Note on social choice allocation in an exchange economy

Takeshi Momi *
Department of Economics, Doshisha University

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Abstract

In this paper we show that in a pure exchange economy there exists no strategy-proof, Pareto optimal social choice allocation function which ensuring positive consumptions to all consumers. We further show that if there exists three consumers, then the allocation given by strategy-proof, Paret optimal, and non-dictatorial social choice function depends only on one consumer's preference who always receives zero consumption. That is, we prove Zhou (1991)'s conjecture in a three-consumer economy and show that a strategy-proof and Pareto optimal social choice allocation function in such an economy should be Satterthwaite and Sonnenschein (1981) type.

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1 Introduction

2 The model

We consider an economies with N consumers indexed by $\mathbf{N} = \{1, \dots, N\}$ where $N \geq 2$ and L goods indexed by $\mathbf{L} = \{1, \dots, L\}$ where $L \geq 2$. The concumption set for each consumer is R^L_+. A consumption bundle for consumer $i \in \mathbf{N}$ is a vector $x^i = (x^i_1, \dots, x^i_L) \in R^L_+$. The total endowment of good for the economy is $\Omega \in R^L_{++}$. An allocation is a vector

*Address: Department of Economics, Doshisha University, Kamigyo-ku, Kyoto 602-8580 Japan; Phone: +81-75-251-3647; E-mail: tmomi@mail.doshisha.ac.jp

$\mathbf{x} = (x^1, \ldots, x^N) \in R_+^{LN}$. The set of feasible allocation for the economy with N consumers and L goods is thus

$$X = \left\{ \mathbf{x} \in R_+^{LN} \middle| \sum_{i \in \mathbf{N}} x^i \leq \Omega \right\}$$

A preference R is a complete, reflexive, and transitive binary relation on R_+^L. The corresponding strict preference P_R and indifference I_R are defined in the usual way. For any x and x' in R_+^L, xP_Rx' implies that $xRx'$and not $x'Rx$, and xI_Rx' implies that xRx' and $x'Rx$. Given a preference R and a consumption bundle $x \in R_+^L$, the upper contour set of R at x is $UC(R, x) = \{x' \in R_+^L | x'Rx\}$ and the lower contour set of R at x is $LC(R, x) = \{x' \in R_+^L | xRx'\}$. Further we let $I(x; R) = \{x' \in R_+^L | x'Rx \text{ and } x'Rx\}$ denote the indifference set of R at x and $P(x; R) = \{x' \in R_+^L | x'Rx \text{ and not } xRx'\}$ denote the strictly prefered set of R at x. A preference R is continuous if $UC(R, x)$ and $LC(R, x)$ are both closed for any $x \in R_+^L$. A preference R is strictly convex on R_{++}^L if $UC(R, x)$ is strictly convex for any $x \in R_{++}^L$. A preference R is strictly monotonic on R_{++}^L if for any x and x' in R_{++}^L, $x > x'$ implies xP_Rx'. A preference R is homothetic if for any x and x' in R_+^L and any $t > 0$, xRx' implies $(tx)R(tx')$. A preference R is smooth if for any $x \in R_{++}^L$ there exists a unique vector $p \in S^{L-1}$ such that p is the normal of a supporting hyperplane to $UC(R, x)$ at x. We call such an vector p as gradient vector of the preference R at x

We let $\mathcal{R}$ denote the set of preferences R that is continuous on R_+^L, strictly convex and strictly monotonic on R_{++}^L, smooth, and homothetic.

A preference profile is an N-tuple $\mathbf{R} = (R^1, \ldots, R^N) \in \mathcal{R}^N$. The subprofile obtained by removing R^i from $\mathbf{R}$ is $\mathbf{R}^{-i} = (R^1, \ldots, R^{i-1}, R^{i+1}, \ldots, R^N)$. It is sometimes convenient to write the profile $(R^1, \ldots, R^{i-1}, \bar{R}^i, R^{i+1}, \ldots, R^N)$ as $(\bar{R}^i, \mathbf{R}^{-i})$.

A social choice function $f : \mathcal{R}^N \to X$ assigns a feasible allocation to each preference profile in $\mathcal{R}^N$. The set $\mathcal{R}^N$ is the domain of the social choice function. For a preference profile $\mathbf{R} \in \mathcal{R}^N$, the outcome chosen can be written as $f(\mathbf{R}) = (f^1(\mathbf{R}), \ldots, f^N(\mathbf{R}))$ where $f^i(\mathbf{R})$ is the consumption bundle allocated to consumer i by f.

Definition 1. A social choice function f is strategy-proof if $f^i(\mathbf{R})R^if^i(\bar{R}^i, \mathbf{R}^{-i})$ for any $i \in \mathbf{N}$, any $\mathbf{R} \in \mathcal{R}^N$, and any $\bar{R}^i \in \mathcal{R}$.

A feasible allocation is Pareto optimal if there is no other feasible allocation that would benefit someone without worsening anyone else. That is $\mathbf{x} \in X$ is Pareto optimal for the preference profile $\mathbf{R}$ if there exists no $\bar{\mathbf{x}} \in X$ such that $\bar{x}^iR^ix^i$ for any $i \in \mathbf{N}$ and $\bar{x}^jP_{R^j}x^j$ for some $j \in \mathbf{N}$. We say a social choice function is Pareto optimal when it always assigns Pareto optimal allocations.

Definition 2. A social choice function f is Pareto optimal if $f(\mathbf{R})$ is Pareto optimal for any $\mathbf{R} \in \mathcal{R}^N$.

We say a social choice function guarantee positive consumption if any consumer's consumption is always non-zero. Note that the posiitive consumption guarantee is just a little weaker condition than the minumum consumption guarantee in Serizawa and Weymark (2003).

Definition 3. A social choice function f is positive consumption guarantee if $f^i(\mathbf{R}) \neq 0$ for any $i \in \mathbf{N}$ and any $\mathbf{R} \in \mathcal{R}^N$.

Definition 4. A social choice function f is dictatorial if there exists $i \in \mathbf{N}$ such that $f^i(\mathbf{R}) = \Omega$ for any $\mathbf{R} \in \mathcal{R}^N$.

Following Satterthwaite and Sonnenschein (1981), we define SS mechanism, which includes the dictatorial sochial choice functions as special cases, as follows.

Definition 5. A social choice function f is an SS mechanism if following conditions are satisfied.

(i) There exists $i \in \mathbf{N}$ such that $f^i(\mathbf{R}) = 0$ for any $\mathbf{R} \in \mathcal{R}^N$.

(ii) For each $\mathbf{R} \in \mathcal{R}^N$, there exists some $j \in \mathbf{N}$ such that $f^j(\mathbf{R}) = \Omega$.

(iii) For any $j \neq i$, $f^j(R^i, \bar{\mathbf{R}}^{-i}) = f^j(R^i, \bar{\mathbf{R}}^{-i})$ for any $\mathbf{R}^{-i}$ and $\bar{\mathbf{R}}^{-i}$ in $\mathcal{R}^{N-1}$, where i is a consumer satisfying (i)

Theorem 1. If a social choice function $f : \mathcal{R}^N \to X$ is strategy-proof and Pareto optimal, then it violates positive consumption guarantee.

Theorem 2. Suppose that $N = 3$. If a social choice function $f : \mathcal{R}^N \to X$ is strategy-proof and Pareto optimal, then it is an SS mechanism.

3 Preliminary result I

In the following two sections we show two results which would be useful to investigate the property of social choice functions. In fact this paper's theorems can be proved by combining these two results. The first in this section is a slight generalization of a result proved by Hashimoto (2008) and sophisticated by Momi (2010). They proved that in an two-consumer economy where the preferences are represented by Cobb-Douglas utility functions, if a social choice function is strategy-proof and Pareto optimal, then any change of one consumer's preference should not affect the other's utility level. Next proposition insists that if one consumer's preference is changed while the other consumers have the same preference $\tilde{R}$, then the sum of the new consumptions of the others should be indifferent to the sum of the old ones with respect to the preference $\tilde{R}$ under a strategy-proof and Pareto-optimal social choice function.

Proposition 1. Suppose that $f : \mathcal{R}^N \to X$ is a strategy-proof and Pareto optimal social choice function. For any $i \in \mathbf{N}$, any $R^i, \bar{R}^i \in \mathcal{R}$ and any $\tilde{\mathbf{R}}^{-i} = (\tilde{R}, \dots, \tilde{R}) \in \mathcal{R}^{N-1}$, $(\sum_{j \neq i} f^j(R^i, \tilde{\mathbf{R}}^{-i})) I_{\tilde{R}} (\sum_{j \neq i} f^j(\bar{R}^i, \tilde{\mathbf{R}}^{-i}))$.

