Games with Discontinuous Payoffs: a Strengthening of Reny's Existence Theorem*

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Abstract: We provide a pure Nash equilibrium existence theorem for games with discontinuous payoffs whose hypotheses are in a number of ways weaker than those of the theorem of Reny (1999). Our result subsumes a prior existence result of Nishimura and Friedman (1981) that is not covered by his theorem. In comparison with Reny's argument, our proof is brief.

1 Introduction

Many important and famous games in economics (e.g, the Hotelling location game, Bertrand competition, Cournot competition with fixed costs, and various auction models) have discontinuous payoffs, and consequently do not satisfy the hypotheses of Nash's existence proof or its infinite dimensional generalizations, but nonetheless have a nonempty set of pure Nash equilibria. Using an argument that is quite ingenious and involved, Reny (1999) establishes a result that subsumes earlier equilibrium existence results covering many such examples. His theorem's hypotheses are simple and weak, and in many cases easy to verify. The result has been applied in novel settings many times since then. (See for example Monteiro and Page (2008).) Largely in response to his work, a number of papers on discontinuous games have appeared recently (Bagh and Jofre (2006); Bich (2006, 2008); Monteiro and Page (2007); Carmona (2008); Prokopovich (2008); Barelli and Soza (2009); de Castro (2009); Tian (2009)). Of these, Barelli and Soza (2009) deserves special mention because it adopts many of our techniques from an earlier version of this paper.

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As Reny explains, the results of Nishimura and Friedman (1981) and Roberts and Sonnenschein (1976) seemingly have a different character, and are not obvious consequences of his result. Here we provide a generalization of Reny's theorem that easily implies the Nishimura and Friedman result. The concepts introduced in Section 3 were inspired by the hope of developing a generalization that could by used to prove the existence result of Novshek (1985), which is the most refined result asserting existence of Cournot equilibrium in the stream of literature following Roberts and Sonnenschein. This was eventually accomplished for the case of two firms, but extending the argument to an arbitrary number of firms proved to be quite difficult, and subsequently we learned of the work of Kukushkin (1994), which presents a brief proof of the general case. As with various proofs of topological fixed point theorems, Kukushkin's argument combines a nontrivial combinatoric result with a straightforward limiting argument. The combinatoric component of Kukushkin's argument is quite different from the analogous component of proofs of Brouwer's fixed point theorem, suggesting that the Novshek-Kukushkin fixed point theorem has a fundamentally different character.

The next section reviews Reny's theorem and states a result whose hypotheses are less restrictive than Reny's, but more restrictive than our main result. We also explain how this result implies the Nishimura-Friedman existence theorem. Section 3 states our main result Theorem 3.3. Reny introduces two concepts, payoff security and reciprocal upper semicontinuity, which together imply the hypotheses of his main result, and which are often easy to verify in applications. Section 3 also explains the extension of those concepts to our setting. Section 4 discusses an example illustrating the application of our result, and Section 5 gives its proof.

2 Preliminaries: Reny and Nishimura-Friedman

Our system of notation is largely taken from Reny (1999). There is a fixed *normal form game*

$$G = (X_1, \ldots, X_N, u_1, \ldots, u_N)$$

with N players, where, for each $i = 1, \ldots, N$, the i-th player's *strategy set* X_i is a non-empty compact convex subset of a topological vector space, and the i-th player's *payoff function* u_i is a function from the set of *strategy profiles* $X = \prod_{i=1}^N X_i$ to $\mathbb{R}$.

We adopt the usual notation for "all players other than i." Let $X_{-i} = \prod_{j \neq i} X_j$. If $x \in X$ is given, x_{-i} denotes the projection of x on X_{-i}. For given $x_{-i} \in X_{-i}$ (or $x \in X$) and $y_i \in X_i$ we write (y_i, x_{-i}) for the strategy profile $z \in X$ satisfying $z_i = y_i$ and $z_j = x_j$ if $j \neq i$. We endow X and each X_{-i} with their product topologies.

