EXPOSURE PROBLEM IN MULTI-UNIT AUCTIONS

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Abstract

We characterize the optimal bidding strategies of local and global bidders for two heterogenous licenses in a multi-unit simultaneous ascending auction. The global bidder wants to win both licenses to enjoy synergies; therefore, she bids more than her stand-alone valuation of a license. This exposes her to the risk of losing money. We determine the optimal bidding strategies in the presence of an exposure problem. We show that the global bidder may accept a loss even when she wins all licenses.

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1 Introduction

In a typical American or Canadian spectrum license auction, hundreds of (heterogenous) licenses are sold simultaneously. Each of these licences gives the spectrum usage right of a geographical area to the winning firm. Some *'local'* firms are interested in winning only specific licenses in order to serve in local markets, while other *'global'* firms are interested in winning all licenses in order to serve nationwide.[1] The global firms enjoy synergies if they win all licenses. This gives them an incentive to bid over their stand alone valuations for some licenses. As a result, there is a risk of accepting losses. This is known as the exposure problem.

In a model simplifying the American and the recent Canadian spectrum license auctions, we derive the optimal bidding strategies of local and global firms in a simultaneous ascending auction. We mainly focus on the optimal bidding strategies when there is the possibility of an exposure problem.

The multi-unit auction literature generally assumes that global bidders have either equal valuations (Englmaier et. al (2009), Albano et. al. (2001), Kagel and Levin (2005), Katok and Roth (2004), Rosenthal and Wang (1996), and Krishna and Rosenthal (1996)) or very large synergies (Albano et al. (2006)). The spectrum licenses for different geographic areas are not homogenous objects; hence, the equal valuation assumption does not fit for the Canadian and the American spectrum license auction. Moreover, in a heterogeneous license environment, bidders may not drop out of both auctions simultaneously. This enables us to analyze bidding behavior in the remaining auction, and hence, the exposure problem in detail.

We allow for moderate synergies, and our focus is on the exposure problem unlike Albano et. al (2006).[2] In our paper, global bidders will lower their bids because of the exposure

[1] In the recent Canadian Advanced Wireless Spectrum auction, firms such as Globalive and Rogers were interested in all licenses whereas firms such as Bragg Communication and Manitoba Telecom Services (MTS) were interested in East Coast and Manitoba licenses, respectively.

[2] They assume large synergies so no exposure problem exists in equilibrium. Our results coincide with theirs when we also assume large synergies.

problem; however, their optimal strategy still requires them to bid over their stand alone valuation for at least one license. If they win this license by receiving a potential loss, then they may need to stay in the other license auction to minimize their loss. Therefore, there are cases in which the exposure problem may arise even when the bidder wins all licenses.

Kagel and Levin (2005) and Krishna and Rosenthal (1996) show that bidders bid more aggressively as the number of bidders decreases in multi-unit auctions with synergies. Chow and Yavas (2009) test this experimentally in a simultaneous second-price auction setting. We also find the same result in our simultaneous ascending auction; the global bidders' optimal drop out price increases as the other bidders drop out.

Almost all proofs are included in the Appendix.

2 The Model

There are 2 licenses, license A and B for sale.[3] There are one global bidder who demands both licenses and $m_j = m - 1$ local bidders who demand only license $j = A, B$.[4] Both local bidders and the global bidder have a private stand alone valuation for a single license, v_{ij}, where i and j represent the bidder and the license, respectively. The valuations v_{ij} are drawn from the continuous distribution function $F(v_{ij})$ with support on $[0, 1]$ and probability density function $f(v_{ij})$ which is positive everywhere with the only exception of $f(0) \geq 0$ is allowed. The type of bidders, global or local, is publicly known.

We consider a situation where the licenses are auctioned off simultaneously through an ascending multi-unit auction. Prices start from zero for all licenses and increase simultaneously and continuously at the same rate. When only one bidder is left on a given license, the clock stops for that license; hence, he wins the license at the price that the last bidder drops. At the same time, if there are more than one bidder on the remaining license, its price will continue to increase. If n bidders drop out at the same price and nobody is left in the auction, then each one of them will win the license with probability $\frac{1}{n}$.

[3]We use two licenses like Albano et. al. (2001 and 2006), Brusco and Lopomo (2002), Chow and Yavas (2009), and Menucicci (2003).

[4]Allowing different number of firms on different licenses will not change our qualitative results.

