tag{17}$$

By Bayes' rule, $A(b, \bar{x}, \bar{y}) > B(b, \bar{x}, \bar{y})$. Dividing (17) by $A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})$ gives

$$\frac{V_{t+1}(A(b, \bar{x}, \bar{y})) - V_{t+1}(B(b, \bar{x}, \bar{y}))}{A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})} = \frac{A(b, \bar{x}, \bar{y}) + B(b, \bar{x}, \bar{y})}{A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})} > 1.$$

This shows that $A(b, \bar{x}, \bar{y}) > b_L$. By Bayes' rule, $A(b, x, \bar{y})$ is monotonically increasing in $x$, so for any $x \geq \bar{x}$ and $\Delta > 0$ we have $A(b, x, \bar{y}), A(b, x + \Delta, \bar{y}) > b_L$ and consequently

$$\begin{aligned}
&\big(V_{t+1}(A(b, x + \Delta, \bar{y})) - A(b, x + \Delta, \bar{y})\big) - \big(V_{t+1}(A(b, x, \bar{y})) - A(b, x, \bar{y})\big) \\
&\quad = (A(b, x + \Delta, \bar{y}) - A(b, x, \bar{y})) \left( \frac{V_{t+1}(A(b, x + \Delta, \bar{y}) - V_{t+1}(A(b, x, \bar{y}))}{A(b, x + \Delta, \bar{y})) - A(b, x, \bar{y})} - 1 \right) > 0.
\end{aligned}$$

That is, $V_{t+1}(A(b, x, \bar{y})) - A(b, x, \bar{y})$ is an increasing function of $x$. On the other hand, Bayes' rule and the monotonicity condition $(\mathbf{M})$ imply that $B(b, x, \bar{y}) + V_{t+1}(B(b, x, \bar{y}))$ is a decreasing function of $x$. As

$$D_L(b, x, \bar{y}) = B(b, x, \bar{y}) + V_{t+1}(B(b, x, \bar{y})) + A(b, x, \bar{y}) - V_{t+1}(A(b, x, \bar{y})),$$

the result follows. Second, consider $D_H(b, x, \bar{y})$. Suppose that we have:

$$J_{t+1}(A(b, \bar{x}, \bar{y})) - J_{t+1}(B(b, \bar{x}, \bar{y})) = A(b, \bar{x}, \bar{y}) + B(b, \bar{x}, \bar{y}) - 2. \tag{18}$$

Similarly, dividing (18) by $A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})$ gives

$$\frac{J_{t+1}(A(b, \bar{x}, \bar{y})) - J_{t+1}(B(b, \bar{x}, \bar{y}))}{A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})} = \frac{A(b, \bar{x}, \bar{y}) + B(b, \bar{x}, \bar{y}) - 2}{A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})} < -1.$$

This shows that $B(b, \bar{x}, \bar{y}) < b_H$. Therefore, $(\mathbf{SH})$ implies that for any $\Delta > 0$ and $x \leq \bar{x}$,

$$\begin{aligned}
&J_{t+1}(B(b, x + \Delta, \bar{y})) - J_{t+1}(B(b, x, \bar{y})) + B(b, x + \Delta, \bar{y}) - B(b, x, \bar{y}) \\
&\quad = (B(b, x + \Delta, \bar{y}) - B(b, x, \bar{y})) \left( \frac{J_{t+1}(B(b, x + \Delta, \bar{y})) - J_{t+1}(B(b, x, \bar{y}))}{B(b, x + \Delta, \bar{y}) - B(b, x, \bar{y})} + 1 \right) < 0,
\end{aligned}$$

as by Bayes' rule, $B(b, x + \Delta, \bar{y}) < B(b, x, \bar{y}) < b_H$. That is, $J_{t+1}(B(b, x, \bar{y})) - B(b, x, \bar{y})$ is a decreasing function of $x$. On the other hand, Bayes' rule and $(\mathbf{M})$ imply that $J_{t+1}(A(b, x, \bar{y})) - A(b, x, \bar{y})$ is an increasing function of $x$. As

$$D_H(b, x, \bar{y}) = \big(1 - A(b, x, \bar{y}) + J_{t+1}(A(b, x, \bar{y}))\big) - \big(B(b, x, \bar{y}) - 1 + J_{t+1}(B(b, x, \bar{y}))\big),$$

the result follows. $\square$

*Proof of Lemma 3.* Let $\bar{x} = 1$ and $\bar{y} = 1$. Then, the results follow from Lemma A1. $\square$

*Proof of Lemma 4.* Suppose that $\tilde{y}_\theta$ is well defined. Continuity of $D_\theta$ indicates that $\tilde{y}_\theta$ is also continuous for each $\theta \in \{H, L\}$. Suppose that $x_1 > x_2$ and $\tilde{y}_L(x_1) \geq \tilde{y}_L(x_2)$. By Lemma A1,

$$0 = D_L(b, x_1, \tilde{y}_L(x_1)) < D_L(b, x_2, \tilde{y}_L(x_1)) \leq D_L(b, x_2, \tilde{y}_L(x_2)),$$

which is a contradiction to $0 = D_L(b, x_2, \tilde{y}_L(x_2))$. By symmetry, the same holds for $\tilde{y}_H$. $\square$

*Proof of Proposition 2.* We give a proof of each statement below.
*Proof of (a).* Suppose that Regime $HL$ arises at $b$. Then, there must exist $(\bar{x}, \bar{y}) \in \mathcal{E}(b)$ with $\bar{x} < 1$ and $\bar{y} < 1$, such that $D_H(b, \bar{x}, \bar{y}) = 0$ and $D_L(b, \bar{x}, \bar{y}) = 0$. By Lemma A1 and Lemma 3, we have:

$$0 = D_L(b, \bar{x}, \bar{y}) > D_L(b, \bar{x}, 1) > D_L(b, 1, 1). \tag{19}$$

By symmetry, we can also prove $0 > D_L(b, 1, 1)$. $\square$
*Proof of (b).* By (a) of this proposition, Regime $HL$ does not arise. Aiming at a contradiction, suppose that Regime $H$ arises. Then there exists an $\bar{x} < 1$ to satisfy $D_H(b, \bar{x}, 1) = 0$. Then, by Lemma 3, we must have $D_H(b, 1, 1) < 0$, which contradicts our assumption. By symmetry, we can prove that Regime $L$ does not arise. $\square$
*Proof of (c).* First, as $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$, Regime $\emptyset$ does not arise because taking an honest strategy is not optimal for the low type. Also, by (a) of this proposition, Regime $HL$ does not arise. Now suppose that Regime $H$ arises. Then there exists an $\bar{x} < 1$ to satisfy $D_H(b, \bar{x}, 1) = 0$. Then, by Lemma 3, we must have $D_H(b, 1, 1) < 0$, which contradicts our assumption.