We now review Reny's theorem. A *Nash equilibrium* of G is a point $x^* \in X$ satisfying $u_i(x^*) \geq u_i(y_i, x^*_{-i})$ for all i and all $y_i \in X_i$. (This would usually be described as a "pure Nash equilibrium," but we never refer to mixed equilibria, so we omit the qualifier 'pure.') For each player i let $B_i\colon X\times\mathbb{R}\to X_i$ be the set valued mapping

$$B_i(x,\alpha_i)=\{y_i\in X_i\colon u_i(y_i,x_{-i})\geq\alpha_i\}\,.$$

Definition 2.1. A player i can *secure* a payoff $\alpha_i\in\mathbb{R}$ at $x\in X$ if there is a neighborhood U of x in X such that

$$\bigcap_{z\in U}B_i(z,\alpha_i)\neq\emptyset.$$

That is, there is some $y_i\in X_i$ such that $u_i(y_i,z_{-i})\geq\alpha_i$ for all $z\in U$.

Throughout we assume that each u_i is bounded. Let

$$u=(u_1,\dots,u_N):X\to\mathbb{R}^N.$$

For $x\in X$ let $A(x)$ be the set of $\alpha\in\mathbb{R}^N$ such that (x,α) is in the closure of the graph of u. Since u is bounded, each $A(x)$ is compact. The game G is *better reply secure* at $x\in X$ if, for any $\alpha'\in A(x)$, there is some player i and $\varepsilon>0$ such that player i can secure $\alpha'_i+\varepsilon$ at x. The game G is *better reply secure* if it is better reply secure on every strategy profile that is not a Nash equilibrium.

Theorem 2.2 (Reny (1999)). *Suppose that for each i and each $x_{-i}\in X_{-i}$ the function $u_i(\cdot,x_{-i})\colon X_i\to\mathbb{R}$ is quasiconcave. If G is better reply secure, then it has a Nash equilibrium.*

To better understand Reny's result in the context of this note we reformulate better reply security.

Definition 2.3. The game is *A-secure* at $x\in X$ if there is $\alpha\in\mathbb{R}^N$ and $\varepsilon>0$ such that:

(a) every player i can secure $\alpha_i+\varepsilon$ at x;

(b) there is a neighborhood U of x such that for any $z\in U$ there exists some player i with $u_i(z_i)<\alpha_i-\varepsilon$, i.e., $z_i\notin B_i(z,\alpha_i-\varepsilon)$.

Lemma 2.4. *For each $x\in X$ the game is better reply secure at x if and only if it is A-secure at x.*

Proof. First assume that the game is A-secure at x, with α, ε and U as in the definition. Each $\alpha'\in A(x)$ is the limit of values of u along some sequence or net converging to x, so there is some i with $\alpha'_i\leq\alpha_i-\varepsilon$. This i can secure $\alpha'_i+\varepsilon$ at x, which shows that the game is better reply secure at x.

Now assume that the game is better reply secure at x. For each i let τ_i be a neighborhood base of x_{-i}, and let

$$\beta_i=\sup_{y_i\in X_i}\ \sup_{U\in\tau_i}\ \inf_{z_{-i}\in U}u_i(y_i,z_{-i})\,.$$

Then $\alpha'_i < \beta_i$ if and only if there is some $\varepsilon > 0$ such that $\alpha'_i + \varepsilon$ can be secured by player i at x. Since the game is better reply secure at x, for each $\alpha' \in A(x)$ there is some player i such that $\beta_i > \alpha'_i$, which implies that the inequality $\beta_i > \alpha''_i + \varepsilon$ holds for some $\varepsilon > 0$ and all α'' in some neighborhood of α'. Since $A(x)$ is compact, it is covered by finitely many such neighborhoods, so we may choose $\varepsilon > 0$ such that for any $\alpha' \in A(x)$ there is some i with $\beta_i > \alpha'_i + 2\varepsilon$. Define $\alpha \in \mathbb{R}^N$ by setting

$$\alpha_i = \beta_i - \varepsilon.$$

In view of the definition of β_i, for each i, player i can secure $\alpha_i + \varepsilon$ at x, as per (a).

