Monetary Equilibria and Knightian Uncertainty

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Abstract: This paper considers a pure-endowment stationary stochastic overlapping generations economy, in which agents have maximin expected utility preferences à la Gilboa and Schmeidler (1989). It proves two main results. First, we show that multiple stationary monetary equilibria exist and hence real as well as price indeterminacy arises under the assumption that the aggregate shock exists. Second, we show that each of these stationary monetary equilibria is conditionally Pareto optimal, that is, no other stationary allocations strictly Pareto dominate the equilibrium allocations.

1. Introduction

This paper studies a pure-endowment stationary stochastic overlapping generations economy (OLG economy) under uncertainty. In each single period, one of shocks in a finite state space realizes, one new agent is born, and she lives for two periods and dies. In the economy, there are available a single physical (consumption) good at each period and an infinitely-lived outside asset with no dividend payment: fiat money. In the first period of her life, an agent divides an initial endowment into a consumption at this period and money holdings. In her second period, she buys goods from a next generation (in its first period) and consume both these and an second-period endowment which realizes depending only on the shock in her second period. There are no storage technology and production.

So far, the model is quite similar with stochastic OLG models in the literature. However, our model is quite different from those in that each agent evaluates her consumption streams over two periods by the maximin expected utility à la Gilboa and Schmeidler (1989). Each agent faces uncertainty about her second-period consumption which is represented by not a *single* probability distribution but a *set* of probability distributions. Such uncertainty is sometimes referred to as Knightian uncertainty after an economist who distinguished such a multi-prior situation from risk where uncertainty is summarized by a single probability distribution (Knight, 1921). Then, each agent is supposed to use the "worst" probability distribution to calculate her final evaluation of consumption streams. Such a behavior of decision-makers was characterized by some behavioral axioms by Gilboa and Schmeidler (1989) for an infinite state space and by Casadesus-Masanell, Klibanoff and Ozdenoren (2000) for a finite state space, which is the case in this paper.

Under these schemes, this paper proves two main results. First, we show that an open set of "economy" exists in which multiple stationary monetary equilibria exist. Here, "economy" is defined by a pair of initial endowments in the agent's first and second period. Since we assume that each generation is identical and that the state space is finite, the economy is defined by a point in a finite-dimensional Euclidean space (apart from the preference structure). It is shown that for some open set of the economy, a continuum of equilibrium prices exists. We also provides several numerical examples which indicate that an increase in uncertainty (in the sense that the set of probability measures of each agent expands) enlarges not only the degree of indeterminacy of equilibrium prices and allocations but also the range of the economy in which indeterminacy is observed. It is well-known (Gottardi, 1996) that when the agent's preference is differentiable, each equilibrium price is locally isolated (although there might be many). In contrast, the preference in our model, which is represented by the maximin expected utility, may *not* be differentiable and this is why the continuum of equilibria shows up in our model.

Second, we conduct some welfare analysis and prove that each of these stationary monetary equilibria is conditionally Pareto optimal, that is, no other stationary allocations strictly Pareto dominate the equilibrium allocations. In particular, this is somewhat surprising since any equilibrium in the continuum of equilibria cannot be dominated by another one in it.

We finally mention the relationship of our model to the existing literature. Epstein and Wang (1994, 1995) extend Lucas' (1978) dynamic asset pricing model by assuming that a representative agent evaluates consumption streams by maximin expected utility. By definition, at an equilibrium, the representative agent does not hold any asset and consume all of her initial endowment in each period. Then, the equilibrium (supporting) prices are those prices under which the optimal decision of the representative agent is to hold no asset. Epstein and Wang (1994, 1995) show that there exists uncountably many such equilibrium prices. However, in their model, indeterminacy of equilibrium *allocation* cannot happen because of the representative-agent setting. On the other hand, in our exchange economy, indeterminacy of allocation (real indeterminacy) as well as price indeterminacy takes place.

Dana (2004) considers a static exchange economy model with maximin-expected-utility-maximizing agents and shows that the price, and hence, real indeterminacy arises under the assumption of no aggregate shock. Here, no aggregate shock means that the total endowment does not change over the state space. On the other hand, ours is a dynamic model and show that the real indeterminacy arises when the aggrregate shock *does* exists, that is, when the sum of initial endowments of the old generation and the young generation in the same period changes over the state space.

2. The Economy

We consider a stationary stochastic overlapping generations economy with a single physical good, in which agents assign multiple priors to uncertainty.

2.1. Fundamentals

Time is divided into discrete periods and indexed by t. While this index can have values between $-\infty$ and $+\infty$, we use convention of calling the current date (or equivalently, the initial period) period 1. That is, $t = 1, 2, 3, \dots$. Uncertainty is modeled by a realization of a shock s in the *finite* set of shocks, S, at the beginning of each single period. The past of the economy, that is, the history of shocks up to period 0 is treated as given and denoted by σ_0. We denote by $\Delta(S)$ or, more simply, Δ_S the set of all probability measures on S.

In each period, one new agent enters the economy after the realization of the shock and lives for two periods. Thus, agents are essentially indexed by (t, s_t), which is a pair of the period and the shock at which they are born. Agent (t, s_t)'s endowment is assumed to depend on the shocks which occurred in the first and the second period of their life, $s_t, s_{t+1} \in S$, but not on the period t itself nor on the past history of shocks. Let $\omega^1_{s_t}$ and $(\omega^2_{s_{t+1}})_{s_{t+1}\in S}$ denote the first- and the second-period endowments of the physical good of agent (t, s_t), respectively. We assume that $(\omega^1_s, (\omega^2_{s'})_{s'\in S}) \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}$ for all $s \in S$. Note that the second-period endowment is assumed to be independent of the shock in the first period.

We denote by $c^t_{s_t} = (c^{1t}_{s_t}, (c^{2t}_{s_t s'})_{s'\in S})$ the contingent consumption stream of agent (t, s_t). Agent (t, s_t) is assumed to rank the consumption streams $c^t_{s_t}$ according to her lifetime utility function $U_{s_t} : \mathbb{R}_{++} \times \mathbb{R}^S_{++} \to \mathbb{R}$.