Lemma 4 indicates that there is no $y < 1$ to satisfy $D_H(b, 1, y) = 0$. As $D_H$ is continuous in $y$, $D_H(b, 1, y) \geq 0$ must hold for all $y \in [0, 1]$. Now, as $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$, (4) and Lemma 4 imply that there exists a $\bar{y} < 1$ to satisfy

$$D_L(b, 1, \bar{y}) = 0 \quad \text{and} \quad D_H(b, 1, \bar{y}) \geq 0.$$

Therefore, we can see that Regime $L$ arises. In addition, by Lemma 3, there is only one $\bar{y}$ to satisfy $D_L(b, 1, \bar{y}) = 0$. $\square$
*Proof of (d).* Done symmetrically with (c) of this proposition. $\square$
*Proof of (e).* Suppose that there is one element in $\mathcal{E}(b)$ that belongs to Regime $H$. Then, by Lemma 3, there is no other element in $\mathcal{E}(b)$ that also belongs to Regime $H$. Symmetrically the same holds for Regime $L$. $\square$ $\square$

**Lemma A2.** *The following results hold:*

*(a) a period-$t$ strategy $\sigma_t$ is continuous and piecewise differentiable in $b$ on $(0, 1)$;*

*(b) ask and bid prices $\alpha_t$ and $\beta_t$ are continuous and piecewise differentiable in $b$ for all $b \in [0, 1]$;*

*(c) the current-period value functions $J_t$ and $V_t$ satisfy $(\mathbf{C})$.*

*Proof of Lemma A2.* First, we prove the continuity of the equilibrium strategies, the bid and ask prices and the value functions. Proposition 2 and Proposition 4 ensure that the equilibrium strategy is unique within each regime in period $t$. Now, $\mathcal{E}$ is a function of prior belief $b$. By the proof of Proposition A2, the equilibrium correspondence $\mathcal{E}$ is upper semicontinuous. Therefore, we conclude that it is continuous within each regime. Now take a sequence $\{b^k\}$ that converges to $b$. Suppose that for a sufficiently small $\epsilon$, Regime $L$ arises at $b^k - \epsilon$ and Regime $\emptyset$ arises at $b$. Take a sequence of the equilibrium strategy at each $b^k$ in Regime $L$, which we denote by $\{\hat{\sigma}^k\}$. We assert that $\hat{\sigma}^k$ converges to the equilibrium strategy in Regime $\emptyset$ as $b^k$ goes to $b$. Suppose not, and then there is a distinct strategy $\hat{\sigma}$ with $\hat{\sigma}_L \neq 1$ and $\hat{\sigma}_H = 1$ at $b$. Then,

$$D_L(b, \hat{\sigma}_L, 1) = 0 \text{ and } D_L(b, 1, 1) \geq 0.$$

This is a contradiction, given Lemma 3, as $\hat{\sigma}_L < 1$. By symmetry, we can prove that at the boundary belief where Regime $H$ shifts to Regime $\emptyset$ as $b$ changes, the equilibrium strategy is continuous. Therefore, by Bayes' rule, the bid and ask prices are also continuous. As such, both of the value functions are a sum of the continuous functions in $b$; that is, the bid and ask prices, next-period value functions and current-period value functions are continuous.

To prove that piecewise differentiability for the equilibrium strategies, the bid and ask prices and the value functions holds, note that by continuity, each function $\sigma_{Ht}$ or $\sigma_{Lt}$ does not have a jump. So, for some interval, if they are not equal to one, the period-$t$ equilibrium strategy $\sigma_H$ or $\sigma_L$ solves each of the following equations:

$$\begin{aligned} &1 - \tfrac{b\times[\mu\,\sigma_H+(1-\mu)\,\gamma]}{\mu\,[b\times\sigma_H]+(1-\mu)\,\gamma} + J_{t+1}(\tfrac{b\times[\mu\,\sigma_H+(1-\mu)\,\gamma]}{\mu\,b\times\sigma_H+(1-\mu)\,\gamma}) \\ &\quad= \tfrac{b\times[(1-\mu)\,(1-\gamma)+\mu(1-\sigma_H)]}{\mu\,[b\times(1-\sigma_H)+(1-b)]+(1-\mu)\,(1-\gamma)} - 1 + J_{t+1}(\tfrac{b\times[\mu\,(1-\sigma_H)+(1-\mu)\,(1-\gamma)]}{\mu\,[b\times(1-\sigma_H)+(1-b)]+(1-\mu)\,(1-\gamma)});\ \text{or} \\ &-\tfrac{b\times[\mu+(1-\mu)\,\gamma]}{\mu\,[b+(1-b)\times\sigma_L]+(1-\mu)\,\gamma} + V_{t+1}(\tfrac{b\times[\mu+(1-\mu)\,\gamma]}{\mu\,[b+(1-b)\times\sigma_L]+(1-\mu)\,\gamma}) \\ &\quad= \tfrac{b\times+(1-\mu)\,(1-\gamma)}{\mu\,(1-b)\times(1-\sigma_L)+(1-\mu)\,(1-\gamma)} + V_{t+1}(\tfrac{b\times(1-\mu)\,(1-\gamma)}{\mu\,(1-b)\times(1-\sigma_L)+(1-\mu)\,(1-\gamma)}). \end{aligned} \tag{20}$$

Obviously, if they are constant at one, they are differentiable. By the implicit function theorem, $\sigma_H$ or $\sigma_L$ are piecewise differentiable in terms of $b$. Bid and ask prices are continuous and piecewise differentiable in terms of $b$ or $\sigma_H$ or $\sigma_L$. Therefore, we conclude that the bid and ask prices are piecewise differentiable. For the same reason with the proof for the continuity in (c), the result follows. $\square$

**Lemma A3.** *The following results hold:*

*(a) $\alpha_{t,\mu}(b)$ and $\beta_{t,\mu}(b)$ converge to $b$ as $\mu$ goes to zero for all $b \in [0, 1]$;*

*(b)* $\partial_+\alpha_{t,\mu}(b)$ *and* $\partial_+\beta_{t,\mu}(b)$ *converge to* $1$ *as* $\mu$ *goes to zero for all* $b \in [0, 1]$*;*

*(c) a family of tame equilibrium value functions* $\{J_{t,\mu}, V_{t,\mu}\}$ *satisfies* $(\mathbf{DH})$ *and* $(\mathbf{DL})$.