For simple exposition we let $\tilde{U} : R^L_+ \to R$ be a differentiable utility function representing the preference $\tilde{R}$. We let $t \to R_t$ be a continuous map mapping a parameter $t \in (0, 1)$ to a preference $R_t \in \mathcal{R}$. For such a map, We let $f(t) = f(R_t, \tilde{\mathbf{R}}^{-i})$ denote the allocation given by f when consumer i's preference is R_t and others' is $\tilde{R}$. All we have to prove is that $\tilde{U}(\sum_{j \neq i} f^j(t)) = \tilde{U}(\sum_{j \neq i} f^j(t))$ or equivalently that $\tilde{U}(\Omega - f^i(t)) = \tilde{U}(\Omega - f^i(t))$. Note that $\mathcal{R}$ is connected, hence that for any $R^i \in \mathcal{R}$ and $\bar{R}^i \in \mathcal{R}$, we can pick a continuous mapping $t \mapsto R_t$ such that $R_{t'} = R^i$ and $R_{t''} = \bar{R}^i$ at some parameter values t' and t'' in $(0, 1)$.

Lemma 1. When $t \mapsto R_t \in \mathcal{R}$ is a continuous map defined on $(0, 1) \subset R$ and $\tilde{\mathbf{R}}^{-i} = (\tilde{R}, \dots, \tilde{R})$. If f is a strategy-proof and Pareto efficient social choice function, then $f^i(t) = f^i(R_t; \tilde{\mathbf{R}}^{-1})$ is a continuous function of t.

Proof. We suppose $t \to \bar{t} \in (0, 1)$. Since X is compact, $f(t)$ converges as $t \to \bar{t}$. We let $f(t) \to \bar{\mathbf{x}} = (\bar{x}^1, \dots, \bar{x}^N)$. All we have to show is that $\bar{x}^i = f^i(\bar{t})$.

We let $U^i(\cdot; t) : R^L_+ \to R$ be a utility function representing the preferenec R_t of consumer i. Since f is strategy-proof, $U^i(f^i(t); t) \geq U^i(f^i(\bar{t}); t)$ holds for any t. Especially at the limit of $t \to \bar{t}$, $U^i(\bar{x}^i; \bar{t}) \geq U^i(f^i(\bar{t}); \bar{t})$ holds. If this equation holds with strict inequality, then the consumer would announce $R_{\tilde{t}}$ where $\tilde{t}$ is sufficiently close to $\bar{t}$ when his true preference is $R_{\bar{t}}$ because $f^i(\tilde{t})$ is close to $\bar{x}^i$, and hence $U^i(f^i(\tilde{t}); \bar{t})$ is close to $U^i(\bar{x}^i; \bar{t})$. This violate the stratefy-proofness of f. Therefore the equation should hold with equality: $U^i(\bar{x}^i; \bar{t}) = U^i(f^i(\bar{t}); \bar{t})$.

We next show that $\bar{\mathbf{x}}$ should be a Pareto optimal allocation in the economy where one consumer's preference is $R_{\bar{t}}$ and others' are $\tilde{R}$. Suppose that $\bar{\mathbf{x}}$ is not Pareto optimal. Then in the economy with preferences $R_{\bar{t}}$ and $\tilde{R}$, which are both strictly convex, there exists $\hat{\mathbf{x}} = (\hat{x}^1, \dots, \hat{x}^N) \in X$ such that $U^i(\hat{x}^i; \bar{t}) > U^i(\bar{x}^i; \bar{t})$ and $\tilde{U}^j(\hat{x}^j) > \tilde{U}^j(\bar{x}^j)$ for any $j \neq i$. When $\tilde{t}$ is sufficiently close to $\bar{t}$, $f(\tilde{t})$ is sufficiently close to $\bar{\mathbf{x}}$ and $R_{\tilde{t}}$ is sufficiently close to $R_{\bar{t}}$. Therefore $U^i(\hat{x}^i; \tilde{t}) > U^i(f^i(\tilde{t}); \tilde{t})$ and $\tilde{U}(\hat{x}^j) > \tilde{U}(f^j(\tilde{t}))$ hold. This violates the Pareto optimality of f. Thus $\bar{\mathbf{x}}$ is a Pareto optimal allocation.

It is easy to observe that in the Edgeworth Box with consumer i's preferences R_t and others' $\tilde{R} \in \mathcal{R}$, which are both homothetic, the set of Pareto optimal allocations intersects consumer i's one indifference surface only once. Therefore If $U^i(\bar{x}^i; \bar{t}) = U^i(f^i(\bar{t}); \bar{t})$ and $\bar{\mathbf{x}}$ and $f(\bar{t})$ are both Pareto optimal allocations, then $\bar{x}^i = f^i(\bar{t})$ holds. ■

Lemma 2. $\tilde{U}(\Omega - f^i(\tilde{t})) = \tilde{U}(\Omega - f^i(\hat{t}))$ for any $\tilde{t}, \hat{t} \in (0, 1)$.

Proof. We suppose that there exists t' and t'' such that $\tilde{U}(\Omega - f^i(t')) \neq \tilde{U}(\Omega - f^i(t''))$. Without loss of generality we assume $t' < t''$.

We first consider the case where $\tilde{U}(\Omega - f^i(t')) > \tilde{U}(\Omega - f^i(t''))$. Note that $\tilde{U}(\Omega - f^i(t))$ is a continuous function of t by Lemma 1 proved above.[1] Then there exist $\bar{t} \in (t', t'')$ and a sequence $\{\epsilon_n\}$ which converges to 0 from the right hand side, $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{\tilde{U}(\Omega - f^i(\bar{t}+\epsilon_n)) - \tilde{U}(\Omega - f^i(\bar{t}))}{\epsilon_n} < 0.^2$$

Since the utility function $\tilde{U}$ is differentiable, the equation becomes

$$\sum_{l=1}^{L} \frac{\partial \tilde{U}(\Omega - f^i(\bar{t}))}{\partial x_l} \lim_{n\to\infty} \frac{-f^i_l(\bar{t}+\epsilon_n) + f^i_l(\bar{t})}{\epsilon_n} < 0.$$

Since f is Pareto optimal, $(\frac{\partial \tilde{U}(\Omega - f^i(t))}{\partial x_1}, \ldots, \frac{\partial \tilde{U}_2(\Omega - f^i(t))}{\partial x_L})$ is parallel to $(\frac{\partial U^i(f^i(t);t)}{\partial x_1}, \ldots, \frac{\partial U^i(f^i(t);t)}{\partial x_L})$. Therefore we have

$$\sum_{l=1}^{L} \frac{\partial U^i(f^i(\bar{t});\bar{t})}{\partial x_l} \lim_{n\to\infty} \frac{f^i_l(\bar{t}+\epsilon_n) - f^i_l(\bar{t})}{\epsilon_n} > 0,$$

hence,

$$\lim_{n\to\infty} \frac{U^i(f^i(\bar{t}+\epsilon_n);\bar{t}) - U^i(f^i(\bar{t});\bar{t})}{\epsilon_n} > 0,$$

This implies $U^i(f^i(\bar{t}+\epsilon_n);\bar{t}) > U^i(f^i(\bar{t});\bar{t})$ with sufficiently large n because $\epsilon_n > 0$. This violates the strategy-proofness of f because consumer i whould announce $\bar{t}+\epsilon_n$ when his true parameter is $\bar{t}$.

Next, we consider the case where $\tilde{U}(\Omega - f^i(t')) < \tilde{U}(\Omega - f^i(t''))$. Then there exist $\bar{t} \in (t', t'')$ and a sequence $\{\epsilon_n\}$ which converges to 0 from the left hand side, $\epsilon_n < 0$ and $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\tilde{U}(\Omega - f^i(\bar{t}+\epsilon_n)) - \tilde{U}(\Omega - f^i(\bar{t}))}{\epsilon_n} > 0.$$

By the same discussion, we have

$$\lim_{n\to\infty} \frac{U^i(f^i(\bar{t}+\epsilon_n);\bar{t}) - U^i(f^i(\bar{t});\bar{t})}{\epsilon_n} < 0.$$

This implies $U^i(f^i(\bar{t}+\epsilon_n);\bar{t}) > U^i(f^i(\bar{t});\bar{t})$ with sufficiently large n because $\epsilon_n < 0$. This again violates the strategy-proofness of f. ■

[1] Note that this might not be a differentiable function at this stage

[2] To the contrary, suppose that $\lim_{n\to\infty} \frac{\tilde{U}(\Omega - f^i(\bar{t}+\epsilon_n)) - \tilde{U}(\Omega - f^i(\bar{t}))}{\epsilon_n} > 0$ for any $\bar{t} \in (t', t'')$ and any sequence $\{\epsilon_n\}$ converging 0 from right hand side. It clearly contradicts to that $\tilde{U}(\Omega - f^i(\cdot))$ is a continuous function and $\tilde{U}(\Omega - f^i(\tilde{t})) > \tilde{U}(\Omega - f^i(t))$.

