The Vickrey-Target Strategy and the Core in Ascending Combinatorial Auctions

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Abstract

This paper considers a general class of combinatorial auctions with ascending prices, which includes the Vickrey-Clarke-Groves mechanism and core-selecting auctions. We analyze incentives in ascending combinatorial auctions under complete and perfect information. We show that in every ascending auction, the "Vickrey-target strategy" constitutes a subgame perfect equilibrium with a restricted strategy space. The equilibrium outcome is in the bidder-optimal core and unique under some criteria. This implies equilibrium selection is done by an ascending price scheme from many equilibria of sealed-bid auctions. The equilibrium outcome is "unfair" in the sense that winners with low valuations tend to earn high profits. The payoff non-monotonicity may lead to inefficiency in the equilibrium under unrestricted strategy space.

JEL classification: D44, C7

Keywords: combinatorial auction, ascending price, the Vickrey auction, core-selecting auction, core

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1 Introduction

This paper formulates a broad class of auction mechanisms with ascending prices. We introduce the "ascending Vickrey-reserve auctions" and analyze the subgame perfect equilibrium under complete information. The Vickrey-reserve auctions are a class of combinatorial (or package) auctions, which includes the Vickrey-Clarke-Groves mechanism and core-selecting auctions.

Since the U.S. Federal Communications Commission conducted a spectrum license auction in 1994, the theory of multi-object auctions has been attracted a great deal of attention. In recent decades, a considerable number of studies have been conducted on the designs and analyses of multi-object auctions, especially combinatorial auctions. Combinatorial auctions are those in whcih bidders can make bids for bundles or packages of goods, not just each individual good. Although such auctions are generally complicated in practical uses, they are actually being implemented in spectrum license auctions in several countries and proposed for auctions of airport landing slots in these days.[1]

In the literature of combinatorial auction, incentives and equilibria in several auction mechanisms have been examined under sealed-bid formats, or direct revelation mechanisms. The Vickrey-Clarke-Groves mechanism (the Vickrey auction) is an important benchmark. The Vickrey auction is known as an efficient mechanism such that is incentive compatible in dominant strategy (Green and Laffont, 1977; Holmstrom, 1979). However, some studies point that the Vickrey auction has several disadvantages such as low revenue and vulnerability to collusive biddings (Ausubel and Milgrom, 2006). In practical combinatorial auction designs, the Vickrey auction has hardly been used ever.

Core-selecting auctions are recent attractive alternatives to the Vickrey auction. A core-selecting auction selects an outcome in the core with respect to the reported valuations. The core-selecting property avoids some disadvantages of the Vickrey auctions. Although core-selecting auctions are not incentive compatible, Day and Milgrom (2008) show that they achieve an outcome in the core in a Nash equilibrium under complete information. This fact forms a theoretical foundation for applying core-selecting auctions. Recently, they have been used for spectrum licenses auctions in the U.K. and several European countries.

[1]See Cramton (2009) for the applications of combinatorial auctions to spectrum license auctions.

On the other hand, from the viewpoint of practical auction design, ascending-price auctions are frequently preferred to sealed-bid auctions. Dynamic, open-bid format is transparent and economizes revealed information about valuations during the auction. For example, the U.S. Federal Communication Commission first adopted the simultaneous ascending auction (SAA) for spectrum auctions, in which items are put on sale simultaneously using an ascending-price rule. Bidders can submit new bids for any item if a new bid is submitted for some item. An ascending-price format is standard for spectrum license auctions in many countries. The ascending-price, open-bid format is more important when package bids are allowed, since a sealed-bid format often requires bidders to submit an exponential number of package bids. Many studies investigate and propose various combinatorial auction designs with ascending-price formats.[2]

Most studies try to formulate "ascending-price Vickrey auctions" for multiple objects; i.e., ascending auctions which terminate with the Vickrey-Clarke-Groves outcome. Such auctions correspond to a standard English auction of sigle object, and bidders reveal true information on valuations in the equilibrium. Parkes and Ungar (2000), Ausubel and Milgrom (2002), and de Vries et al. (2007) formulate ascending combinatorial auctions with non-linear and non-anonymous prices. Their auctions are ascending Vickrey auctions for substitute goods, whereas they are not for general valuations. However, they are ascending core-selecting auctions for general valuations.

These auctions will be desirable if we prefer ascending-price formats and core-selecting auctions. However, a natural question is what the subgame perfect equilibrium of an ascending auction is, if it is not an ascending Vickrey auction. Unfortunately, the equilibrium of those ascending combinatorial auctions has never been examined. In their seminal paper, Ausubel and Milgrom (2002) formulate an ascending combinatorial auction, however, they consider proxy bidding in the equilibrium analysis and do not examine the incentives during the ascending-price procedure. Is there a subgame perfect equilibrium which achieves an outcome in the core in ascending core-selecting auctions?

This paper answers this question and shows that every ascending core-selecting auction has a subgame perfect equilibrium in the bidder-optimal core with a limited

[2]See Parkes (2006) for a review of several designs and the advantages of ascending auctions over sealed-bid auctions.

strategy space. Moreover, we show that the identical equilibrium exists in a broader class of ascending combinatorial auctions. We consider a general form of ascending price combinatorial auction with a single price path of non-linear and non-anonymous price vector. We allow arbitrary ascending price scheme and possible final discounts; i.e., payments may be different from the terminal prices. We introduce “ascending Vickrey-reserve auctions,” in which bidders pay at least their Vickrey payments with respect to the revealed information on valuations. Our model includes most of auctions in the literature, such as Parkes and Ungar (2000), Ausubel and Milgrom (2002), de Vries et al. (2007), and Mishra and Parkes (2007).[3]

We focus on a class of dynamic strategy, *semi-truthful strategy*, which corresponds to the truncation strategy by Day and Milgrom (2008) in sealed-bid auctions. In a semi-truthful strategy, a bidder either reports his true demand or stops bidding at each period. The stopping timings of bidders generally depend on the others' behavior. In most of the paper, we consider the subgame perfect equilibrium (SPE) with the strategy space restricted to semi-truthful strategies.

We have three main results. First, we show that a particular strategy, which we call the Vickrey-target strategy, constitutes an subgame perfect equilibrium. In this strategy, a bidder aims to bid up to his constrained Vickrey price. This strategy is free from the specifications of the auction rules. The equilibrium outcome is in the bidder-optimal core[4] with respect to the true valuations. This result is similar to that of Day and Milgrom (2008), who show a particular strategy profile as a Nash equilibrium of *every* core-selecting auction.

Second, we show that the specified equilibrium outcome is a unique equilibrium outcome under certain conditions in every *strict* Vickrey-reserve auction, in which winners pay amounts strictly more than the Vickrey price. This result contrasts with the fact that there are possibly many Nash equilibria in sealed-bid Vickrey-reserve auctions. Equilibrium selection is done to some extent by introducing an ascending-price format and subgame perfection.

Third, although the outcome by the Vickrey-target strategies is in the bidder-optimal core, it is “unfair” in the sense that the lower the valuation of a winner, the higher are the profits he tends to earn. The payoff non-monotonicity leads to a possibility that the Vickrey-target strategy may not constitute an SPE with unre-

[3]Ausubel's (2006) auction uses multiple price paths, and it is an exception.

[4]A core outcome is bidder-optimal if it is Pareto-optimal among bidders.

stricted strategy space. Moreover, we show that an SPE with unrestricted strategy space may be inefficient.

The intuition of these results is simple. In an ascending auction, the prices of goods increase gradually from the initial low values. Bidders decide whether to continue bidding or not at each period. Note that there exists a best core outcome for each bidder, in which he obtains the Vickrey payoff. If a bidder stops bidding at his Vickrey price, and if the auction finally selects the efficient allocation, he will be able to win the goods with the Vickrey price. Hence, by stopping at the Vickrey price, he will definitely earn the Vickrey payoff, which is the best payoff in the core. The Vickrey payments of the winners with lower valuations are generally lower as compared to those of the high-value winners. Hence, the prices first reach the Vickrey prices of low-value winners, and hence, low-value winners achieve their most preferred outcomes. When a winner stops bidding, the remaining bidders need to raise their bids even further to win. High-value bidders tend to pay dearly and earn little net profit.

It is quite restrictive to focus only on semi-truthful strategies. However, the analysis of this paper is applied under the unrestricted strategy space if bidders are single-minded, i.e., they are interested only in a particular bundle and place only bids for that bundle.

The contribution of the paper is as follows. First, we consider a general class of combinatorial auctions with ascending price and show an equivalence in an equilibrium strategy under complete information. With a restricted strategy space, every ascending combinatorial auction has a subgame perfect equilibrium in bidder-optimal core with respect to true valuations. This corresponds to the preceding results on sealed-bid combinatorial auctions by Bernheim and Whinston (1986), Ausubel and Milgrom (2002), Day and Milgrom (2008). Second, we show that the equilibrium outcome is unique with some criteria. This contrasts with the multiple equilibria of sealed-bid combinatorial auctions. As Milgrom (2007) discusses, the preceding analyses are not satisfying even if we accept the strong assumption of complete information, since there are many plausible equilibria. Our result can be interpreted as an equilibrium selection and indicate which outcome in the core is the most plausible, if we still assume complete information. Finally, we show some negative properties such as non-monotonicity of the equilibrium payoff and possible inefficiency in an equilibrium with unrestricted strategy space. In practical uses, we need to consider that the ascending price formats do not always perform well.

1.1 Related Literature

As we have mentioned, various ascending-price auctions are proposed by Parkes and Ungar (2000), Ausubel and Milgrom (2002), Ausubel (2006), de Vries et al. (2007), and Mishra and Parkes (2007). All these auctions terminate with the Vickrey outcome and incentive compatible for substitutes goods. Even for general valuations, they are core-selecting auctions except Ausubel (2006) and Mishra and Parkes (2007). de Vries et al. (2007) show that it is impossible to design an ascending auction which converges to the Vickrey outcome under general valuations. Mishra and Parkes (2007) introduce final discounts after ascending price procedure and provide a class of the ascending Vickrey auctions for general valuations. The conditions for the Vickrey outcome being in the core are studied by Bikhchancani and Ostroy (2002) and Ausubel and Milgrom (2002).