The dropout is irreversible; once a bidder drops out of bidding for a given license, he cannot bid for this license later.[5] The number of active bidders and the drop-out prices are publicly known. We also assume that there is no budget constraints for the bidders.

We assume that there are homogeneous positive synergies for the global bidder, and denote this kind of synergy by $\alpha > 0$ and α is public information (as in Albano et. al. (2006)). Then, the global bidder, denoted by Firm 1's total valuation, given that it wins two licenses is, $V_1 = v_{1A} + v_{1B} + \alpha$. His stand-alone valuation of license A or B is given by v_{1A} or v_{1B}. Firm iA, $i = 2, 3, ...m$ is only interested in license A; her private valuation is v_{iA} and Firm iB is only interested in license B; her private valuation is v_{iB}.[6]

We will derive a symmetric perfect Bayesian equilibrium with the help of lemmas that follow. First, we describe the equilibrium strategy of the local bidder.

Lemma 1 *: Each local bidder has a weakly dominant strategy to stay in the auction until the price reaches his stand alone valuation.*

This is a well-known result so we skip the proof.

Lemma 2 *If the global bidder wins license B (or A) first, then it will stay in license A (or B) auction until the price reaches $v_{1A} + \alpha$ (or $v_{1B} + \alpha$)*

The proof is as follows. Suppose the local bidder drops out of license B auction, and hence, the global bidder wins license B at the price p_B (in equilibrium, this price would be $p_B = max\{v_{2B}, ..., v_{(m)B}\}$ by lemma 1). Then, as the price for license A increases, the global bidder will compare the payoff from dropping out from license A auction at the clock price p (which is $v_{1B} - p_B$) and the payoff from winning license A at price p (which is $v_{1A} + v_{1B} + \alpha - p_B - p$. The updated optimal drop out price (denote it as p_A) is found by equating these two equations: $v_{1A} + v_{1B} + \alpha - p_B - p_A = v_{1B} - p_B \Rightarrow p_A = v_{1A} + \alpha$. If global bidder wins license A first, the updated optimal drop out price can be found symmetrically.

[5]In the real-world auctions, there is activity rule. If the bidders do not have enough highest standing bids, then the number of licenses they may bid is decreased (in the next rounds). Hence, when there are two licenses, this translates into an irreversible drop-out.

[6]A local bidder who is interested in license j participates only on license j auction. We do not assume that $v_{iA} > v_{iB}$ since local firms are different; hence, their efficiency may differ.

Lemma 3 *: a) The global bidder stays in both license auctions at least until the price reaches the minimum of his/her stand alone valuations.*

b) If his average valuation is greater than 1, his optimal strategy is to stay in until the price reaches his average valuation.[7]

The result comes from comparing the expected profits from dropping before the minimum of the stand alone valuations and dropping out at the minimum stand alone valuation. If the global bidder drops out before the minimum of its stand alone valuation, it loses the possibility of winning both licenses and enjoying the synergy. We skip the proof of this lemma.

When his average valuation, $\frac{V_1}{2} = \frac{v_{1A}+v_{1B}+\alpha}{2}$, exceeds 1; the global bidder will bid up to his average valuation, $\frac{V_1}{2}$. This will shut out the local bidders since local bidders' stand alone valuation can be at most 1.[8]

How to calculate the optimal drop out price for the global bidder? Let us look at the case in which $v_{1A} > v_{1B}$.[9] The global bidder must compare the payoffs for two cases at each price p as the clock is running: **Case 1** is the payoff from dropping out from license B auction at price p and optimally continuing on license A auction. **Case 2** is the payoff from winning license B at price p and optimally continuing on license A auction.[10] At the beginning of the auction, that is $p = 0$, the second case payoff is higher so the global bidder will start staying in the auction. We show that the difference between these two cases are monotonic in p ; therefore, there is a unique price that makes the global bidder indifferent between these two cases (assuming that the two local bidders are still active). This is the optimal drop out price. Let p_1^* denote this price and we show that this price can be calculated at the beginning of the auction. Note that according to Lemma 3, $p_1^* \geq v_{1B}$ and the optimal [updated] drop out price for license A -after winning license B at price p- is $v_{1A} + \alpha$ (by lemma 2).

[7]Note that this lemma is also valid for the general case when there are n global bidders.

[8]If α is large enough, this condition will always be satisfied. In such a case, the global bidder always wins both licenses in equilibrium.

[9]The other case can be calculated symmetrically.