4 Preliminary result II

In this section we show an application of Maskin monotonic transformation. Consider a preference $R \in \mathcal{R}$ and a consumption bundle $x \in R^L_+$. A preference $\bar{R}$ is called Maskin monotonic transformation of R at x if $\bar{x} \in UC(\bar{R}, x)$ and $\bar{x} \neq x$ implies $\bar{x} P_R x$. If an individual receives the commodity bundle x at the profile $\mathbf{R}$, strategy-proofness implies that this consumer receives the same commodity bundle when his preference is subject to a Maskin monotonic transformation at x.

Lemma 3. Suppose that $f : \mathcal{R}^N \to X$ is a strategy proof social choice function. For any $\mathbf{R} \in \mathcal{R}^N$ and any $i \in \mathbf{N}$, if $\bar{R}^i \in \mathcal{R}$ is a Maskin monotonic transformation of R^i at $f^i(R)$, then $f^i(\bar{R}^i; \mathbf{R}^{-1}) = f^i(\mathbf{R})$.

In addition to the preference R and the consumption bundle x, consider another preference $\tilde{R}$ and another consumption bundle $\tilde{x}$. If these are as in Figure 1, it would be possible to image a preference which is a Maskin monotonic transformation of R at x and also a Maskin monotonic transformation of $\tilde{R}$ at $\tilde{x}$. Next proposition shows when such a transformation is possible. For $x \in R^L_+ \setminus 0$, we let $[x]$ denote the ray in the consumption set R^L_+ starting from the origin and passing through x. Keep in mind two preliminary facts about homothetic preferences we deal with.

Lemma 4. For $R, \tilde{R} \in \mathcal{R}$, if $UC(x; R) = UC(x; \tilde{R})$ at a consumption bundle $x \in R^L_{++}$, then R and $\tilde{R}$ are the same preference.

Lemma 5. If $\tilde{R} \in \mathcal{R}$ is a Maskin monotonic transformation of $R \in \mathcal{R}$ at $x \in R^L_{++}$, then $\tilde{R}$ is a Maskin Monotonic transformation of R at any non-zero consumption $x' \in [x]$.

Proposition 2. For any $R, \tilde{R} \in \mathcal{R}$ and any $x, \tilde{x} \in R^L_{++}$, if $x \in P(I(x; R) \bigcap [\tilde{x}]; \tilde{R})$ and $\tilde{x} \in P(I(\tilde{x}; \tilde{R}) \bigcap [x]; R)$, then there exists a preference $\bar{R} \in \mathcal{R}$ that is a Maskin monotonic transformation of R at x and of $\tilde{R}$ at $\tilde{x}$.

Proof. Figure 2 (i) describes an example of R, $\tilde{R}$, x and $\tilde{x}$ satisfying the condition in the proposition. We first consider a special case where $\tilde{x} \in P(x; R)$ and $x \in P(\tilde{x}; \tilde{R})$, and observe that there exist a preference $\bar{R}$ which is a Maskin transformation of R at x and $\tilde{R}$ at $\tilde{x}$. Figure 2 (ii) draws this situation. It is easy to image a desired Maskin monotonic transformation. A rigorous discussion follows.

Suppose $\tilde{x} \in P(x; R)$ and $x \in P(\tilde{x}; \tilde{R})$. We pick a closed subset $Y \subset R^L_+$ such that (i) $Y \subset UC(x; R) \bigcap UC(\tilde{x}; \tilde{R})$, (ii)for any $x \in Y$, $x + R^L_+ \subset Y$, (iii) the boundary of Y, ∂Y, is smooth, (iv) $x \in \partial Y$, and $UC(x; R)$ and Y have the same hyperplane at x, and (v) $\tilde{x} \in \partial Y$, and $UC(\tilde{x}; \tilde{R})$ and Y have the same hyperplane at $\tilde{x}$. To obtain such a set Y. For example, let $B_\epsilon(y)$ be a closed ball with center y and radius ϵ. Fix sufficiently small

ϵ and let Y be a sum of $B_\epsilon(y)$ over y's such that $B_\epsilon(y) \subset UC(x;R) \bigcap UC(\tilde{x};\tilde{R})$. That is, $Y = \{x \in R^L_+ | x \in B_\epsilon(y)$ for some y such that $B_\epsilon(y) \subset UC(x;R) \bigcap UC(\tilde{x};\tilde{R})\}$. If the ϵ is sufficiently small, Y is a desirable set satisfying (i)–(v). We let $\bar{R} \in \mathcal{R}$ be a preference such that $Y$is an upper contour set of $\bar{R}$ at x. As in Lemma 4, $\bar{R}$ is thus determined uniquely. It is clear that $\bar{R}$ is a Maskin monotonic transformation of R at x and $\tilde{R}$ at $\tilde{x}$ because $UC(x;\bar{R}) = UC(\tilde{x};\bar{R}) = Y \subset UC(x;R) \bigcap UC(\tilde{x};\tilde{R})$ from the construction.

We next consider a general case. Suppose $R, \tilde{R} \in \mathcal{R}$ and $x, \tilde{x} \in R^L_+$ satisfy the condition in the proposition. We pick $\hat{x} \in [\tilde{x}]$ such that $\hat{x} \in P(x;R)$ and $x \in P(\hat{x};\tilde{R})$. From the above discussion there exist a preference $\bar{R} \in \mathcal{R}$ that is a maskin monotonic transformation of R at x and $\tilde{R}$ at $\hat{x}$. Then, as in Lemma 5, $\bar{R}$ is also a Maskin monotonic transformation of $\tilde{R}$ at $\tilde{x}$ because $\tilde{x}$ and $\hat{x}$ is on the same ray. ■

5 Proof of Theorem 1

Suppose f is a strategy-proof and Pareto optinal social choice fuinction which guarantees positive consumptions. When all consumers have an identical preference, all consumers should receive a posiitive portion of Ω: $f^i(\mathbf{R}) = \lambda^i \Omega$ with some $0 < \lambda^i < 1$ for $\mathbf{R} = (R, \dots, R)$. Pick two different preference R and $\tilde{R}$ and consider the allocations given by f at $\mathbf{R} = (R, \dots, R)$ and $\tilde{\mathbf{R}} = (\tilde{R}, \dots, \tilde{R})$.

We let $A(x;R)$ be the set of consumption bundle x' such that $\Omega - x'$ is indifferent to $\Omega - x$ with respect to R

$$A(x;R) = \{x' \in R^L_+ | (\Omega - x') I_R (\Omega - x)\}$$

and let $A^+(x;R) = \{x' \in R^L_+ | (\Omega - x) P_R (\Omega - x')\}$, which is the upper right part of the consumption set partitioned by $A(x;R)$ and let $A^-(x;R) = \{x' \in R^L_+ | (\Omega - x') P_R (\Omega - x)\}$, which is the lower left part.

Without loss of generality we assume $f^1(\mathbf{R}) \geq f^1(\tilde{\mathbf{R}})$. We pick $\bar{x}^1 \in A(f^1(\tilde{\mathbf{R}});\tilde{R})$ in the neighborhood of $f^1(\tilde{\mathbf{R}})$ so that $\bar{x}^1$ is in $A(f^1(\mathbf{R});R)^-$ and $\bar{x}^1$ is not parallel to Ω. Next, let x' be the intersection of $A(f^1(\mathbf{R});R)$ and the segment $[\bar{x}^1, \Omega]$ and pick $\hat{x}^1 \in A(f^1(\mathbf{R});R)$ in the neighborhood of x' so that $\hat{x}^1 \in A^-(x';\tilde{R})$. See the Edgeworth Box in Figure 3, where the consumption of consumer 1 is mesured from the lower left vertex and sum of consumptions of the other consumers is measured from the upper right vertex.

As we observed in Proposition 1, consumption of consumer 1 is on $A(f^1(\mathbf{R});R)$ (resp. $A(f^1(\tilde{\mathbf{R}});\tilde{R})$) when other consumers' preference is R (resp. $\tilde{R}$) and the preference of consumer 1 is changed. There exist preferences $\bar{R}$ and $\hat{R}$ such that $f^1(\bar{R}, \mathbf{R}^{-1}) = \bar{x}^1$ and $f^1(\hat{R}, \tilde{\mathbf{R}}^{-1}) = \hat{x}^1$.

We let $\check{R}$ be a preference which is a Maskin monotonic transformation of R at $\Omega - \hat{x}^1$ and of $\tilde{R}$ at $\Omega - \bar{x}$. Observe that our choice of $\bar{x}^1$ and $\hat{x}^1$ ensures the condition in Proposition 2 and supports the existence of such a Maskin monotonic transformation.

