Non-cooperative and Axiomatic Characterizations of the Average Lexicographic Value

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August 13, 2010

Abstract

We give a non-cooperative and an axiomatic characterization of the Average Lexicographic value (AL-value) on the class of balanced games. The AL-value is a single-valued solution for balanced TU-games. It is defined as the average of lexicographic maximum of the core of the game with respect to all orders on the player set, and it can be seen as a core selection based on the priority orders on the

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players. In both of our characterizations of the AL-value, a consistency property which we call average consistency plays an important role, and the property is obtained by the consistency property à la Davis and Maschler of the lexicographic vectors with respect to any order of players.

JEL classification: C71, C72

Keywords: cooperative games, average lexicographic value, average consistency, characterization

1 Introduction

The core (Gillies 1953) is a central solution concept in cooperative game theory. If an allocation belongs to the core, no coalition has an incentive to deviate from it; thus, it is coalitionally stable. It goes without saying that the stability is very important in allocation problems; however if there are multiple core allocations, the question "how to select one of them?" arises. One of the well-known core selection is the nucleolus (Schmeidler 1969), which is a lexicographic center of the core (see Maschler et al. 1979). Recently, González-Díaz and Sánchez-Rodríguez (2007) have introduced the core-center, which is a mathematical expectation of the uniform distribution defined over the core.

In some economic situations where there are multiple agents and they negotiate how to divide the outcome obtained by their cooperation, the agents may fix a priority order among them and, following the priority, agents request the best outcome among the remaining ones. For example, the run-to-the-bank rule in bankruptcy problems (O'Neill 1982) and the serial dictatorship rule in house allocation problems (Hylland and Zeckhauser 1979 and Svensson 1994). The Average Lexicographic value (henceforth, AL-value, Tijs 2005 and Branzei et al. 2008) is a one-point solution for cooperative game theory which reflects the use of priority orders. By definition, the AL-value is the weighted average of some vertices of the core (more precisely, the weighted average of vertices obtained as leximals with respect to some orders for the set of players), and it is different from the above mentioned two core selections.[1]

In this paper, we characterize the AL-value in both non-cooperative and axiomatic approaches. In both characterizations a consistency property which we call average consistency property of the AL-value plays a key role. It is obtained by the fact that leximals satisfy consistency à la Davis and Maschler (1965). Our results are closely related to existing non-cooperative and axiomatic characterizations of the well-known one-point solution the Shapley value (Shapley 1953). Similarities and differences between the AL-value and the Shapley value are well-captured by them.

In the non-cooperative approach, given a cooperative game, we construct a non-cooperative game in which the AL-value is obtained as the unique

[1]An example illustrating that the AL-value, the nucleolus, and the core-center are different can be obtained from the authors upon request.

subgame perfect equilibrium payoff. Our non-cooperative game is inspired by the bidding mechanism of Pérez-Castrillo and Wettstein (2001), however we focus more on the core property of the outcome. The differences between our non-cooperative game and the bidding mechanism lie in the game played by the remaining players when a proposer's offer is rejected. In the bidding mechanism the game is based on the restricted game defined on the set of the remaining players, whereas in our non-cooperative game, the game is defined based on the Davis and Maschler's reduced game with respect to the proposer's most desirable core allocation. By assuming the existence of the core and thus the formation of the grand coalition, our approach corresponds to remaining players' behaviors based on their pessimistic outlook for the division after a player obtains an advantageous position, a proposer. Our non-cooperative game can be seen as a bargaining based on core allocations among players and it works well in "verifiable" environment (see Ju and Wettstein 2009). To the best of our knowledge, the non-cooperative foundations of the other core selections such as the nucleolus and the core-center are limited in the literature. For the nucleolus, Serrano (1993) mentions a non-cooperative implementation of it for 3-person super-additive games and Montero (2006) provides a non-cooperative interpretation of it for proper simple games. There are no non-cooperative foundations of the core-center.

In the axiomatic approach, we give some variations of the balanced contributions property of Myerson (1980). As mentioned in Kamijo and Kongo (2009), Myerson's balanced contributions property is rather a restrictive one.

Hence, first we consider a weaker version of Myerson's balanced contributions property. We call the new property the balanced *average* contributions property and we show that this new property also characterizes the Shapley value as Myerson (1980) did. Then, we consider two new properties which are based on the special reduced game of Davis and Maschler (1965): the balanced DM-contributions property and the balanced average DM-contributions property. Unlike the case of the Shapley value, only the latter property characterizes the AL-value and there is no efficient value that satisfies the former property.

The outline of this paper is as follows. Section 2 gives preliminaries. Section 3 provides a consistency property of leximals and an average consistency property of the AL-value. Section 4 gives a set of non-cooperative games, in which the AL-value is obtained in any subgame perfect equilibrium. Section 5 presents an axiomatic characterization of the AL-value. Section 6 concludes the paper.

2 Preliminaries

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of players with $n = |N|$, and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on 2^N such that $v(\emptyset) = 0$. For any coalition $S \subset N$, $v(S)$ is called the *worth* of coalition S. This is what the members of coalition S can obtain by agreeing to cooperate. We denote the class of all TU-games by $\mathcal{G}$. Then the set of games with player set N is denoted by $\mathcal{G}^N$. We also denote $I^*(N,v) = \{x \in \mathbb{R}^N | x(N) = v(N)\}$, where $x(N) = \sum_{i \in N} x_i$, as the set of *pre-imputations* of (N, v).

The very basic solution concept of this paper is the *core* given (cf. Gillies 1953) by

$$C(N,v) = \{x \in \mathbb{R}^N | x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\},$$

for each $(N,v) \in \mathcal{G}^N$, where we denote $x(S) = \sum_{i \in S} x_i$. A game with a non-empty core is called a *balanced game* and the set of all balanced games is denoted by $\mathcal{G}_C$. Then, we denote the set of balanced games with player set N by $\mathcal{G}_C^N$.

Let $(N,v) \in \mathcal{G}_C^N$ and let $\Pi(N)$ be the set of all orders on N, that is, one to one onto mappings from $\{1,2,...,n\}$ to N. We also denote $\sigma = \{\sigma(1), \sigma(2), ..., \sigma(n)\}$. For each $\sigma \in \Pi(N)$, the lexicographic vector $L^\sigma(N,v)$ is inductively defined by, for $i \in N$,

$$L^\sigma_{\sigma(i)}(N,v) = \max\{x_{\sigma(i)} | x \in C(N,v), L^\sigma_{\sigma(j)}(N,v) = x_{\sigma(j)} \text{ for each } j \in N \text{ with } j < i\}.$$

We note that $L^\sigma(N,v)$, $\sigma \in \Pi(N)$, is an extreme point of the core $C(N,v)$: each $L^\sigma(N,v)$ is the lexicographic maximum of the core of (N,v) with respect to the ordering σ. In the sequel, we refer to each $L^\sigma(N,v)$, $\sigma \in \Pi(N)$, as the *leximal* of v with respect to σ, whereas we refer to L^σ as the leximal (operator) with respect to σ.