Throughout this paper, we assume that agents have the maximin expected utility (MMEU) preferences[1], that is, there exist an increasing[2], strictly concave, and continuously differentiable real-valued function u on $\mathbb{R}^2_{++}$ and a family of compact and convex subsets of Δ_S, $(\mathcal{P}_s)_{s\in S}$, such that

$$(\forall s \in S)(\forall c \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}) \quad U_s(c) = \min_{\pi\in\mathcal{P}_s} \sum_{s'\in S} u(c^1, c^2_{s'})\pi_{s'}. \tag{1}$$

Since the MMEU preference crucially depends on $\mathcal{P}_s$ and since s affects U only through $\mathcal{P}_s$, we often write $U_s(c)$ as $U(\mathcal{P}_s)(c)$. We denote by $U(\pi)(c)$ the MMEU preference rather than by $U(\{\pi\})(c)$ when $\mathcal{P}_s \equiv \{\pi\}$ for some $\pi \in \Delta_S$. Since $U(\cdot)(c)$ is clearly continuous on Δ_S for each c and since each $\mathcal{P}_s \subseteq \Delta_S$ is compact by the assumption, it turns out that the minimum in (1) is actually achieved. Hence, we may define

$$(\forall s \in S)(\forall c \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}) \quad \mathcal{M}(\mathcal{P}_s)(c) := \arg\min_{\pi\in\mathcal{P}_s} U(\pi)(c),$$

which is the nonempty set of priors minimizing the expected utility given the consumption stream $c \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}$. We can prove the strict concavity of $U(\mathcal{P}_s)(c)$ in c :[3]

Theorem 1 $U(\mathcal{P}_s)(\cdot)$ *is strictly concave for all* $s \in S$.

We will invoke Theorem 1 when we characterize the equilibria and when we show their conditional optimality.

[1] An axiomatization of the maximin expected utility preferences over Savage acts with a finite state space is given, for example, by Casadesus-Masanell, Klibanoff and Ozdenoren (2000).

[2] Let H be a nonempty finite set. A real-valued function f on $X \subseteq \mathbb{R}^H$ is *nondecreasing* if $f(x) \geq f(y)$ for all $x, y \in X$ such that $x_h \geq y_h$ for all $h \in H$ and is *increasing* if $f(x) > f(y)$ for all $x, y \in X$ such that $x_h \geq y_h$ for all $h \in H$ and $x_k > y_k$ for some $k \in H$.

[3] One can also show the (quasi-/strict quasi-) concavity of $U(\mathcal{P}_s)(\cdot)$ under the assumption of the (quasi-/strictly quasi-) concavity of u.

2.2. Equilibria

To describe the intergenerational trade, we introduce an infinitely-lived outside asset, which yields no dividends. This asset is so-called fiat money. The stock of fiat money is constant over date-events and is denoted by $M > 0$.

A pair of a contingent real money balance $q^* \in \mathbb{R}^S_+$ and a contingent consumption stream $c^* = (c^{*1}_s, (c^{*2}_{ss'})_{s' \in S})_{s \in S}$, (q^*, c^*), is a *stationary competitive equilibrium* if there exists a $m^* \in \mathbb{R}^S$ such that: for all $s \in S$, (i) (c^*_s, m^*_s) belongs to the set

$$\operatorname*{arg\,max}_{(c^1,(c^2_{s'})_{s'\in S},m)\in\mathbb{R}_{++}\times\mathbb{R}^S_{++}\times\mathbb{R}} \left\{ U_s(c) \;\middle|\; \begin{array}{c} c^1 = \omega^1_s - q^*_s m/M, \\ (\forall s' \in S) \;\; c^2_{s'} = \omega^2_{s'} + q^*_{s'} m/M \end{array} \right\}$$

and (ii) $m^*_s = M$. While the condition (i) requires that the pair of the consumption stream $(c^{*1}_s, (c^{*2}_{ss'})_{s' \in S})$ and money holding m^*_s must be the solution of agent (t, s_t)'s (lifetime) utility-maximizing problem, the condition (ii) is the market clearing condition of fiat money.[4] From (i), (c^*_s, m^*_s) satisfies $c^{*1}_s = \omega^1_s - q^*_s m^*_s/M$ and $(\forall s')$ $c^{*2}_{ss'} = \omega^2_{s'} + q^*_{s'} m^*_s/M$, which together with (ii) implies that $c^{*1}_s = \omega^1_s - q^*_s$ and $(\forall s')$ $c^{*2}_{ss'} = \omega^2_{s'} + q^*_{s'}$. Therefore, it follows that $c^{*2}_{ss'}$ is independent of s (which we write as $c^{*2}_{s'}$) and that

$$(\forall s \in S) \quad c^{*1}_s + c^{*2}_s = \omega^1_s + \omega^2_s \, .$$

That is, we obtain the market-clearing conditions of the contingent consumption good (the Warlas law).

A nonnegative-valued function q^* on S is a *stationary equilibrium* if there exists a contingent consumption streams c^* with which (q^*, c^*) is a stationary competitive equilibrium. Note that we often identify a stationary competitive equilibrium with a stationary equilibrium since they have a one-to-one relationship with each other. A stationary equilibrium q^* is *monetary* if it is positive-valued. We also say that a stationary equilibrium q^* with its corresponding consumption streams $c^* = (c^{*1}_s, (c^{*2}_{s'})_{s' \in S})_{s \in S}$ is *fully-insured* (with respect to the second period consumptions) if, for all $s', s'' \in S$, $c^{*2}_{s'} = c^{*2}_{s''}$.

3. Characterization of Stationary Equilibria

[4] In (i), though the first budget constraint holds with an equality by the increase of u, the other budget constraints may hold with inequalities. Here, we simply assume that all the constraints hold with equalities.