*Proof of Lemma A3.* The first statement (a) is proved by substituting $\mu = 0$ into Bayes' rule. Second, by selection, there is at most one equilibrium strategy for each $b$. We write $\partial_+\sigma_H$ or $\partial_+\sigma_L$ to denote the right limit of $\sigma_H$ and $\sigma_L$ with respect to $b$. By (a) of Lemma A2, they are well defined. Notice that

$$\begin{aligned}
\partial_+\alpha_{t,\mu}(b) &= \tfrac{[(1-\mu)\gamma+\mu\sigma_H+\mu\partial_+\sigma_H]}{(1-\mu)\gamma+\mu b\sigma_H+\mu(1-b)(1-\sigma_L)} - \tfrac{\mu b[(1-\mu)\gamma+\mu\sigma_H][\sigma_H+\sigma_L-1+b\partial_+\sigma_H-(1-b)\partial_+\sigma_L]}{[(1-\mu)\gamma+\mu b\sigma_H+\mu(1-b)(1-\sigma_L)]^2}; \\
\partial_+\beta_{t,\mu}(b) &= \tfrac{[(1-\mu)(1-\gamma)+\mu(1-\sigma_H-\partial_+\sigma_H)]}{(1-\mu)(1-\gamma)+\mu b(1-\sigma_H)+\mu(1-b)\sigma_L} - \tfrac{\mu b[(1-\mu)(1-\gamma)+\mu(1-\sigma_H)][1-\sigma_H-\sigma_L-b\partial_+\sigma_H-b\partial_+\sigma_L]}{[(1-\mu)(1-\gamma)+\mu b(1-\sigma_H)+\mu(1-b)\sigma_L]^2}.
\end{aligned}$$

Substituting $\mu = 0$ into the above equations, we obtain the second result (b). To complete the proof, note that

$$\partial_+V_{t,\mu}(b) = \mu\partial_+\beta_{t,\mu}(b) + (1-\mu)\left(\gamma\partial_+\alpha_{t,\mu}(b)\partial_+V_{t+1,\mu}(\alpha_{t,\mu}(b)) + (1-\gamma)\partial_+\beta_{t,\mu}(b)\partial_+V_{t+1,\mu}(\beta_{t,\mu}(b))\right),$$

and our induction hypothesis $(\mathbf{DL})$, together with (a) and (b) of this lemma, completes the proof for $\partial_+V_{t,\mu}(b)$. By symmetry, we can also prove the statement for $\partial_+J_{t,\mu}(b)$. □

**Lemma A4.** *When* $\mu$ *is sufficiently small, the current-period value functions* $J_t$ *and* $V_t$ *satisfy* $(\mathbf{SH})$ *and* $(\mathbf{SL})$.

*Proof of Lemma A4.* We only prove that $V_t$ satisfies $(\mathbf{SL})$; the rest follows by symmetry. If $\mu(\lfloor\frac{1}{\mu^r}\rfloor - t + 1) < 1$ or $\mu(\lfloor\frac{1}{\mu^r}\rfloor - t + 1) > 1$, for a sufficiently small $\mu$, (c) of Lemma A3 completes the proof. Consequently, we focus on the case of $\mu_0(\lfloor\frac{1}{\mu_0^r}\rfloor - t + 1) = 1$ for some sufficiently small $\mu_0$. Suppose that $(\mathbf{DL})$ does not hold when $\mu = \mu_0$. Take $\mu_1$ sufficiently close to $\mu_0$ with $\lfloor\frac{1}{\mu_1^r}\rfloor = \lfloor\frac{1}{\mu_0^r}\rfloor$ so that $\mu_1(\lfloor\frac{1}{\mu_1^r}\rfloor - t + 1) \neq 1$.

As $V_t$ is continuous and piecewise differentiable by Lemma A2, there must be $b_0, b_1, b_2$ with $b_0 < b_1 < b_2$ such that $V'_{t,\mu_0}(b_0) < 1$, $V'_{t,\mu_0}(b_1) > 1$ and $V'_{t,\mu_0}(b_2) < 1$. Then, together with $(\mathbf{DL})$, we can find $b' \in (b_0, b_1)$, $b'' \in (b_1, b_2)$ and $\bar{d} > 0$ such that

$$V'_{t,\mu_0}(b') > 1 + 2\bar{d} \quad \text{and} \quad V'_{t,\mu_0}(b'') < 1 - 2\bar{d};$$

and also

$$\left|V'_{t,\mu_0}(b') - V'_{t,\mu_1}(b')\right| \leq \bar{d} \quad \text{and} \quad \left|V'_{t,\mu_0}(b'') - V'_{t,\mu_1}(b'')\right| \leq \bar{d}.$$

<u>Case 1: $1 < \mu_1(\lfloor\frac{1}{\mu_1^r}\rfloor - t + 1)$.</u> As $\left|V'_{t,\mu_0}(b'') - V'_{t,\mu_1}(b'')\right| \leq \bar{d}$,

$$\begin{aligned}
& \left|\mu_1(\lfloor\tfrac{1}{\mu_1^r}\rfloor - t + 1) - V'_{t,\mu_1}(b'')\right| \\
= & \left|\mu_1(\lfloor\tfrac{1}{\mu_1^r}\rfloor - t + 1) - 1 + 1 - V'_{t,\mu_0}(b'') + V'_{t,\mu_0}(b'') - V'_{t,\mu_1}(b'')\right| \\
> & \ \bar{d}.
\end{aligned}$$

Thus, we obtain a contradiction to (c) of Lemma A3 in this case.

Case 2: $1 > \mu_1(\lfloor \frac{1}{\mu^r} \rfloor - t + 1)$. Similarly to the first case, as $\left| V'_{t,\mu_0}(b') - V'_{t,\mu_1}(b') \right| \leq \bar{d}$, we obtain a contradiction because

$$
\begin{aligned}
& \left| \mu_1(\lfloor \tfrac{1}{\mu_1^r} \rfloor - t + 1) - V'_{t,\mu_1}(b') \right| \\
= & \left| \mu_1(\lfloor \tfrac{1}{\mu_1^r} \rfloor - t + 1) - 1 + 1 - V'_{t,\mu_0}(b') + V'_{t,\mu_0}(b') - V'_{t,\mu_1}(b') \right| \\
> & \ \bar{d}.
\end{aligned}
$$

For a sufficiently small $\mu$, $(\mathbf{SL})$ is satisfied. □

**Lemma A5.** *Suppose that $D_H(b, x, y) = 0$ and $D_L(b, x, y) = 0$ do not intersect. Then either Regime $H$ or Regime $L$ arises, uniquely. Moreover, if $x^L > x^H$, then $(\sigma_H, \sigma_L) = (x^H, 1)$, and if $x^L < x^H$, then $(\sigma_H, \sigma_L) = (1, y^L)$.*

*Proof.* Suppose that the two curves do not intersect. By symmetry and continuity, we can assume that

$$
x^L > x^H \quad \text{and} \quad y^L > y^H. \tag{21}
$$

Then, by Lemma 3, we obtain:

$$
\begin{aligned}
D_L(b, x^H, 1) > 0 \qquad & \text{and} \qquad D_H(b, x^H, 1) = 0 \\
D_L(b, 1, y^L) = 0 \qquad & \text{and} \qquad D_H(b, 1, y^L) < 0.
\end{aligned} \tag{22}
$$

Therefore, we conclude that Regime $H$ arises. Notice that Regime $L$ does not arise because of the second line of (22), and Regime  does not arise as the honest strategy is not optimal. Moreover, by Lemma 3, there is no other $x$ except for $x^H$ to satisfy $D_H(b, x, 1) = 0$. This completes the proof and we can prove the result for the second case symmetrically. □

**Lemma A6.** *When $\mu$ is sufficiently small, for $r \in (0, 2)$, the following hold:*

*(a) Ask and bid prices $\alpha_t(b)$ and $\beta_t(b)$ are increasing in $b$ for all $b \in [0, 1]$;*

*(b) The current-period value functions $J_t$ and $V_t$ satisfy $(\mathbf{M})$.*

*Proof of Lemma A6.* We provide the proof for each statement below.