On information requirement, Mishra and Parkes (2007) show the necessary and sufficient condition for computing the Vickrey outcome from the auction outcome. Matsushima (2011) provides another necessary and sufficient condition for implementing the Vickrey outcome using a general price-based scheme. He also shows the necessary and sufficient condition for implementing a strategy-proof and interim individually rational mechanism using a price-based scheme. Blumrosen and Nisan (2010) show that we need non-linear and non-anonymous prices to achieve efficiency by ascending auctions in general valuations.

Equilibrium analyses of combinatorial auctions are conducted mainly under complete information and sealed-bid formats. Bernheim and Whinston (1986) consider the "first-price" combinatorial auction called menu auction. They show that there exist possibly many full-information Nash equilibria in the core. Ausubel and Milgrom (2002) consider the ascending proxy auction, where bidders report their valuations in advance to their proxy agents. They show that Bernheim and Whinston's (1986) Nash equilibrium is also a Nash equilibrium in the ascending proxy auction. Day and Milgrom (2008) generalize these results to every sealed-bid core-selecting auction. Sano (2011b) further generalize Day and Milgrom (2008). Sano shows that Bernheim and Whinston's (1986) Nash equilibrium exists if and only if bidders pay at least their Vickrey payments.

There are several studies on the analysis of ascending-price non-package auctions. Ausubel and Schwartz (1999) and Grimm et al. (2003) study the subgame perfect equilibrium in a multi-unit ascending auction with complete information. They consider a multi-unit, ascending uniform-price auction with two bidders. They show that there is a unique low-price subgame perfect equilibrium. Their low-price equilibrium stems from the demand reduction or implicit collusions by bidders in multi-unit uniform-price auctions (Engelbrecht-Wiggans and Kahn, 1998; Ausubel and Cramton, 2002).

The remainder of this paper proceeds as follows. In section 2, we provide a simple example and explain the intuition of the results. In section 3, we formulate the model and the auction. We define a Vickrey-reserve auction and introduce an ascending-price format. In section 4, we show that the Vickrey-target strategies constitute an equilibrium and lead to an outcome in the bidder-optimal core. Then, we examine the uniqueness of the equilibrium and the equilibrium selection. In section 5, we discuss the results. If an ascending auction is core-selecting, it is robust to collusive overbiddings. In addition, we show the non-monotonicity of the equilibrium payoff. The payoff non-monotonicity leads to inefficiency in the SPE with unrestricted strategy space.

2 An Illustration

We first look at a simple example of two goods and three bidders.

Example 1. There are two goods $\{A, B\}$, and three bidders $\{1, 2, 3\}$. Suppose that bidder 1 wants only good A, whereas bidder 2 wants only B. Bidder 3 wants both A and B. Bidder 1's willingness to pay for A is 7, and 2's willingness to pay for B is 8. Bidder 3's willingness to pay is 10 for AB, and 0 for each good. In the efficient allocation, bidders 1 and 2 get A and B, respectively. The core of the auction game is described as follows: $p_1(A) \leq 7$, $p_2(B) \leq 8$, and $p_1(A)+p_2(B) \geq 10$, where $p_i(k)$ denotes the payment of i for good(s) k. In the bidder-optimal core, $p_1(A) + p_2(B) = 10$.

First, consider a sealed-bid core-selecting auction. Assuming bidder 3 truthfully places a bid 10 for the package AB, every bid profile (b_1, b_2) such that $b_1 + b_2 = 10$, $b_1 \leq 7$, and $b_2 \leq 8$, is a Nash equilibrium. In the equilibrium, each winning bidder pays b_i $(i = 1, 2)$ by the core-selecting pricing rule. Thus, any bidder-optimal core

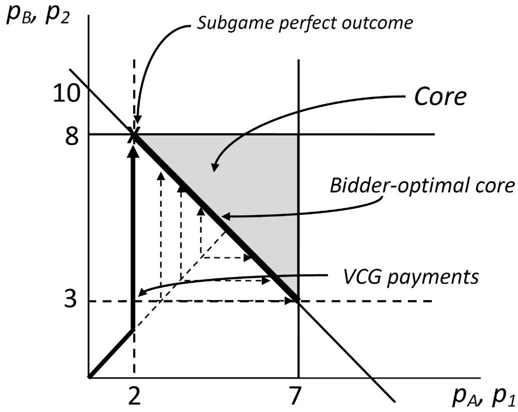

Figure 1: The subgame perfect equilibrium path

outcome is achieved in a Nash equilibrium (Day and Milgrom, 2008).[5] Notably, these strategy profiles are also Nash equilibria of the Vickrey auction (Sano, 2011b).

Next, consider an ascending auction by Parkes and Ungar (2000) and Ausubel and Milgrom (2002). The auction starts from zero prices, and bidders gradually raise the bids.[6] Suppose that bidders 1 and 2 submit bids for each single good, $p_1^t(A)$ and $p_2^t(B)$, and that bidder 3 can submit package bids for AB, $p_3^t(AB)$. At period 1, each bidder places bids of 1 for his interest: $p_1^1(A) = p_2^1(B) = p_3^1(AB) = 1$. Then, bidders 1 and 2 are tentative winners, so that bidder 3 raises the bid at period 2: $p_3^2(AB) = 2$. If bidder 3 becomes the tentative winner at period 2, bidders 1 and 2 raise the bids at period 3, and so on.

Suppose that all bidders behave truthfully and raise the bids until their true values. Then, the auction terminates when $p_1(A) = p_2(B) = 5$ and $p_3(AB) = 10$. Bidders 1 and 2 win goods A and B, respectively with the price of 5. Note that the outcome is in the core.[7]

Let us consider the subgame perfect equilibrium of the auction. It is natural to assume that once a bidder stops bidding at t, he can no longer raise the bids. We can easily obtain the equilibrium by the standard manner of the backward induction. Suppose that bidder 3 behaves truthfully and raises the bids until $p_3(AB) = 10$.

[5]The specification of a pricing rule does not matter. The core with respect to the reported bids is uniquely determined in the equilibrium.

[6]Bidders are not allowed to jump bids.

[7]The final prices may differ by the bid increment. However, we ignore it.

Consider a subgame in which bidder 2 first stops bidding at $3 \leq p_2(B) < 5$. Since $10 - p_2(B) \leq 7$, bidder 1 successfully wins the good by bidding until $10 - p_2(B)$. Note that the price $10 - p_2(B)$ is 1's Vickrey price, given that 2's value is $p_2(B)$. Hence, for bidder 1, the "Vickrey-target strategy" is to bid up until $10 - p_2(B)$, and it is optimal when $p_2(B) \geq 3$. When bidder 2 stops bidding at $p_2(B) < 3$, bidder 1 has to pay $10 - p_2(B) > 7$ in the case of winning. Hence, it is optimal for 1 to stop at $p_1(A) = 7$ and lose. Similarly, for bidder 2, the Vickrey-target strategy, bidding until $10 - p_1(A)$, is optimal when $p_1(A) \geq 2$.

Now, go back to a subgame where nobody has stopped yet. Applying the consideration above, bidder 1 will win as long as he bids until $p_1(A) \geq 10 - 8 = 2$. On the other hand, bidder 2 will win when he first stops at $p_2(B) \geq 10 - 7 = 3$. To minimize the payment, bidder 1's best strategy is to bid until $p_1(A) = 2$, and 2's until $p_2(B) = 3$. These prices are their Vickrey prices given true values. Thus, it is a perfect equilibrium to stop at the Vickrey prices that are computed from bidders' true values and stopping prices. In the equilibrium outcome, bidder 1 stops first at $p_1(A) = 2$ and bidder 2 raises bids until $p_2(B) = 8$. This outcome is in the bidder-optimal core. In addition, by inspection, this is a unique subgame perfect equilibrium outcome as long as bidder 3 behaves truthfully. Figure 1 illustrates the equilibrium path of this example.

3 The Model

A seller wants to allocate multiple indivisible objects, and K denotes the set of the goods. Let $N \equiv \{0, 1, 2, \ldots, n\}$ be the set of all players. $I = \{1, \ldots, n\}$ is the set of all bidders and 0 denotes the seller. Let $X_i \subseteq 2^K$ be the set of admissible bundles for bidder i. For each $i \in I$, a null bundle is denoted by $\underline{x}_i$, and $\underline{x}_i \in X_i$. $X \subseteq X_1 \times \cdots \times X_n$ denotes the set of feasible allocations. All bidders have quasi-linear utilities. Suppose that valuations for bundles of goods are integer-valued. Let $u_i : X_i \to \mathbb{Z}_+$ be a bidder i's valuation function. Suppose each u_i is monotone and $u_i(\underline{x}_i) = 0$ for all $i \in I$. Bidder i earns a payoff $\pi_i = u_i(x_i) - p_i$ where $x_i \in X_i$ denotes goods allocated to i and p_i is the monetary transfer to the seller. The seller's payoff is the revenue from the auction: $\pi_0 = \sum_{i \in I} p_i$.

Let $X^*(u) \subseteq X$ be the set of efficient allocations with respect to the profile of valuation functions $u = (u_i)_{i \in I}$:

$$X^*(u) \equiv \arg\max_{x \in X} \sum_{i \in I} u_i(x_i). \tag{1}$$

The *coalitional value function* V is defined by

$$V(J, u) = \begin{cases} \max_{x \in X} \sum_{i \in J} u_i(x_i) & \text{if } 0 \in J \\ 0 & \text{if } 0 \notin J \end{cases}, \tag{2}$$

where $J \subseteq N$ and $u_0(\cdot) \equiv 0$. $V(J, u)$ denotes the total surplus that a coalition J can produce. We sometimes use the notation of $V(\cdot)$ instead of $V(\cdot, u)$. Given a valuation profile u, a payoff profile $\pi \in \mathbb{R}^{n+1}$ is feasible if $\sum_{i \in N} \pi_i \leq V(N)$. A payoff profile π is *individually rational* if $\pi \geq 0$. The *core of the auction game* is

$$Core(N, V) = \big\{\pi \geq 0 | \sum_{i \in N} \pi_i = V(N) \text{ and } (\forall J \subseteq N) \sum_{i \in J} \pi_i \geq V(J)\big\}. \tag{3}$$

A payoff profile $\pi \in Core(N, V)$ is *bidder-optimal* if there is no $\pi' \in Core(N, V) \backslash \{\pi\}$ such that $\pi'_i \geq \pi_i$ for all $i \in I$. Let $BOC(N, V) \subseteq Core(N, V)$ be the set of bidder-optimal core payoff profiles.