[10]Remember that we assume $v_{1A} > v_{1B}$; hence, the global bidder will drop out of license B first -assuming that he has not won license A yet.

We denote the expected profit of Firm 1 for Case 1 by $E\Pi_1^1$ and the expected profit for Case 2 by $E\Pi_1^2$, respectively. The superscript represents which case Firm 1 plays and the subscript represents the global bidder, Firm 1. In the following equations, p_A denotes the price of license A, if Firm 1 wins license A. We have $p_A = \max\{v_{2A}, ..., v_{(m+1)A}\}$. Payoffs are as follows:

$$E\Pi_1^1 = Max\{0, \int_p^{v_{1A}} (v_{1A} - p_A) g(p_A|p) dp_A\} \tag{1}$$

$$E\Pi_1^2 = \int_p^{Min\{v_{1A}+\alpha,1\}} (V_1 - p - p_A) g(p_A|p) dp_A + \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p) g(p_A|p) dp_A \tag{2}$$

The explanation of equation 1 is as follows. After the global bidder, Firm 1, drops out of the auction for license B at p, it becomes just like a local bidder, and hence, will continue to stay in the auction for license A until v_{1A}. If he wins, he will pay p_A since the local bidder with highest valuation of license A will drop out last (by lemma 1). In order to calculate his expected profit, global bidder will be using $G(p_A|p)$ (highest order statistic) which is the distribution function of the local bidders' highest valuation p_A for license A given p. The function $G(p_A|p) = (F(p_A|p))^{m-1} = (\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-1}$ and the corresponding density function $g(p_A|p) = (m-1)(\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-2}(\frac{f(p_A)}{\int_p^1 f(v)dv})$.

The first term of $E\Pi_1^2$ is Firm 1's expected profit of winning two licenses (assuming that he wins license B at the price p). If the highest local bidder's valuation p_A is less than Firm 1's (updated) willingness to pay, $v_{1A} + \alpha$, then Firm 1 wins license A and pays p_A. Since $p_A < 1$, we use the minimum function in the upper limit of the first integral. The second term of $E\Pi_1^2$ is Firm 1's expected profit of winning only license B which can happen only if $p_A > v_{1A} + \alpha$. Note that the second term is non-positive by lemma 3 (which is the exposure problem arising from winning only one license).

In Lemma 4 below, we characterize the global bidder's optimal bids. It can be found from $E\Pi_1^1 = E\Pi_1^2$.

Lemma 4 : *Suppose that the average valuation of Firm 1 is less than 1 and no local bidders* *have dropped out yet.*

Firm 1, assuming that he has valuations such that $v_{1A} > v_{1B}$,[11] *will drop out of license B auction at the unique optimal drop-out price* $p_1^* \in [0, 1]$ *that satisfies* $E\Pi_1^1 = E\Pi_1^2$.

a) If $\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A + (v_{1B} - v_{1A}) < 0$, *then* $p_1^* < v_{1A}$ *and Firm 1 will stay in license A auction until* v_{1A} *(after dropping out from license B auction).*

b) If $\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A + (v_{1B} - v_{1A}) > 0$, *then* $p_1^* > v_{1A}$ *and Firm 1 will also drop out of license A auction at* p_1^*.

Proof. See the Appendix.

We are ready to summarize our Perfect Bayesian equilibrium.

Proposition 5 *(Perfect Bayesian Equilibrium)*

a) Out-of-equilibrium-path beliefs: If the global bidder, Firm 1, drops out of license A before license B then other firms' beliefs will not change. Specifically, they will continue to believe that Firm l will act like a local bidder from now on and its valuation for license B is drawn from distribution F on $[0, 1]$.

b) Lemma 1, 2, 3, 4 and out of equilibrium path beliefs constitute a Perfect Bayesian Nash Equilibrium.

At the beginning of the game, each firm calculates its optimal drop-out price. For local bidders, the optimal drop out prices are their valuations, respectively. In equilibrium, it is optimal for the global bidder to stay in the auctions for both licenses up to his optimal drop-out price calculated in lemma 4. When his average valuation exceeds 1, it will stay until this average valuation (and will win both licenses for sure). When the price reaches the minimum of these optimal drop put prices, that firm drops out of license auction. If, for example, the highest local bidder for license B dropped out before Firm 1, Firm 1 would continue to stay in the auction for license A until the price reaches $v_{1A} + \alpha$. At this price, it finds that the payoff from winning only license B is more than winning both licenses even though it will enjoy synergy; hence, it drops out.