*Proof of (a).* When nobody manipulates, by Bayes' rule we can show the result and so it suffices to show that the result holds in Regime $H$ and $L$. As the argument is symmetric, we only prove the result for Regime $L$. Suppose that Regime $L$ arises at $b$. As a strategy is continuous by Lemma A2, we can take $b + \epsilon$ for an arbitrarily small $\epsilon$ at which Regime $L$ also arises. Then, by the type-$L$ trader's indifference condition, we obtain:

$$
\partial_+ \alpha_t(b)(-1 + \partial_+ V_{t+1}(\alpha_t(b))) = \partial_+ \beta_t(b)(1 + \partial_+ V_{t+1}(\beta_t(b))). \tag{23}
$$

By Lemma 2 and (M), (23) indicates that $\partial_+\alpha_t(b) > 0$ if and only if $\partial_+\beta_t(b) > 0$. Let $F_L(b) = \sigma_{Lt}(b) - \partial_+\sigma_{Lt}(b)b(1-b)$. Then, we have:

$$\partial_+\alpha_t(b) = \frac{((1-\mu)\gamma+\mu)\cdot F_L(b)}{(1-\mu)\gamma+\mu b+\mu(1-b)(1-\sigma_{Lt}(b))} \text{ and } \partial_+\beta_t(b) = \frac{(1-\mu)(1-\gamma)\cdot(1-F_L(b))}{(1-\mu)(1-\gamma)+\mu(1-b)\sigma_{Lt}(b)}.$$

Thus, if $\partial_+\alpha_t(b) \leq 0$, we must have $1 - F_L(b) > 1$, which implies $\partial_+\beta_t(b) > 0$. We obtain a contradiction. □

*Proof of (b).* As the argument is symmetric, we only prove the case of the low type. Note that the next-period value function and bid and ask prices are all monotonically increasing, starting at the origin. Thus, the summation of these functions is also monotonic. The case for $J_t$ is proved similarly. □ □

# B Existence of a Symmetric Equilibrium

Let $\tilde{b} = 1-b$ and $\tilde{\gamma} = 1-\gamma$. Consider the same situation as our original economy, except that now the liquidity trader buys with probability $\tilde{\gamma}$ and the market maker's belief is set as $\tilde{b}$. We refer to this economy as the "mirror economy." In what follows, ˜ stands for variables associated with the mirror economy.

**Proposition B1.** *Fix time $t$ and prior belief $b$.*

*(a) Let $\sigma \in \mathcal{E}(b)$ and $\tilde{\sigma}_L = 1-\sigma_H$, $\tilde{\sigma}_H = 1-\sigma_L$. Then we have: $\tilde{\sigma} \in \tilde{\mathcal{E}}(\tilde{b})$.*

*(b) Let $(\alpha,\beta)$ denote the equilibrium prices associated with $\sigma$ in the original economy and let $(\tilde{\alpha},\tilde{\beta})$ be the equilibrium prices associated with $\tilde{\sigma}$ in the mirror economy. Then, we have: $\alpha = 1-\tilde{\beta}$, $\beta = 1-\tilde{\alpha}$.*

*(c) $V_t(b) = \tilde{J}_t(\tilde{b})$ and $J_t(b) = \tilde{V}_t(\tilde{b})$.*

*Proof of Proposition B1.* By definition, the period-$t$ value of the game for each type in the mirror economy is expressed as for $\tilde{b} = 1-b$, and in response to prices $(\tilde{\alpha},\tilde{\beta})$

$$\begin{aligned}\tilde{J}_t(\tilde{b}) = \max_{\tilde{\sigma}_H\in[0,1]} \Big(&\mu\tilde{\sigma}_H(1-\tilde{\alpha}+\tilde{J}_{t+1}(\tilde{\alpha})) \\ &+\mu(1-\tilde{\sigma}_H)(\tilde{\beta}-1+\tilde{J}_{t+1}(\tilde{\beta}))+(1-\mu)\times\Big[\tilde{\gamma}\tilde{J}_{t+1}(\tilde{\alpha})+(1-\tilde{\gamma})\tilde{J}_{t+1}(\tilde{\beta})\Big]\Big),\end{aligned} \tag{24}$$

and

$$\begin{aligned}\tilde{V}_t(\tilde{b}) \;=\; \max_{\tilde{\sigma}_L\in[0,1]} &\Big(\mu\tilde{\sigma}_L(-\tilde{\alpha}+\tilde{V}_{t+1}(\tilde{\alpha}))+\mu(1-\tilde{\sigma}_L)(\tilde{\beta}+\tilde{V}_{t+1}(\tilde{\beta}))\Big) \\ &+(1-\mu)\times\Big(\tilde{\gamma}\tilde{V}_{t+1}(\tilde{\alpha})+(1-\tilde{\gamma})\tilde{V}_{t+1}(\tilde{\beta})\Big)\Big).\end{aligned} \tag{25}$$

In addition, Bayes' rule dictates:

$$\tilde{\alpha} = \frac{\mu\tilde{\sigma}_H + (1-\mu)\tilde{\gamma}}{(1-\mu)\tilde{\gamma} + \mu\tilde{\sigma}_L(1-\tilde{b}) + \mu\tilde{\sigma}_H\tilde{b}} \cdot \tilde{b}; \tag{26}$$

and

$$\tilde{\beta} = \frac{\mu(1-\tilde{\sigma}_H) + (1-\mu)(1-\tilde{\gamma})}{(1-\mu)(1-\tilde{\gamma}) + \mu(1-\tilde{\sigma}_L)\cdot(1-\tilde{b}) + \mu(1-\tilde{\sigma}_H)\cdot\tilde{b}} \cdot \tilde{b}. \tag{27}$$

Having the description of the equilibrium in the mirror economy, we now consider the relationship between the two equilibria in the original economy and the mirror economy recursively. When $t = T$, we have $\tilde{\sigma}_L = (1-\sigma_H) = 1$ and $1-\tilde{\sigma}_H = \sigma_L = 0$ because they do not manipulate in the last period, and so (a) is proved. Then by Bayes' rule, (26) and (27), we have $\alpha = 1-\tilde{\beta}$, $\beta = 1-\tilde{\alpha}$, which proves (b), and as there is no more opportunity to trade, the equalities of those prices and the comparison of the value functions in the original economy stated in (M3) and (25) and (24) give us: $V_T(b) = \tilde{J}_T(\tilde{b})$ and $J_T(b) = \tilde{V}_T(\tilde{b})$. This gives us (c) and completes the proof for this case. □