3.1 Vickrey-Reserve Auctions

Before we define a class of ascending-price auctions, we introduce sealed-bid auctions or direct revelation mechanisms. In a sealed-bid auction $(\bar{g}, \bar{p})$, each bidder reports a valuation function $\hat{u}_i$. For a profile of valuation functions $\hat{u} = (\hat{u}_i)_{i \in I}$, the outcome of the auction is $(\bar{g}(\hat{u}), (\bar{p}_i(\hat{u}))_{i \in I}) \in (X, \mathbb{R}^n_+)$, which specifies the choice of an allocation $x = \bar{g}(\hat{u})$ and payments $\bar{p}_i(\hat{u}) \in \mathbb{R}_+$. A sealed-bid auction $(\bar{g}, \bar{p})$ is *efficient* if for all $\hat{u}$, $\bar{g}(\hat{u}) \in X^*(\hat{u})$. In addition, $(\bar{g}, \bar{p})$ is *individually rational* if for all $\hat{u}$ and $x = \bar{g}(\hat{u})$, $\bar{p}_i(\hat{u}) \leq \hat{u}_i(x_i)$ for all $i \in I$.[8] Bidder i is said to be a winner if $\bar{g}_i(\hat{u}) \neq \underline{x}_i$. Conversely, i is said to be a loser if $\bar{g}_i(\hat{u}) = \underline{x}_i$.

Let $\hat{V}(\cdot) \equiv V(\cdot, \hat{u})$, which is the coalitional value function with respect to $\hat{u}$. Given an auction mechanism $(\bar{g}, \bar{p})$ and a report profile $\hat{u}$, let $\hat{\pi}_i \equiv \hat{u}_i(\bar{g}_i(\hat{u})) - \bar{p}_i(\hat{u})$ for each bidder and $\hat{\pi}_0 \equiv \pi_0 = \sum \bar{p}_i(\hat{u})$ for the seller. The auction mechanisms in the existing literature are defined as follows.

[8] Let $\hat{u}_i(\underline{x}_i) \equiv 0$ for all $i \in I$. Since $\bar{p}_i \in \mathbb{R}_+$, individual rationality implies that every bidder assigned the null bundle pays 0.

Definition 1 A sealed-bid auction $(\bar{g}, \bar{p}^V)$ is *the Vickrey auction* if it is efficient and for each $i \in I$,

$$\bar{p}_i^V(\hat{u}) = \hat{V}(N_{-i}) - \sum_{j \neq i} \hat{u}_j(\bar{g}_j(\hat{u})). \tag{4}$$

In addition, $\bar{\pi}_i$ denotes bidder i's *Vickrey payoff*:

$$\bar{\pi}_i \equiv u_i(\bar{g}_i(u)) - \bar{p}_i^V(u) = V(N, u) - V(N_{-i}, u).$$

Definition 2 A sealed-bid auction $(\bar{g}, \bar{p})$ is *core-selecting* if it satisfies $\forall \hat{u}$, $\hat{\pi} \in Core(N, \hat{V})$.

Definition 3 A sealed-bid auction $(\bar{g}, \bar{p})$ is of *Vickrey-reserve* if it is efficient, individually rational, and $\forall \hat{u}$, $\bar{p}(\hat{u}) \geq \bar{p}^V(\hat{u})$. In addition, it is a *strict Vickrey-reserve auction* if it satisfies $\bar{p}_i(\hat{u}) > \bar{p}_i^V(\hat{u})$ as long as $\bar{p}_i^V(\hat{u}) < \hat{u}_i(\bar{g}_i)$.

Note that each bidder's payoff in the truth-telling equilibrium in the Vickrey auction $\bar{\pi}_i$ is the upper bound of the payoffs achieved in the core.[9] This implies that every core-selecting auction is a Vickrey-reserve auction. Bikhchandani and Ostroy (2002) show that if goods are substitutes, core-selecting auctions and Vickrey-reserve auctions are equivalent.[10] When goods may be complements, the equivalence does not hold.

3.2 Ascending Auctions

An ascending-price format is introduced to Vickrey-reserve auctions. Following Parkes and Ungar (2000), de Vries et al. (2003), and Mishra and Parkes (2007), we consider complex prices, which are non-linear and non-anonymous. This means that a price of a bundle x_i for i, which is denoted by $p_i(x_i)$, does not have to be the sum of prices of each individual object. Moreover, the price for a bundle can be different between bidders. A non-linear and non-anonymous price vector p is in $\mathbb{R}_+^{\sum|X_i|}$. We suppose $p_i(\underline{x}_i) \equiv 0$ for all i. Blumrosen and Nisan (2010) show that we need a complex price vector to conduct an ascending auction which finds an efficient allocation.

Given a price vector p, let $D_i(p)$ be i's (true) demand set:

$$D_i(p) \equiv \{x_i \in X_i | u_i(x_i) - p_i(x_i) \geq u_i(y_i) - p_i(y_i) \ \forall y_i \in X_i\}. \tag{5}$$

[9] See Ausubel and Milgrom (2002) and Bikhchandani and Ostroy (2002).

[10] To be precise, the equivalence holds if and only if "bidders are substitutes."

In an ascending auction, the auctioneer proposes a price vector p^t at each period t. Each bidder responds with his demand set $\hat{D}_i(p^t)$. The auctioneer then adjusts the price vector and repeats the process. Bidder i is said to be *active at* t if for all $\tau \leq t$, $\underline{x}_i \notin \hat{D}_i(p^\tau)$. Let $I^t \subseteq I$ be the set of all active bidders at t. Active bidders are defined above because if $\underline{x}_i \notin D_i$, he has a non-null bundle x_i which earns a positive payoff under the current price: $u_i(x_i) - p_i(x_i) > 0$. Thus, he can afford to pay more for that bundle.

In this paper, we define ascending combinatorial auctions in a general form in the sense that we do not specify the detail of the rule in the following three ways. First, although we fix the price increment by unity, the selections of bidders facing price increases at each period are arbitrary. Second, we do not specify when the auction terminates. We allow various conditions for stopping price increases to consider both the Vickrey and core-selecting pricing. Third, bidders' payments can be different from the prices in the terminal period. Bidders' payments may be discounted from the terminal prices, and the discounting rule is arbitrary with mild conditions. Our definition of the auction extends Mishra and Parkes (2007) in the third way.

Now we define ascending combinatorial auctions in a general form. Our definition follows that of Mishra and Parkes (2007).

1. The auctioneer initializes the price vector as $p^1 = (0, \dots, 0)$.

2. At each period $t = 1, 2, \dots$, each bidder reports his demand set $\hat{D}_i(p^t)$. The auctioneer chooses a set of active bidders $J^t \subseteq I^t$. If $i \in J^t$ and if $x_i \in \hat{D}_i(p^t)$, then $p_i^{t+1}(x_i) = p_i^t(x_i) + 1$. Else, let $p_i^{t+1}(x_i) = p_i^t(x_i)$.

3. Repeat the process. It terminates at $T \leq \bar{T}$, when $I^{\bar{T}} = \emptyset$. The auctioneer selects an allocation $x \in X$ and determines bidders' payments $p \in \mathbb{R}^n_+$.

Let $(g, (p_i)_{i \in I})$ be the mechanism of the ascending auction, which decides the final allocation $g(h) \in X$ and the payments $(p_i(h))_{i \in I} \in \mathbb{R}^n_+$, where $h \in H$ denotes a history throughout the ascending auction.[11] Note that bidders' payments do not have to be the posted prices at the terminal period.

We focus on auctions which lead to an efficient allocation with respect to reported information. Although we do not specify the condition for the termination

[11] Mishra and Parkes (2007) and de Vries et al. (2007) define an ascending auction as (g, p) is determined from only $(p^T, (D_i(p^T))_{i \in I})$. Our definition allows the auctioneer to determine an outcome using all the information during the auction.

of the price adjustment, we need at the terminal period T, the auctioneer specifies a competitive equilibrium from the history of prices and demand sets (Mishra and Parkes, 2007). This means that the auctioneer can select an allocation $x \in X$ such that for all $i \in I$,

$$x_i \in \hat{D}_i(p^T)$$

and

$$x \in \arg\max_X \sum p_i^T(x_i).$$

However, it is not sufficient to implement the Vickrey outcome, and the auctioneer needs to continue price increase even if p^t is a competitive equilibrium.[12] Thus, different terminal conditions will be adopted by different goals.

Define $\hat{u}_i : X_i \to \mathbb{R}_+$ for each $i \in I$ as follows. If $\underline{x}_i \in \hat{D}_i(p^T)$, let $\hat{u}_i(\cdot) \equiv p_i^T(\cdot)$. If $\underline{x}_i \notin \hat{D}_i(p^T)$, then let

$$\hat{u}_i(x_i) \equiv \begin{cases} p_i^T(x_i) + 1 & \text{if } x_i \in \hat{D}_i(p^T) \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

Obviously, each $\hat{u}_i$ is consistent with the all demand sets reported by i, and $\hat{u}_i$ can be interpreted as the "representative valuation function" (Matsushima, 2011). The efficiency and the individual rationality in the ascending auction are defined with respect to $\hat{u}$ similarly to sealed-bid auctions. In addition, $\hat{V}$ and $\hat{\pi}$ are also similarly defined.

Definition 4 An ascending auction is an *ascending Vickrey auction* if it is efficient and $p(h) = \bar{p}^V(\hat{u})$ for all $h \in H$. An ascending auction is *core-selecting* if $\forall h$, $\hat{\pi} \in Core(N, \hat{V})$. An ascending auction is of *Vickrey-reserve* if it is efficient, individually rational, and $p(h) \geq \bar{p}^V(\hat{u})$ for all h.

The definition of $\hat{u}$ follows from the termination of the auction. Hence, for each auction, a proper terminal condition is applied. In order to avoid confusion regarding the terminal conditions and the corresponding definitions of $\hat{u}$, it is convenient to modify the definition of the termination of the ascending prices as follows:

Modified Terminal Condition. The auction terminates at $\bar{T}$ if $I^{\bar{T}} = \emptyset$.

[12] To implement the Vickrey outcome, the auctioneer needs to find a "universal competitive equilibrium" (Mishra and Parkes, 2007).

We allow the final outcome to be determined from all the information during the auction. Hence, the formerly defined auction mechanisms are compatible to the modified definition by interpreting the periods after the "true" termination, $T+1, T+2, \ldots, \bar{T}$, as a fictitious game which is irrelevant to the final outcome. Under the modified termination, we suppose that all bidders have to fully reveal their valuation functions. Our definition of ascending auctions are motivated not by proposing a specific ascending auction design, but rather by providing a general model for analyzing the proposed designs.