This section provides a characterization of stationary equilibria. Under the current framework, the agents' optimization problems degenerate into the simple form: for all $s \in S$, the agent solves

$$\max_{m \in \mathbb{R}} U(\mathcal{P}_s)(\omega_s^1 - q_s^* m/M, (\omega_{s'}^2 + q_{s'}^* m/M)_{s' \in S}). \tag{2}$$

We denote by $V(\mathcal{P}_s)(m)$ the objective function in (2) for the notational convenience. When $\mathcal{P}_s$ is a singleton for all $s \in S$, the objective function can be differentiable with respect to m, and hence, a stationary equilibrium can be characterized by a system of *equations*, which is derived from the first order condition of the optimization problem and the money market clearing condition $m_s = M$ for all $s \in S$. However, if $\mathcal{P}_s$ has multiple elements, the objective function may not be differentiable. This is the main difficulty in characterizing a stationary equilibrium under MMEU preferences. In such cases, we can no longer characterize a stationary equilibrium by a system of equations. To overcome this difficulty, we will use the following theorem, which is adapted from Aubin (1979, p.118, Proposition 6).

Theorem 2 (Aubin, 1979) *Let Δ be a nonempty subset of a metric space and let $\{f_\pi\}_{\pi \in \Delta}$ be a collection of functions from $\mathbb{R}$ to $\mathbb{R}$. For each $x \in \mathbb{R}$, define*

$$g(x) := \inf_{\pi \in \Delta} f_\pi(x) \quad \textit{and} \quad \mathcal{M}(x) := \{\pi \in \Delta |\, g(x) = f_\pi(x)\}.$$

Let $x \in \mathbb{R}$. If (a) Δ is compact, (b) there exists a neighborhood X of x such that functions $\pi \mapsto f_\pi(y)$ are continuous (in the metric topology) for all $y \in X$, and (c) for all $\pi \in \Delta$, f_π is concave and differentiable, then g is differentiable at x both from left and right and it holds that

$$D_- g(x) = \max_{\pi \in \mathcal{M}(x)} Df_\pi(x) \quad \textit{and} \quad D_+ g(x) = \min_{\pi \in \mathcal{M}(x)} Df_\pi(x).$$

By Theorem 1, $U(\mathcal{P}_s)(\cdot)$ is strictly concave. Hence, a real number $m \in \mathbb{R}$ is a solution of the degenerate optimization problem (2) if and only if

$$(\forall s \in S) \quad D_+ V(\mathcal{P}_s)(m) \leq 0 \leq D_- V(\mathcal{P}_s)(m),$$

where $D_- V(\mathcal{P}_s)(m)$ and $D_+ V(\mathcal{P}_s)(m)$ are the left and the right derivatives of $V(\mathcal{P}_s)(m)$ taken with respect to m, whose existence is guaranteed by Theorem 2.

Given any $q^* \in \mathbb{R}^S_+$ and any $s \in S$, let $c^m_s(q^*) := (\omega^1_s - q^*_s m/M, (\omega^2_{s'} + q^*_{s'} m/M)_{s' \in S})$. We are now ready to characterize a stationary equilibrium by the system of *inclusions*:

Theorem 3 *A nonnegative-valued function q^* on S is a stationary equilibrium if and only if*

$$(\forall s \in S) \quad 0 \in \left\{ -\sum_{s' \in S} q^*_s u_1(c^M_s(q^*))\pi_{s'} + \sum_{s' \in S} q^*_{s'} u_2(c^M_s(q^*))\pi_{s'} \,\middle|\, \pi \in \mathcal{M}(\mathcal{P}_s)(c^M_s(q^*)) \right\}, \qquad (3)$$

where $c^M_s(q^*) = (\omega^1_s - q^*_s, (\omega^2_{s'} + q^*_{s'})_{s' \in S})$.

This is a natural extension of the characterization of a stationary equilibrium in the standard OLG models as considered by Magill and Quinzii (2003) and Ohtaki (2010), in which agents have a unique prior. In fact, if $\mathcal{P}_s$ is singleton for all $s \in S$, this system of inclusions degenerates into a system of equations, which is the same result with the standard stochastic OLG model with a unique prior.

4. Existence and Indeterminacy of Stationary Monetary Equilibria

In the previous section, we have observed that a stationary equilibrium can be characterized by a system of not equations but inclusions. However, we have no idea on the existence and the number of stationary *monetary* equilibria yet. This section will present the existence of a continuum of stationary monetary equilibria.

Let $\omega^2 \in \mathbb{R}^S_{++}$ be arbitrarily given. Consider an equilibrium price $q^* \in \mathbb{R}^S_{++}$ such that for any $s', s'' \in S$, $\omega^2_{s'} + q^*_{s'} \neq \omega^2_{s''} + q^*_{s''}$ whenever $s' \neq s''$. In this case, there exists a neighborhood of q^* on which $\mathcal{M}(\mathcal{P}_s)(c^M_s(q))$ is constant and a singleton for each s. We call this unique measure by μ_s (which depends on s since $\mathcal{P}_s$ depends on s). Then the system of inclusions, (3), turns out to be the system of simultaneous equations:

$$(\forall s \in S) \quad 0 = -\sum_{s' \in S} q^*_s u_1(\omega^1_s - q^*_s, \omega^2_{s'} + q^*_{s'})\mu_{ss'} + \sum_{s' \in S} q^*_{s'} u_2(\omega^1_s - q^*_s, \omega^2_{s'} + q^*_{s'})\mu_{ss'}.$$

Any solution to this system is an equilibrium of the economy and its local nature, including whether or not there exists a continuum of solutions, is the same as in the standard stochastic OLG model. There is nothing new here and we do not pursue this line any more.

Next, we turn to a *partially-insured* equilibrium. Let S' be a subset of S with $|S'| \geq 2$. Consider an equilibrium price $q^* \in \mathbb{R}^S_{++}$ such that for any $s', s'' \in S'$, $\omega^2_{s'} + q^*_{s'} = \omega^2_{s''} + q^*_{s''}$. Then $\mathcal{M}(\mathcal{P}_s)(c^M_s(q^*))$ includes all probability measures in $\mathcal{P}_s$ which assign the same probability to S'. However, since q^* need *not* be constant over S' (although $\omega^2 + q^*$ need), the set in (3) is not necessarily a singleton. Therefore, a continuum of solutions to the system of inclusions, (3), may arise. To spell out the configuration of endowments which allows such indeterminacy will become very complicated. But importantly, we can choose q^* conveniently so as to make $\omega^2 + q^*$ to be constant over some set. This "endogenized flatness" can be further exploited to show indeterminacy of *fully-insured* equilibria. We do this in a new subsection.