The *AL-value*, defined by Tijs (2005) (see also Branzei et al. 2008) as the average of the leximals, is a solution concept on the domain $\mathcal{G}_C^N$, which is uniquely determined by the core. The AL-value is the function $AL : \mathcal{G}_C^N \to \mathbb{R}^N$ defined by

$$AL(N,v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(N,v)$$

for each $(N,v) \in \mathcal{G}_C^N$.

On special classes of balanced games the AL-value coincides with specific solutions on those classes (for details, see Branzei et al. 2008, Lohmann 2006, and Lohmann et al. 2007). In this paper we refer to its relations with the Shapley value.

The coincidence of the Shapley value and the AL-value on some classes of balanced games does not seem unexpected because both values use the principle of averaging entities based on orderings of players. The Shapley value is the average of the marginals $m^\sigma : \mathcal{G}^N \to \mathbb{R}^N$, $\sigma \in \Pi(N)$, defined by

$$m^\sigma_{\sigma(i)}(N,v) = v(\{\sigma(j)|j \in N, j \leq i\}) - v(\{\sigma(j)|j \in N, j < i\}) \quad \text{for each } i \in N.$$

Then the Shapley value $\phi(N,v)$ of $(N,v) \in \mathcal{G}$ is given (cf. Shapley 1953) by

$$\phi(N,v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N,v).$$

Some basic differences between the marginal worth vectors of a balanced game and its leximals on the one hand, and between the Shapley value of such game and its AL-value on the other hand deserve to be mentioned here. Whereas a marginal worth vector of a balanced game might not be a core allocation of the game, each leximal is an extreme point of the core. Furthermore, whereas the Shapley value of a balanced game might not belong to the core of the game (even it might not be an imputation), the AL-value is a selection of the core. Moreover, the property of invariance with respect to exactification (proved for the AL-value in Branzei et al. 2008) and the average consistency property with respect to the core (which we prove for the AL-value in Section 3) are not satisfied by the Shapley value on the class of balanced games. We notice that for real life applications balanced games and their cores are of particular interest, because if the core of the game is the empty set the grand coalition might not form. Finally, we point out that the marginals and the Shapley value are defined for arbitrary cooperative games, whereas the leximals and the AL-value are defined only for balanced games.

The class of convex games is an important subclass of balanced games on which the Shapley value and the AL-value coincide (see Branzei et al. 2008). Recall that a game $(N, v) \in \mathcal{G}$ is *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subset N.$$

3 Average consistency of the AL-value

Our goal in this section is to prove an average consistency property of the AL-value. To do this, we first need to study a consistency property of the leximals operators. We use notations $N \setminus j$ instead of $N \setminus \{j\}$ hereafter.

Consistency with respect to a reduced game is one of the very important properties of solutions of a game, which requires the coincidence of the payoffs in the original game and its reduced game. Peleg (1986) shows that the core satisfies consistency with respect to the reduced game à la Davis and Maschler. Núñez and Rafels (1998) show that each extreme point of the core satisfies the same consistency property.

Since each leximal is one of the extreme points of the core, the payoffs of the leximal of the original game coincide with the payoffs of an extreme point of the core of the reduced game. However, the payoffs of the leximal of the original game might not be equal to the payoffs of the leximal of the reduced game. More precisely, we have to show the coincidence of the payoffs of a leximal with respect to an order σ of the players in the original game with the payoffs of the leximal with respect to an induced order from order σ in the reduced game à la Davis and Maschler.

The *reduced game à la Davis and Maschler*, in short the *DM-reduced game* $(N \setminus j, v^x)$, is for $x \in I^*(N, v)$ and $j \in N$, defined (cf. Davis and Maschler

1965) by

$$v^x(N \setminus j) = v(N) - x_j,$$
$$v^x(S) = \max\{v(S \cup \{j\}) - x_j, v(S)\} \text{ for all } S \subsetneq N \setminus j,$$
$$v^x(\emptyset) = 0.$$

Next, we will show the consistency of the leximals with respect to the reduced game. To prove this, following Caprari et al. (2006), we define a function $\mathcal{L}^\sigma$ with respect to $\sigma \in \Pi(N)$, $\mathcal{L}^\sigma : \mathcal{K} \to \mathbb{R}^N$, by

$$\mathcal{L}^\sigma_{\sigma(i)}(K) = \max\{x_{\sigma(i)} | x \in K, \mathcal{L}^\sigma_{\sigma(j)}(K) = x_{\sigma(j)} \text{ for each } j < i\}$$

for each $K \in \mathcal{K}$, where $\mathcal{K} = \{K | K \subset \mathbb{R}^N, K \text{ is convex and compact}\}$. It holds $L^\sigma(N, v) = \mathcal{L}^\sigma(C(N, v))$.[2]

For any subset $S \subset N$ and any order $\sigma \in \Pi(N)$, take $T \subset \{1, 2, ..., n\}$ such that $\sigma(T) = S$; then, we can define a function σ_S on T, $\sigma_S : T \to S$, by $\sigma_S(i) = \sigma(i)$ for $i \in T$. We also denote the set of such functions by $\tilde{\Pi}(S)$. Then σ_S is not in $\Pi(S)$, but it induces a natural order on S. Let $\mathcal{K}_S = \{K_S | \exists K \in \mathcal{K} \text{ s.t. } K_S = K \cap \mathbb{R}^S\}$. We define $(\mathcal{L}^{\sigma_S})|_{\mathcal{K}_S}$ by

$$(\mathcal{L}^{\sigma_S}|_{\mathcal{K}_S})_{\sigma_S(i)}(K_S) = \max\{x_{\sigma_S(i)} \mid x \in K_S, (\mathcal{L}^{\sigma_S}|_{\mathcal{K}_S})_{\sigma_S(j)}(K_S) = x_{\sigma_S(j)}, \forall j < i \text{ with } j \in \sigma^{-1}(S)\}$$

for $i \in \sigma^{-1}(S)$ and for any compact convex set $K_S \in \mathcal{K}_S$. We also denote

[2]In Caprari et al. (2006), K is a share set, not a general compact convex set.

$\mathcal{L}^{\sigma_S}|_{\mathcal{K}_S}$ by $\mathcal{L}^{\sigma}$, and σ_S by σ if there is no confusion. For leximals, we also use a similar notation, that is, for any $\sigma \in \Pi(N)$, any $S \subset N$, and any game (S, w), we denote $L^{\sigma_S}(S, w) = \mathcal{L}^{\sigma_S}(C(S, w))$ by $L^{\sigma}(S, w)$. Then we obtain the following.