Aiming at a contradiction, suppose (b) is false, so for each $U \in \tau_i$ there is some $z_U \in U$ such that $u_i(z_U) \geq \alpha_i - \varepsilon$ for all i. Since τ_i is a directed set (ordered by reverse inclusion) the boundedness of the image of u implies that there is a convergent subnet, so there is $\alpha' \in A(x)$ such that $\alpha'_i \geq \alpha_i - \varepsilon = \beta_i - 2\varepsilon$ for all i, contrary to what we showed above. ■

For each i let $C_i \colon X \times \mathbb{R} \to X_i$ be the set valued mapping

$$C_i(x, \alpha_i) = \operatorname{con} B_i(x, \alpha_i)\,.$$

Definition 2.5. The game is *B-secure* at $x \in X$ if there is $\alpha \in \mathbb{R}^N$ such that:

(a) every player i can secure α_i at x;

(b) there is a neighborhood U of x such that for any $z \in U$ there exists some player i with $z_i \notin C_i(z, \alpha_i)$.

Because there is no $\varepsilon > 0$, in one respect B-secure is more easily satisfied than A-security. A very simple maximization example illustrates how this may matter: let $N = 1$ and $X_1 = [0, 1]$ with

$$u_1(x_1) = \begin{cases} 0, & x_1 = 0, \\ (x_1 - \frac{1}{2})^2, & 0 < x_1 \leq 1. \end{cases}$$

The game is B-secure at every $x \in [0, 1)$ for $\alpha_1 = \frac{1}{4}$. But the game is not A-secure at $x_1 = 0$. Proposition 2.6 below implies that it has an equilibrium, which obviousely is $x_1 = 1$.

On the other hand, in part (b) Definition 2.5 replaces $B_i(z, \alpha_i)$ with $C_i(z, \alpha_i)$, which makes it harder to satisfy. However, $B_i(z, \alpha_i) = C_i(z, \alpha_i)$ when each $u_i(\cdot, x_{-i})$ is quasiconcave, so the net effect is to make the hypotheses of Proposition 2.6 below weaker than those of Reny's Theorem 2.2. The hypotheses of the main result Theorem 3.3 are weaker still.

Proposition 2.6. *If the game is B-secure at each $x \in X$ that is not a Nash equilibrium, then G has a Nash equilibrium.*

Nishimura and Friedman (1981) prove the existence of a Nash equilibrium when each X_i is a nonempty, compact, convex subset of a Euclidean space, u is continuous (but not necessarily quasiconcave) and for any x that is not a Nash equilibrium there is an agent i, a coordinate index k, and an open neighborhood U of x, such that

$$(y^1_{ik} - x^1_{ik})(y^2_{ik} - x^2_{ik}) > 0$$

whenever $x^1, x^2 \in U$ and y^1_i and y^2_i are best responses for i to x^1 and x^2 respectively. Using compactness, and the continuity of u_i, it is not difficult to show that this is equivalent to $(y^1_{ik} - x_{ik})(y^2_{ik} - x_{ik}) > 0$ for any two best responses y^1_i, y^2_i to x. A more general condition that does not depend on the coordinate system, or the assumption of finite dimensionality, is that there is a hyperplane that strictly separates x_i from the set of i's best responses to x.

We now show that if G satisfies the hypotheses of Nishimura and Friedman's result, then it is B-secure and thus satisfies the hypotheses of Proposition 2.6. Consider an $x \in X$ that is not a Nash equilibrium. For each $i = 1, \dots, N$ let β_i be the utility for i when other agents play their components of x_{-i} and i plays a best response to x. Since u is continuous, for any $\varepsilon > 0$ player i can secure $\beta_i - \varepsilon$ at x by playing such a best response. For any neighborhood V of $B_i(x, \beta_i)$ it is the case that $B_i(x, \beta_i - \varepsilon) \subset V$ when ε is sufficiently small, and in turn it follows that there is a neighborhood U of x such that $B_i(z, \beta_i - \varepsilon) \subset V$ for all $z \in U$. It follows that if there is a hyperplane strictly separating x_i from $B_i(x, \beta_i)$, then for sufficiently small $\varepsilon > 0$ and a sufficiently small neighborhood U of x, this hyperplane also strictly separates z_i and $B_i(z, \beta_i - \varepsilon)$ for all $z \in U$, in which case $z_i \notin C_i(z, \beta_i)$. Setting $\alpha = (\beta_1 - \varepsilon, \dots, \beta_N - \varepsilon)$ gives the required property.