4.1. Fully-insured Equilibria

Henceforth, we consider a *fully*-insured stationary equilibrium such that

$$(\exists d > 0)(\forall s \in S) \quad \omega^2_s + q^*_s \equiv d\,. \tag{4}$$

where $d > 0$ is some constant. Then, we first observe that

$$\mathcal{M}(\mathcal{P}_s)(c^M_s(q^*)) = \mathcal{P}_s\,.$$

Furthermore, we have the following theorem.

Theorem 4 *Suppose that q^* satisfies* (4) *for some d. Then, q^* is a stationary equilibriujm if and only if, for all $s \in S$,*

$$d - \max_{\pi \in \mathcal{P}_s} \sum_{s' \in S} \omega^2_{s'} \pi_{s'} \leq (d - \omega^2_s)\frac{u_1(\omega^1_s + \omega^2_s - d, d)}{u_2(\omega^1_s + \omega^2_s - d, d)} \leq d - \min_{\pi \in \mathcal{P}_s} \sum_{s' \in S} \omega^2_{s'} \pi_{s'}\,. \tag{5}$$

For a notational ease, we define a function $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ by

$$(\forall (x, y) \in \mathbb{R}^2_{++}) \quad f(x, y) := \frac{u_1(x, y)}{u_2(x, y)}\,.$$

Note that f is continuous, since u is continuously differentiable. We can then demonstrate the indeterminacy of full-insured stationary monetary equilibria.

Theorem 5 *Suppose that it holds that, for all $s \in S$,*

$$\max_{\pi \in \mathcal{P}_s} \sum_{s' \in S} \omega^2_{s'} \pi_{s'} > \min_{\pi \in \mathcal{P}_s} \sum_{s' \in S} \omega^2_{s'} \pi_{s'} \,. \tag{6}$$

and that the function $f(\cdot, y)$ is surjective for each $y > 0$. Then, there exists an open set $\Omega^1 \subseteq \mathbb{R}^{|S|}_{++}$ such that, for any $\omega^1 \in \Omega^1$, there exists an open interval $D(\omega^1) \in \mathbb{R}_{++}$ such that (5) holds for any $d \in D(\omega^1)$ and for any $s \in S$.

When the function u is time-separable, the assumptions of Theorem 5 may be further simplified.

Corollary 1 *Suppose that (6) holds. Also, assume that there exist continuously differentiable functions $v, w : \mathbb{R}_{++} \to \mathbb{R}$ such that*

$$(\forall c^1, c^2 \in \mathbb{R}_{++}) \quad u(c^1, c^2) = v(c^1) + w(c^2) \,.$$

Finally, assume that v satisfies the Inada conditions, that is,

$$\lim_{c \downarrow 0} v'(c) = +\infty \quad \text{and} \quad \lim_{c \uparrow +\infty} v'(c) = 0 \,.$$

Then, all the assumptions of the previous theorem are satisfied.

Precisely, we have found indeterminacy of the second-period consumption d. For any d found in Theorem 5, define $q^*(d) : S \to \mathbb{R}_{++}$ by $q^*_s(d) := d - \omega^2_s > 0$ for all $s \in S$. One can easily find that $q^*(d) \neq q^*(d')$ for $d \neq d'$ and that $q^*(d)$ is a full-insured stationary monetary equilibrium for any d found in Theorem 5. These imply the indeterminacy of full-insured stationary monetary equilibria, and hence, the *real* indeterminacy (indeterminacy of equilibrium allocation) also arises.

When agents have a unique prior, *i.e.*, when $\mathcal{P}_s$ is singleton for all $s \in S$, the lifetime utility function can be differentiable. Then, according to Gottardi (1996), a stationary monetary equilibrium generically exists and is generically regular and hence locally isolated.[5] On the other

[5]Moreover, if preferences are separable as in Corollary 1 and if the relative risk aversion of w is less than or equal to unity, the number of stationary monetary equilibrium is at most one. See Ohtaki (2010) for more details.

hand, we have shown the existence of a continuum of stationary monetary equilibria when agents assigns not a single prior but the set of priors.

Dow and Werlang (1992), Epstein and Wang (1994, 1995), and Dana (2003) explored implications of Knightian uncertainty. While they also showed indeterminacy, it may disappear in their model when aggregate shocks exist. On the other hand, if one of ω and ω^2 is constant over S and the other is not, then there is no $d > 0$ which satisfies (5). Thus, in our model, indeterminacy of stationary monetary equilibria requires aggregate shocks and shocks on the second period endowments.

Also note that sufficient conditions for indeterminacy of stationary monetary equilibria, which we have found, are imposed on not the pair of ω^1 and ω^2 but only ω^2. The robustness of Theorem 5 or its corollary depends on the size of the set Ω^1. To see this and other aspects of indeterminacy, we consider a simple two-state example in the next section.

5. A Two-state Example

Let $S = \{L, H\}$ and $\omega_L^2 < \omega_H^2$. Assume that $v(x) = w(x) = \ln x$ for all $x > 0$ and

$$\mathcal{P}_s = \{\, \{1-p, p\} \,:\, \varepsilon \le p \le \delta\}$$

for all $s \in S$, where $0 \le \varepsilon \le \delta \le 1$. Then, (5) can be rewritten as, for all $s \in S$,

$$(d - \omega_s^2)A_\varepsilon(d) \le \omega_s^1 \le (d - \omega_s^2)B_\delta(d),$$

where

$$A_\varepsilon(d) := \frac{2d - ((1-\varepsilon)\omega_L^2 + \varepsilon\omega_H^2)}{d - ((1-\varepsilon)\omega_L^2 + \varepsilon\omega_H^2)} \quad \text{and} \quad B_\delta(d) := \frac{2d - ((1-\delta)\omega_L^2 + \delta\omega_H^2)}{d - ((1-\delta)\omega_L^2 + \delta\omega_H^2)}.$$