We observe that the consumption allocated to consumer 1 should not be changed when the preferences of consumers other than consumer 1 are changed to $\check{R}$ from the profile $(\bar{R}, \mathbf{R}^{-1})$ or the profile $(\hat{R}, \tilde{\mathbf{R}}^{-1})$

Since f is Pareto optimal and positive consumption guarantee, at the profile $(\bar{R}, \mathbf{R}^{-1})$, consumer 1 receives $\bar{x}^1$ and each of the other consumers $i = 2, \ldots, N$, receives a positive portion of $\Omega - \bar{x}^1$: $\bar{\lambda}^i(\Omega - \bar{x}^1)$, $i = 2, \ldots, N$, where $0 < \bar{\lambda}^i < 1$ and $\sum_{i=2}^N \bar{\lambda}^i = 1$. Note that since we have chosen $\bar{x}^1$ not parallel to Ω, $\bar{x}^1$ and $\Omega - \bar{x}^1$ are independent vectors. Now, let us change the consumer 2's preference to $\check{R}$ from R. Write the new profile as $(\bar{R}, \check{R}, \mathbf{R}_{-2})$ where consumer 1's profile is $\bar{R}$, consumer 2's $\check{R}$ and others' R.

Since $\check{R}$ is a Maskin monotonic transformation of R at $\Omega - \bar{x}^1$, it is so at consumer 2's consumption. Therefore consumer 2's consumption shopuld not be changed and her gradient vector at the consumption should not be changed. Because of the Pareto optimarity, all consumers' gradient vectors at their consumptions should be the same. Hence, all consumers have the same gradient vector at the both profiles $(\bar{R}, \mathbf{R}^{-1})$ and $(\bar{R}, \check{R}, \mathbf{R}_{-2})$. Since the preferences are homothetic, the equality of the gradient vectors implies that each consumer's consumptions at the both profiles should be parallel. That is, at the new profile, consumer 1's consumptions is parallel to $\bar{x}^1$ and the other consumer's consumptions are parallel to $\Omega - \bar{x}^1$. Because of the pareto optimarity, the consumptions at the new profile should sum up to the total endowment Ω. Then consumer 1's consumption should be still $\bar{x}^1$.

Next we further change consumer 3's preference to $\check{R}$ from R. Discussions are the same. Because $\tilde{R}$ is a Maskin monotonic transformation of R at her consumption, consumer 3's consumptions at the new profile $(\bar{R}, \check{R}, \check{R}, \mathbf{R}_{-3})$, where consumer 1's preference is $\bar{R}$, consumer 2's and 3's $\check{R}$ and others' R, and at the old one $(\bar{R}, \check{R}, \mathbf{R}_{-2})$ are the same and the gradient vectors at the consumption are the same. Then, all consumers have the same gradient vector at their consumptions at the profiles $(\bar{R}, \check{R}, \check{R}, \mathbf{R}_{-3})$ and $(\bar{R}, \check{R}, \mathbf{R}_{-2})$, and their consumptions are parallel at the both profiles. Thus consumption of consumer 1 is parallel to $\bar{x}^1$ and others are parallel to $\Omega - \bar{x}^1$. Then consumer 1's consumption at the profile $(\bar{R}, \check{R}, \check{R}, \mathbf{R}_{-3})$ should be still $\bar{x}^1$.

By applying the discussions repeatedly untill we change the preferences of all consumers but consumer 1 to $\check{R}$, we finally obtain that $f^1(\bar{R}, \check{\mathbf{R}}^{-1}) = \bar{x}^1$. Discussions are the same for the profile $(\hat{R}, \tilde{\mathbf{R}}^{-1})$ and we obtain $f^1(\hat{R}, \check{\mathbf{R}}^{-1}) = \hat{x}^1$.

Remember our choice of $\hat{x}^1$ and $\bar{x}^1$. From the definition, x' is strictly prefered to $\bar{x}$ with respect to any prefrence and $\hat{x}^1$ can be chosen arbitrarily close to x'. Therefore $\hat{x}^1$ could have been choosen to be prefered to $\bar{x}^1$ under the consumer 1's preference $\bar{R}$. This violates the strategy-proofness of f. This ends the proof of Theorem 1.

6 Proof of Theorem 2

We first prove a lemma.

Lemma 6 Let $N = 3$ and f be a strategy-proof and Pareto optimal social choice function. Let $\mathbf{R} = (R, R, R)$ and $\bar{\mathbf{R}} = (\bar{R}, \bar{R}, \bar{R})$ be preference profiles where all agents have the same preferences R and $\bar{R}$ respectively. If one consumer is given the total endowment Ω by f at $\mathbf{R}$ and another consumer is given Ω at $\bar{\mathbf{R}}$, then there exists no preference $\tilde{R}$ such that the other consumer is given Ω at $\tilde{\mathbf{R}} = (\tilde{R}, \tilde{R}, \tilde{R})$.

Proof Without loss of generality we assume $f(\mathbf{R}) = (\Omega, 0, 0)$ and $f(\bar{\mathbf{R}}) = (0, \Omega, 0)$ and prove $f(\tilde{\mathbf{R}}) \neq (0, 0, \Omega)$. To the contrary, we suppose $f(\tilde{\mathbf{R}}) = (0, 0, \Omega)$.

$f(\mathbf{R}) = (\Omega, 0, 0)$ implies $f(\bar{R}, R, R) = (\Omega, 0, 0)$ because of the strategy-proofness of f. This, again by the strategy-proofness of f, implies $f^2(\bar{R}, \bar{R}, R) = 0$. On the other hand, $f(\bar{\mathbf{R}}) = (0, \Omega, 0)$ implies $f^3(\bar{R}, \bar{R}, R) = 0$. Therefore we obtain $f(\bar{R}, \bar{R}, R) = (\Omega, 0, 0)$, hence $f(\tilde{R}, \bar{R}, R) = (\Omega, 0, 0)$.

By the same discussions, $f(\tilde{\mathbf{R}}) = (0, 0, \Omega)$ implies $f(\tilde{R}, \tilde{R}, \bar{R}) = (0, 0, \Omega)$, which implies $f^2(\tilde{R}, \bar{R}, \bar{R}) = 0$. $f(\bar{\mathbf{R}}) = (0, \Omega, 0)$ implies $f^1(\tilde{R}, \bar{R}, \bar{R}) = 0$. Therefroe $f(\tilde{R}, \bar{R}, \bar{R}) = (0, 0, \Omega)$, hence $f(\tilde{R}, \bar{R}, R) = (0, 0, \Omega)$. This is a contradiction. ■

Part 1. In this part, we prove that if f is a strategy-proof and Pareto optimal social choice function, then one consumer is given the total endowment Ω at each profile (R, R, R) where all consumers have the same preference.

(1) We pick a preference R such that at least two consumers receive non-zero consumptions at the profile (R, R, R).[3] In this step, we show that there exists another preference $\tilde{R}$ such that at least two consumers receives non-zero consumptions at the profile $(\tilde{R}, \tilde{R}, \tilde{R})$.

We suppose that there exists no $\tilde{R}$ different from R such that at least two consumers receives non-zero consumption at $\tilde{\mathbf{R}} = (\tilde{R}, \tilde{R}, \tilde{R})$. In other words, we suppose that some consumer receives Ω at each $\tilde{\mathbf{R}} = (\tilde{R}, \tilde{R}, \tilde{R})$, $\tilde{R} \neq R$.

Because of Lemma 6, all consumers 1,2 and 3 cannot be the receiver of the total endowment at some profiles where all consumers have same preferences. Without loss of generality, we assume consumer 1 or 2 receives the total endowment at each profiles $\tilde{\mathbf{R}} = (\tilde{R}, \tilde{R}, \tilde{R})$, $\tilde{R} \neq R$ and consumer 1 receives the total endowment at least one such profile. We consider the following two cases separately: (i) $f^3(R, R, R) = 0$ and (ii) $f^3(R, R, R) \neq \{0, \Omega\}$. Note that $f^3(R, R, R) \neq \Omega$ because of Lemma 6.

(i) We conider the case $f^3(R, R, R) = 0$. Let R' be a preference such that $f(R', R', R') = (\Omega, 0, 0)$. Then $f(R'', R', R') = (\Omega, 0, 0)$ for any R'', and hence $f^2(R'', R'', R) = 0$ because

[3] If there exists no such preference, then one consumer receives the total endowment at each profile where all consumers have the same preference as we desired in this part.

of the strategy-proofness of f. On the other hand, if $R'' \neq R$, then $f^3(R'', R'', R'') = 0$ by assumption, and hence $f^3(R'', R'', R') = 0$ because of the strategy-proofness of f. Therefore, if $R'' \neq R$, then $f(R'', R'', R') = (\Omega, 0, 0)$. Then, $f(\bar{R}, R'', R') = (\Omega, 0, 0)$ for any $\bar{R}$ and any $R'' \neq R$, and hence $f^3(\bar{R}, R'', R) = 0$ because of the strategy-proofness of f. Remember that at the profile (R, R, R) consumer 3 receives zero consumption, and consumers 1 and 2 receive non-zero consumptions. This implies that $f(\cdot, \cdot, R)$ is a strategy-proof, Pareto optimal, and non-dictatorial social choice function in the two-consumer economy with the consumers 1 and 2. This is a contradiction.