Theorem 1. *For any $\sigma \in \Pi(N)$, the leximal L^{σ} satisfies the DM-consistency, that is, for any $(N, v) \in \mathcal{G}_C$ and $j \in N$, the DM-reduced game $(N \setminus j, v^{L^{\sigma}(N,v)})$ belongs to $\mathcal{G}_C$, and*

$$L_i^{\sigma}(N, v) = L_i^{\sigma}(N \setminus j, v^{L^{\sigma}(N,v)}) \quad \textit{for each } i \in N \setminus j.$$

Proof. Without loss of generality, we assume that $N = \{1, 2, ..., n\}$. Take any $j \in N$, $\sigma \in \Pi(N)$, and let $y = L^{\sigma}(N, v)$. Consider the reduced game $(N \setminus j, v^y)$ for $j \in N$. Let l be such that $\sigma(l) = j$. We distinguish two cases.

First, we consider the case when $i < l$. Let $i = 1$; then

$$\begin{aligned} y_{\sigma(1)} &= \mathcal{L}^{\sigma}_{\sigma(1)}(C(N, v)) \\ &= \max\{x_{\sigma(1)} \in \mathbb{R} | x \in C(N, v)\} \\ &= \max\{x_{\sigma(1)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N\} \\ &= \max\{x_{\sigma(1)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(l)} = y_{\sigma(l)}\}, \end{aligned}$$

where the last equality holds because y belongs to $C(N, v)$.

Now,

$$\{x \in \mathbb{R}^{N \setminus j} | x(N) = v(N), x(S) \geq v(S),\ \forall S \subset N, x_j = y_j\}$$

$$
\begin{aligned}
&= \{x \in \mathbb{R}^{N\setminus j} | x(N \setminus j) + y_j = v(N), x(S) \geq v(S)\ \forall S \subseteq N \setminus j, \\
&\qquad\qquad x(S \setminus j) + y_j \geq v(S)\ \forall S \subseteq N \text{ with } S \ni j\} \\
&= \{x \in \mathbb{R}^{N\setminus j} | x(N \setminus j) = v(N) - y_j, x(S) \geq v(S)\ \forall S \subseteq N \setminus j, x(S) \geq v(S \cup \{j\}) - y_j \\
&\qquad\qquad \forall S \subseteq N \setminus j\} \\
&= \{x \in \mathbb{R}^{N\setminus j} | x(N \setminus j) = v^y(N \setminus j), x(S) \geq \max\{v(S), v(S \cup \{j\}) - y_j\}\ \forall S \subseteq N \setminus j\} \\
&= \{x \in \mathbb{R}^{N\setminus j} | x(N \setminus j) = v^y(N \setminus j), x(S) \geq v^y(S),\ \forall S \subseteq N \setminus j\} = C(N \setminus j, v^y).
\end{aligned}
$$

So,

$$\{x \in \mathbb{R}^{N\setminus j} | x(N) = v(N), x(S) \geq v(S),\ \forall S \subset N, x_j = y_j\} = C(N \setminus j, v^y). \quad (1)$$

Then, we have

$$y_{\sigma(1)} = \max\{x_{\sigma(1)} \in \mathbb{R} | x \in C(N \setminus j, v^y)\} = \mathcal{L}^{\sigma}_{\sigma(1)}(C(N \setminus j, v^y)) = L^{\sigma}_{\sigma(1)}(N \setminus j, v^y).$$

Next consider $i = 2$. Based on $y_{\sigma(1)} = \mathcal{L}^{\sigma}_{\sigma(1)}(C(N \setminus j, v^y))$, we have

$$
\begin{aligned}
y_{\sigma(2)} &= \mathcal{L}^{\sigma}_{\sigma(2)}(C(N, v)) \\
&= \max\{x_{\sigma(2)} \in \mathbb{R} | x \in C(N, v), x_{\sigma(1)} = y_{\sigma(1)}\} \\
&= \max\{x_{\sigma(2)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(1)} = y_{\sigma(1)}\} \\
&= \max\{x_{\sigma(2)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(l)} = y_{\sigma(l)}, x_{\sigma(1)} = y_{\sigma(1)}\}.
\end{aligned}
$$

Further, by (1) we obtain,

$$\{x \in \mathbb{R}^{N\setminus j} | x(N) = v(N), x(S) \geq v(S),\ \forall S \subset N, x_j = y_j, x_{\sigma(1)} = y_{\sigma(1)}\}$$
$$= \{x \in \mathbb{R}^{N\setminus j} | x \in C(N \setminus j, v^y), x_{\sigma(1)} = y_{\sigma(1)}\}.$$

This implies that

$$\begin{aligned} y_{\sigma(2)} &= \max\{x_{\sigma(2)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(l)} = y_{\sigma(l)}, x_{\sigma(1)} = y_{\sigma(1)}\} \\ &= \max\{x_{\sigma(2)} \in \mathbb{R} | x \in C(N \setminus j, v^y), x_{\sigma(1)} = y_{\sigma(1)}\} \\ &= \mathcal{L}^{\sigma}_{\sigma(2)}(C(N \setminus j, v^y)) = L^{\sigma}_{\sigma(2)}(N \setminus j, v^y). \end{aligned}$$

Based on $y_{\sigma(1)} = \mathcal{L}^{\sigma}_{\sigma(1)}(C(N \setminus j, v^y))$, $y_{\sigma(2)} = \mathcal{L}^{\sigma}_{\sigma(2)}(C(N \setminus j, v^y))$,..., $y_{\sigma(i-1)} = \mathcal{L}^{\sigma}_{\sigma(i-1)}(C(N \setminus j, v^y))$, by (1) we obtain $y_{\sigma(i)} = L^{\sigma}_{\sigma(i)}(N \setminus j, v^y)$ for the case $i < l$.

Second, consider the case when $i > l$. Based on the fact that $y_{\sigma(s)} = \mathcal{L}^{\sigma}_{\sigma(s)}(C(N \setminus j, v^y))$ for $s < i$, by (1) we obtain

$$\begin{aligned} y_{\sigma(i)} &= \mathcal{L}^{\sigma}_{\sigma(i)}(C(N, v)) \\ &= \max\{x_{\sigma(i)} \in \mathbb{R} | x \in C(N, v), x_{\sigma(t)} = y_{\sigma(t)} \text{ for } t < i\} \\ &= \max\{x_{\sigma(i)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(t)} = y_{\sigma(t)} \text{ for } t < i\} \\ &= \max\{x_{\sigma(i)} \in \mathbb{R} | x \in C(N \setminus j, v^y), x_{\sigma(t)} = y_{\sigma(t)} \text{ for } t < i\} \\ &= \mathcal{L}^{\sigma}_{\sigma(i)}(C(N \setminus j, v^y)) = L^{\sigma}_{\sigma(i)}(N \setminus j, v^y). \end{aligned}$$

□

Next, we cope with a consistency property of the AL-value which we call the average consistency of the AL-value with respect to the core. We consider a special type of reduced games. Let $k \in N$ and let $\sigma^k \in \Pi(N)$ be an order which satisfies $\sigma^k(1) = k$. We denote the set of such orders by $\Pi^k(N)$. We also denote $z^k = \max\{x_k \in \mathbb{R} | x \in C(N,v)\} = L^{\sigma^k}_{\sigma^k(1)}(N,v)$.