3 The Main Result and Payoff Security

For each i fix a set valued mapping $\mathcal{X}_i \colon X \to X_i$, and let $\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_N)$. We call $\mathcal{X}$ a *restriction operator*. For each i define the set valued mappings $B^{\mathcal{X}}_i \colon X \times \mathbb{R} \to X_i$ and $C^{\mathcal{X}}_i \colon X \times \mathbb{R} \to X_i$ by setting

$$B^{\mathcal{X}}_i(x, \alpha_i) = \{y_i \in \mathcal{X}_i(x) \colon u_i(y_i, x_{-i}) \geq \alpha_i\} \quad \text{and} \quad C^{\mathcal{X}}_i(x, \alpha_i) = \operatorname{con} B^{\mathcal{X}}_i(x, \alpha_i).$$

This machinery can be motivated by considering Cournot oligopoly with fixed costs. It will often happen that the quantities that allow a firm to obtain at least a certain level of profits α_i, for a given vector x_{-i} of quantities of its rivals, will include both increases and reductions to zero, in which case the current quantity will be in $C_i(x, \alpha_i)$, contrary to (b) of Definition 2.5. Introducing a restriction operator allows the analyst to restrict attention to one possibility or the other at each point, and if this is done in a "sufficiently continuous" manner the conditions below will be satisfied.

We should expect the restriction operator to be defined in different ways in different regions of X, and consequently the power of this technique should be enhanced if we allow multiple securing strategies to be employed at points in the boundaries of these regions. Our main result allows this possibility, but it should be mentioned that it does not figure in the analysis of the example in the next section.

Definition 3.1. Player i can *$\mathcal{X}$-secure* a payoff $\alpha_i \in \mathbb{R}$ at $x \in X$ if there is a finite closed cover $F^1, \dots, F^J$ of a neighborhood of x such that for each j we have $\bigcap_{z \in F^j} B_i^{\mathcal{X}}(z, \alpha_i) \neq \emptyset$.

As in Reny (1999) X need not be a Hausdorff space. Thus, for $x \in X$ the set $\{x\}$ need not be closed, and we let $[x]$ denote the closure of $\{x\}$.

Definition 3.2. If $x \in X$, $U \subset X$, and $\alpha \in \mathbb{R}^N$, then the game G is *$(\mathcal{X}, \alpha, U)$-secure at x* if:

(a) each player i can $\mathcal{X}$-secure α_i at x;

(b) for any $z \in U$ there is some player i for whom $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$.

The game is *$(\mathcal{X}, \alpha)$-secure at x* if it is $(\mathcal{X}, \alpha, U)$-secure at x for some neighborhood U of x, and it is *$\mathcal{X}$-secure at x* if there is some α such that it is $(\mathcal{X}, \alpha)$-secure at x. We say that G is *$\mathcal{X}$-secure* if it is $\mathcal{X}$-secure at every x such that $[x]$ does not contain a Nash equilibrium, and we say that G is *restrictionally secure* if there is a restriction operator $\mathcal{X}$ such that it is $\mathcal{X}$-secure.

Our main result, which is proved in Section 5, is as follows:

Theorem 3.3. *The game G is restrictionally secure if and only if it has a Nash equilibrium.*

Reny points out that a combination of two conditions, payoff security and reciprocal upper semicontinuity, imply the hypotheses of his existence result, and in applications it is typically relatively easy to verify them when they hold. We now define natural generalizations of these notions in our setting, and establish that together they imply that the game is restrictionally secure.

Reny's notion of payoff security requires that for each x and $\varepsilon > 0$ each player i can secure $u_i(x) - \varepsilon$ at x. We say that $\mathcal{X}$ is *universal* if $\mathcal{X}_i(x) = X_i$ for all i and $x \in X$. Even when this is the case the following definition is more easily satisfied than Reny's notion because of the possibility of using different securing strategies in different cells of a finite closed cover of a neighborhood of x.

Definition 3.4. The game G is *$\mathcal{X}$-payoff secure at $x \in X$* if, for each $\varepsilon > 0$, each player i can $\mathcal{X}$-secure $u_i(x) - \varepsilon$ at x. The game G is *$\mathcal{X}$-payoff secure* if it is $\mathcal{X}$-secure at each $x \in X$.