(ii) We consider the case $f^3(R, R, R) \neq \{0, \Omega\}$. As in the above case, we let R' be a preference such that $f(R', R', R') = (\Omega, 0, 0)$ and obtain $f(\bar{R}, R'', R') = (\Omega, 0, 0)$ for any $\bar{R}$ and any $R'' \neq R$. Note that this implies $f^2(\bar{R}, R, R') = 0$ for any $\bar{R}$ because of the strategy-proofness of f. $f^3(R, R, R) \neq \{0, \Omega\}$ implies $f^3(R, R, R') \neq \{0, \Omega\}$, and this implies $f^1(R, R, R') \neq \{0, \Omega\}$ because consumer 2 receives zero-consumption at the profile, and hence $f^1(\bar{R}, R, R') =\neq \{0, \Omega\}$ for any $\bar{R}$. Therefore at the profile $(\bar{R}, R, R')$ for any $\bar{R}$, consumer 2 receives zero-consumption and consumers 1 and 3 receive non-zero consumptions. Hence, $f^3(\bar{R}, R, \hat{R}) \neq 0$ for any $\bar{R}$ and any $\hat{R}$

Now suppose there exists $\hat{R}$ different from R' such that $f(\hat{R}, \hat{R}, \hat{R}) = (0, \Omega, 0)$, then by the symmetric discussion, we have $f(R'', \bar{R}, \hat{R}) = (0, \Omega, 0)$ for any $\bar{R}$ and any $R'' \neq R$. Especially, $f(R'', R, \hat{R}) = (0, \Omega, 0)$ for any $R'' \neq R$. This contradicts to the conclusion in the above paragraph.

From the discusssions in the above paragraphs, we have $f(R', R', R') = (\Omega, 0, 0)$ for any $R' \neq R$ and $f^2(\bar{R}, R, R') = 0$, $f^1(\bar{R}, R, R') \neq \{0, \Omega\}$ and $f^3(\bar{R}, R, R') \neq \{0, \Omega\}$ for any $\bar{R}$. This implies that $f(\cdot, R, \cdot)$ defined on the profile set $\mathcal{R} \times (\mathcal{R} \setminus R)$ is a strategy-proof, Pareto optimal, and non-dictatorial social choice function in the two-consumer economy with the consumers 1 and 3. This is a contradiction.

(2) Since there exists three consumers, there exist at least one consumer who receives non-zero consumption at the both profile $\mathbf{R} = (R, R, R)$ and $\tilde{\mathbf{R}} = (\tilde{R}, \tilde{R}, \tilde{R})$. Without loss of generality we let consumer 1 is such a consumr: $f^1(\mathbf{R}) \neq \{0, \Omega\}$ and $f^1(\tilde{\mathbf{R}}) \neq \{0, \Omega\}$. Without loss of generality we assume $f^1(\mathbf{R}) \geq f^1(\tilde{\mathbf{R}})$.

We change the preference of consumer 1 as we did in the proof of Theorem 1. We pick $\bar{x}^1 \in A(f^1(\tilde{\mathbf{R}}); \tilde{R})$ in a neighborhood of $f^1(\tilde{\mathbf{R}})$ so that $\bar{x}^1$ is in $A(f^1(\mathbf{R}); R)^-$ and $\bar{x}^1$ is not parallel to Ω. Next, let x' be the intersection of $A(f^1(\mathbf{R}); R)$ and the segment $[\bar{x}^1, \Omega]$ and pick $\hat{x}^1 \in A(f^1(\mathbf{R}); R)$ in a neighborhood of x' so that $\hat{x}^1 \in A^-(x'; \tilde{R})$. As observed in Proposition 1, there exists a preference $\bar{R}$ such that $f^1(\bar{R}; \mathbf{R}^{-1}) = \bar{x}^1$ and a preference $\hat{R}$ such that $f^1(\hat{R}; \tilde{\mathbf{R}}^{-1}) = \hat{x}^1$. (Remember Figure 2.)

(3) Since $f^1(\bar{R}, \tilde{R}, \tilde{R}) = \bar{x}^1 \neq \{0, \Omega\}$, consumer 2 or consumer 3 receives non-zero consumption at the profile. In this step, we prove that even if one consumer receives zero

consumption at the profile, there exists a profile near the old one such that all consumers receive non-zero consumptions.

We first consider new profiles $(R', \tilde{R}, \tilde{R})$ where consumer 1's preference R' is a Maskin monotonic transformation of $\bar{R}$ at $\bar{x}$ and is close to $\bar{R}$. If consumer 2 and 3 receive non-zero consumptions at such a profile, we achieve the desired result. Further observe that, since R' is a Maskin monotonic transformation of $\bar{R}$ at $\bar{x}^1$, $f^1(R', \tilde{R}, \tilde{R}) = \bar{x}^1$, $f^2(R', \tilde{R}, \tilde{R})$ and $f^3(R', \tilde{R}, \tilde{R})$ are both parallel to $\Omega - \bar{x}^1$ which is not parallel to $\bar{x}^1$.

Next we suppose that there exist no such Maskin monotonic transformation R' in the neighborhood of $\bar{R}$. Without loss of generality, we let R' and R'' be preferences such that (i) R' and R'' are Maskin monotonic transformations of $\bar{R}$ at $\bar{x}$ in the neighborhood of $\bar{R}$, (ii) R'' is a Maskin monotonic transformation of R' at $\bar{x}$, and (iii) $f^3(R', \tilde{R}, \tilde{R}) = 0$ and $f^3(R'', \tilde{R}, \tilde{R}) = 0$. Note that then $f^1(R', \tilde{R}, \tilde{R}) = f^1(R'', \tilde{R}, \tilde{R}) = \bar{x}^1$ and $f^2(R', \tilde{R}, \tilde{R}) = f^2(R'', \tilde{R}, \tilde{R}) = \Omega - \bar{x}^1$ hold.

We let $t \mapsto R_t$ be a continuous map such that $R_{\bar{t}} = \tilde{R}$ and consider the profiles $(R', R_t, \tilde{R})$ and $(R'', R_t, \tilde{R})$. Suppose there exists no t in the neighborhood of $\bar{t}$ such that $f^3(R', R_t, \tilde{R}) \neq 0$ or $f^3(R'', R_t, \tilde{R}) \neq 0$. Then, as shown in Proposition 1, $f^1(R', R_t, \tilde{R})$ (resp. $f^1(R'', R_t, \tilde{R})$) is indifferent to $\bar{x}^1$ with respect to the preference R' (resp. R'') for t in the neighborhood of $\bar{t}$ because the total endowment Ω should be allcated to the two consumers 1 and 2. In other words, $f^2(R', R_t, \tilde{R})$ (resp. $f^2(R'', R_t, \tilde{R})$) is on $A(\Omega - \bar{x}^1; R')$ (resp. $A(\Omega - \bar{x}^1; R'')$) for t in the neighborhood of $\bar{t}$. This, however, implies that $f^1(R'', R_t, \tilde{R})$ is prefered to $f^1(R', R_t, \tilde{R})$ because R'' is a Maskin monotonic transformation of R'. This contradicts to the strategy-proofness of f. Therefore there should exists $\hat{t}$ in any neighborhood of $\bar{t}$ such that $f^3(R', R_{\hat{t}}, \tilde{R}) \neq 0$ or $f^3(R'', R_{\hat{t}}, \tilde{R}) \neq 0$.

Consider the case $f^3(R', R_{\hat{t}}, \tilde{R}) \neq 0$ for $\hat{t}$ sufficiently close to $\bar{t}$. We can prove that $f^2(R', R_t, \tilde{R}) \to f^2(R', \tilde{R}, \tilde{R})$ as $t \to \bar{t}$. Therefore $f^2(R', R_{\hat{t}}, \tilde{R})$ is close to $f^2(R', \tilde{R}, \tilde{R}) = \Omega - \bar{x}^1$ and the gradient vectors at the consumptions are also close to each other. The closeness of the gradient vectors implies that $f^1(R', R_{\hat{t}}, \tilde{R})$ is on the ray close to $[\bar{x}^1]$ because of the homoceticity of R'. Further, the closeness of $R_{\hat{t}}$ and $\tilde{R}$ implies that $f^3(R', R_{\hat{t}}, \tilde{R})$ is on the ray close to $[f^2(R', R_{\hat{t}}, \tilde{R})]$ hence is on the ray close to $[\Omega - \bar{x}^1]$. These implies that $f^1(R', R_{\hat{t}}, \tilde{R})$ is close to $\bar{x}^1$ because $f^1(R', R_{\hat{t}}, \tilde{R}) + f^2(R', R_{\hat{t}}, \tilde{R}) + f^3(R', R_{\hat{t}}, \tilde{R}) = \Omega$, where $f^1(R', R_{\hat{t}}, \tilde{R})$ is on the ray close to $[\bar{x}^1]$, $f^2(R', R_{\hat{t}}, \tilde{R})$ and $f^3(R', R_{\hat{t}}, \tilde{R})$ are on the rays close to $[\Omega - \bar{x}^1]$ and $f^2(R', R_{\hat{t}}, \tilde{R})$ is close to $\Omega - \bar{x}^1$. The discussion is the same for the case $f^3(R'', R_{\hat{t}}, \tilde{R}) \neq 0$.