It is easy to verify that $A_\varepsilon(d) < B_\delta(d)$ if $\varepsilon < \delta$. Moreover, both an increase in δ and a decrease in ε enlarge the interval between $A_\varepsilon(d)$ and $B_\delta(d)$. Finally, Ω^1 is given by

$$\Omega^1 := \bigcup_{d > \omega_H^2} \left[((d-\omega_L^2)A_\varepsilon(d), (d-\omega_L^2)B_\delta(d)) \times ((d-\omega_H^2)A_\varepsilon(d), (d-\omega_H^2)B_\delta(d)) \right].$$

Figures 1 and 2 plot the region of ω^1 at which the indeterminacy arises for some specific values of ω^2, ε and δ. In particular, Figure 2 shows that, when uncertainty increases in accordance with the sense of Ghirardato and Marinacci (2002), the "economy" that exhibits indeterminacy of stationary monetary equilibria is enlarged. Figure 3 depicts the "Edgeworth Box" when $\varepsilon = \delta = 1/2$, that is, when uncertainty is reduced to risk, in which indifference curves and the contract curve are drawn. Figure 4 depicts typical indifference curves of both types (the old who is born at state H and the old who is born at state L), which exhibit kinks on 45 degree line. Figure 5 depicts the "Edgeworth Box" when uncertainty is present, from which we see how indeterminacy arises. In that figure, we add two budget lines at two of many stationary monetary equilibria. (Note that in Figures 3-5, (w_L^1, w_H^1) is chosen from the region depicted in Figure 1 and that the "Edgeworth Box" is drawn under this endowment configuration.)

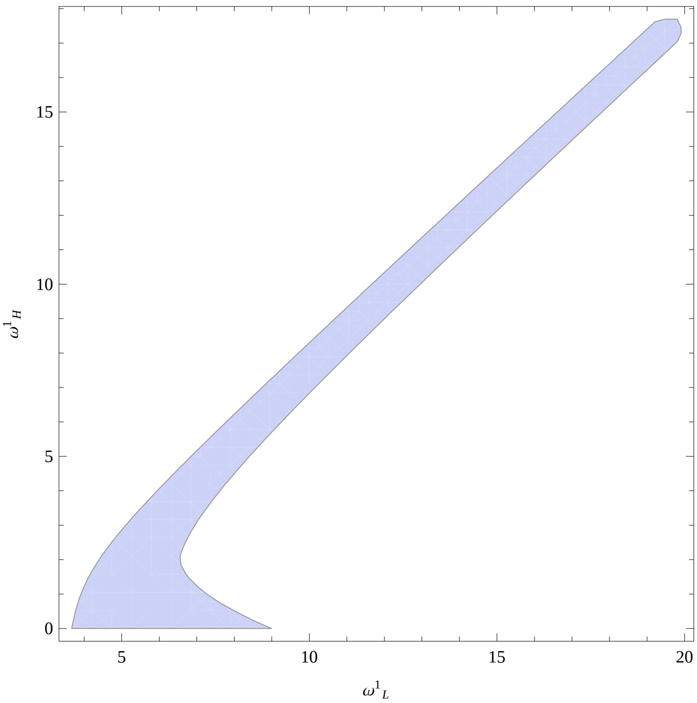

Figure 1: Range of ω^1: $(\omega_L^2, \omega_H^2) = (1, 2)$, $(\varepsilon, \delta) = (1/4, 3/4)$

6. Efficiency of Stationary Monetary Equilibrium Allocations

We have found multiple stationary monetary equilibria. A natural question is that allocations corresponding to stationary monetary equilibria are optimal or not. To investigate equilibrium welfare, we begin with definition of stationary feasible allocations.

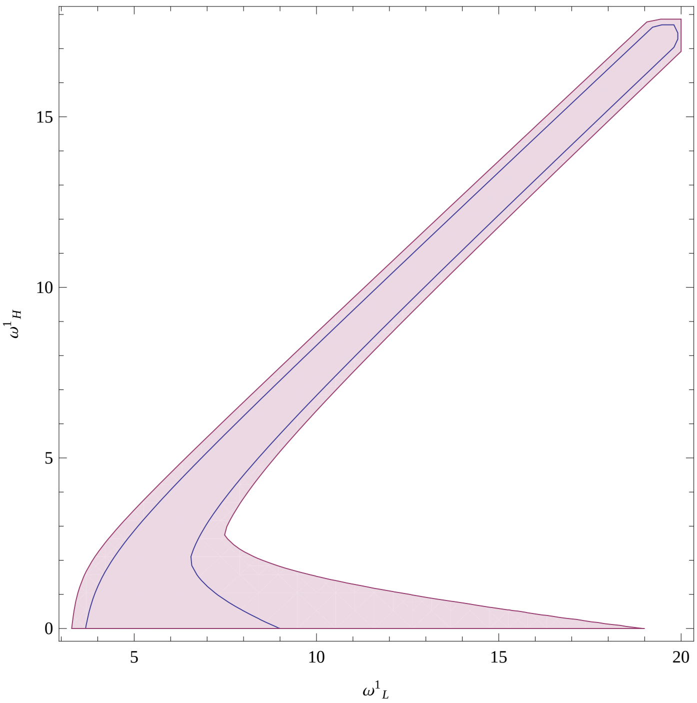

Figure 2: Increase in Uncertainty: $(\omega_L^2, \omega_H^2) = (1, 2)$, $(\varepsilon, \delta) = (1/8, 8/9)$