We can consider the reduced game à la Davis and Maschler with respect to z^k, $(N \setminus k, v^{-k})$, given by

$$v^{-k}(N \setminus k) = v(N) - z^k,$$
$$v^{-k}(S) = \max\{v(S \cup \{k\}) - z^k, v(S)\} \text{ for all } S \subsetneq N \setminus k,$$
$$v^{-k}(\emptyset) = 0.$$

We call the above game the *lexicographically DM reduced game* with respect to $k \in N$. This game reflects remaining players' pessimistic outlook. Now, given a balanced game (N,v), we assume the grand coalition N is formed and consider the division of the worth $v(N)$ among players in N, based on the characteristic function v. Then, players are not able to obtain any imputation that is not an element of the core of the game (N,v). This is because, for such an imputation, there exists a coalition that is willing to deviate from the grand coalition N, in the hope of getting more among them, and thus it conflicts the assumption that N is formed. In this sense, we consider the division of $v(N)$ as a selection among all core allocations of (N,v). Further, assume that one of the players, k, obtains an advantageous position among all players.[3] A coalition S of remaining players $N \setminus k$ reconsiders the situation in the following manner: First, S can obtain the worth $v(S)$ by only themselves. In addition, they are able to win the cooperation with k if they offer k's maximal payoff among core allocation, z^k, to k. Therefore, the worth of S is at least $\max\{v(S \cup \{k\}) - z^k, v(S)\}$, as defined above.

Theorem 1 implies that for any $k \in N$,

$$L_i^{\sigma^k}(N, v) = L_i^{\sigma^k}(N \setminus k, v^{-k}) \text{ for any } i \in N \setminus k.$$

Definition 1 (Average Consistency)**.** *Let* $(N, v) \in \mathcal{G}_C^N$*. For any* $k \in N$*, let* $z^k = \max\{x_k \in \mathbb{R} | x \in C(N, v)\}$*. Then, a value* $\varphi : \mathcal{G}_C^N \to \mathbb{R}^n$ *satisfies* average consistency *with respect to the core if and only if for any* $i \in N$*,*

$$\varphi_i(N, v) = \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \varphi_i(N \setminus k, v^{-k}).$$

Theorem 1 easily implies the following theorem. Caprari et al. (2008) also consider the average consistency of the AL-value for share sets, but not for TU games. Though TU games are related to share sets, the relationship between the reduced games à la Davis and Maschler and the corresponding share sets is not clear. Thus the next theorem could not be obtained directly from their result.

[3] Now we do not mention this point in detail. We give a clear interpretation of it as "being chosen as an proposer of a division to the other players" in the non-cooperative characterization of the AL value, presented in the next section.

Theorem 2. *The AL-value satisfies the average consistency property with respect to the core, that is for any $i \in N$,*

$$AL_i(N, v) = \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} AL_i(N \setminus k, v^{-k}).$$

Proof. For any $k \in N$ and $i \in N \setminus k$,

$$\begin{aligned}
AL_i(N, v) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L_i^{\sigma}(N, v) \\
&= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \frac{1}{(n-1)!} \sum_{\sigma^k \in \Pi^k(N)} L_i^{\sigma^k}(N, v) \\
&= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \frac{1}{(n-1)!} \sum_{(\sigma^k)_{N \setminus k} \in \Pi(N \setminus k)} L_i^{(\sigma^k)_{N \setminus k}}(N \setminus k, v^{-k}) \\
&= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \frac{1}{(n-1)!} \sum_{\sigma \in \Pi(N \setminus k)} L_i^{\sigma}(N \setminus k, v^{-k}) \\
&= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} AL_i(N \setminus k, v^{-k}).
\end{aligned}$$

□

Corollary 1. *Let $(N, v) \in \mathcal{G}^N$ and $i \in N$. Take any $\tau^i \in \Pi^i(N)$ and $\sigma^j \in \Pi^j(N)$ for all $j \neq i$. Then*

$$AL_i(N, v) = \frac{1}{n} \sum_{j \neq i} \left(\frac{1}{n-1} L_i^{\tau^i}(N \setminus j, v^{L^{\tau^i}(N,v)}) + AL_i(N \setminus j, v^{L^{\sigma^j}(N,v)}) \right).$$

Proof. By the consistency property of the leximals, we have

$$z^i = L_{\tau^i(1)}^{\tau^i}(N, v) = L_i^{\tau^i}(N, v) = L_i^{\tau^i}(N \setminus j, v^{L^{\tau^i}(N,v)}) \quad \text{for each } j \in N \setminus i.$$

Then, we have

$$z^i = \frac{1}{n-1}\sum_{j\neq i} z^i = \frac{1}{n-1}\sum_{j\neq i} L_i^{\tau^i}(N\setminus j, v^{L^{\tau^i}(N,v)}).$$

We plug this into Theorem 2. Note that $v^{-j} = v^{L^{\sigma^j}(N,v)}$. □

4 Non-cooperative characterization of the AL-value

In this section, we develop a non-cooperative characterization of the AL-value.

Given a TU-game $(N, v) \in \mathcal{G}_C$, the non-cooperative game $\Gamma(N, v)$ is defined in the following recursive manner.

In the case $|N| = 1$, player $i \in N$ obtains $v(\{i\})$ and the game is over.

Assume that the non-cooperative game is known when there are less than n players. We consider the non-cooperative game where there are n players.

t=1 Each player $i \in N$ makes *bids* $b^i_j \in \mathbb{R}$ for every player $j \neq i$.

For each $i \in N$, the *net bid* B^i is the sum of the bids he made minus the sum of the bids the others made to him, that is, $B^i = \sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i$. Let $\alpha = argmax_i B^i$ be the player chosen as the proposer, where in the case of multiple maximizers one of them is randomly chosen. The chosen player α pays b^α_j to every player $j \neq \alpha$.

t=2 Player α makes an offer $x_j \in \mathbb{R}$ to every player $j \in N \backslash \alpha$.

t=3 Players in $N \backslash \alpha$ respond to the proposer's offer in a sequential manner, say $(j_1, \ldots, j_{n-1})$. An ordering of the players makes no matter. Response is either "accept it" or "reject it".

In the case player j_h accepts the offer, the next player j_{h+1} responds to it. If every j_h accepts the offer, the players come to an agreement. If there is some rejection, an agreement is not reached.

When an agreement is reached, proposer α pays the proposed payoff x_j for any $j \in N \backslash \alpha$ in return for obtaining the value of their total cooperation, $v(N)$. Thus, the final payoff distribution for responder j is $b_j^\alpha + x_j$ and the payoff for proposer α is $-\sum_{j \neq \alpha} b_j^\alpha + v(N) - \sum_{j \neq \alpha} x_j$, and the game is over.