Selections of J^t specifies the ascending price procedure in detail. A specification of J^t is selecting "tentative losing bidders." The auctioneer selects a revenue-maximizing allocation $x(t) \in X \cap \left((\hat{D}_1 \cup \{\underline{x}_1\}) \times \cdots \times (\hat{D}_n \cup \{\underline{x}_n\})\right)$ at each period. Then J^t is defined as $J^t = \{j \in I^t | x_j(t) = \underline{x}_j\}$. This specification is intuitive and proposed by Parkes and Ungar (2000) and Ausubel and Milgrom (2002). Other studies specify J^t as the "minimally undersupplied bidders" (de Vries et al. 2007).

During the auction, bidders are restricted by the following activity rule in order that there exists a valuation function consistent with a bidder's behavior. We follow the activity rule considered by Mishra and Parkes (2007).

Assumption 1 (Activity Rule) Each bidder must satisfy the followings:

1. If $p_i^s = p_i^t$, $\hat{D}_i(p^s) = \hat{D}_i(p^t)$.
2. For all t, $\hat{D}_i(p^t) \subseteq \hat{D}_i(p^{t+1})$.
3. If $x_i \subseteq x_i'$ and $x_i \in \hat{D}_i(p^t)$, then $x_i' \in \hat{D}_i(p^t)$.

The first rule requires that if the prices remain the same for a bidder, he must report the same demand set. Equivalently, only bidders who face price increases make new decisions at each period. The second one should be satisfied when there is a valuation function $\hat{u}$ consistent with the collection of demand sets. Every bundle demanded at t has to be demanded at $t+1$ because the price of the bundle is increased by only the minimum increment. The third one requires that reports have to be consistent with monotonicity of valuation functions.[13]

To simplify the analysis, we assume that bidders make choices sequentially. This assumption is crucial for the uniqueness result of the equilibrium. However, as we will discuss later, it will not be essential to the results.

[13]See Mishra and Parkes (2007) for the sufficiency of this activity rule.

Assumption 2 (Sequential Decisions) Bidders make choices sequentially from 1 to n. Each bidder observes all actions made before his decision at each period.

3.3 Strategy and Equilibrium

In sealed-bid auctions, preceding studies (Bernheim and Whinston, 1986; Ausubel and Milgrom, 2002; Day and Milgrom, 2008) focus on a class of strategies: *truncation strategies*.[14] A strategy $\hat{u}_i$ is said to be α_i *truncation of* u_i if $\exists \alpha_i \geq 0$, $\forall x_i \in X_i$, $\hat{u}_i(x_i) = \max\{u_i(x_i) - \alpha_i, 0\}$. That is, a bidder understates a value for each bundle of the goods by a fixed amount α_i. For every Vickrey-reserve auction, a profile of truncation strategy constitutes a Nash equilibrium.

Proposition 1 (Sano, 2011b) *For every u and every $\pi \in BOC(N, V)$, the profile of π_i truncations of u_i is a Nash equilibrium of every sealed-bid Vickrey-reserve auction. The associated equilibrium payoff profile is π.*

In an ascending auction, bidder i's (pure) strategy σ_i is a mapping from his information sets or i's decision nodes E_i to 2^{X_i}. A *feasible* strategy is one satisfying the Activity Rule. Σ_i denotes the set of feasible strategies for i. Let $\Sigma \equiv \Sigma_1 \times \cdots \times \Sigma_n$ be the set of profiles of feasible strategies.

We focus on the following *semi-truthful strategies*, which corresponds to the concept of truncation strategy.[15]

Definition 5 A strategy $\sigma_i \in \Sigma_i$ is *semi-truthful* if it satisfies $\forall t$,

$$\hat{D}_i(p^t) \in \{D_i(p^t), X_i\}. \tag{7}$$

Let $\Sigma_i^* \subseteq \Sigma_i$ be the set of semi-truthful strategies, and let $\Sigma^* \equiv \Sigma_1^* \times \cdots \times \Sigma_n^*$.

The terminology "truthful" is adopted since bidder i reports his true demand set as long as he is active. Once i reports $\hat{D}_i = X_i$ ($\ni \underline{x}_i$), he cannot renew his demand set any longer. Hence, bidder i is said to *stop at t* if $\hat{D}_i(p^{t-1}) \neq X_i$ and if $\hat{D}_i(p^t) = X_i$. A semi-truthful strategy does not necessarily report the true valuations, since bidders

[14] The terminology of truncation strategy is adopted in Day and Milgrom (2008). The truncation strategy is also called truthful strategy (Bernheim and Whinston, 1986), semi-sincere strategy, and profit-target strategy (Ausubel and Milgrom, 2002).

[15] Ausubel and Milgrom (2002) use the terminology of "limited straightforward bidding" for our semi-truthful strategy.

may stop before prices reach their true valuations. Semi-truthful strategies in an ascending auction correspond to truncation strategies in a sealed-bid auction.

Lemma 1 *A bidder i follows $\sigma_i \in \Sigma_i^*$ if and only if there exists $\alpha_i \geq 0$ and for $\forall x_i \in X_i$,*

$$\hat{u}_i(x_i) = \max\{u_i(x_i) - \alpha_i, 0\}. \tag{8}$$

Proof. See Appendix B.

In every semi-truthful strategy, bidders report their true valuations or understate and never bid over their true values. We allow bidders playing overbidding strategies consistent with semi-truthful strategies as follows: $\hat{D}_i(p^t) \in \{X_i \setminus \{\underline{x}_i\}, X_i\}$ if $\underline{x}_i \in D_i(p^t)$. Let $\Sigma_i^{*+} \supset \Sigma_i^*$ be the set of semi-truthful strategies and overbidding strategies consistent with semi-truthful strategies, and let $\Sigma^{*+} \equiv \Sigma_1^{*+} \times \cdots \times \Sigma_n^{*+}$.

In the most of the paper, we restrict each bidder's strategy space to Σ_i^* or Σ_i^{*+}. We consider the *truthful perfect equilibrium* as an equilibrium concept. A truthful perfect equilibrium is an SPE with respect to Σ^{*+}.

Definition 6 A strategy profile $\sigma \in \Sigma^{*+}$ is a *truthful perfect equilibrium (TPE)* if it is a subgame perfect equilibrium under the condition which each bidder's strategy space is restricted to Σ_i^{*+}.

4 Main Results

Since each bidder makes choices sequentially, the auction is a perfect information game. To make it clear, we relabel the time by each bidder's decision node. We refer to bidder i's desicion node at period s as "period t $(= n(s-1)+i)$."

4.1 Further Notations and Assumptions

Let $u_i^t : X_i \to \mathbb{R}_+$ be the *provisional valuation function at t*, which is the possible valuation function given bidding behavior up to t: for each $x_i \in X_i$

$$u_i^t(x_i) \equiv \begin{cases} \max\{u_i(x_i), p_i^t(x_i) + \mathbf{1}_{\{x_i \in \hat{D}_i(p^t)\}}\} & \text{if } i \text{ is active at } t \\ p_i^t(x_i) & \text{otherwise} \end{cases}. \tag{9}$$

When the strategy space is restricted to Σ_i^*, the price vector never exceeds the true valuation function. Then, the provisional valuation function is equivalent to i's true valuation function if he is active at t, and otherwise it coincides with the reported valuation $\hat{u}_i$. Given $u^t = (u_i^t)_{i \in I}$, let $V^t(\cdot) \equiv V(\cdot, u^t)$ for simplicity. In addition, let $\bar{\pi}_i^t$ be bidder i's Vickrey payoff with respect to u^t: $\bar{\pi}_i^t = V^t(N) - V^t(N_{-i})$. Let $X^t \equiv X^*(u^t)$ be the set of efficient allocations with respect to u^t, and let $X_i^t \equiv \{x_i \in X_i | x \in X^t\}$.

We impose two additional assumptions. One is regarding the auction rule. To simplify the analysis and sharpen the results, we consider the following tie-breaking rule.

Assumption 3 For each $x \in X^*(\hat{u})$, define $t(x) \equiv \min\{t | (\forall s \geq t)\ x \in X^s\}$. Then, $g(h) \in \arg\min_{x \in X^*(\hat{u})} t(x)$.

In auction models with complete information, ties are likely to occur, and an equilibrium may fail to exist with random tie-breaking when strategy space is *continuous*. Hence, ties are traditionally broken in a way that depends on bidders' values and not only on their bids. For example, in a first-price auction of a single object, the highest two bidders submit the same bid in a Nash equilibrium, which is the value of the second highest bidder. In the analysis, we assume that the bidder with the higher value is chosen in the result of tie-breaking. This practice is accepted because the selected outcome is the limit of an equilibrium of an auction in which bidding is discrete with an increment $\epsilon > 0$.[16] Since the strategy space in our model is discrete, we do not have to care about ties actually. However, from a viewpoint that our model can be converted into a continuous case by taking a limit of small price increment, we follows this practice. Indeed, with Assumption 3, we sharpen the results and have a striking property with respect to bidder-optimality. In Appendix A, we construct a TPE without Assumption 3.

Another assumption is regarding bidders' behavior.

Assumption 4 Let (x_i, p_i) indicate obtaining x_i with a payment p_i. Suppose that for any non-null bundle x_i, there is a set of alternatives $C \supseteq \{(\underline{x}_i, 0), (x_i, u_i(x_i))\}$. Then, every bidder chooses $(\underline{x}_i, 0)$ with probability 0.

Assumption 4 implies that if a bidder expects that he can win a bundle x_i by placing the bid of $u_i(x_i)$, he actually does. We do not need Assumption 4 for the existence

[16] For the ways of tie-breaking and the related topics, see Reny (1999), Simon and Zame (1990), Ausubel and Milgrom (2002), and Day and Milgrom (2008).

of an equilibrium (Theorem 1). However, it is critical for the uniqueness of the equilibrium outcome. It can be justified when we additionally require trembling-hand perfection for the equilibrium concept. A bidder may win some goods with a lower price with a small probability when other bidders stop earlier than the prediction.

4.2 The Vickrey-Target Strategy

The following proposition states that any efficient allocation according to u^t, $x \in X^t$, remains efficient later on in any TPE. It simplifies the backward induction. Bidders never choose an action which changes X^t if no bidder is restricted to bid over the true values.