[11] If $v_{1A} < v_{1B}$, then the proposition has to be written symmetrically

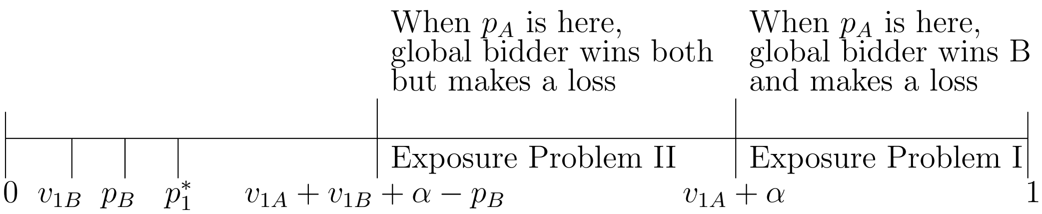

Figure 1: EXPOSURE PROBLEM

If the licenses were identical (e.g. Albano et. al. (2001)), the global firm would drop out of both licenses at the same time.

Let us give an example for the optimal drop out price by assuming that $F(.)$ is a uniform distribution, the optimal drop-out prices are given in corollary 6.

Corollary 6 *: Assume that valuations are drawn from a uniform distribution with a support* $[0, 1]$*. In addition, assume that* $v_{1A} > v_{1B}$ *(other case is symmetrically found by exchanging* v_{1A} *with* v_{1B}*).*

$$p_1^* = \begin{cases} \frac{1}{2}\{v_{1B} + \alpha + 1 - (v_{1B}^2 + 1 - 2v_{1B} - \alpha^2 + 2v_{1B}\alpha + 2\alpha - 4v_{1A}\alpha)^{\frac{1}{2}}\}, & \\ \qquad \textit{if } 0 < v_{1A} < 1 - \alpha \textit{ and } 2(1 - v_{1A})(v_{1A} - v_{1B}) > \alpha^2; & \\ \frac{1}{3}\{v_{1A} + v_{1B} + \alpha + 1 - ((v_{1A} + v_{1B} + \alpha + 1)^2 - 3(v_{1A} + \alpha)^2 - 6v_{1B})^{\frac{1}{2}}\}, & \\ \qquad \textit{if } 0 < v_{1A} < 1 - \alpha \textit{ and } 2(1 - v_{1A})(v_{1A} - v_{1B}) \leq \alpha^2; & \\ \frac{1}{2}\{v_{1B} + \alpha + 1 - \{(v_{1B} + \alpha + 1)^2 - 4(v_{1A} + v_{1B} + \alpha) + 2 + 2v_{1A}^2\}^{\frac{1}{2}}\}, & \\ \qquad \textit{if } 1 - \alpha \leq v_{1A} < 1 \textit{ and } 1 + v_{1A} > 2(v_{1B} + \alpha); & \\ \frac{2(v_{1A} + v_{1B} + \alpha) - 1}{3}, & \\ \qquad \textit{if } 1 - \alpha \leq v_{1A} < 1 \textit{ and } 1 + v_{1A} \leq 2(v_{1B} + \alpha). & \end{cases} \tag{3}$$

The optimal drop-out price is a function that takes a unique value defined in the corollary above. For example, case $0 < v_{1A} < 1 - \alpha$ and $2(1 - v_{1A})(v_{1A} - v_{1B}) > \alpha^2$ implies that $p_1^* < v_{1A}$.

2.1 Exposure Problem

Now we can discuss the exposure problem with the help of figure 1. In the first type of exposure problem, the global bidder may win license B at a price above his stand alone valuation (i.e., $v_{1B} < p_B < p_1^*$) and loses the other license (i.e., $p_A > v_{1A} + \alpha$, since he will continue to bid until $v_{1A} + \alpha$). This is the type of exposure problem Chakraborty (2004) focuses on.

In the second type of exposure problem, the global bidder wins both licenses but makes a loss. This is the case when he wins license B at $v_{1B} < p_B < p_1^*$ and wins license A at $v_{1A}+\alpha > p_A > v_{1A}+\alpha+v_{1B}-p_B$. Note that if he wins license A at the price $v_{1A}+\alpha+v_{1B}-p_B$, his payoff is zero. The global bidder stays in the auction for license A to minimize its loss once the price passes $v_{1A} + \alpha + v_{1B} - p_B$.

If objects were homogenous, second type of exposure problem would not be observed since the bidder would drop out of both license auctions at the same time.