On the other hand, when an agreement is not reached, the proposer is weakly split off by the other players. He leaves the game with obtaining $z^\alpha = \max\{x_i \in \mathbb{R} | x \in C(N, v)\}$, and the remaining players $N \backslash \alpha$ continue the non-cooperative game $\Gamma(N \backslash \alpha, v^{-\alpha})$ where $(N \backslash \alpha, v^{-\alpha})$ is the lexicographically DM-reduced game with respect to α (defined in Section 3.) As we mentioned before in the previous section, the core satisfies the *DM-reduced game consistency* (see Peleg 1986). Thus, the game $(N \setminus \alpha, v^{-\alpha})$ is balanced and hence, $\Gamma(N \setminus \alpha, v^{-\alpha})$ is well-defined.

This non-cooperative game is inspired by the *bidding mechanism* presented in Pérez-Castrillo and Wettstein (2001). The differences between our non-cooperative game and the bidding mechanism lie in the treatment of the players in the case of rejection at t=3. In our non-cooperative game, the rejected proposer obtains his maximal payoff among core allocations and the remaining players play the non-cooperative game defined on the lexicographically DM-reduced game $(N \setminus \alpha, v^{-\alpha})$. On the other hand, in the bidding mechanism, the rejected proposer obtains the worth of his stand-alone coalition and the remaining players play the non-cooperative game defined on the *restricted game* $(N \setminus \alpha, v|_{N\setminus\alpha})$, where $v|_{N\setminus\alpha}(T) = v(T)$ for any $T \subseteq N \setminus \alpha$.

Note that in our non-cooperative game, the "designer" of the non-cooperative game does *not* have to get information on the value z^i for every $i \in N$, in advance, but has to have a chance to obtain it. For the designer, the information on z^i is needed only when someone rejects the offer from i. In "verifiable" environments (Ju and Wettstein 2009), the players can prove the worth of coalition through an outside authority *if necessary.*[4] Thus, in the verifiable environments the designer can obtain the information on z^i after a rejection. Moreover, in the subgame perfect equilibrium strategies that we will mention in the proof of Theorem 3, players reach an agreement at t=3 and the AL-value of the game is obtained even if the designer has no information on z^i for every $i \in N$.

Pérez-Castrillo and Wettstein (2001)'s bidding mechanism produces the Shapley value payoff for any zero-monotonic game in any subgame perfect equilibrium. That is, for any zero-monotonic game, players share the worth

[4]Such kind of environments are also studied by Serrano (1995) and Dagan et al. (1997).

of grand coalition as equilibrium outcomes. However, if the outcome of the mechanism, the Shapley value of the original zero-monotonic game, is not an element of the core of the game, players may fail to form the grand coalition. By assuming the existence of the core (and that players form the grand coalition), our non-cooperative game can be seen as a bargaining based on core allocations. It produces the AL-value payoff for any balanced game in any subgame perfect equilibrium.

Theorem 3. *The non-cooperative game $\Gamma(N, v)$ produces the AL-value payoff for $(N, v) \in \mathcal{G}_C$ in any subgame perfect equilibrium.*

Proof. The proof proceeds by induction with respect to the number of players. If $|N| = 1$, the AL-value is equal to the value of the stand-alone coalition; hence, the theorem holds. Assume that the theorem holds in the case there are less than n players and consider the case when there are n players.

First, we show that there exists a subgame perfect equilibrium (hereafter, SPE) whose payoff coincides with the AL-value of the game (N, v). Consider the following strategy for each player.

t=1 Each player $i \in N$ announces $b^i_j = AL_j(N, v) - AL_j(N \backslash i, v^{-i})$ for every player $j \neq i$.

t=2 The proposer α offers $x_j = AL_j(N \backslash \alpha, v^{-\alpha})$ for every $j \in N \backslash \alpha$.

t=3 A responder j accepts the offer if $x_j \geq AL_j(N \backslash \alpha, v^{-\alpha})$ and rejects it otherwise.

If all players take the above strategies, an agreement is formed at t=3 and the game is over. It is clear that the above strategy profile yields the AL-value for any player who is not the proposer α since $b_j^\alpha + x_j = AL_j(N, v)$ for any $j \neq \alpha$. The proposer obtains $v(N) - \sum_{j\neq\alpha} b_j^\alpha - \sum_{j\neq\alpha} x_j = v(N) - \sum_{j\neq\alpha} AL_j(N, v) = AL_\alpha(N, v)$. Note that each player obtains his AL-value whether or not the player is the proposer. In other words, given the strategies an outcome is the same regardless of whom is chosen as the proposer.

To check whether the above strategies constitute an SPE, first, we show that the strategies at t=3 are best responses for each of the players. Let j_{n-1} be the last player who has to decide whether to accept or reject the offer. If no other players reject an offer, player j_{n-1}'s best response is to accept the offer if $x_{j_{n-1}} \geq AL_{j_{n-1}}(N\backslash\alpha, v^{-\alpha})$ and reject it otherwise.[5] Knowing the above mentioned reaction of the last player, the second last player j_{n-2}'s best response is to accept the offer if $x_{j_{n-2}} \geq AL_{j_{n-2}}(N\backslash\alpha, v^{-\alpha})$ and reject it otherwise. Using the same argument when going backward, we can show that the strategies mentioned above constitute an SPE of the subgame starting from t=3.

Next, we prove that the strategies at t=2 are best responses for each of them. By the strategies, the proposer incrementally obtains $v(N) - \sum_{j\neq\alpha} AL_j(N\backslash\alpha, v^{-\alpha}) = v(N) - v^{-\alpha}(N\backslash\alpha) = z^\alpha$ in the subgame starting from t=2. If he offers some player j a value $\bar{x}_j$ less than $AL_j(N\backslash\alpha, v^{-\alpha})$, the offer is rejected by the player. In the case of rejection, the proposer obtains

[5] Note that it is not a unique best response.

z^{α}; hence it does not make the proposer strictly better off.

If he offers some player j a value $\hat{x}_j$ larger than $AL_j(N\backslash\alpha, v^{-\alpha})$ without lowering the offer to the other players, the offer is accepted but the share of the proposer is strictly worse off. Thus, the above mentioned strategies constitute an SPE of the subgame starting from t=2.