Proposition 2 *Suppose Assumptions 1, 2, 3, and 4. Suppose that each bidder's strategy space is restricted to Σ_i^*. Then, any TPE satisfies $X^{t-1} \subseteq X^t$ for all t, both on and off equilibrium paths.*

Proof. See Appendix B.

Proposition 2 implies $X^*(u) = X^0 \subseteq X^T = X^*(\hat{u})$. Hence, any TPE is efficient as long as no one overbids.

Suppose that in an efficient allocation, bidder i obtains a non-null bundle x_i^*. By Proposition 2, given u_{-i}^t, it is optimal for i to stop bidding at the least price p^t such that $x^* \in X^*(p_i^t, u_{-i}^t)$, since i's payment never exceed $p_i^t(x_i^*)$ by the individual rationality. Such a price vector satisfies

$$\sum_{j \neq i} u_j^t(x_j^*) + p_i^t(x_i^*) = \max_X \sum_{j \neq i} u_j^t(x_j), \tag{10}$$

hence,

$$p_i^t(x_i^*) = V^t(N_{-i}) - \sum_{j \neq i} u_j^t(x_j^*), \tag{11}$$

which is the Vickrey payment. Thus, it is a TPE for each bidder to stop at the Vickrey payment with respect to u^t. Formally, we define the "Vickrey-target strategy" as follows.

Definition 7 A semi-truthful strategy $\sigma_i^* \in \Sigma_i^*$ is said to be the *Vickrey-target* *strategy* if $\forall t \geq 1$ and $\forall p^t$,

$$\hat{D}_i(p^t) = \begin{cases} D_i(p^t) & \text{if } p_i^t(x_i) < u_i(x_i) - \bar{\pi}_i^{t-1} \text{ for all } x_i(\neq \underline{x}_i) \in X_i^{t-1}, \\ & \text{ or if } X_i^{t-1} = \{\underline{x}_i\} \\ X_i & \text{otherwise} \end{cases} . \tag{12}$$

Theorems 1 and 2 are our main theorems of the current paper. The Vickrey-target strategy constitutes a TPE of every ascending Vickrey-reserve auction. Moreover, the equilibrium outcome is in the bidder-optimal core with respect to the true valuations. Let π^* be the corresponding payoff allocation associated with σ^*.

Theorem 1 *Suppose Assumptions 2 and 3.*[17] *The profile of the Vickrey-target strategies $\sigma^* \in \Sigma^{*+}$ is a TPE of every ascending Vickrey-reserve auction.*

Proof. See Appendix B.

Theorem 2 *Suppose that $\bar{p}_i^V(u) > 0$ for all winners. Then, the outcome associated with σ^*, π^*, is in the bidder-optimal core with respect to the true values.*

Proof. See Appendix B.

At the initial period, every bidder is active and $\bar{\pi}^t = \bar{\pi}$. Hence, bidders first seek to stop bidding at their Vickrey payments. Note that once a bidder stops, he cannot renew the bids any longer. Hence, the stopping bidder's reported utility function is revealed. Each bidder recomputes his Vickrey payoff, regarding the price vector for the stopping bidder as his true valuation function. This recomputation weakly decreases the Vickrey payoffs of bidders. Remaining bidders continue bidding and aim for the revised Vickrey prices.

When the Vickrey outcoome is in the core, it is a unique bidder-optimal outcome. Hence, the TPE outcome coincides with the Vickrey outcome.

Note that σ^* is a TPE regardless of any specification of J^t, the terminal condition, and final discounts. Theorem 1 shows an equivalence in equilibrium strategy of ascending Vickrey-reserve auctions. It is similar to the results of Day and Milgrom (2008) and Sano (2011b), which show a particular strategy profile is a Nash equilibrium of every core-selecting or Vickrey-reserve auctions. However, the equilibrium outcome π^* can differ between the rules.

[17] Assumption 1 is automatically satisfied as long as we focus on semi-truthful strategies.

We consider a general valuations structure and a restricted strategy space. Another way of the analysis is to formulate a restricted valuations domain with unrestricted strategies. If bidders are single-minded; i.e., they are interested only in a specific bundle of goods, then Theorems 1 and 2 hold with unrestricted strategies. A bidder is *single-minded* if there is a non-null bundle $y_i \in X_i$ and if

$$u_i(x_i) = \begin{cases} v_i & \text{if } y_i \subseteq x_i \\ 0 & \text{otherwise} \end{cases}.$$

If a bidder is single-minded, he can make profit by bidding for y_i (or larger bundles). It is meaningless to bid for bundles that do not contain y_i. Hence, it is obvious that the bidder's stratgy must be semi-truthful.

Corollary 1 *If each bidder is single-minded, Theorems 1 and 2 hold with unrestricted strategy space.*

Remark 1 Assumption 2 is not crucial for Theorem 1. We have Theorem 1 without Assumption 2 by slightly modifying the Vickrey-target strategy. If two or more bidders (say, $\{i, j, \dots\} \equiv M$) simultaneously reach their stopping prices at t with $\bar{\pi}_i^{t-1}$, $\bar{\pi}_j^{t-1}, \dots$, then we take a maximal set $M^* \subseteq M$ that satisfies the followings: (a) Each $i \in M^*$ stops at t, (b) the others remain active at t, and (c) $X^{t-1} \subseteq X^t$.

4.3 Equilibrium Selection

Under certain criteria, π^* is a unique TPE outcome. When goods complementarities exist, there are many outcome in the bidder-optimal core in general. As Day and Milgrom (2008) and Proposition 1 show, any payoff profile in the bidder-optimal core is achieved in a Nash equilibrium. Subgame perfection (restricted to Σ^*) selects one from the set of those Nash equilibria.

We focus on the equilibrium outcome in which *losers behave truthfully.* There are in fact many equilibrium outcomes since it is optimal for losers to stop at any period in the auction as long as they lose. The restriction is natural and some preceding studies also focus on such an equilibrium in sealed-bid formats (Bernheim and Whinston, 1986; Ausubel and Milgrom, 2002). Also, this restriction can be justified when we additionally require trembling-hand perfection.[18] We assume losers

[18] If a loser stops under the true values, he loses a chance to win with a small probability. Conversely, if he bids over the true values, he may suffer a loss.

follow the Vickrey-target strategy σ_i^*.

Theorem 3 *Suppose Assumption 1, 2, 3, and 4. Further suppose that* $\bar{p}_i^V(u) > 0$ *for all winners. If each bidder's strategy space is resticted to* Σ_i^* *and if all losing bidders follow* σ_i^**, then* π^* *is a unique TPE outcome in every ascending strict Vickrey-reserve auction.*

Proof. Under the assumptions, any TPE is efficient by Proposition 2. Since all losers reveal true utility functions, for any winning bidder i, $\bar{\pi}_i^t$ is nonincreasing in t in any equilibrium by the argument in the proof of Theorem 1.

By Proposition 2, every winner i must stop when $p_i^t(x_i) \geq u_i(x_i) - \bar{\pi}_i^{t-1}$. If $\hat{u}_i(x_i) > u_i(x_i) - \bar{\pi}_i^{t-1}$, i's payment is $p_i > u_i(x_i) - \bar{\pi}_i^{t-1}$ by the strict Vickrey-reserve pricing and monotonicity of $\bar{\pi}_i^t$. On the other hand, if bidder i follows σ_i^*, his payment $p_i = u_i(x_i) - \bar{\pi}_i^{t-1}$. Hence, σ_i^* is a unique optimal strategy for each winning bidder. ■

The intuition of Theorem 3 is straightforward. By Proposition 2, each bidder always chooses X^t-preserving actions. Hence, each winner i can minimize the payment by stopping at the earliest period such that X^t unchanges even if i stops. Such a strategy is the Vickrey-target strategy. Assumption 2 is critical to Theorem 3. As we discussed in Remark 1, we need to coordinate the behavior if two or more bidders simultaneously reach their target prices at t. There will be several possible selections of M^*, and each of them will lead to different equilibrium outcome.

Although we focus on only strict Vickrey-reserve auctions, Theorem 3 is applied to ascending auctions without final discounts, such as Parkes and Ungar (2000), Ausubel and Milgrom (2002), de Vries et al. (2007). If the payments are equal to the final prices of the bundles, it is clearly suboptimal to bid over the true values and win. Hence, Theorem 3 holds without restricting to Σ_i^*.

Corollary 2 *Suppose Assumptions 1, 2, 3, and 4, and suppose that* $\bar{p}_i^V(u) > 0$ *for all winners. If all losing bidders follow* σ_i^**, then* π^* *is a unique TPE outcome in every ascending auctions with no final discount.*

5 Discussions

5.1 Resistance to Collusive Overbidding

Theorem 3 holds for every ascending strict Vickrey-reserve auction when bidders are not allowed to overbid. As Day and Milgrom (2008) point, there may exist an inefficient equilibrium in which some bidders collusively overstate their values and outbid the efficient allocation.

For example, consider the same situation as Example 1. Suppose that bidder 1, who wants good A, values 4 and that bidder 2 values 4 for good B. Suppose that bidder 3 wants the package of A and B and values 10. In this case, in the subgame perfect equilibrium σ^*, bidder 3 wins both goods with the payment 8. This outcome coincides with the Vickrey outcome. However, there is an inefficient Nash equilibrium in the sealed-bid Vickrey auction. Suppose that all the bidders submit 10. Then, bidders 1 and 2 wins each good with zero payment, and it is an equilibrium. Such an equilibrium exists in some strict Vickrey-reserve auctions. In addition, similar TPE exists in some ascending strict Vickrey-reserve auctions as well.

Such a collusive overbidding equilibrium is excluded by imposing core-selecting pricing (Day and Milgrom, 2008). Similarly, ascending core-selecting auctions prevent bidders from bidding over the true valuations collusively. Let G^* be the set of winning bidders associated with σ^*. let $\sigma_J = (\sigma_j)_{j \in J}$.

Theorem 4 *In any ascending core-selecting auction, there is no group of bidders $G \not\subseteq G^*$ such that $\exists \sigma_G \in \Sigma_G^{*+}$, $\pi_i^\sigma > \pi_i^*$, and $\hat{u}_i(\hat{x}_i) > 0$ for all $i \in G$ under $\sigma = (\sigma_G, \sigma^*_{I \setminus G})$.*

Proof. See Appendix B.

5.2 Ascending Vickrey Auction

Ascending Vickrey-reserve auctions include ascending Vickrey auctions. Clearly, truth-telling is also an SPE in their auctions.