Reny's reciprocal upper semicontinuity requires that $u(x) = \alpha$ whenever (x, α) is in the closure of the graph of u and $\alpha \geq u(x)$. That is, for each $\delta > 0$ there is a neighborhood U of x and an $\varepsilon > 0$ such that for each $z \in U$, if $u_i(z) > u_i(x) + \delta$ for some j, then there must be a player j with $u_j(z) < u_j(x) - \varepsilon$. Our generalization is as follows:

Definition 3.5. The game G is *$\mathcal{X}$-reciprocally upper semicontinuous at $x \in X$* if for every $\delta > 0$ there exists $\varepsilon > 0$ and a neighborhood U of x such that for each $z \in U$, if there is a player i with $z_i \in C_i^{\mathcal{X}}(z, u_i(x) + \delta)$, then there is a player j such that $z_j \notin C_j^{\mathcal{X}}(z, u_j(x) - \varepsilon)$. The game G is *$\mathcal{X}$-reciprocally upper semicontinuous* if it is reciprocally upper semicontinuous at each $x \in X$.

The next result generalizes Reny's Proposition 3.2, which asserts that payoff security and reciprocal upper semicontinuity imply better reply security. We say that $\mathcal{X}$ is *other-directed* if, for each i, $\mathcal{X}_i(x)$ depends only on x_{-i}, so that $\mathcal{X}_i(y_i, x_{-i}) = \mathcal{X}_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $y_i, z_i \in X_i$.

Proposition 3.6. *If $\mathcal{X}$ is other-directed and G is $\mathcal{X}$-payoff secure and $\mathcal{X}$-reciprocally upper semicontinuous, then it is $\mathcal{X}$-secure. Thus, the game has a Nash equilibrium.*

Proof. Fixing an $x \in X$ that is not a Nash equilibrium, our goal is to show that the game is $\mathcal{X}$-secure at x. Let $\mathcal{I}$ be the set of players such that x_i is not a best response to x_{-i}. For each $i \in \mathcal{I}$, choose y_i such that $u_i(y_i, x_{-i}) > u_i(x)$, and let $\delta > 0$ be small enough that $u_i(y_i, x_{-i}) > u_i(x) + \delta$ for all $i \in \mathcal{I}$. Since the game is $\mathcal{X}$-payoff secure, each $i \in \mathcal{I}$ can $\mathcal{X}$-secure $\alpha_i = u_i(x) + \delta$ at (y_i, x_{-i}). Since $\mathcal{X}$ is other-directed, player i can also $\mathcal{X}$-secure α_i at x.

Since the game is $\mathcal{X}$-reciprocally upper semicontinuous, there is an $\varepsilon > 0$ and a neighborhood U of x such that for all $z \in U$, if $z_i \in C_i^{\mathcal{X}}(z, u_i(x) + \delta)$ for some i, then $z_j \notin C_j^{\mathcal{X}}(z, u_j(x) - \varepsilon)$ for some j. Since the game is $\mathcal{X}$-payoff secure, each $j \notin \mathcal{I}$ can $\mathcal{X}$-secure $\alpha_j = u_j(x) - \varepsilon$ at x. It is now the case that for each $z \in U$, either $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$ for all $i \in \mathcal{I}$ or $z_j \notin C_j^{\mathcal{X}}(z, \alpha_j)$ for some $j \notin \mathcal{I}$. ■

Reny points out (Corollary 3.4) that if each u_i is lower semicontinuous in the strategies of the other players, then the game is necessarily payoff secure because for each x, $\varepsilon > 0$, and i, player i can use x_i to secure $u_i(x) - \varepsilon$. Since it is possible that $x_i \notin \mathcal{X}_i(x)$, this logic does not extend to our setting. However, the game is evidently $\mathcal{X}$-payoff secure if:

(a) for each each $x \in X$, i, and $\epsilon > 0$, the set $B_i^{\mathcal{X}}(x, u_i(x) - \epsilon)$ is nonempty,

(b) for each i and $y_i \in X_i$ the set $\{x \in X : y_i \in \mathcal{X}_i(x)\}$ is open, and

(c) each u_i is lower semicontinuous in the strategies of the other players.