It remains to show that the strategies at t=1 are best responses for each of the players. Given the strategies, for any $i \in N$,

$$\begin{aligned}
B^i &= \sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i \\
&= \sum_{j\neq i}(AL_j(N,v) - AL_j(N\backslash i, v^{-i})) - \sum_{j\neq i}(AL_i(N,v) - AL_i(N\backslash j, v^{-j})) \\
&= v(N) - AL_i(N,v) - v^{-i}(N\backslash i) - (n-1)AL_i(N,v) + \sum_{j\neq i} AL_i(N\backslash j, v^{-j}) \\
&= -nAL_i(N,v) + z^i + \sum_{j\neq i} AL_i(N \setminus j, v^{-j}) = 0,
\end{aligned}$$

where the last equality holds by Theorem 2. Hence, all players can be chosen to be the proposer with probability $\frac{1}{n}$. As seen before, the outcome is the same regardless of whom is chosen as the proposer. Given the above mentioned strategies, consider the case when player i changes his strategy to $\bar{b}^i_j = b^i_j + a_j$ for each of $j \neq i$. If $\sum_{j\neq i} a_j < 0$, i is not chosen as the proposer; hence, his final payoff is unchanged. If $\sum_{j\neq i} a_j = 0$, i may be chosen to be the proposer; in the case when he is not chosen as the proposer his final payoff is unchanged; in the case when he is chosen as the proposer his final payoff is

$$v(N)-\sum_{j\neq i}\bar{b}_j^i-\sum_{j\neq i}AL_j(N\backslash i,v^{-i})=v(N)-\sum_{j\neq i}b_j^i-\sum_{j\neq i}AL_j(N\backslash i,v^{-i})=AL_i(N,v),$$

which means that his final payoff is also unchanged. If $\sum_{j\neq i}a_j>0$, i must be chosen to be the proposer. However, by the previous result, he obtains

$$v(N)-\sum_{j\neq i}\bar{b}_j^i-\sum_{j\neq i}AL_j(N\backslash i,v^{-i})<v(N)-\sum_{j\neq i}b_j^i-\sum_{j\neq i}AL_j(N\backslash i,v^{-i})=AL_i(N,v).$$

Thus, his share is strictly worse off. Therefore, the above mentioned strategies constitute an SPE.

Next, we prove, by the following series of claims, that in any SPE the AL-value payoff is obtained.

Claim 1: In any subgame starting from t=2, the proposer α incrementally obtains z^α and each of the other players incrementally obtains his AL-value of the game $(N\backslash\alpha,v^{-\alpha})$ in any SPE.

Let α be the proposer. We consider two types of SPEs. One type is of those SPEs in which some player rejects the offer at t=3: here the proposer α obtains z^α and each of other players obtains his AL-value by the induction hypothesis. The other type is of those SPEs in which all players accept the offer. For such SPEs, by the induction hypothesis, each responder $j\neq\alpha$ surely obtains $AL_j(N\setminus\alpha,v^{-\alpha})$ by rejecting the offer. It means that in SPEs in which all players accepts the offer, each responder obtains at least $AL_j(N\setminus$

$\alpha, v^{-\alpha})$ and thus the proposer α obtains at most $v(N) - \sum_{j \in N \setminus \alpha} AL_j(N \setminus \alpha, v^{-\alpha}) = v(N) - v^{-\alpha}(N \setminus \alpha, v^{-\alpha}) = z^{\alpha}$. By the fact that the offer is accepted, the sum of the payoffs of all players is $v(N)$, and thus, we obtain the desired result.

Claim 2: In any SPE, $B^i = \sum_{j \neq i} b^i_j - \sum_{j \neq i} b^j_i = 0$ for any $i \in N$.

Let $\Omega = \{i \in N | i = argmax_j B^j\}$. If $\Omega = N$, the claim holds since $\sum_{i \in N} B^i = 0$. If $\Omega \neq N$, take any $i \in \Omega$ and any $j \in N \setminus \Omega$. Let

$$\bar{b}^i_k = \begin{cases} b^i_k + \epsilon & \text{for any } k \in \Omega \setminus i \\ b^i_k - |\Omega|\epsilon & \text{if } k = j \\ b^i_k & \text{for any } k \in N \setminus \Omega, k \neq j, \end{cases}$$

where $\epsilon > 0$. If player i changes his bid from b^i_j to $\bar{b}^i_j$ for all j and the other players keep their bids unchanged, $\bar{B}^i = \sum_{k \neq i} \bar{b}^i_k - \sum_{k \neq i} b^k_i = B^i - \epsilon$, $\bar{B}^h = \sum_{k \neq i} b^i_k - \sum_{k \neq h,i} b^k_h - \bar{b}^i_h = B^h - \epsilon$ for each $h \in \Omega \setminus i$, $\bar{B}^h = B^h$ for each $h \in (N \setminus \Omega) \setminus j$, and $\bar{B}^j = B^j + |\Omega|\epsilon$. Since $B^i > B^j$, if ϵ is small enough, $B^j + |\Omega|\epsilon < B^i - \epsilon$, that is $\bar{B}^h > \bar{B}^j$ for each $h \in \Omega$. It means that Ω does not change. However, without affecting the probability of winning, i can decrease the sum of his payments in the case he wins since $\sum_{k \neq i} \bar{b}^i_k < \sum_{k \neq i} b^i_k$. It is a contradiction.

Claim 3: In any SPE each player's payoff is the same regardless of whom is chosen as the proposer.

By Claim 2, B^i are the same for any $i \in N$. If some player increases his payoff by being chosen as the proposer, he should slightly increase one of his bids. Similarly, if some player increases his payoff when another player being chosen as the proposer, he should slightly decrease his bid to the player. Claim 2 implies that, in any SPE, there are no players who have such incentives.

Claim 4: In any SPE the final payoff coincides with the AL-value.

Let u_i^j be the player i's equilibrium payoff when the proposer is j at t=1. By Claim 1,

$$u_i^i = -\sum_{k \neq i} b_k^i + z^i,$$

and for each $j \neq i$,

$$u_i^j = b_i^j + AL_i(N \backslash j, v^{-j}).$$

Thus,

$$\sum_{j \in N} u_i^j = -\sum_{k \neq i} b_k^i + z^i + \sum_{j \neq i} b_i^j + \sum_{j \neq i} AL_i(N \backslash j, v^{-j}).$$

By Claim 2, the above equality is equivalent to

$$\sum_{j \in N} u_i^j = z^i + \sum_{j \neq i} AL_i(N \backslash j, v^{-j}).$$

By Claim 3, $\sum_{j \in N} u_i^j = n u_i^k$ for each $k \in N$. Therefore, for each $k \in N$,

$$u_i^k = \frac{1}{n} z^i + \frac{1}{n} \sum_{j \neq i} AL_i(N \backslash j, v^{-j}).$$

By Theorem 2, the right-hand side of the above equality coincides with $AL_i(N, v)$. $\square$

5 Axiomatic characterization of the AL-value

In this section, we present an axiomatic characterization of the AL-value. As we mentioned before, the AL and Shapley values are closely related. Thus, we start with recalling some relevant axiomatic characterizations of the Shapley value.

Let $\mathcal{G}$ be a class of all games. The following axiom is the basic one.

Efficiency: For each $(N, v) \in \mathcal{G}$, $\sum_{i\in N} \varphi_i(N, v) = v(N)$.

The next axiom is introduced in Myerson (1980).