Proposition 3 *Suppose that the Vickrey outcome is not in the core. Then, there are at least two TPE outcomes in every ascending Vickrey auction: the Vickrey outcome $\bar{\pi}$ and the core-implementing outcome π^*.*

Auction designers expect that in an ascending Vickrey auction, bidders behave truthfully, and thus that the Vickrey outcome is actually implemented. However, when the Vickrey outcome is not in the core, ascending Vickrey auctions have another equilibrium that leads to an outcome in the core. Moreover, the "core-implementing" equilibrium seems more robust in the following senses. First, the Vickrey-target strategy σ_i^* is obviously a best response among Σ_i^* to both σ_{-i}^* and truth-telling strategies. Conversely, the truth-telling strategy is not the best response if the other players follow σ_{-i}^*. Second, σ^* is an equilibrium even if the auction is slightly different from the Vickrey auctions. Truth-telling, however, is not an equilibrium of such an almost-Vickrey auction.

5.3 Payoff Non-Monotonicity and Free-Rider Problem

It seems to be a positive result that the ascending auctions have a unique TPE lying in the core. However, the equilibrium outcome may not be necessarily desirable. In the TPE σ^*, when a winner has a low valuation, he tends to obtain a large profit. This is because the Vickrey payments for the low-value bidders are low and their prices reach the Vickrey prices earlier. The higher the value a winner has, the lower are the profits he tends to get in the equilibrium.

Example 1 (continued). Remember Parkes and Ungar's (2000) auction with 2 goods and 3 bidders. When bidders' actual values are $(7, 8, 10)$, the equilibrium payoff allocation is $(\pi_0, \pi_1, \pi_2, \pi_3) = (10, 5, 0, 0)$. Note that bidder 1, who has a lower value than bidder 2, earns all the gains, whereas bidder 2 earns zero net payoff. Suppose that bidder 1's value for A is 9, with everything else remaining unchanged. Then, in the equilibrium, bidder 2 stops at the price of 1 and bidder 1 behaves truthfully. The equilibrium payoff allocation is now $(10, 0, 7, 0)$. The equilibrium payoff of bidder 1 decreases as his valuation increases (Figure 2).

As this example shows, the TPE outcome is on the corner of the bidder-optimal core. Moreover, the winner with a low value earns the Vickrey payoff, while the high-value winner earns 0.[19]

This situation is quite similar to a standard free-rider problem. Suppose a private

[19] The equilibrium payoff allocation depends on the auction rule, and payoff non-monotonicity does not necessarily arise.

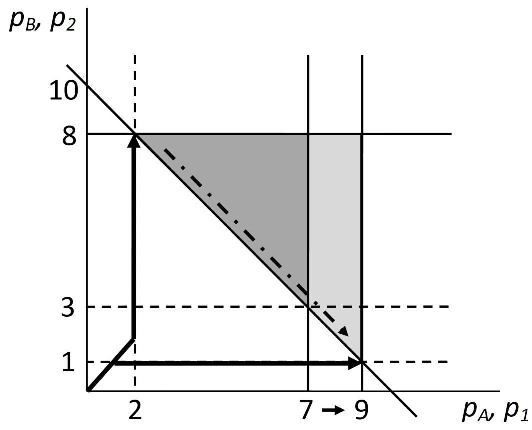

Figure 2: Non-Monotonic Equilibrium Payoffs

provision game of a public good, in which marginal values for a public good are heterogeneous among agents. Then in a unique equilibrium, only one agent with the highest value provides the good and the others do not provide at all.[20] In our auction situation, bidders with low values free-ride on other bidders with higher values. Indeed, the incentive problem in core-selecting auctions is referred to as a kind of free-rider problem (Milgrom, 2000), while it is called the "threshold problem." In a sealed-bid format, the threshold problem is often interpreted as a kind of coordination failure by bidders (Bykowsky et al., 2000). However, in an ascending price open-bid format, it seems more appropriate to interpret the incentive problem as a free-rider problem.

This free-rider problem appears in a striking form when valuations are private information of each bidder. Suppose the same situation as Example 1 and asymmetric information. Even when bidder 1 has a low value, he will have a certain amount of expected payoff because he can free-ride on bidder 2. Conversely, when bidder 1 has a high value, his expected payoff may be low because bidder 2 may have a low value and free-ride on him. Hence, it may be good for bidder 1 to behave as a low value bidder even when he has a high value. Thus, both bidders 1 and 2 behave as a low value bidder. This will lead to inefficiency and low revenue.[21]

[20] See Mas-Collel et al. (1995) for the free-rider problem in the public goods game.
[21] For the incomplete information case, see Sano (2011b).

5.4 Inefficiency under the Unrestricted Strategies

Even with complete information, the payoff non-monotonicity provides inefficiency in the case where strategy space is unrestricted. In a sealed-bid format, a truncation strategy is a best response among all strategies (Day and Milgrom, 2008; Sano 2011b). However, in the ascending auctions, the Vickrey-target strategy is not necessarily a best response among Σ_i. The TPE σ^* is not an SPE in general. Moreover, an SPE may be inefficient. The following example shows an SPE is not efficient.

Example 2. Suppose that there are 3 goods $\{A, B, C\}$ and 7 bidders. All bidders except bidder 7 are interested only in a unique bundle of the goods. Each of the values for these six bidders is $u_1(ABC) = 12$, $u_2(A) = 7$, $u_3(B) = u_4(B) = 1$, and $u_5(C) = u_6(C) = 1$ respectively. Bidder 7 is interested in goods B and C. His valuation function is such that $u_7(B) = u_7(BC) = 8$ and $u_7(C) = 6$. In the efficient allocation, bidders 2, 5 (or 6), and 7 win the single item A, C, B, respectively. The auction is Parkes and Ungar's (2000) ascending auction.

Since bidders 3, 4, 5, and 6 are completely competitive, they bid until their true values in any equilibrium. The Vickrey payments for bidders 2 and 7 are 3 and 4, respectively. Hence, in the TPE σ^*, bidder 2 stops earlier than bidder 7. In the equilibrium, $p_2(A) = 3$, $p_5(C) = 1$, and $p_7(B) = 8$. Bidder 7's TPE payoff is 0.

Now, consider that bidder 7 follows the Vickrey-target strategy with respect to the following valuation function: $\tilde{u}_7(B) < \tilde{u}_7(C) = \tilde{u}_7(BC) = 6$. Under $(u_{-7}, \tilde{u}_7)$, bidder 7 obtains good C in the efficient allocation. And, bidder 2's Vickrey payment changes to 5, whereas that of bidder 7 for C remains the same. In the TPE outcome, bidder 7 wins item C with $p_7(C) = 4$, whereas bidder 2 pays 7 for item A. By inspection, this is an SPE and the equilibrium outcome is inefficient.

This inefficiency stems from the payoff non-monotonicity. In an efficient allocation, a bidder obtains some goods whose value is sufficiently large. However, the true value is so large that other bidders stop earlier and he may have to pay too much. On the other hand, if he focuses on another good whose value is not so high, he may win it with a lower price, and it may be more profitable.

Proposition 4 *The TPE σ^* is not an SPE with the unrestricted strategy domain in general. Moreover, An SPE is not efficient in general.*

Remark 2 Inefficient subgame perfect equilibrium can exist because of another logic. Suppose that there are 2 goods, A and B, and 2 bidders. Both bidders have identical valuations of $u_i(A) = u_i(B) = 6$ and $u_i(AB) = 12$. In any TPE outcome, both bidders have to earn zero payoff. However, if both bidders report $\hat{D}_i = \{A, B, AB\}$ at the initial period, then both can get one good with zero payment. And, this constitutes an equilibrium. This is an implicit collusion, in which bidders split up goods between each other and end up the auction with low prices. Such an equilibrium is considered also by Ausubel and Schwartz (1999) and Grimm et al. (2003) in an ascending auction without package bidding.

6 Conclusion

We formulate a general class of ascending-price auctions. The Vickrey-target strategy constitutes a perfect equilibrium of every ascending Vickrey-reserve auction with restricted strategy space. The equilibrium outcome is in the bidder-optimal core and unique in ascending strict Vickrey-reserve auctions if losing bidders follow the Vickrey-target strategy. These results are positive findings, as sealed-bid Vickrey-reserve auctions may have multiple Nash equilibria. Although ascending Vickrey auctions have both truth-telling and core-implementing equilibria, the core-implementing equilibrium seems more robust under complete information and restricted strategy space.

The equilibrium outcome, however, can be "unfair" in the sense that bidders with lower values tend to obtain higher payoffs. This situation is similar to a standard free-rider problem, and it will lead to inefficiency under incomplete information. The payoff non-monotonic property also provides inefficiency in the case of unrestricted strategy space. An interesting future research is to constitute an SPE with unrestricted strategy space. It is also an open question what class of valuation functions assures SPE in the core.

A Theorem 1 without Assumption 3

In this paper, we suppose that bidders' strategies are discrete, however, ties are broken in a way that particular bidders are favored. This assumption is made for analytical purpose. We also have a corresponding result when the special tie-breaking rule is not assumed. We redefine the Vickrey-target strategy as one decreasing the

Vickrey payoff by unity. In this section, we assume that there is a unique efficient allocation with respect to true valuations: $X^*(u) = \{x^*\}$.

Suppose that bidder i has a non-null bundle $x_i \in X_i^{t-1}$ for some period $t-1$, and that he stops at t under the Vickrey-target strategy. Then, the provisional coalitional value $V^t(N) = V^{t-1}(N) - \bar{\pi}_i^{t-1} = V^{t-1}(N_{-i}) = V^t(N_{-i})$. Hence, there is an allocation $\tilde{x} \in X^t$ such that $\tilde{x}_i = \underline{x}_i$ and ties occur.

Let λ_i^t be the *approximate Vickrey payoff with respect to* u^t:

$$\lambda_i^t = \max\{\bar{\pi}_i^t - 1, 0\}, \tag{13}$$

Then, we redefine the Vickrey-target strategy with λ_i^t as follows.

Definition 8 A semi-truthful strategy $\sigma_i^* \in \Sigma_i^*$ is said to be the *(approximate) Vickrey-target strategy* if $\forall t \geq 1$ and $\forall p^t$,

$$\hat{D}_i(p^t) = \begin{cases} D_i(p^t) & \text{if } p_i^t(x_i) < u_i(x_i) - \lambda_i^{t-1} \text{ for all } x_i(\neq \underline{x}_i) \in X_i^{t-1}, \\ & \text{ or if } X_i^{t-1} = \{\underline{x}_i\} \\ X_i & \text{otherwise} \end{cases}. \tag{14}$$

Then, we have Theorem 1 with the approximate Vickrey-target strategy.