When $t \neq T$, suppose that $\sigma \in \mathcal{E}(b)$ and $(\alpha, \beta)$ are the equilibrium prices associated with $\sigma$ in the original economy. Moreover, suppose that the next-period value functions satisfy the property that (c) describes. Let $\tilde{\sigma}_{LB} = (1-\sigma_H)$, $\tilde{\sigma}_{HS} = \sigma_L$. Then we have (b) because:

$$\alpha = 1-\tilde{\beta} \;\text{ and }\; \beta = 1-\tilde{\alpha}. \tag{28}$$

By substituting (b) into (25) and $\tilde{V}_{t+1}$, and applying (c) to $\tilde{V}_{t+1}$, we obtain:

$$\begin{aligned}(25) = \max_{\sigma_H\in[0,1]} &(\mu\sigma_H(1-\alpha_t + J_{t+1}(\alpha_t)) \\ &+\mu(1-\sigma_H)(\beta_t - 1 + J_{t+1}(\beta_t)) + (1-\mu)\times[\gamma J_{t+1}(\alpha_t) + (1-\gamma)J_{t+1}(\beta_t)]) = J_t(b),\end{aligned} \tag{29}$$

and similarly, by substituting (b) into (24) and $\tilde{J}_{t+1}$, and applying (c) to $\tilde{J}_{t+1}$, we obtain:

$$\begin{aligned}(24) = \max_{\sigma_L\in[0,1]} &(\mu\sigma_L(-\alpha_t + V_{t+1}(\alpha_t)) + \mu(1-\sigma_L)(\beta_t + V_{t+1}(\beta_t))) \\ &+(1-\mu)\times(\gamma V_{t+1}(\alpha_t) + (1-\gamma)V_{t+1}(\beta_t))) = V_t(b).\end{aligned} \tag{30}$$

This shows that the current-period value functions also satisfy (c), and it remains to show that (a) is satisfied. If $\tilde{\sigma} \notin \tilde{\mathcal{E}}(\tilde{b})$, then there must be a different strategy profile $\overline{\sigma} \in \tilde{\mathcal{E}}(\tilde{b})$, which indicates that there is a different strategy profile $\underline{\sigma} \in \mathcal{E}(b)$. This is a contradiction to our assumption. □

As the results hold for the last period $T$, by mathematical induction we conclude that the results hold for all of the periods. □

# C The Calibration Method

As we make use of an approximation, we set out a different notation for the purpose of calibration. Bold-faced letters denote approximated variables in our simulation. For example, in the calibration, we denote the probability that the type-$H$ trader buys in the high state at period $t$ by $\mathbf{h}_t$ and the probability that the type-$L$ trader sells in the low state by $\mathbf{l}_t$. Moreover, let

$$\mathbf{H}_t = (1-\mu)\gamma + \mu\mathbf{h}_t \quad \text{and} \quad \mathbf{L}_t = (1-\mu)(1-\gamma) + \mu\mathbf{l}_t.$$

Then, $\mathbf{H}_t$ is the probability that a buy occurs in the high state in period $t$ and $\mathbf{L}_t$ is the probability that a sell occurs in the low state. We can write:

$$\boldsymbol{\alpha}_t = \frac{\mathbf{H}_t b}{\mathbf{H}_t b + (1-\mathbf{L}_t)(1-b)} \quad \text{and} \quad \boldsymbol{\beta}_t = \frac{(1-\mathbf{H}_t)b}{(1-\mathbf{H}_t)b + \mathbf{L}_t(1-b)}. \tag{31}$$

When the type-$L$ trader manipulates, we write the bid price as a function of the ask price and the probability that a buy will occur in the high state. Then, we obtain:

$$\boldsymbol{\beta}_t = \frac{\boldsymbol{\alpha}_t b(1-\mathbf{H}_t)}{(\boldsymbol{\alpha}_t - b\mathbf{H}_t)}. \tag{32}$$

In the computer program, we inspect each interval of $b$ to check whether there is a pair of ask and bid prices that satisfies the following indifference condition for the low type:

$$-\boldsymbol{\alpha}_t + \mathbf{V}_{t+1}(\boldsymbol{\alpha}_t) = \boldsymbol{\beta}_t + \mathbf{V}_{t+1}(\boldsymbol{\beta}_t), \tag{33}$$

where $\boldsymbol{\beta}_t$ satisfies (32). In our procedure, the new function $\mathbf{V}_{t+1}$ is constructed through a linear interpolation from $V_{t+1}$, which is: for $\boldsymbol{\alpha}_t \in [b_k, b_{k+1}]$,

$$\mathbf{V}_{t+1}(\boldsymbol{\alpha}_t) = (\boldsymbol{\alpha}_t - b_k)\frac{V_{t+1}(b_{k+1}) - V_{t+1}(b_k)}{(b_{k+1} - b_k)} + V_{t+1}(b_k), \tag{34}$$

and for $\boldsymbol{\beta}_t \in [b_j, b_{j+1}]$,

$$\mathbf{V}_{t+1}(\boldsymbol{\beta}) = (\boldsymbol{\beta}_t - b_j)\frac{V_{t+1}(b_{j+1}) - V_{t+1}(b_j)}{(b_{j+1} - b_j)} + V_{t+1}(b_j). \tag{35}$$

Similarly, when the type-$H$ trader manipulates, we write the ask price as a function of the bid price and the probability that a buy will occur in the low state. First, we solve $\mathbf{H}_t$ as a function of the ask price $\boldsymbol{\alpha}_t$. Then we have:

$$\mathbf{H}_t = \frac{\boldsymbol{\alpha}_t(1-b)(1-\mathbf{L}_t)}{(1-\boldsymbol{\alpha}_t)b}. \tag{36}$$

Then, we substitute $\mathbf{H}$ into the bid price. Then, we obtain:

$$\boldsymbol{\beta}_t = \frac{(b-\boldsymbol{\alpha}_t) + \boldsymbol{\alpha}_t\mathbf{L}_t(1-b)}{(b-\boldsymbol{\alpha}_t) + \mathbf{L}_t(1-b)}. \tag{37}$$