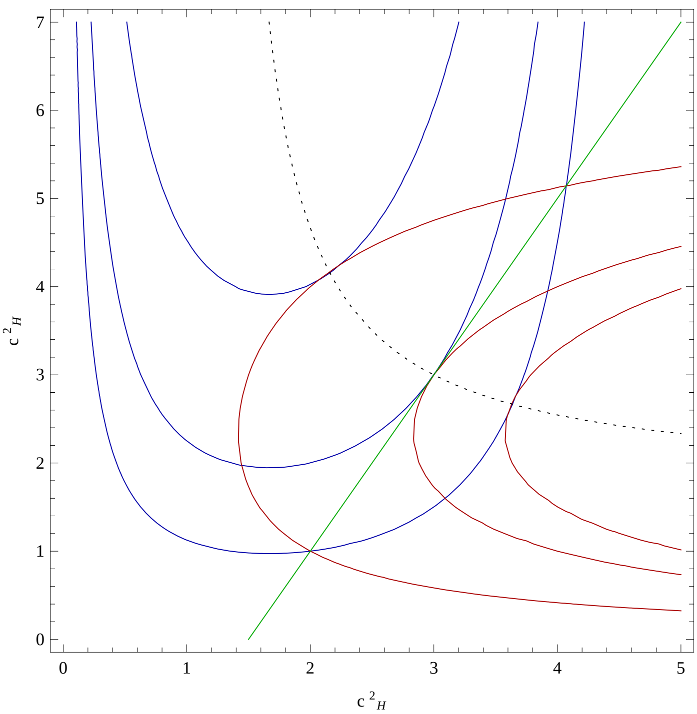

Figure 3: "Edgeworth Box": $(\omega_L^1, \omega_H^1) = (6, 3)$, $(\omega_L^2, \omega_H^2) = (1, 2)$, $(\varepsilon, \delta) = (1/2, 1/2)$

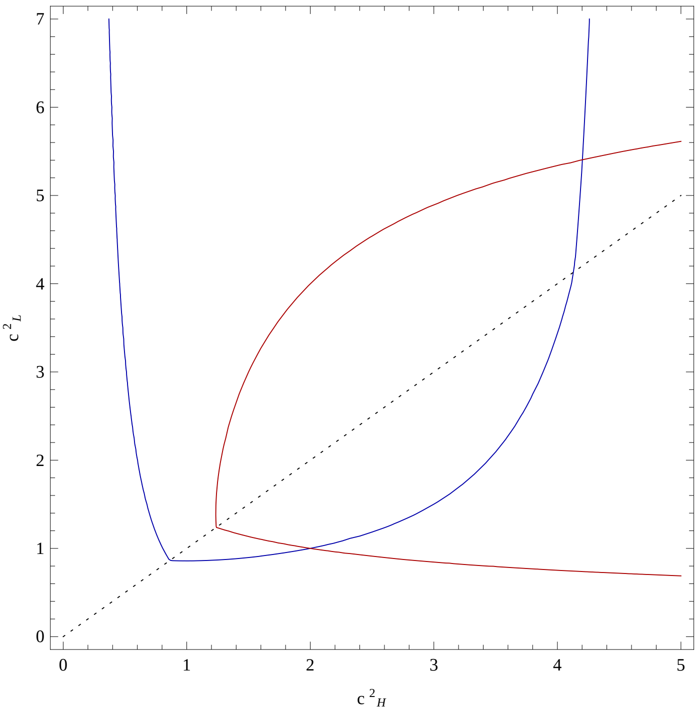

Figure 4: Indifference Curves at the Initial Endowment: $(\omega^1_L, \omega^1_H) = (6, 3)$, $(\omega^2_L, \omega^2_H) = (1, 2)$, $(\varepsilon, \delta) = (1/4, 3/4)$

Let $S_0 := \{\sigma_0\} \cup S$. A *stationary feasible allocation* is a pair of $c^1 : S \to \mathbb{R}_+$ and $c^2 : S_0 \times S \to \mathbb{R}_+$, which satisfies that

$$(\forall (s, s') \in S_0 \times S) \quad c^1_{s'} + c^2_{ss'} = \omega_{s'}.$$

It is easy to verify that the stationary feasible allocations are independent of the predecessor events, that is, $c^2_{ss'}$ is independent of s. To see this, let $c = (c^1, c^2)$ be a stationary feasible allocation. By its feasibility, it follows that $(\forall (s, s') \in S_0 \times S)$ $c^1_{s'} + c^2_{ss'} = \omega_{s'}$ and $(\forall s' \in S)$ $c^1_{s'} + c^2_{\sigma_0 s'} = \omega_{s'}$. Therefore, we obtain that $(\forall s, s' \in S)$ $c^2_{\sigma_0 s'} = c^2_{ss'}$, which verifies the claim.

A stationary feasible allocation $b = (b^1, b^2)$ is *conditionally Pareto superior* to a stationary feasible allocation $c = (c^1, c^2)$ if

$$(\forall s \in S) \quad U_s(b^1_s, b^2_s) \geq U_s(c^1_s, c^2_s) \quad \text{and} \quad b^2_{\sigma_0 s} \geq c^2_{\sigma_0 s}$$

with strict inequality somewhere. A stationary feasible allocation c is *conditionally Pareto optimal* if there is no other stationary feasible allocation that is conditionally Pareto superior to c. The proof of the following theorem is essentially the same as that of Sakai (1988) and hence, is omitted.