Using λ_i^t instead of $\bar{\pi}_i^t$, ties do not occur. We can observe this as follows. Suppose that X^{t-1} is singleton and $X^{t-1} = \{x^{t-1}\}$. Then, $V^{t-1}(N) \geq V^{t-1}(N_{-i}) + 1$ for each $i \in \{i \in I | x_i^{t-1} \neq \underline{x}_i\}$. Since $\bar{\pi}_i^{t-1} \geq 1$, $\lambda_i^{t-1} = \bar{\pi}_i^{t-1} - 1$. When bidder i stops at t by the approximate Vickrey-target strategy, then his reported valuation function is

$$\hat{u}_i(x_i) = \max\{u_i(x_i) - \lambda_i^{t-1}, 0\}.$$

Hence,

$$\begin{aligned} V^t(N) &= \max_X \big[\max\{u_i(x_i) - \lambda_i^{t-1}, 0\} + \sum_{j \neq i} u_j^{t-1}(x_j)\big] \\ &= V^{t-1}(N) - \lambda_i^{t-1} \\ &= V^{t-1}(N) - (V^{t-1}(N) - V^{t-1}(N_{-i}) - 1) \\ &= V^{t-1}(N_{-i}) + 1. \end{aligned} \tag{15}$$

Since x^{t-1} is a unique efficient allocation, we have $X^t = \{x^{t-1}\}$.

B Proofs

B.1 Proof of Lemma 1

(Only if part.) Suppose that bidder i follows $\sigma_i \in \Sigma_i^*$ and stops at t. When bidder i is active at T, we suppose a fictitious period $T+1$, and let i stop at $T+1$. Then $\hat{u}_i = p_i^t$ for all i.

It is trivial in the case of $t = 1$. Hence, suppose $t \geq 2$. Take arbitrary $\hat{x}_i \in \hat{D}_i(p^{t-1}) = D_i(p^{t-1})$, and let $\alpha_i \equiv u_i(\hat{x}_i) - p_i^t(\hat{x}_i) \geq 0$. Then $\hat{u}_i(\hat{x}_i) = u_i(\hat{x}_i) - \alpha_i \geq 0$.

By the activity rule, $p_i^{t-1} \neq p_i^t$ and $p_i^t(x_i) = p_i^{t-1}(x_i) + 1$ for all $x_i \in \hat{D}_i(p^{t-1})$. For every $x_i \in \hat{D}_i(p^{t-1})$,

$$u_i(x_i) - p_i^{t-1}(x_i) = u_i(\hat{x}_i) - p_i^{t-1}(\hat{x}_i) = \alpha_i + 1.$$

Therefore,

$$\hat{u}_i(x_i) = p_i^t(x_i) = u_i(x_i) - \alpha_i.$$

On the other hand, by the activity rule, $x_i \notin \hat{D}_i(p^{t-1})$ implies $x_i \notin \hat{D}_i(p^s)$ for all $s \leq t-1$. Hence, $\hat{u}_i(x_i) = p_i^t(x_i) = 0$. Since $x_i \notin D_i(p^{t-1})$, $u_i(x_i) < \alpha_i + 1$. Since u_i is integer, $\max\{u_i(x_i) - \alpha_i, 0\} = 0$.

(If part.) Suppose that $\hat{u}_i$ has a form of (8) under some $\sigma_i \in \Sigma_i$. Suppose for contradiction there exists some period t and $\hat{D}_i(p^t) \notin \{D_i(p^t), X_i\}$.

Suppose $x_i \in D_i(p^t)$ and $x_i \notin \hat{D}_i(p^t)$. Then, for any $x_i' \in \hat{D}_i(p^t)$,

$$u_i(x_i) - p_i^t(x_i) \geq u_i(x_i') - p_i^t(x_i'),$$

hence,

$$u_i(x_i) - u_i(x_i') \geq p_i^t(x_i) - p_i^t(x_i'). \tag{16}$$

On the other hand, i's report implies

$$\hat{u}_i(x_i) - p_i^t(x_i) < \hat{u}_i(x_i') - p_i^t(x_i'). \tag{17}$$

Since $\hat{u}_i(x_i') \geq 0$, $\hat{u}_i(x_i') = u_i(x_i') - \alpha_i$. Hence, we have

$$\begin{aligned} \hat{u}_i(x_i) &< \hat{u}_i(x_i') + p_i^t(x_i) - p_i^t(x_i') \\ &\leq \hat{u}_i(x_i') + u_i(x_i) - u_i(x_i') \\ &= u_i(x_i) - \alpha_i \\ &\leq \max\{u_i(x_i) - \alpha_i, 0\}, \end{aligned} \tag{18}$$

which is a contradiction.

We also have a contradiction in the same manner when $x_i \notin D_i(p^t)$ and $x_i \in \hat{D}_i(p^t)$. ■

B.2 Proof of Proposition 2

We prove by induction. Suppose there are m active bidders at t.

Step 1. Suppose $m = 1$ and bidder i is active. Since $p_i^s(\cdot) \leq u_i(\cdot)$ for all s by assumption, $X^s = X^{s-1}$ for all $s \geq t$ as long as i is active.

If $X_i^t = \{\underline{x}_i\}$, clearly $X^s = X^t$ for all $s \geq t$. Hence, suppose that there exists $x^{t-1} \in X^{t-1}$, $x_i^{t-1} \neq \underline{x}_i$, and that $p_i^t(x_i^{t-1}) \leq u_i(x_i^{t-1}) - \bar{\pi}_i^{t-1}$. Consider that bidder i reports X_i at $s \geq t$ and that $\hat{u}_i(x_i) = \max\{u_i(x_i) - d, 0\}$. Note that every other bidder has already stopped, so that $\hat{u}_{-i}$ is determined. By Day and Milgrom (2008) and Sano (2011b), $\hat{u}_i = u_i - \bar{\pi}_i$ is among best responses given $\hat{u}_{-i}$ and that any $\tilde{u}_i < \hat{u}_i$ is not. Hence, $d \leq \bar{\pi}_i^{t-1}$ in every TPE. Then,

$$\max_{x \in X} \big[\max\{u_i(x_i) - d, 0\} + \sum_{j \neq i} \hat{u}_j(x_j)\big] = V^{t-1}(N) - d. \tag{19}$$

Equality holds since $V^{t-1}(N) - d \geq V^{t-1}(N_{-i})$. On the other hand,

$$u_i(x_i^{t-1}) - d + \sum_{j \neq i} \hat{u}_j(x_j^{t-1}) = V^{t-1}(N) - d. \tag{20}$$

Therefore, $x^{t-1} \in X^s$.

It is trivial in the case that $p_i^t(x_i^{t-1}) > u_i(x_i^{t-1}) - \bar{\pi}_i^{t-1}$.

Step 2. Suppose $m \geq 2$ and the proposition is true for $\forall m' \leq m - 1$. Let i be the bidder making the decision at t. Hence, $u_{-i}^t = u_{-i}^{t-1}$.

Step 2.1. Suppose $X_i^{t-1} = \{\underline{x}_i\}$. Then,

$$\max\{u_i(\tilde{x}_i) - d, 0\} + \sum_{j \neq i} u_j^t(\tilde{x}_j) < \max \sum_{j \neq i} u_j^t(x_j) = V^t(N_{-i}) \tag{21}$$

for all $d \geq 0$ and for all $\tilde{x} \in \{x \in X | x_i \neq \underline{x}_i\}$. Hence, as long as i follows a semi-truthful strategy, $X_i^t = \{\underline{x}_i\}$ and $X^t = X^{t-1}$.

Step 2.2. Suppose that there exists a non-null bundle $x_i^{t-1} \in X_i^{t-1}$. Suppose that bidder i stops at t: $\hat{D}_i(p^t) = X_i$. Let $\hat{u}_i$ be the reported valuation function and $\hat{u}_i = \max\{u_i - d, 0\}$.

If $d > \bar{\pi}_i^{t-1}$, then for any $\tilde{x} \in \{x \in X | x_i \neq \underline{x}_i\}$,

$$\max\{u_i(\tilde{x}_i) - d, 0\} + \sum_{j \neq i} u_j^t(\tilde{x}_j) \leq V^{t-1}(N) - d < V^{t-1}(N_{-i}) = V^t(N_{-i}). \tag{22}$$

Hence, $X_i^t = \{\underline{x}_i\}$. Induction hypothesis and Assumption 3 imply that in any TPE, i must obtain $\underline{x}_i$.

Now suppose $d \leq \bar{\pi}_i^{t-1}$. Then,

$$\max_{x \in X} \left[\max\{u_i(x_i) - d, 0\} + \sum_{j \neq i} u_j^t(x_j)\right] = V^{t-1}(N) - d. \tag{23}$$

On the other hand,

$$u_i(x_i^{t-1}) - d + \sum_{j \neq i} u_j^t(x_j^{t-1}) = V^{t-1}(N) - d. \tag{24}$$

Therefore, $x^{t-1} \in X^t$. By induction hypothesis, $x^{t-1} \in X^*(\hat{u})$.

Hence, it is not optimal to report X_i in the case of $d > \bar{\pi}_i^{t-1} > 0$. In addition, by Assumption 4, bidder i does not report X_i in the case of $d > \bar{\pi}_i^{t-1} = 0$ either. Therefore, $x^{t-1} \in X^s$ for all $s \geq t$. ■

B.3 Proof of Theorem 1

We prove by induction. Suppose there are m active bidders at t.

Step 1. Suppose $m = 1$. Suppose that bidder i is active at t. Note that every other bidder has stopped, and that $\hat{u}_{-i}$ is determined. By Day and Milgrom (2008) and Sano (2011b), $\hat{u}_i = u_i - \bar{\pi}_i$ is among best responses given $\hat{u}_{-i}$. Hence, σ_i^* obviously constitutes an equilibrium.

Step 2. Suppose that there are $m \geq 2$ active bidders at t. Further, suppose that every active bidder follows the Vickrey-target strategy σ_i^* after $m' \leq m - 1$ bidders remain active, and that this constitutes an equilibrium for $m - 1$ bidders.