We inspect each interval of $b$ to check whether there is a pair of ask and bid prices that satisfies the following indifference condition for the high type:

$$1 - \boldsymbol{\alpha}_t + \mathbf{J}_{t+1}(\boldsymbol{\alpha}_t) = \boldsymbol{\beta}_t - 1 + \mathbf{J}_{t+1}(\boldsymbol{\beta}_t). \tag{38}$$

| | |
|---|---|
| $m_k^\theta :=$ | $\frac{F_{t+1}^\theta(b_{k+1})-F_{t+1}^\theta(b_k)}{b_{k+1}-b_k}$ |
| $m_j^\theta :=$ | $\frac{F_{t+1}^\theta(b_{j+1})-F_{t+1}^\theta(b_j)}{(b_{j+1}-b_j)}$ |
| $A_k^\theta :=$ | $m_k^\theta - 1$ |
| $B_j^\theta :=$ | $m_j^\theta + 1$ |
| $C^L :=$ | $(b_j m_j^L - \mathbf{V}_{t+1}(b_j)) - (b_k m_k^L - \mathbf{V}_{t+1}(b_k))$ |
| $C^H :=$ | $(b_j m_j^H - \mathbf{J}_{t+1}(b_j)) - (b_k m_k^H - \mathbf{J}_{t+1}(b_k)) + 2$ |
| $K(\theta) :=$ | $-A_k^\theta + B_j^\theta - C^\theta$ |
| $G(\theta, \mathbf{L}_t, b) :=$ | $B_j^\theta \left[(1-\mathbf{L}_t)(1-b) - b\right] - A_k^\theta \left[(1-\mathbf{L}_t)(1-b) - 1\right] + C^\theta \left[1 - 2(1-\mathbf{L}_t)(1-b)\right]$ |
| $N(\theta, \mathbf{L}_t, b) :=$ | $B_j^\theta b - C^\theta \left[1 - (1-\mathbf{L}_t)(1-b)\right]$ |
| $T_\theta :=$ | $\mathbf{H}_t b \left(B_j^\theta - A_k^\theta - 2C^\theta\right) + \left(-B_j^\theta b + C^\theta\right)$ |
| $M_\theta :=$ | $\mathbf{H}_t b \left(-bB_j^\theta + A_k^\theta + C^\theta\right)$ |

Table 2: Summary of Abbreviated Notations

* Each $\theta$ belongs to $\{H, L\}$ and for each $F^\theta$, $F^H = \mathbf{J}$ and $F^L = \mathbf{V}$.

By applying the method of linear interpolation, we construct a function $\mathbf{J}_{t+1}$ that approximates $J_{t+1}$ such that for $\boldsymbol{\alpha}_t \in [b_k, b_{k+1}]$,

$$\mathbf{J}_{t+1}(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}_t - b_k)\frac{J_{t+1}(b_{k+1}) - J_{t+1}(b_k)}{(b_{k+1} - b_k)} + J_{t+1}(b_k), \tag{39}$$

and for $\boldsymbol{\beta} \in [b_j, b_{j+1}]$,

$$\mathbf{J}_{t+1}(\boldsymbol{\beta}) = (\boldsymbol{\beta}_t - b_j)\frac{J_{t+1}(b_{j+1}) - J_{t+1}(b_j)}{(b_{j+1} - b_j)} + J_{t+1}(b_j). \tag{40}$$

Finally, we obtain the following propositions. To keep the notation simple, we use some abbreviations; these are summarised in Table 2.

**Proposition C1.** *When the type-$L$ trader manipulates, an equilibrium ask price $\boldsymbol{\alpha}_t$ solves:*

$$\boldsymbol{\alpha}_t^2 A_k^L + \boldsymbol{\alpha}_t \left(-b\mathbf{H}_t A_k^L + b(\mathbf{H}_t - 1)B_j^L + C^L\right) - C^L b\mathbf{H}_t = 0,$$

*subject to $\mathbf{H}_t = (1-\mu)\gamma + \mu$ and*

$$\mathbf{L}_t = \frac{\boldsymbol{\alpha}_t(1-b) - b(1-\boldsymbol{\alpha}_t)\mathbf{H}_t}{\boldsymbol{\alpha}_t(1-b)} \leq (1-\mu)(1-\gamma) + \mu.$$

*Proof of Proposition C1.* From (33),

$$-\boldsymbol{\alpha}_t + (\boldsymbol{\alpha}_t - b_k)m_k^L + \mathbf{V}_{t+1}(b_k) = \frac{\boldsymbol{\alpha}_t b(-1+\mathbf{H}_t)}{(-\boldsymbol{\alpha}_t + b\mathbf{H}_t)} + (\frac{\boldsymbol{\alpha}_t b(-1+\mathbf{H}_t)}{(-\boldsymbol{\alpha}_t + b\mathbf{H}_t)} - b_j)m_j^L + \mathbf{V}_{t+1}(b_j).$$

Reorganizing terms, we can obtain the desired equation. □

**Proposition C2.** *When the type-$H$ trader manipulates, then an equilibrium ask price $\boldsymbol{\alpha}_t$ solves*

$$\boldsymbol{\alpha}_t^2 A_k^H + X\boldsymbol{\alpha}_t + Y = 0,$$

*where*

$$\begin{aligned} X &= 1 + b + 2\mathbf{L}_t(1-b) - [b_k + b + \mathbf{L}_t(1-b)]\, m_k^H + [b_j - 1 + \mathbf{L}_t(1-b)]\, m_j^H + \\ &\quad + \mathbf{J}_{t+1}(b_k) - \mathbf{J}_{t+1}(b_j); \\ Y &= -\mathbf{L}(1-b) + [b + \mathbf{L}_t(1-b)]\left[b_k m_k^H - 1 + \mathbf{J}_{t+1}(b_j) - \mathbf{J}_{t+1}(b_k)\right] + \\ &\quad + [b(1-b_j) - b_j \mathbf{L}_t(1-b)]\, m_j^H, \end{aligned}$$

*subject to* $\mathbf{L}_t = (1-\mu)(1-\gamma) + \mu$ *and*

$$\mathbf{H}_t = \frac{\boldsymbol{\alpha}_t(1-\mathbf{L}_t)(1-b)}{b(1-\boldsymbol{\alpha}_t)} \leq (1-\mu)\gamma + \mu.$$

*Proof of Proposition C2.* From (38),

$$\begin{aligned} &1 - \boldsymbol{\alpha}_t + (\boldsymbol{\alpha}_t - b_k)m_k^H + \mathbf{J}_{t+1}(b_k) = \\ &\qquad = \tfrac{(b-\boldsymbol{\alpha}_t)+\boldsymbol{\alpha}_t\mathbf{L}_t(1-b)}{(b-\boldsymbol{\alpha}_t)+\mathbf{L}_t(1-b)} - 1 + \left(\tfrac{(b-\boldsymbol{\alpha}_t)+\boldsymbol{\alpha}_t\mathbf{L}_t(1-b)}{(b-\boldsymbol{\alpha}_t)+\mathbf{L}_t(1-b)} - b_j\right)m_j^H + \mathbf{J}_{t+1}(b_j). \end{aligned}$$

Similarly to Proposition C1, we can obtain the desired equation by calculation. □