The conclusion of this step follows. There exist preferences $\bar{R}^1$, $\tilde{R}^2$ and $\tilde{R}^3$ respectively in the neighborhoods of $\bar{R}$, $\tilde{R}$, and $\tilde{R}$ such that all consumers are given non-zero consumptions at the profile $(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$ and $f^1(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$ is close to $f^1(\bar{R}, \tilde{R}, \tilde{R}) = \bar{x}^1$ and is parallel to neither $f^2(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$ nor $f^2(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$.

(4) The discussion is the same for the profile $(\hat{R}, R, R)$. There exist preferences $\hat{R}^1$,

R^2 and R^3 respectively in the neighborhoods of $\hat{R}$, R, and R such that all consumers are given non-zero consumptions at the profile $(\hat{R}^1, R^2, R^3)$ and $f^1(\hat{R}^1, R^2, R^3)$ is close to $f^1(\hat{R}, R, R) = \hat{x}^1$ and is parallel to neither $f^2(\hat{R}^1, R^2, R^3)$ nor $f^2(\hat{R}^1, R^2, R^3)$.

(5) Let $\check{R}^2$ be a Maskin monotonic transformation of $\tilde{R}^2$ at $f^2(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$ and of R^2 at $f^2(\hat{R}^1, R^2, R^3)$. Note that there exists a Maskin monotonic transformation of $\tilde{R}$ at $\Omega - \bar{x}^1$ and of R at $\Omega - \hat{x}^1$ as we observed in the proof of Theorem 1. (See Figure 3.) Then there exists a desired transformation $\check{R}^2$ for the preferences $\tilde{R}^2$ and R^2 close to $\tilde{R}$ and R respectively and the consumptions $f^2(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$ and $f^2(\hat{R}^1, R^2, R^3)$ close to $\Omega - \bar{x}^1$ and $\Omega - \hat{x}^1$.

By the discussion similar to that in the proof of Theorem 1 this transformation does not change consumptions of any consumers.

Next, let $\check{R}^3$ be a Maskin monotonic transformation of R^3 at $f^3(\bar{R}^1, \check{R}^2, R^3)$ and of $\tilde{R}^3$ at $f^3(\hat{R}^1, \check{R}^2, \tilde{R}^3)$. The existence of this transformation is also supported by that fact that $\tilde{R}^3$ and R^3 are close to $\tilde{R}$ and R respectively and the consumptions $f^3(\bar{R}^1, \tilde{R}^2, \tilde{R}^3)$ and $f^3(\hat{R}^1, R^2, R^3)$ are respectively on the rays close to $[\Omega - \bar{x}^1]$ and $[\Omega - \hat{x}^1]$. Again this transformation does not change consumptions of any consumers.

Thus we have that $f^1(\bar{R}^1, \check{R}^2, \check{R}^3)$ is close to $\bar{x}^1$ and $f^1(\hat{R}^1, \check{R}^2, \check{R}^3)$ is close to $\hat{x}^1$. This contradicts to the strategy-proofness of f because $\hat{x}$ is chosen to be prefered to $\bar{x}$ with respect to the prefrence $\bar{R}$ in step (2), and $\bar{R}^1$ is choosen to be sufficiently close $\bar{R}$ in step (3). This ends the proof of Part 1.

Part 2. We have proved that one consumer receives all endowments at each preference profile $\mathbf{R} = (R, R, R)$ where all consumers have the same preference. Because of Lemma 6, there exists a consumer who receives zero consumption at any such profile $\mathbf{R} = (R, R, R)$. In this part, we prove that the allocation given by f should depend only on the preference of the consumer.

Without loss of generality we assume consumer 3 be the consumer receiving zero consumptions at such profiles : $f^3(R, R, R) = 0$ for any $\mathbf{R} = (R, R, R)$.

Pick any preference R. We know $f(R, R, R)$ is $(\Omega, 0, 0)$ or $(0, \Omega, 0)$. We prove that if $f(R, R, R) = (\Omega, 0, 0)$, then $f(\tilde{R}, \bar{R}, R) = (\Omega, 0, 0)$ for any $\bar{R}, \tilde{R} \in \mathcal{R}$, and symmetrically if $f(R, R, R) = (0, \Omega, 0)$, then $f(\tilde{R}, \bar{R}, R) = (0, \Omega, 0)$ for any $\bar{R}, \tilde{R} \in \mathcal{R}$.

We assume $f(R, R, R) = (\Omega, 0, 0)$. Then $f(\bar{R}, R, R) = (\Omega, 0, 0)$. Hence $f^2(\bar{R}, \bar{R}, R) = 0$.

$f^3(\bar{R}, \bar{R}, \bar{R}) = 0$ implies $f^3(\bar{R}, \bar{R}, R) = 0$. Thus we have $f(\bar{R}, \bar{R}, R) = (\Omega, 0, 0)$. This implies $f(\tilde{R}, \bar{R}, R) = (\Omega, 0, 0)$. The discussion is symmetric for the case $f(R, R, R) = (0, \Omega, 0)$. This ends the proof of Theprem 2.

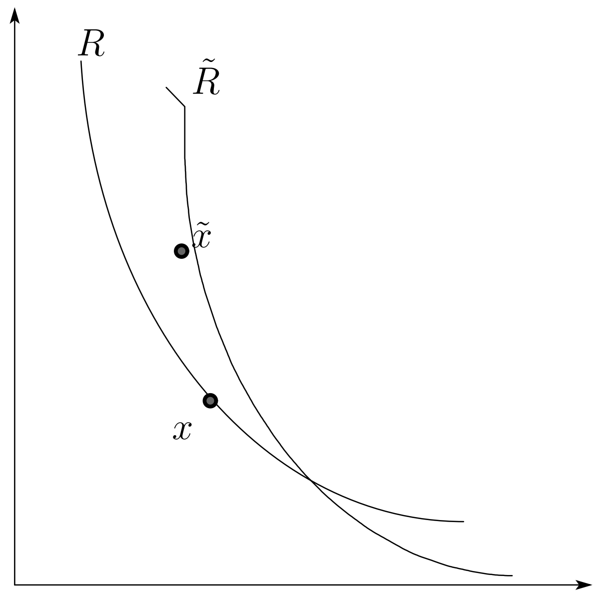

Figure 1

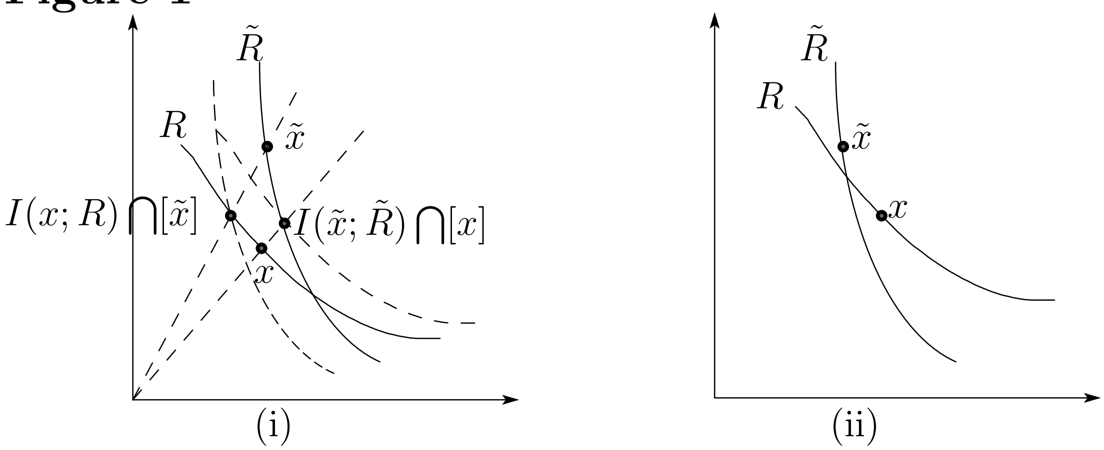

Figure 2

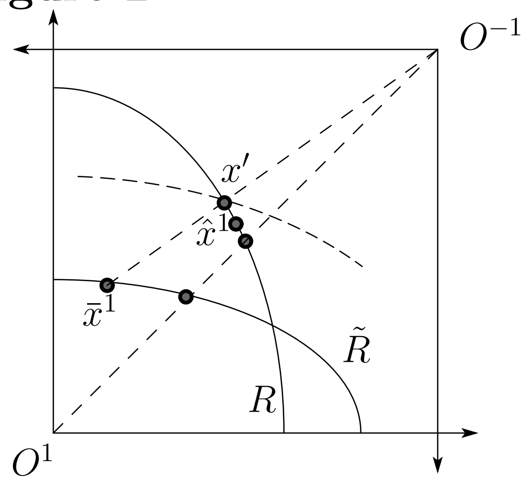

Figure 3

Note on social choice allocation in an economy with Cobb-Douglas utility

Takeshi Momi *
Department of Economics, Doshisha University

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Abstract

In this note we show that in a pure exchange economy with two consumers and a finite number of goods, there exists no strategy-proof, Pareto-efficient and non-dictatorial social choice allocation function on any local Cobb-Douglas preference domain. This is a slight extension of a result proved by Hashimoto (2008).