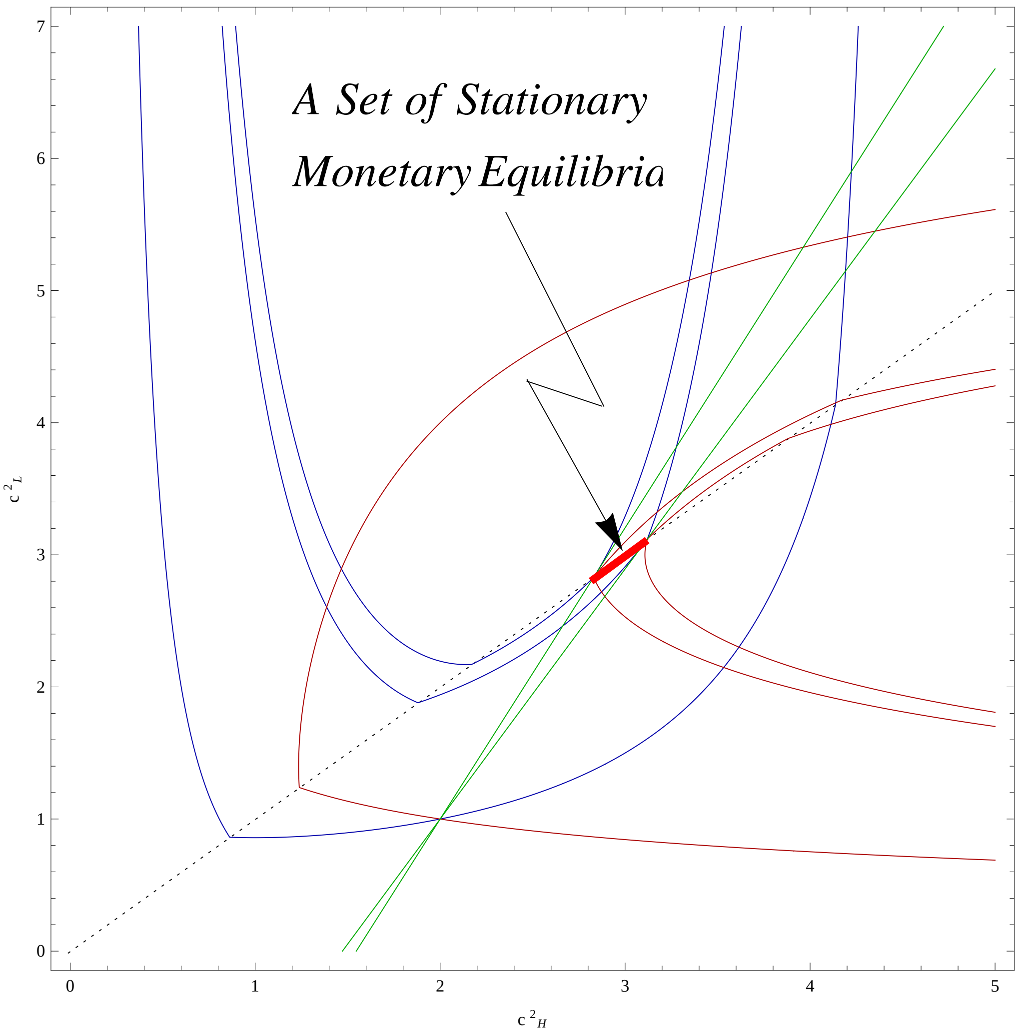

Figure 5: "Edgeworth Box:" $(\omega_L^1, \omega_H^1) = (6, 3)$, $(\omega_L^2, \omega_H^2) = (1, 2)$, $(\varepsilon, \delta) = (1/4, 3/4)$

Theorem 6 *If $U_s : \mathbb{R}_{++} \times \mathbb{R}^S_{++} \to \mathbb{R}$ is monotone and strictly quasi-concave for all $s \in S$, then, for any stationary feasible allocation c corresponding to a stationary monetary equilibrium, there exists no other stationary feasible allocation b that is conditionally Pareto superior to c.*

By Theorem 1, $U(\mathcal{P}_s)(\cdot)$ is strictly concave, so that it is also strictly quasi-concave.[6] Theorefore, thanks to Theorem 7 we have established

Theorem 7 *For any stationary feasible allocation c corresponding to a stationary monetary equilibrium, there exists no other stationary feasible allocation b that is conditionally Pareto superior to c.*

One might be surprised at the statement of this theorem. While there exists a continuum of stationary monetary equilibria under MMEU preferences, Theorem 7 says that all of them are conditionally Pareto optimal.[7]

7. Proofs

Proof of Theorem 1. Let $s \in S$. Note that $U(\cdot)(c)$ is continuous for any given $c \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}$. Also let $c, b \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}$ and $\alpha \in [0,1]$. We denote by $c_\alpha b$ the convex combination of c and b, *i.e.*, $c_\alpha b = \alpha c + (1-\alpha) b$. We first claim that

$$U(\mathcal{P}_s)(c_\alpha b) > \min_{\pi \in \mathcal{P}_s} \left[\alpha U(\pi)(c) + (1-\alpha) U(\pi)(b) \right].$$

Suppose the contrary that

$$U(\mathcal{P}_s)(c_\alpha b) \leq \min_{\pi \in \mathcal{P}_s} \left[\alpha U(\pi)(c) + (1-\alpha) U(\pi)(b) \right]. \tag{7}$$

By compactness of $\mathcal{P}_s$ and continuity of $U(\cdot)(d)$ for all $d \in \mathbb{R}_{++} \times \mathbb{R}^S_{++}$, $\mathcal{M}(\mathcal{P}_s)(c_\alpha b)$ and the set

$$\mathcal{N}(\mathcal{P}_s) = \arg\min_{\pi \in \mathcal{P}_s} \left[\alpha U(\pi)(c) + (1-\alpha) U(\pi)(b) \right]$$

[6] See, for example, Theorem 1.E.1(i) of Takayama (1974)

[7] Notice that we can show Theorem 7 without continuous differentiability of u. Also notice that we can replace the strict concavity of u with its strict quasi-concavity. See also Footnote 3.

are nonempty. Hence, it follows from Eq.(7) that there exist $\mu \in \mathcal{M}(\mathcal{P}_s)(c_\alpha b)$ and $\nu \in \mathcal{N}(\mathcal{P}_s)$ such that

$$\begin{aligned} U(\mu)(c_\alpha b) &= U(\mathcal{P})(c_\alpha b) \\ &\leq \min_{\pi \in \mathcal{P}} [\alpha U(\pi)(c) + (1-\alpha)U(\pi)(b)] \\ &= \alpha U(\nu)(c) + (1-\alpha)U(\nu)(b), \end{aligned}$$

which implies that

$$\alpha U(\mu)(c) + (1-\alpha)U(\mu)(b) < U(\mu)(c_\alpha b) \leq \alpha U(\nu)(c) + (1-\alpha)U(\nu)(b)$$

by strict concavity of u. This contradicts with the definition of ν, which should minimize $\alpha U(\cdot)(c) + (1-\alpha)U(\cdot)(b)$ in $\mathcal{P}_s$. This completes the claim.