Lastly, we consider the case where both types manipulate. To compute the equilibrium in this case, we simultaneously solve the two equations (33) and (38). Then, (33) and (38) can be re-written as: for each $\theta \in \{H, L\}$, Let $\mathbf{x}_H = \mathbf{H}_t b$ and $\mathbf{x}_L = (1-\mathbf{L}_t)(1-b)$. Then, for each $\theta \in \{H, L\}$, the indifference conditions can be rewritten as:

$$\begin{aligned} &\left[-A_k^\theta + B_j^\theta - C^\theta\right]\mathbf{x}_H^2 + \mathbf{x}_H\left(-bB_j^\theta + A_k^\theta + C^\theta\right) + \\ &\qquad + \mathbf{x}_L\left[\mathbf{x}_H\left(B_j^\theta - A_k^\theta - 2C^\theta\right) + \left(-B_j^\theta b + C^\theta\right)\right] - C^\theta \mathbf{x}_L^2 = 0. \end{aligned} \tag{41}$$

**Proposition C3.** *When both types manipulate,* $\mathbf{x}_H = \mathbf{H}_t b$ *and* $\mathbf{x}_L = (1-\mathbf{L}_t)(1-b)$ *satisfy* (41) *for each* $\theta \in \{H, L\}$. *Moreover,* $\mathbf{H}_t$ *and* $\mathbf{L}_t$ *must satisfy:*

$$(1-\mu)\gamma < \mathbf{H}_t < (1-\mu)\gamma + \mu\gamma \quad \textit{and} \quad (1-\mu)(1-\gamma) < \mathbf{L}_t < (1-\mu)(1-\gamma) + \mu. \tag{42}$$

Simultaneously solving for $\mathbf{H}_t$ and $\mathbf{L}_t$ is somehow tricky as the procedure has to find a two-dimensional fixed point. First, we identify a pair of strategies that derives bid and ask prices so as to make both types indifferent for each belief. Given $\mathbf{H}_t$, we can find an interval for $\mathbf{L}_t$ that (38) holds and given $\mathbf{L}_t$, (33) holds. By using this point as an initial point, we use the Newton–Raphson method to obtain the solution to the above equations. We denote the LHS of (41) by $f_\theta(\mathbf{x}_H, \mathbf{x}_L)$. Then, keeping all of the coefficients fixed, we obtain:

$$\begin{aligned} \tfrac{\mathrm{d}f_\theta}{\mathbf{x}_H} &= \left[-A_k^\theta + B_j^\theta - C^\theta\right]2\mathbf{x}_H + \left(-bB_j^\theta + A_k^\theta + C^\theta\right) + \mathbf{x}_L\left(B_j^\theta - A_k^\theta - 2C^\theta\right) \\ \tfrac{\mathrm{d}f_\theta}{\mathbf{x}_L} &= \left[\mathbf{x}_H\left(B_j^\theta - A_k^\theta - 2C^\theta\right) + \left(-B_j^\theta b + C^\theta\right)\right] - 2C^\theta \mathbf{x}_L. \end{aligned}$$

Let $\mathbf{x} = \begin{pmatrix} \mathbf{x}_H \\ \mathbf{x}_L \end{pmatrix}$, $f(\mathbf{x}) = \begin{pmatrix} f_H(\mathbf{x}) \\ f_L(\mathbf{x}) \end{pmatrix}$ and $\mathcal{J} = \begin{pmatrix} \frac{\mathrm{d}f_H}{\mathrm{d}\mathbf{x}_H} & \frac{\mathrm{d}f_H}{\mathrm{d}\mathbf{x}_L} \\ \frac{\mathrm{d}f_L}{\mathrm{d}\mathbf{x}_H} & \frac{\mathrm{d}f_L}{\mathrm{d}\mathbf{x}_L} \end{pmatrix}$. By the Newton–Raphson method,

$$f(\mathbf{x} + \delta\mathbf{x}) = f(\mathbf{x}) + \mathcal{J}\delta\mathbf{x}.$$

Assuming $f(\mathbf{x} + \delta\mathbf{x}) \approx 0$ yields:

$$\delta\mathbf{x} = -\mathcal{J}^{-1} f(\mathbf{x}). \tag{43}$$

We obtain a convergent point $\mathbf{x}^*$ by using (43).

*Proof of Proposition C3.* Let $\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b) = \mathbf{P}$. Then, from (33),

$$\begin{aligned}(m_k^L - 1)(1 - \mathbf{P})\mathbf{H}_t b - b_k \mathbf{P}(1 - \mathbf{P}) m_k^L + \mathbf{P}(1 - \mathbf{P})\mathbf{V}_{t+1}(b_k) \\ = (m_j^L + 1)\mathbf{P}(1 - \mathbf{H}_t) b - \mathbf{P}(1 - \mathbf{P}) b_j m_j^L + \mathbf{P}(1 - \mathbf{P})\mathbf{V}_{t+1}(b_j).\end{aligned}$$

Then,

$$\left(A_k^L \mathbf{H} - B_j^L \mathbf{P}\right) b - \mathbf{P}\mathbf{H}_t b (A_k^L - B_j^L) + \mathbf{P}(1 - \mathbf{P}) C^L = 0.$$

Reorganizing terms, resubstituting $\mathbf{P} = [\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b)]$, and again reorganizing terms, we obtain:

$$\begin{aligned}&\left[-A_k^L + B_j^L - C^L\right] \mathbf{H}_t^2 b^2 \\ &+\mathbf{H}_t b \left[B_j^L \left[(1 - \mathbf{L}_t)(1 - b) - b\right] - A_k^L \left[(1 - \mathbf{L}_t)(1 - b) - 1\right] + C^L \left[1 - 2(1 - \mathbf{L}_t)(1 - b)\right]\right] \\ &-(1 - \mathbf{L}_t)(1 - b) \left[B_j^L b - C^L \left[1 - (1 - \mathbf{L}_t)(1 - b)\right]\right] = 0.\end{aligned} \tag{44}$$

By symmetry, we obtain:

$$\begin{aligned}&\left[-A_k^H + B_j^H - C^H\right] \mathbf{H}_t^2 b^2 \\ &+\mathbf{H}_t b \left[B_j^H \left[(1 - \mathbf{L}_t)(1 - b) - b\right] - A_k^H \left[(1 - \mathbf{L}_t)(1 - b) - 1\right] + C^H \left[1 - 2(1 - \mathbf{L}_t)(1 - b)\right]\right] \\ &-(1 - \mathbf{L}_t)(1 - b) \left[B_j^H b - C^H \left[1 - (1 - \mathbf{L}_t)(1 - b)\right]\right] = 0.\end{aligned} \tag{45}$$