Consider $\bar{\pi}_i^s = V^s(N) - V^s(N_{-i})$. Suppose that all the remaining bidders except i follow the Vickrey-target strategy. Then, $V^s(N)$ decreases at s by $\bar{\pi}_j^{s-1}$ if and only if someone j stops bidding. In addition, $V^s(N_{-i})$ decreases at s by at most $\bar{\pi}_j^{s-1}$. Hence, $\bar{\pi}_i^s$ is nonincreasing in s as long as every other bidder follows the Vickrey-target strategy.

Suppose that $X_i^{t-1} = \{\underline{x}_i\}$. No bidder overstates the values when he follows the Vickrey-target strategy. Hence, by the proof of Proposition 2, $x_i = \underline{x}_i$ for $\forall x \in X^*(\hat{u})$

for any strategy such that $\hat{u}_i \leq u_i$. Suppose that bidder i continues bidding until $p_i^t > u_i$ and that for some t, $\tilde{x}_i$ ($\neq \underline{x}_i$) $\in X_i^t$. Let t be the minimum of such t. Then, $\{\underline{x}_i, \tilde{x}_i\} \subseteq X_i^t$, since bid increment is unity. Hence, $V^t(N) = V^t(N_{-i})$. Then, bidder i has to pay at least $\bar{p}_i^V(\hat{u})$ for $\tilde{x}_i$ and his payoff is at most

$$\begin{aligned} u_i(\tilde{x}_i) - \bar{p}_i^V(\hat{u}) &= u_i(\tilde{x}_i) - \hat{u}_i(\tilde{x}_i) + \hat{V}(N) - \hat{V}(N_{-i}) \\ &\leq u_i(\tilde{x}_i) - p_i^t(\tilde{x}_i) + V^t(N) - V^t(N_{-i}) \\ &< 0. \end{aligned} \tag{25}$$

Weak inequality comes from the fact that $\bar{\pi}_i^\tau$ does not increase in τ in which bidders except i make decisions. Hence, it is never optimal to bid over the true valuations, and thus, σ_i^* is an optimal strategy.

Suppose that $\exists x^{t-1} \in X^{t-1}$ and $x_i^{t-1} \neq \emptyset$. Suppose that bidder i stops bidding at t. If $d \equiv u_i(x_i^{t-1}) - \hat{u}_i(x_i^{t-1}) > \bar{\pi}_i^{t-1}$, then $X_i^t = \{\underline{x}_i\}$ by the consideration in Proposition 2. By induction hypothesis and Assumption 3, i obtains $\underline{x}_i$ in the end and $\pi_i = 0$. On the other hand, if $d \leq \bar{\pi}_i^{t-1}$, then $x^{t-1} \in X^t$.

By the Vickrey-reserve pricing, bidder i's payoff π_i is

$$\begin{aligned} \pi_i = u_i(x_i) - p_i &\leq u_i(x_i) - \bar{p}_i^V(\hat{u}) \\ &= u_i(x_i) - \hat{u}_i(x_i) + \hat{V}(N) - \hat{V}(N_{-i}) \\ &\leq V^{t-1}(N) - V^{t-1}(N_{-i}) = \bar{\pi}_i^{t-1}. \end{aligned} \tag{26}$$

The second inequality comes from the weak monotonicity of $\bar{\pi}_i^\tau$ in τ. If $d = \bar{\pi}_i^{t-1}$, then $x^{t-1} \in X^*(\hat{u})$ and bidder i wins a non-null bundle x_i^{t-1}.[22] Bidder i earns at least $\bar{\pi}_i^{t-1}$ by the individual rationality. Since $\bar{\pi}_i^s$ is nonincreasing in s, it is suboptimal to stay active whenever m bidders are active. Hence, the Vickrey-target strategy is among best responses for i. ■

B.4 Proof of Theorem 2

Let $x^* \in X^*(\hat{u})$ be the resulting allocation. By construction of σ^*, $\hat{\pi}_i = 0$ for all $i \in I$. Hence, for all $J \subseteq N$,

$$\hat{V}(J) \leq \hat{V}(N) = \pi_0^*. \tag{27}$$

[22] If there is another allocation $\tilde{x} \in X^{t-1}$, where i obtains $\underline{x}_i$, and if it is selected by the tie-breaking rule, then $\bar{\pi}_i^{t-1} = 0$. It is still optimal to follow σ_i^* by the same consideration as the case of $X_i^{t-1} = \{\underline{x}_i\}$.

Note that each bidder's true payoff $\pi_i^* = u_i(x_i^*) - \hat{u}_i(x_i^*)$. Therefore, for any coalition J including the seller,

$$\begin{aligned}\sum_{j\in J}\pi_j^* &\geq \hat{V}(J) + \sum_{j\in J_{-0}}\pi_j^* \\ &= \max_{x\in X}\Big[\sum_{j\in J_{-0}}\hat{u}_j(x_j)\Big] + \sum_{j\in J_{-0}}\pi_j^* \\ &\geq \max_{x\in X}\Big[\sum_{j\in J_{-0}}(u_j(x_j)-\pi_j^*)\Big] + \sum_{j\in J_{-0}}\pi_j^* = V(J)\end{aligned} \quad . \qquad (28)$$

The first inequality is from (27). The second inequality comes from $\hat{u}_i(x_i) = \max\{u_i(x_i) - \pi_i^*, 0\}$.[23] By Theorem 1, any winner i is blocked (will obtaion nothing) if he stops one-period earlier. This implies that if a bidder's payment is decreased by unity, then the seller chooses a different revenue-maximizing allocation. Hence, π^* is bidder-optimal. ■

B.5 Proof of Theorem 4

Suppose for contradiction there is a group of bidders $G \not\subseteq G^*$ and $\sigma_G \in \times_{i\in G}\Sigma_i^{*+}$, and $\pi_i^\sigma > \pi_i^*$, where $\sigma = (\sigma_G, \sigma^*_{I\setminus G})$ and π^σ is the corresponding payoff allocation. Since $\pi_i^\sigma > \pi_i^* \geq 0$, each $i \in G$ is a winner under σ. Let $\hat{G} \supseteq G$ be the set of winners under σ. Let $\hat{x}$ be the corresponding goods allocation associated with σ.

Since each $i \in I \setminus G$ follow σ_i^*, every decision node for them satisfies $X^{t-1} \subseteq X^t$. Hence,

$$\sum_{i\in G^*\setminus G}\hat{u}_i^\sigma(x_i^*) + \sum_{i\in G\cap G^*}u_i(x_i^*) \geq V(\hat{G}+0,u) \geq \sum_{\hat{G}}u_i(\hat{x}_i), \qquad (29)$$

where $\hat{u}_i^\sigma$ denotes the reported valuation function for i under σ. Let $d_i \equiv u_i(x_i) - \hat{u}_i^\sigma(\hat{x}_i)$. Then, by assumption $\hat{u}_i^\sigma(\hat{x}_i) = u_i(\hat{x}_i) - d_i > 0$ for $i \in G$. Since $\sigma_i \in \Sigma_i^{*+}$, $\hat{u}_i(x_i^*) = \max\{u_i(x_i^*) - d_i, 0\}$. Hence,

$$u_i(x_i^*) - \hat{u}_i^\sigma(x_i^*) \leq d_i = u_i(\hat{x}_i) - \hat{u}_i^\sigma(\hat{x}_i) \qquad (30)$$

[23] If $\bar{p}_i^V(u) = 0$ for some i, then i stops at the initial period and $\hat{u}_i(x_i) = 0 \leq \max\{u_i(x_i) - \pi_i^*, 0\}$ for all x_i.

for all $i \in G$. Therefore,

$$\begin{aligned}
\hat{V}^\sigma(G^*+0) &\geq \sum_{i\in G^*} \hat{u}_i^\sigma(x_i^*) \\
&= \sum_{i\in G^*\setminus G} \hat{u}_i^\sigma(x_i^*) + \sum_{i\in G\cap G^*} u_i(x_i^*) - \sum_{i\in G\cap G^*} \big(u_i(x_i^*) - \hat{u}_i^\sigma(x_i^*)\big) \\
&\geq \sum_{i\in \hat{G}} u_i(\hat{x}_i) - \sum_{i\in G\cap G^*} \big(u_i(\hat{x}_i) - \hat{u}_i^\sigma(\hat{x}_i)\big).
\end{aligned} \tag{31}$$

Feasibility requires

$$\pi_0^\sigma + \sum_{i\in\hat{G}} \pi_i^\sigma \leq \sum_{i\in\hat{G}} u_i(\hat{x}_i). \tag{32}$$

Hence,

$$\begin{aligned}
\hat{V}^\sigma(G^*+0) &\geq \sum_{i\in \hat{G}} u_i(\hat{x}_i) - \sum_{i\in G\cap G^*} \big(u_i(\hat{x}_i) - \hat{u}_i^\sigma(\hat{x}_i)\big) \\
&\geq \pi_0^\sigma + \sum_{i\in\hat{G}\setminus(G\cap G^*)} \pi_i^\sigma + \sum_{i\in G\cap G^*} \big(\hat{u}_i^\sigma(\hat{x}_i) - p_i(\sigma)\big) \\
&> \pi_0^\sigma + \sum_{i\in(\hat{G}\cap G^*)\setminus G} \pi_i^\sigma + \sum_{i\in G\cap G^*} \hat{\pi}_i^\sigma \\
&\geq \pi_0^\sigma + \sum_{i\in\hat{G}\cap G^*} \hat{\pi}_i^\sigma \\
&= \pi_0^\sigma + \sum_{i\in G^*} \hat{\pi}_i^\sigma,
\end{aligned} \tag{33}$$

where $p(\sigma)$ and $\hat{\pi}^\sigma$ denote the payments and the reported payoffs under σ. Strict inequality comes from $\pi_i^\sigma > 0$ for $i \in G \setminus G^*$ and $\pi_i^\sigma \geq 0$ for $i \in \hat{G} \setminus (G \cup G^*)$, for $i \in \hat{G} \setminus (G \cup G^*)$ follows σ_i^* and earns a nonnegative payoff. The fourth inequality comes from $\pi_i^\sigma \geq \hat{\pi}_i^\sigma$ for $i \in (\hat{G} \cap G^*) \setminus G$, for $i \in (\hat{G} \cap G^*) \setminus G$ follows σ_i^* and $\hat{u}_i^\sigma(\cdot) \leq u_i(\cdot)$. This contradicts to the core-selecting property. ■

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