In the next section, we will show that bid-withdrawal eliminates the exposure problem.

3 Conclusion and Discussion

We showed the optimal bidding strategies of global bidders when there are moderate synergies and the licenses are heterogeneous. We also analyzed exposure problem.

We used one global bidder like Kagel and Levin (2005). We were able to show exposure problem can occur even when the global bidder wins all licenses. Extending this to n global bidders would be very complicated since the optimal strategies of global bidders (optimal drop out prices) should be determined jointly which in turn would depend on how many local and how many global bidders are still in the auction.

4 Appendix

Proof of Lemma 4:

We will prove that there is a unique optimal drop out price by solving $E\Pi_1^1 = E\Pi_1^2$. We have two cases.

Case I: In this case, we will assume $\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|v_{1A})dp_A + (v_{1B} - v_{1A}) < 0$ implies $v_{1A} \geq p_1^*$ (which in turn implies $E\Pi_1^1 > 0$).

First, we show that there exists a unique solution that makes equations 1 and 2 equal, and this is the optimal drop out price p_1^*. We define a new function, $J(p) = E\Pi_1^1 - E\Pi_1^2$. To prove uniqueness, we will show that this function is monotonically increasing and it is negative when $p = v_{1B}$ (by lemma 2 p cannot be less than v_{1B}) and is positive when $p = v_{1A}$. Hence, there must be a unique root at the interval $v_{1B} < p < v_{1A}$.

$J(p,m) = \int_p^{v_{1A}} (v_{1A} - p_A) g(p_A|p) dp_A - \int_p^{Min\{v_{1A}+\alpha,1\}} (V_1 - p - p_A) g(p_A|p) dp_A$
$- \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p) g(p_A|p) dp_A$.

By using $(v_{1B} - p) \int_p^1 g(p_A|p) dp_A = v_{1B} - p$, we can re-write it as
$\int_p^{v_{1A}} (v_{1A} - p_A) g(p_A|p) dp_A - \int_p^{Min\{v_{1A}+\alpha,1\}} (v_{1A} + \alpha - p_A) g(p_A|p) dp_A - (v_{1B} - p)$

By using integration by parts twice (and using $dv = g(p_A|p)dp_A$), we have
$= (v_{1A} - p_A) G(p_A|p) \mid_p^{v_{1A}} - \int_p^{v_{1A}} G(p_A|p) d(v_{1A} - p_A)$
$- (v_{1A} + \alpha - p_A) G(p_A|p) \mid_p^{Min\{v_{1A}+\alpha,1\}} + \int_p^{Min\{v_{1A}+\alpha,1\}} G(p_A|p) d(v_{1A} + \alpha - p_A) - (v_{1B} - p)$
$= \int_p^{v_{1A}} G(p_A|p) dp_A - \int_p^{Min\{v_{1A}+\alpha,1\}} G(p_A|p) dp_A - (v_{1B} - p)$

We take partial derivative of $J(p,m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = \frac{\partial}{\partial p}[-\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)] + 1 > 0$

It is positive since the term $\frac{\partial}{\partial p}[\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)]$ is negative. As the lower limit of the integral increases, the value of the expression decreases (does not increase) if the term inside is non-negative which is true since it is a cumulative distribution function. We must also show that $\frac{\partial G(p_A|p)}{\partial p} \leq 0$ to prove this. While one can easily see that this is correct (as p increases the cumulative distribution conditional on p decreases), we will give a formal proof by using Leibniz's rule when necessary.

$\Leftrightarrow \frac{\partial G(p_A|p)}{\partial p} = \frac{\partial [(\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-1}]}{\partial p}$
$= -(m-1) f(p) \frac{(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}} + (m-1) f(p) \frac{(\int_p^{p_A} f(v)dv)^{m-1}}{(\int_p^1 f(v)dv)^{m}}$
$= \frac{(m-1) f(p) (\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}} [-1 + \frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv}]$
$= \frac{(m-1) f(p) (\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}} [-1 + F(p_A|p)] < 0$ (≤ 0 only if $p_A = 1$).

Thus, $J(p,m)$ is monotonically increasing function of p, when $v_{1B} \leq p < v_{1A}$.

If $p = v_{1B}$, then $J(v_{1B}) = \int_{v_{1B}}^{v_{1A}} G(p_A|\alpha) dp_A - \int_{v_{1B}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|v_{1B}) dp_A$
$= - \int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|v_{1B}) dp_A < 0$.