□

# D The Slopes of the Value Functions at 0 and 1

**Proposition D4.** *Let*

$$\begin{aligned}Z_1(\mu) &= -\tfrac{\mu[\mu+(1-\mu)\gamma]}{(1-\mu)\gamma}; \\ Z_2(\mu) &= [\tfrac{(\mu+(1-\mu)\gamma)^2}{(1-\mu)\gamma} + \tfrac{[(1-\mu)(1-\gamma)]^2}{\mu+(1-\mu)(1-\gamma)}]; \\ Z_3(\mu) &= \tfrac{\mu[\mu+(1-\mu)(1-\gamma)]}{(1-\mu)(1-\gamma)}; \\ Z_4(\mu) &= \tfrac{\mu[\mu+(1-\mu)(1-\gamma)]}{(1-\mu)(1-\gamma)} + \tfrac{(1-\mu)^2\gamma^2}{\mu+(1-\mu)\gamma} + \mu + (1-\mu)(1-\gamma).\end{aligned}$$

*Then, for all $t \leq T$,*

$$\begin{array}{rcl} J_t'(0) & = & Z_1(\mu) \cdot \sum_{r=0}^{T-t} Z_2^r(\mu); \\ V_t'(0) & = & \frac{(T-t+1)\mu(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)}; \\ V_t'(1) & = & Z_3(\mu) \cdot \sum_{r=0}^{T-t} Z_4^r(\mu); \\ J_t'(1) & = & \frac{-(T-t+1)\mu(1-\mu)\gamma}{\mu+(1-\mu)\gamma}. \end{array}$$

*Proof.* When $b$ is equal to zero there will be no manipulation. Consequently, Bayes' rule gives

$$\alpha_t(b) = \frac{b(\mu+(1-\mu)\gamma)}{b\mu+(1-\mu)\gamma} \quad \text{and} \quad \beta_t(b) = \frac{b(1-\mu)(1-\gamma)}{\mu(1-b)+(1-\mu)(1-\gamma)}.$$

The derivatives of these functions at zero are

$$\alpha_t'(0) = \frac{\mu+(1-\mu)\gamma}{(1-\mu)\gamma} \quad \text{and} \quad \beta_t'(0) = \frac{(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)}.$$

We compute that

$$\begin{array}{rcl} V_t'(0) & = & \mu(\beta_t'(0) + V_{t+1}'(0)\beta_t'(0)) + (1-\mu)(\gamma V_{t+1}'(0)\alpha_t'(0) + (1-\gamma)V_{t+1}'(0)\beta'(0)) \\ & = & \mu\beta'(0) + V_{t+1}'(0)[(1-\mu)\gamma\alpha'(0) + ((1-\mu)(1-\gamma)+\mu)\beta'(0)] \\ & = & \mu\frac{(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)} + V_{t+1}'(0)[(\mu+(1-\mu)\gamma)+(1-\mu)(1-\gamma)] \\ & = & \frac{\mu(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)} + V_{t+1}'(0) = \frac{(T-t+1)\mu(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)}, \end{array} \tag{46}$$

where the last equality is by induction. Similarly, we compute that

$$\begin{array}{rcl} J_t'(0) & = & \mu(-\alpha_t'(0) + J_{t+1}'(0)\alpha_t'(0)) + (1-\mu)(\gamma J_{t+1}'(0)\alpha_t'(0) + (1-\gamma)J_{t+1}'(0)\beta'(0)) \\ & = & -\mu\alpha'(0) + J_{t+1}'(0)[((1-\mu)\gamma+\mu)\alpha'(0) + (1-\mu)(1-\gamma)\beta'(0)] \\ & = & -\mu\frac{\mu+(1-\mu)\gamma}{(1-\mu)\gamma} + J_{t+1}'(0)[\frac{(\mu+(1-\mu)\gamma)^2}{(1-\mu)\gamma} + \frac{[(1-\mu)(1-\gamma)]^2}{\mu+(1-\mu)(1-\gamma)}]. \end{array} \tag{47}$$

Then, we can rewrite (47) as

$$J_t'(0) - \frac{Z_1(\mu)}{1-b(\mu)} = Z_2(J_{t+1}'(0) - \frac{Z_1(\mu)}{1-Z_2(\mu)}). \tag{48}$$

By induction, as $J_T'(0) = Z_1(\mu)$, we obtain

$$\begin{array}{rcl} J_t'(0) & = & Z_2(\mu)^{T-t}(J_T'(0) - \frac{Z_1(\mu)}{1-Z_2(\mu)}) + \frac{Z_1(\mu)}{1-Z_2(\mu)} \\ & = & Z_2(\mu)^{T-t} \cdot Z_1(\mu) + \frac{Z_1(\mu)}{1-Z_2(\mu)}(1-b^{T-t}) \\ & = & Z_1(\mu)\frac{1-Z_2(\mu)^{T-t+1}}{1-Z_2(\mu)} = Z_1(\mu) \cdot \sum_{r=0}^{T-t} Z_2^r(\mu). \end{array} \tag{49}$$

Similarly, when $b$ is very close to one, there will be no manipulation. Therefore, Bayes' rule gives

$$\alpha_t(b) = \frac{b(\mu+(1-\mu)\gamma)}{b\mu+(1-\mu)\gamma} \quad \text{and} \quad \beta_t(b) = \frac{b(1-\mu)(1-\gamma)}{\mu(1-b)+(1-\mu)(1-\gamma)}.$$

The derivatives of these functions at zero are

$$\alpha_t'(1) = \frac{(1-\mu)\gamma}{\mu + (1-\mu)\gamma} \quad \text{and} \quad \beta_t'(1) = \frac{\mu + (1-\mu)(1-\gamma)}{(1-\mu)(1-\gamma)}.$$

We compute that

$$\begin{array}{rcl} V_t'(1) & = & \mu(\beta_t'(1) + V_{t+1}'(1)\beta_t'(1)) + (1-\mu)(\gamma V_{t+1}'(1)\alpha_t'(1) + (1-\gamma)V_{t+1}'(1)\beta'(1)) \\ & = & \frac{\mu[\mu+(1-\mu)(1-\gamma)]}{(1-\mu)(1-\gamma)} + V_{t+1}'(1)\left(\frac{\mu[\mu+(1-\mu)(1-\gamma)]}{(1-\mu)(1-\gamma)} + \frac{(1-\mu)^2\gamma^2}{\mu+(1-\mu)\gamma} + \mu + (1-\mu)(1-\gamma)\right) \\ J_t'(1) & = & \mu(-\alpha_t'(1) + J_{t+1}'(1)\alpha_t'(1)) + (1-\mu)(\gamma J_{t+1}'(1)\alpha_t'(1) + (1-\gamma)J_{t+1}'(1)\beta'(1)) \\ & = & \frac{-\mu(1-\mu)\gamma}{\mu+(1-\mu)\gamma} + J_{t+1}'(1)\left((1-\mu)\gamma + \mu + (1-\mu)(1-\gamma)\right). \end{array} \tag{50}$$

By using (50) recursively, we obtain our desired result. □

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