Balanced contributions property: For each $(N, v) \in \mathcal{G}$ and any $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N\backslash j, v|_{N\backslash j}) = \varphi_j(N, v) - \varphi_j(N\backslash i, v|_{N\backslash i}),$$

where $(N \setminus k, v|_{N\setminus k})$ is a *restricted game* on $N \setminus k$ with $v|_{N\setminus k} : 2^{N\setminus k} \to \mathbb{R}$ given by $v|_{N\setminus k}(S) = v(S)$ for any $S \subseteq N \setminus k$, for $k = i, j$.

In the above property, for any pair of players, a contribution from one player to another is balanced with that from another player to the player. Further, rearranging the above equation, we have that i's value minus j's value in the original game is equal to i's value in the game in which j is deleted minus j's value in the game in which i is deleted. Hence, it can be interpreted as the difference between two players' values is unchanged whether or not the other player exists. Myerson (1980) characterizes the Shapley value by the above two axioms.

Theorem 4 (Myerson 1980)**.** *The Shapley value is the unique value on $\mathcal{G}$ which satisfies Balanced contributions property and Efficiency.*

Following the above result, Kamijo and Kongo (2009) point out that Balanced contributions property is rather a restrictive one, because the efficient value that satisfies the property is only the Shapley value. They study a weaker property than Balanced contributions property. In their weaker property, contributions among all players are balanced in a cyclical manner and their weaker property is called the balanced cycle contributions property. Here, we consider another new weaker property than Balanced contributions property as follows.

Balanced average contributions property: For each $(N, v) \in \mathcal{G}$ with $|N| = n \geq 2$ and any $i \in N$,

$$\frac{1}{n-1} \sum_{j \in N \setminus i} \left(\varphi_i(N, v) - \varphi_i(N \setminus j, v|_{N \setminus j}) \right) = \frac{1}{n-1} \sum_{j \in N \setminus i} \left(\varphi_j(N, v) - \varphi_j(N \setminus i, v|_{N \setminus i}) \right).$$

In the above property, we focus on the *average* of contributions from a player to another. It requires that the average of the other players' contributions to a player is balanced with the average of the player's contributions to each of the other players. Similar to the case of Balanced contributions property, it also be interpreted as the difference between the average of a player's value minus each of the other players' values is unchanged whether or not each of the other player exists. It is straightforward that Balanced contributions property and Balanced average contributions property coincide when we consider two-person games and in general the latter one is weaker than the former one.

Under Efficiency, if a value φ satisfies Balanced *average* contributions property, it is represented by the following recursive manner: For any $i \in N$,

$$\varphi_i(N, v) = \frac{1}{n}(v(N) - v(N \setminus i)) + \frac{1}{n} \sum_{j \in N \setminus i} \varphi_i(N \setminus j, v).$$

The above expression is the same as the well-known recursive representation of the Shapley value (see Maschler and Owen 1989 and Hart and Mas-Colell 1989). Along with Efficiency, the above recursive formula uniquely determines the value of games and it coincides with the Shapley value. Therefore, we can characterize the Shapley value by replacing the Myerson's balanced contributions property with our new weaker property.

Theorem 5. [6] *The Shapley value is the unique value on $\mathcal{G}$ which satisfies Balanced average contributions property and Efficiency.*

Next, we study the axiomatic characterization of the AL-value. As we discussed in the previous sections, the AL-value is closely related to the

[6]Following the above arguments, the proof is obvious. Hence, we omit it.

reduced game à la Davis and Maschler. Replacing restricted games in the above two balanced contributions properties with the lexicographically DM-reduced games, we obtain the following two new properties.

Balanced DM-contributions property: For each $(N, v) \in \mathcal{G}_C$ and any $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \backslash j, v^{-j}) = \varphi_j(N, v) - \varphi_j(N \backslash i, v^{-i}).$$

Balanced average DM-contributions property: For each $(N, v) \in \mathcal{G}_C$ with $|N| = n \geq 2$ and any $i \in N$,

$$\frac{1}{n-1} \sum_{j \in N \backslash i} \left(\varphi_i(N, v) - \varphi_i(N \backslash j, v^{-j}) \right) = \frac{1}{n-1} \sum_{j \in N \backslash i} \left(\varphi_j(N, v) - \varphi_j(N \backslash i, v^{-i}) \right).$$

In the above two properties, we evaluate a player's contribution to another through the lexicographically DM-reduced games instead of restricted games. As we mentioned before, the lexicographically DM-reduced games focus on the bargaining among players based on the core of the game, and the function v^{-i} can be interpreted as the situation in which i is in an advantageous position to the others in the sense that i surely obtains its maximal payoff among core allocations of the original game (N, v). Rearranging the equation in the former property, i's value minus j's value in the original game is equal to i's value in the game in which j is in an advantageous position minus j's value in the game in which i is in an advantageous position. Thus, for example, the former property is interpreted as the difference between two players' values is unchanged regardless of each of the two players discriminatory treatments in the games. It is clear that the above two properties coincide when we consider two-person games and that in general the latter one is weaker than the former one. The following theorem shows that Balanced average DM-contributions property and Efficiency characterize the AL-value.

Theorem 6. *The AL-value is the unique value on $\mathcal{G}_C$ which satisfies Balanced average DM-contributions property and Efficiency.*

Proof. First we show that the AL-value satisfies Balanced average DM-contributions property. If $|N| = n \geq 2$,

$$\frac{1}{n-1}\sum_{j\in N\backslash i}\big(AL_i(N,v) - AL_i(N\backslash j, v^{-j})\big) - \frac{1}{n-1}\sum_{j\in N\backslash i}\big(AL_j(N,v) - AL_j(N\backslash i, v^{-i})\big)$$
$$= \frac{1}{n-1}\left(nAL_i(N,v) - \sum_{j\in N\backslash i} AL_i(N\backslash j, v^{-j}) - (v(N) - AL_i(N,v)) + v^{-i}(N\setminus i)\right)$$
$$= \frac{1}{n-1}\left(nAL_i(N,v) - \sum_{j\in N\backslash i} AL_i(N\backslash j, v^{-j}) - z^i\right) = 0,$$

where the last equality holds by Theorem 2. It is obvious that the AL-value satisfies Efficiency.

For the uniqueness, we use induction with respect to the number of players. Let φ be a value on $\mathcal{G}_C$. In the case of $|N| = 1$, $\varphi_i(N,v) = v(i) =$

$AL_i(N, v)$ for $i \in N$. Let $n \geq 2$ and suppose $\varphi = AL$ in case there are less than n players. Consider the case of n players. By Balanced average DM-contributions property, Efficiency, and the induction hypothesis, we obtain that, for any $i \in N$,

$$\varphi_i(N, v) = \frac{1}{n} z_i + \frac{1}{n} \sum_{j \in N \setminus i} AL_i(N \setminus j, v^{-j}).$$

By Theorem 2, $\varphi_i(N, v) = AL_i(N, v)$. □

Unlike the Shapley value, we cannot characterize the AL-value by Balanced DM-contributions property and Efficiency. Moreover, we obtain the following.