JEL classification: , .

Keywords:

.

1 Introduction

2 The model

We consider pure exchange economies with I agents indexed by $i = 1, \ldots, I$ $(I \geq 2)$ and L goods indexed by $l = 1, \ldots, L$ $(L \geq 2)$.

Each agent i has initial endowment of goods $\omega_i = (\omega_{i1}, \ldots, \omega_{iL}) \in R^L_+$ where ω_{il} is his endowment of l-th good. The total endowments of goods is $\omega = (\omega_1, \ldots, \omega_L) = \sum_{i=1}^{I} \omega_i \in R^L_{++}$. A consumption bundle of agent i is $x_i = (x_{i1}, \ldots, x_{iL}) \in R^L_+$ where x_{il} is his consumption of l-th good. The set of allocations is $X = \{(x_1, \ldots, x_I) \in R^{LI}_+ | \sum_{i=1}^{N} x_i = \omega\}$.

Each agent i has a preference represented by a Cobb-Douglas utility function U_i on the consumption space R^L_+:

$$U_i(x; a^i) = x_1^{a_{i1}} \cdots x_L^{a_{iL}}$$

*Address: Department of Economics, Doshisha University, Kamigyo-ku, Kyoto 602-8580 Japan; Phone: +81-75-251-3647; E-mail: tmomi@mail.doshisha.ac.jp

where $a_i = (a_{i1}, \dots, a_{iL}) \in R^L_{++}$ is the parameter defining the utility function. Clearly a^i can be identified with the utility function; and hence with the preference represented by the utility function. If a^i equals to $\bar{a}^i$ up to normalization ($a^i = t\bar{a}^i$ with positive $t \in R_+$), the preference defined by a^i equals to one by $\bar{a}^i$. We write $a^i \sim \bar{a}^i$ when a_i and $\bar{a}_i$ induce the same preference. A preference profile is a list of preferences of agents $a = (a_1, \dots, a_I) \in R^{LI}_{++}$. To deal with the case where we are interested in restricted set of the preferences, we let $A_i \subset R^L_{++}$ denote the set of a_i we are concerned and write $A = A_1 \times \dots \times A_I$

A social choice function $f : A \to X$ is a map from a preference proofile to an allocation. Let $f_i(a) = (f_{i1}(a), \dots, f_{iL}(a))$ denote the consumption bundle of agent i given by f at a.

Definition 1. An allocation $x \in X$ is Pareto efficient for a if there exists no $\bar{x} \in X$ such that $U_i(\bar{x}_i; a_i) \geq U_i(x_i; a_i)$ for all $i = 1, \dots, I$ and $U(\bar{x}_j; a_j) > U(x_j; a_j)$ for some $j = 1, \dots, I$. A social choice function $f : A \to X$ is Pareto-efficient if $f(a)$ is a Pareto-efficient allocation for any $a \in A$.

For $a = (a_1, \dots, a_I) \in A$, we write a_{-i} to denote the preferences of agents other than agent i and write $f(a'_i, a_{-i})$ to denote the allocation given by f at $(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_I)$.

Definition 2. A social choice function $f : A \to X$ is strategy-proof if $U_i(f_i(a); a_i) \geq U_i(f_i(a'_i, a_{-i}); a_i)$ for any $i = 1, \dots, I$, any $a \in A$ and $a'_i \in A_i$.

Definition 3. A social choice function f is dictatorial if there exists some agent i such that $f^i(a) = \omega$ for any $a \in A$.

3 Economy with two agents

In this section we consider the case where there exists two agents ($I = 2$). We let A_1 and A_2 be any open set of R^L_{++}. Thus a social choice function f is defined for $a = (a_1, a_2) \in A = A^1 \times A^2$. The result proved by Hashimoto (2008) imposed restrictions that A_1 and A_2 include parameters a_1 and a_2 respectively so that they induces the same preference and the preference is symmetric with respect to $L - 1$ goods, that is, $(1, \frac{a_{12}}{a_{11}}, \dots, \frac{a_{1L}}{a_{11}}) = (1, \frac{a_{22}}{a_{21}}, \dots, \frac{a_{2L}}{a_{21}}) = (1, t, \cdots, t)$.

Proposition 1. There exists no social choice function $f : A^1 \times A^2 \to X$ that is strategy-proof, Pareto-efficient and non-dictatorial.

The proof is essentially the same as the proof by Hashimoto (2008). We first show that $f(\cdot, a_j) : A_i \to X$ is a continuous function of a_i. Note that this does not generally imply that $f : A_1 \times A_2 \to X$ is a continuous function

Lemma 1. If $f : A \to X$ is a strategy-proof and Pareto-efficient social choice function, then $f(\cdot, a_j) : A_i \to X$ is a continuous function for any $a_j \in A_j$, $i, j = 1, 2$, $(i \neq j)$.

Proof. We arbitrarily fix $a_2 \in A_2$ and show that the function $f(\cdot, a_2) : A_1 \to X$ is continuous.

We suppose $a_1 \to \bar{a}_1 \in A_1$. Since X is compact, $f(a_1, a_2)$ converges as $a_1 \to \bar{a}_1$. We let $f(a_1, a_2) \to \bar{x}$. All we have to show is that $\bar{x} = f(\bar{a}_1, a_2)$.

Since f is strategy-proof, $U_1(f_1(a_1, a_2); a_1) \geq U_1(f_1(\bar{a}_1, a_2); a_1)$ for any a_1. Especially at the limit of $a_1 \to \bar{a}$, $U_1(\bar{x}_1; \bar{a}_1) \geq U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)$. If this equation holds with strict inequality, then the consumer would announce $\tilde{a}_1$ which is sufficiently close to $\bar{a}_1$ when his true preference is $\bar{a}_1$ because $f_1(\tilde{a}_1, a_2)$ is close to $\bar{x}_1$, and hence $U_1(f_1(\tilde{a}_1, a_2); \bar{a}_1)$ is close to $U_1(\bar{x}_1; \bar{a}_1)$. This violate the stratefy-proofness of f. Therefore the equation should hold with equality: $U_1(\bar{x}_1; \bar{a}_1) = U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)$.

We next show that $\bar{x}$ should be a Pareto-efficient allocation with respect to preferences $\bar{a}_1$ and a_2. Suppose that $\bar{x}$ is not Pareto-efficient. Then in the economy with Cobb-Douglas preferences there exists $x' = (x'_1, x'_2) \in X$ such that $U_1(x'_1; \bar{a}_1) > U_1(\bar{x}_1; \bar{a}_1)$ and $U_2(x'_2; a_2) > U_2(\bar{x}_2; a_2)$. When $\tilde{a}_1$ is sufficiently close to $\bar{a}_1$, $f(\tilde{a}_1, a_2)$ is sufficiently close to $\bar{x}$ and $U_1(f_1(\tilde{a}_1, a_2); \tilde{a}_1)$ is sufficiently close to $U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)$. Therefore $U_1(x'_1; \tilde{a}_1) > U_1(f_1(\tilde{a}_1, a_2); \tilde{a}_1)$ and $U_2(x'_2; a_2) > U_2(f_2(\tilde{a}_1, a_2); a_2)$ hold. This violates the Pareto-efficiency of f.

It is easy to observe that in the Edgeworth Box with two agents of Cobb-Douglas preferences the set of Pareto-efficient allocation intersects each consumer's one indifference surface only once. Therefore If $U(\bar{x}_1; \bar{a}_1) = U(f(\bar{a}_1, a_2); \bar{a}_1)$ and $\bar{x}$ and $f(\bar{a}_1, a_2)$ are both Pareto-efficient allocation, then $\bar{x} = f(\bar{a}_1, a_2)$ holds. ■

Lemma 2. $U_j(f_j(a_i, a_j); a_j) = U_j(f_j(a'_i, a_j); a_j)$ for any $i, j = 1, 2$, $(i \neq j)$ and any $a_i, a'_i \in A_i$ and $a_j \in A_j$.

Proof. We set $i = 1$ and $j = 2$ and prove the lemma for $a_1 = (a_{11}, a_{12} \ldots, a_{1L})$ and $a'_1 = (a'_{11}, a_{12}, \ldots, a_{1L})$. That is, we prove that the utility of the consumer 2 is not changed when a_{11} is changed. The discussion is symmetric for other components $a_{12}, \ldots, a_{1L}$. Since these changes of each component sums up to any changes of the parameter a_1, this is sufficient as the proof of the lemma.