Recall that $\mathcal{N}(Q)$ is nonempty. For any $\nu \in \mathcal{N}(Q)$, we can obtain that

$$\begin{aligned} \min_{\pi \in \mathcal{P}_s} [\alpha U(\pi)(c) + (1-\alpha)U(\pi)(b)] &= \alpha U(\nu)(c) + (1-\alpha)U(\nu)(b) \\ &\geq \alpha U(\mathcal{P}_s)(c) + (1-\alpha)U(\mathcal{P}_s)(b), \end{aligned}$$

by the definition of $U(\mathcal{P})(\cdot)$. Combining this with the previous claim, it follows that

$$\begin{aligned} U(\mathcal{P}_s)(c_\alpha b) &> \min_{\pi \in \mathcal{P}_s} [\alpha U(\pi)(c) + (1-\alpha)U(\pi)(b)] \\ &\geq \alpha U(\mathcal{P}_s)(c) + (1-\alpha)U(\mathcal{P}_s)(b). \end{aligned}$$

Therefore, $U(\mathcal{P}_s)(\cdot)$ is concave. ■

Proof of Theorem 3. By Theorems 1 and 2, we can characterize a stationary equilibrium by the system of inequalities:

$$(\forall s \in S) \quad D_+V(\mathcal{P}_s)(M) \leq 0 \leq D_-V(\mathcal{P}_s)(M).$$

Since

$$\begin{aligned} D_+V(\mathcal{P}_s)(m) &= \min_{\pi \in \mathcal{M}(\mathcal{P}_s)(c_s^m(q^*))} \left(-\sum_{s' \in S} q_s^* u_1(c_s^m(q^*))\pi_{s'} + \sum_{s' \in S} q_{s'}^* u_2(c_s^m(q^*))\pi_{s'} \right), \\ D_-V(\mathcal{P}_s)(m) &= \max_{\pi \in \mathcal{M}(\mathcal{P}_s)(c_s^m(q^*))} \left(-\sum_{s' \in S} q_s^* u_1(c_s^m(q^*))\pi_{s'} + \sum_{s' \in S} q_{s'}^* u_2(c_s^m(q^*))\pi_{s'} \right) \end{aligned}$$

for all $m \in \mathbb{R}$ and $\mathcal{M}(\mathcal{P}_s)(c_s^M(q^*))$ is convex for all $s \in S$, we obtain (3). ■

Proof of Theorem 4. Let $s \in S$. Under (4), (3) is successively rewritten as follows:

$$0 \in \left\{ -\sum_{s' \in S} q_s^* u_1(\omega_s^1 - q_s^*, d)\pi_{s'} \; + \sum_{s' \in S} q_{s'}^* u_2(\omega_s^1 - q_s^*, d)\pi_{s'} \,\middle|\, \pi \in \mathcal{P}_s \right\},$$

which is equivalent to

$$0 \in \left\{ -q_s^* u_1(\omega_s^1 - q_s^*, d) \; + u_2(\omega_s^1 - q_s^*, d) \sum_{s' \in S} q_{s'}^* \pi_{s'} \,\middle|\, \pi \in \mathcal{P}_s \right\}.$$

Hence, it follows that

$$q_s^* \frac{u_1(\omega_s^1 - q_s^*, d)}{u_2(\omega_s^1 - q_s^*, d)} \in \left\{ \sum_{s' \in S} q_{s'}^* \pi_{s'} \,\middle|\, \pi \in \mathcal{P}_s \right\}$$

Because $\mathcal{P}_s$ is compact and convex for each s, the last expression is equivalent to

$$\min_{\pi \in \mathcal{P}_s} \sum_{s' \in S} q_{s'}^* \pi_{s'} \leq q_s^* \frac{u_1(\omega_s^1 - q_s^*, d)}{u_2(\omega_s^1 - q_s^*, d)} \leq \min_{\pi \in \mathcal{P}_s} \sum_{s' \in S} q_{s'}^* \pi_{s'} \,.$$

By substituting $q_s^* = d - \omega_s^2$ to this, we obtain (5). ■

Proof of Theorem 5. Note that $d > \omega_s^2$ for all s. This follows from (4) since we are looking for an equilibrium q^* such that $q_s^* > 0$ for all s. Therefore, by (6) and the assumption on f, for any s and for any d, we can find an open interval $\Omega_s^1(d) \subseteq \mathbb{R}_{++}$ such that, for any $\omega_s^1 \in \Omega_s^1(d)$, ω_s^1 satisfies (5) with strict inequalities.

Let d be given and let $\omega^1 \in \mathbb{R}_{++}^{|S|}$ be such that $\omega_s^1 \in \Omega_s^1(d)$ for each s. Then, for each s, there exists an open interval $D_s(\omega^1)$ which satisfies that $d \in D_s(\omega^1)$ and that, for each $d' \in D_s(\omega^1)$, (5) holds with $d = d'$. This follows because we may take an appropriate neighborhood of d since ω_s^1 satisfies (5) with strict inequalities by the previous paragraph and since f is a continuous function.

Finally, define Ω^1 by

$$\Omega^1 := \bigcup_{d > \max_s \omega_s^2} \prod_{s=1}^{|S|} \Omega_s^1(d) \,.$$

Let $\omega^1 \in \Omega^1$. Then, there exists $d > \max_s \omega_s^2$ such that, for each s, $\omega_s^1 \in \Omega_s^1(d)$. Therefore, by the previous paragraph, there exists an open interval $D_s(\omega^1)$ such that, for any $d' \in D_s(\omega^1)$, (5) holds with $d = d'$. Define $D(\omega^1)$ by $D(\omega^1) := \cap_{s=1}^{|S|} D_s(\omega^1)$. Note that $D(\omega^1)$ is nonempty and open since for all s, $d \in D_s(\omega^1)$ by the previous paragraph. Then, if $d' \in D(\omega^1)$, (5) holds with $d = d'$. Thus, we obtained Ω^1 and $D(\omega^1)$ whose existence is claimed in the theorem. ■

Proof of Corollary 1. Under the stated assumptions, the function f will become $f(x, y) = v'(x)/w'(y)$ for each x and y. By the Inada condition on v, $f(\cdot, y)$ is clearly surjective for each y, which completes the proof. ■

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