Theorem 7. *There is no efficient value on $\mathcal{G}_C$ that satisfies Balanced DM-contributions property.*

Proof. By definition, if a value satisfies Balanced DM-contributions property then it also satisfies Balanced *average* DM-contributions property. By Theorem 6, the AL-value is the unique efficient value on $\mathcal{G}_C$ that satisfies Balanced average DM-contributions property. Thus, it is enough to show that the AL-value does *not* satisfy Balanced DM-contributions property.

Consider the following four-person game defined on $N = \{1, 2, 3, 4\}$.[7]

$$v(N) = 10, \qquad v(S_1) = v(\{1, 2, 3\}) = 6,$$

$$v(S_2) = v(\{1, 2\}) = 2, \qquad v(S_3) = v(\{3, 4\}) = 2,$$

[7]Note that this game is not super-additive but balanced.

$$v(S_4) = v(\{1,3\}) = 0, \qquad v(S_5) = v(\{2,4\}) = 1,$$

$$v(S_6) = v(\{1,4\}) = 5, \qquad v(S_7) = v(\{2,3\}) = 1,$$

$$v(S) = 0 \text{ if } S \neq S_1, \ldots, S_7, N.$$

The AL-value of the games (N, v), $(N \setminus 1, v^{-1})$, and $(N \setminus 4, v^{-4})$ are $AL(N, v) = (\frac{31}{8}, \frac{14}{8}, \frac{15}{8}, \frac{20}{8})$, $AL(N \setminus 1, v^{-1}) = (0, 1, 1)$, and $AL(N \setminus 4, v^{-4}) = (\frac{5}{2}, 2, \frac{3}{2})$, respectively. Thus, $AL_1(N, v) - AL_1(N \setminus 4, v^{-4}) = \frac{11}{8} \neq \frac{12}{8} = AL_4(N, v) - AL_4(N \setminus 1, v^{-1})$. □

6 Concluding Remarks

In this paper, we give two characterizations of the AL-value. One is a non-cooperative characterization and the other is an axiomatic characterization. A consistency property which we call average consistency plays an important role in the both characterizations. Together with the results in Myerson (1980) and Pérez-Castrillo and Wettstein (2001), our results highlight the similarities and differences between the AL-value and the Shapley value.

Finally, we mention an open problem. Let σ be a one to one and onto mapping from $\mathbb{N}$ to $\mathbb{N}$. For any different $i, j \in \mathbb{N}$, we consider the two-person game $(\{i, j\}, v)$ where $v(\{i, j\}) \geq v(\{i\}) + v(\{j\})$. Then, we can define a value p^σ with respect to σ on the class of such games by

$$p^\sigma_i(N, v) = v(\{i, j\}) - v(\{j\}),\ p^\sigma_j(N, v) = v(\{j\})$$

if $\sigma^{-1}(i) < \sigma^{-1}(j)$.

We notice that this value p^σ coincides with L^σ on the class of two-person games v as defined above. An interesting open question is whether L^σ, $\sigma \in \Pi(N)$, is the unique value which coincides with p^σ on the set of two-person balanced games and satisfies the DM-consistency.

Acknowledgments: The authors thank Yoshio Kamijo for his helpful comments.

References

Branzei R, Dimitrov D, Tijs S (2008) Models in Cooperative Game Theory Second Edition Springer-Verlag Berlin Heidelberg

Caprari E, Patrone F, Pusillo L, Tijs S, Torre A (2006) Share opportunity sets and cooperative games. CentER DP 2006-115, Tilburg University, Tilburg, The Netherlands, to appear in International Game Theory Review

Caprari E, S.Tijs, Torre A (2008) Weighted average lexicographic values for share sets and balanced cooperative games. CentER DP 2008, Tilburg University, Tilburg, The Netherlands.

Dagan N, Serrano R, Volij O (1997) A noncooperative view of consistent bankruptcy rules. Games and Economic Behavior 18:55–72

Davis M, Maschler M (1965) The kernel of a cooperative game. Naval Research Logistics Quaterly 12:223–259

Gillies D (1953) Some theorems on n-person games. PhD dissertation, Princeton University Press

González-Díaz J, Sánchez-Rodríguez E (2007) A natural selection from the core of a TU-game: the core-center. International Journal of Game Theory 36:27–46

Hart S, Mas-Colell A (1989) Potential, value and consistency. Econometrica 57:589–614

Hylland A, Zeckhauser R (1979) The efficient allocation of individuals to positions. Journal of Political Economy 87:293–314

Ju Y, Wettstein D (2009) Implementing cooperative solution concepts: A generalized bidding approach. Economic Theory 39:307–330. mimeo 2006

Kamijo Y, Kongo T (2009) Axiomatization of the Shapley value using the balanced cycle contributions property. International Journal of Game Theory. doi:10.1007/s00182-009-0187-0

Lohmann E (2006) A walk with Alexia. Bachelor's Thesis, Department of Econometrics and Operations Research, Tilburg University, Tilburg, The Netherlands

Lohmann E, Borm P, Quant M (2007) A stroll with Alexia. mimeo

Maschler M, Owen G (1989) The consistent Shapley value for hyperplane games. International Journal of Game Theory 18:389–407

Maschler M, Peleg B, Shapley L (1979) Geometric properties of the kernel and nucleolus, and related solution concepts. Mathematics of Operations Research 4:303–338

Montero M (2006) Noncooperative foundations of the nucleolus in majority games. Games and Economic Behavior 54:380–397

Myerson RB (1980) Conference structures and fair allocation rules. International Journal of Game Theory 9:169–182

Núñez M, Rafels C (1998) On extreme points of the core and reduced games. Annals of Operations Research 84:121–133

O'Neill B (1982) A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2:345–371

Peleg B (1986) On the reduced game property and its converse. International Journal of Game Theory 15:187–200

Pérez-Castrillo D, Wettstein D (2001) Bidding for the surplus: A non-cooperative approach to the shapley value. Journal of Economic Theory 100:274–294

Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM Journal of Applied Mathematics 17(6):1163–1170

Serrano R (1993) Non-cooperative implementation of the nucleolus: The 3-player case. International Journal of Game Theory 22:345–357

Serrano R (1995) Strategic Bargaining, Surplus Sharing Problems and the Nucleolus. Journal of Mathematical Economics 24:319–329

Shapley LS (1953) A value for n-person games. In: Kuhn, H, Tucker A (eds.) Contributions to the Theory of Games II, Princeton University Press, pp. 307–317

Svensson LG (1994) Queue allocation of indivisible goods. Social Choice and Welfare 11:323–330

Tijs S (2005) The first steps with Alexia, the average lexicographic value. CentER DP 2005-123, Tilburg University, Tilburg, The Netherlands.