Since $(a_{12}, \ldots, a_{1L})$ and a_2 are fixed in the following discussisons, we simply write $f_1(a_{11})$ and $f_2(a_{11})$ to denote the consumptions given by f at (a_1, a_2).

We suppose that there exists a'_{11} and a''_{11} such that $U_2(f_2(a'_{11}); a_2) \neq U_2(f_2(a''_{11}); a_2)$. Without loss of generality we assume $a'_{11} < a''_{11}$.

We first consider the case where $U_2(f_2(a'_{11}); a_2) > U_2(f_2(a''_{11}); a_2)$. Note that $U_2(f_2(a_{11}); a_2)$ is a continuous function of a_{11} by Lemma 1 proved above. Then there exist $\bar{a}_{11} \in (a'_{11}, a''_{11})$

and a sequence $\{\epsilon_n\}$ which converges to 0 from the right hand side, $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{U_2(f_2(\bar{a}_{11}+\epsilon_n);a_2) - U_2(f_2(\bar{a}_{11});a_2)}{\epsilon_n} < 0.^1$$

Since the utility function $U_2(\cdot;a_2)$ is differentiable, the equation becomes

$$\sum_{l=1}^{L} \frac{\partial U_2(f_2(\bar{a}_{11});a_2)}{\partial x_{2l}} \lim_{n\to\infty} \frac{f_{2l}(\bar{a}_{11}+\epsilon_n) - f_{2l}(\bar{a}_{11})}{\epsilon_n} < 0.$$

Since f is Pareto-efficient, $f_2(a_{11}) = \omega - f(a_{11})$ holds for any a_{11} and $(\frac{\partial U_2(f_2(a_{11});a_2)}{\partial x_{21}}, \dots, \frac{\partial U_2(f_2(a_{11});a_2)}{\partial x_{2L}})$ is parallel to $(\frac{\partial U_1(f_1(a_{11});a_1)}{\partial x_{11}}, \dots, \frac{\partial U_1(f_1(a_{11});a_1)}{\partial x_{1L}})$. Therefore we have

$$\sum_{l=1}^{L} \frac{\partial U_1(f_1(\bar{a}_{11});\bar{a}_1)}{\partial x_{1l}} \lim_{n\to\infty} \frac{f_{1l}(\bar{a}_{11}+\epsilon_n) - f_{1l}(\bar{a}_{11})}{\epsilon_n} > 0,$$

hence,

$$\lim_{n\to\infty} \frac{U_1(f_1(\bar{a}_{11}+\epsilon_n);\bar{a}_1) - U_1(f_1(\bar{a}_{11});\bar{a}_1)}{\epsilon_n} > 0,$$

where $\bar{a}_1 = (\bar{a}_{11}, a_{12} \dots, a_{1L})$. This implies $U_1(f_1(\bar{a}_{11}+\epsilon_n);\bar{a}_1) > U_1(f_1(\bar{a}_{11});\bar{a}_1)$ with sufficiently large n because $\epsilon_n > 0$. This violates the strategy-proofness of f because consumer 1 whould announce $(\bar{a}_{11}+\epsilon_n, a_{12}, \dots, a_{1L})$ when his true parameter is $\bar{a}_1$.

Next, we consider the case where $U_2(f_2(a'_{11});a_2) < U_2(f_2(a''_{11});a_2)$. Then there exist $\bar{a}_{11} \in (a'_{11}, a''_{11})$ and a sequence $\{\epsilon_n\}$ which converges to 0 from the left hand side, $\epsilon_n < 0$ and $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{U_2(f_2(\bar{a}_{11}+\epsilon_n);a_2) - U_2(f_2(\bar{a}_{11});a_2)}{\epsilon_n} > 0.$$

By the same discussion, we have

$$\lim_{n\to\infty} \frac{U_1(f_1(\bar{a}_{11}+\epsilon_n);\bar{a}_1) - U_1(f_1(\bar{a}_{11});\bar{a}_1)}{\epsilon_n} < 0.$$

This implies $U_1(f_1(\bar{a}_{11}+\epsilon_n);\bar{a}_1) > U_1(f_1(\bar{a}_{11});\bar{a}_1)$ with sufficiently large n because $\epsilon_n < 0$. This again violates the strategy-proofness of f. ■

Proof of Proposition 1. We select $a_1 \in A_1$ and $a_2 \in A_2$ such that $f(a_1, a_2)$ is in the interior of X. Such a pair (a_1, a_2) exists because f is non-dictatorial.

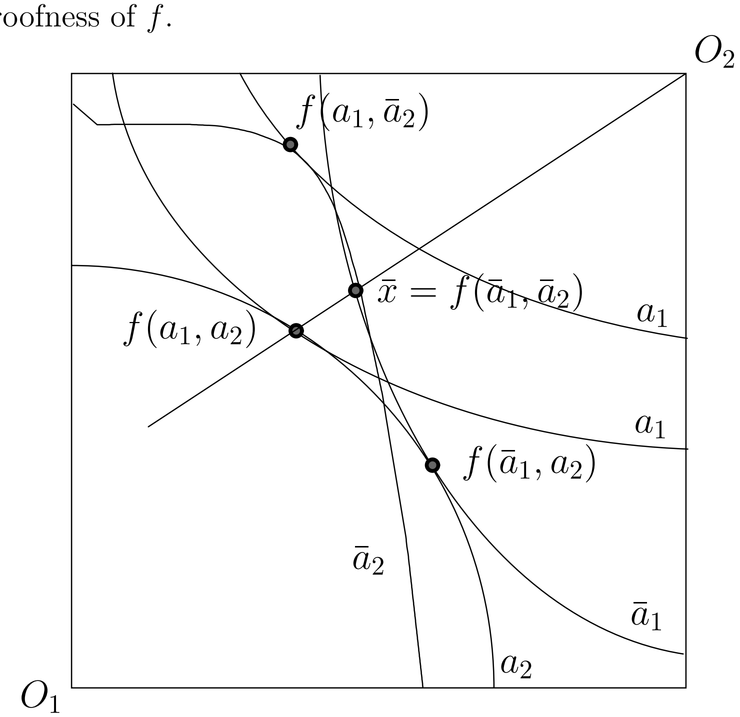
Choose any $\bar{a}_1$ which is sufficiently close to a_1. In the Edgeworth Box, consider a ray starting from the vertex of consumer 2 ($(x_1, x_2) = (\omega, 0)$) and passsing through $f(a_1, a_2)$.

[1] On the contrary, suppose that $\lim_{n\to\infty} \frac{U_2(f_2(\bar{a}_{11}+\epsilon_n);a_2)-U_2(f_2(\bar{a}_{11});a_2)}{\epsilon_n} > 0$ for any $\bar{a}_{11} \in (a'_{11}, a''_{11})$ and any sequence $\{\epsilon_n\}$ converging 0 from right hand side. It clearly contradicts to that $U_2(f(\cdot);a_2)$ is a continuous function and $U_2(f_2(a'_{11});a_2) > U_2(f_2(a''_{11});a_2)$.

Also consider the indifference set of consumer 1 with preference $\bar{a}_1$ at $f_1(\bar{a}_1, a_2)$: $\{x_1 \in R^L_+ | U_1(x_1; \bar{a}_1) = U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)\}$. Let $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ denote the intersection of this ray and this indifference set in the Edgeworth Box. It is easy to observe that the intersection is determined uniquely. It is also easy to observe that $\bar{x}_2 = tf_2(a_1, a_2)$ with some $0 < t < 1$. The reason is that if $t > 0$, then consumer 1 whould announce a_1 when his true parameter is $\bar{a}_1$.

We select parameter $\bar{a}_2$ so that the vector $(\frac{\partial U_2(\bar{x}_2;\bar{a}_2)}{\partial x_{21}}, \ldots, \frac{\partial U_2(\bar{x}_2;\bar{a}_2)}{\partial x_{2L}})$ is parallel to $(\frac{\partial U_1(\bar{x}_1;\bar{a}_1)}{\partial x_{11}}, \ldots, \frac{\partial U_1(\bar{x}_1;\bar{a}_1)}{\partial x_{1L}})$. Because of Lemma 2, $f(\bar{a}_1, \bar{a}_2)$ should be on the consuner 1's indifference set $\{x_1 \in R^L_+ | U_1(x_1; \bar{a}_1) = U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)\}$. Since the indifference set intersects the set of pareto-efficient allocations with preferences $\bar{a}_1$ and $\bar{a}_2$ only once, we have $f(\bar{a}_1, \bar{a}_2) = \bar{x}$.

Finally consider $f(a_1, \bar{a}_2)$. Observe that $f_2(a_1, \bar{a}_2)$ is indifferent to $\bar{x}_2$ with respect to the utility $\bar{a}_2$ because of Lemma 2 and that $f_2(a_1, a_2)$ is prefered to $\bar{x}_2$ with respect to any preference. Thus $U_2(f_2(a_1, a_2); \bar{a}_2) > U_2(f_2(a_1, \bar{a}_2); \bar{a}_2)$, which violates the strategy-proofness of f. ■



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