If $p = v_{1A}$, $J(v_{1A}) = 0 - \int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p) dp_A - (v_{1B} - v_{1A}) > 0$, then our assumption $\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p) dp_A + (v_{1B} - v_{1A}) < 0$ implies that $J(p = v_{1A}) > 0$.

Hence, there is a unique root in the interval $v_{1B} < p < v_{1A}$.

Next, we show that as the number of active firms in license A auction decreases, the optimal drop out price will increase. We will use the implicit function theorem for this:

$\Leftrightarrow \frac{dp_1^*}{dm} = -\frac{\frac{\partial J(p_1^*,m)}{\partial m}}{\frac{\partial J(p_1^*,m)}{\partial p_1^*}} < 0.$

We have already shown that $\frac{\partial J(p_1^*,m)}{\partial p_1^*} > 0.$

Since $J(p,m) = \int_p^{v_{1A}} G(p_A|p)dp_A - \int_p^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A - (v_{1B} - p).$

We take partial derivative of $J(p,m)$ with respect to m, that is,

$\frac{\partial J(p,m)}{\partial m} = \int_p^{v_{1A}} \frac{\partial G(p_A|p)}{\partial m} dp_A - \int_p^{Min\{v_{1A}+\alpha,1\}} \frac{\partial G(p_A|p)}{\partial m} dp_A$

$= -\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} \frac{\partial G(p_A|p)}{\partial m} dp_A = -\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} \ln(F(p_A|p))G(p_A|p)dp_A > 0.$

Since $\frac{\partial G(p_A|p)}{\partial m} = \ln(F(p_A|p))G(p_A|p) < 0$. Hence, we show that $\frac{\partial G(p_A|p)}{\partial m} > 0$ holds.

By the implicit function theorem, we show that the optimal drop out price increases as the number of local firms, m, decreases.

Since $\frac{\partial J(p,m)}{\partial p} > 0$ and $\frac{\partial J(p,m)}{\partial m} > 0$, we have, $\frac{dp_1^*}{dm} = -\frac{\frac{\partial F(p_1^*,m)}{\partial m}}{\frac{\partial F(p_1^*,m)}{\partial p_1^*}} < 0$

Case II: In this case, we will assume that $\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A + (v_{1B} - v_{1A}) > 0$ which implies $v_{1A} \leq p_1^*$. And this condition in turn implies that $E\Pi_1^1 = 0$.

Now let $J(p,m) = E\Pi_1^1 - E\Pi_1^2$

$J(p,m) = 0 - \int_p^{Min\{v_{1A}+\alpha,1\}}(V_1 - p - p_A)g(p_A|p)dp_A - \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p)g(p_A|p)dp_A$

$= -\int_p^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A - (v_{1B} - p).$

When $p \geq v_{1A}$, we take partial derivative of $J(p,m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = -\frac{\partial}{\partial p}[\int_p^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A] + 1 > 0$, since $\frac{\partial G(p_A|p)}{\partial p} < 0$.

Thus, $J(p,m)$ is monotonically increasing function of p, when $v_{1A} \leq p \leq Min\{v_{1A}+\alpha, 1\}$.

Our assumption $\int_{v_{1A}}^{Min\{v_{1A}+\alpha,1\}} G(p_A|p)dp_A + (v_{1B} - v_{1A}) < 0$ implies that $J(p = v_{1A}) < 0$. If $p = Min\{v_{1A} + \alpha, 1\}$, then $J(Min\{v_{1A} + \alpha, 1\}) = 0 - 0 - (v_{1B} - Min\{v_{1A} + \alpha, 1\}) > 0$. Thus, there is a unique solution, p_1^*, in the interval $(v_{1A}, v_{1A} + \alpha)$.

Next, we show that when the number of active firms in license A auction decreases, this optimal drop out price will increase.

We take partial derivative of $J(p,m)$ with respect to m, we have,

$\frac{\partial J(p,m)}{\partial m} = -\int_p^{Min\{v_{1A}+\alpha,1\}} \ln(F(p_A|p))G(p_A|p)dp_A > 0.$

Since $\frac{\partial J(p,m)}{\partial p} > 0$ and $\frac{\partial J(p,m)}{\partial m} > 0$, we have,

$$\frac{dp_1^*}{dm} = -\frac{\frac{\partial J(p_1^*,m)}{\partial m}}{\frac{\partial J(p_1^*,m)}{\partial p_1^*}} < 0$$

■

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