Bagh and Jofre (2006) show that best reply security is implied by payoff security and a condition that is weaker than reciprocal upper semicontinuity. We have not managed to find a suitable analog of that concept. Carmona (2008) shows that existence of equilibrium is implied by a weakening of payoff security and a weak form of upper semicontinuity. Again, we do not know of analogs of these conditions in our setting.

4 An Example

This section presents an example illustrating how restriction operators can be applied. Let $X_1 = X_2 = [0, 1]$, and let $u_1, u_2 : X \to \mathbb{R}$ be bounded utility functions that are upper semicontinuous and continuous in the other agent's strategy, so that for each i and x_i, $u_i(x_i, \cdot)$ is continuous.

For each $i = 1, 2$ let $r_i \colon \mathbb{R} \to \mathbb{R}$ be the set valued function

$$r_i(x_{-i}) = \operatorname*{argmax}_{z_i \in \mathbb{R}} u_i(z_i, x_{-i}).$$

Since u_i is upper semicontinuous, r_i is nonempty valued and has a closed graph, which is to say that it is an upper semicontinuous correspondence. Let

$$r_i^+(x_{-i}) = \max r_i(x_{-i}) \quad \text{and} \quad r_i^-(x_{-i}) = \min r_i(x_{-i}).$$

The key assumption is a version of strategic complementarity: if $x_{-i} < x'_{-i}$, then $r_i^+(x_{-i}) \le r_i^-(x'_{-i})$. Our example is obtained from the two firm Cournot model by reversing the ordering of one of the agent's strategies; Novshek (1985) discusses assumptions on the Cournot model that imply this condition. We also assume that $r_i^-(0) > 0$ and $r_i^+(1) < 1$ for both i. This is without loss of generality: for example, for $\varepsilon > 0$ we can define an extended game with the same set of equilibria by setting $\tilde{X}_1 = \tilde{X}_2 = [0, 1 + \varepsilon]$ and

$$\tilde{u}_i(\tilde{x}) = u_i(\min\{\tilde{x}_1, 1\}, \min\{\tilde{x}_2, 1\}) - \max\{0, \tilde{x}_i - 1\}.$$

We will say that x is a *quasiequilibrium* if $r_i^-(x_{-i}) \le x_i \le r_i^+(x_{-i})$ for both i. Applying the Kakutani fixed point theorem to the correspondence

$$x \mapsto [r_1^-(x_2), r_1^+(x_2)] \times [r_2^-(x_1), r_2^+(x_1)]$$

shows that the set of quasiequilibria is nonempty and compact, and our assumptions imply that it is contained in $(0, 1)^2$. If x and x' are quasiequilibria with $x'_1 > x_1$, then $x'_2 \ge r_2^-(x'_1) \ge r_2^+(x_1) \ge x_2$, and similarly with the two agents reversed. Since the set of quasiequilibria is compact, there is a quasiequilibrium x^* such that $x_1 \le x_1^*$ and $x_2 \le x_2^*$ for all quasiequilibria x. Let

$$\Omega = \{\, x \in [0, 1]^2 : x_1 \le x_1^* \text{ or } x_2 \le x_2^* \,\}.$$

The restriction operator $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2)$ is defined by setting

$$\mathcal{X}_i(x) = \begin{cases} [r_i^-(x_{-i}), 1], & \text{if } x \notin \Omega, \\ (x_i, 1], & \text{if } x \in \Omega \text{ and } x_i \neq 1, \\ \{1\}, & \text{if } x \in \Omega \text{ and } x_i = 1. \end{cases}$$

We will show that if there is no Nash equilibrium, then the game is $\mathcal{X}$-secure. An important point is that $r_2^-(x_1) > x_2^*$ whenever $x_1 > x_1^*$. This is certainly the case if $r_1^+(x_2^*) = x_1^*$ because then $r_2^+(x_1^*) > x_2^*$. (Otherwise x^* is a Nash equilibrium.) On the other hand, if $r_1^+(x_2^*) > x_1^*$ and $r_2^-(x_1) = x_2^*$, then $r_2^-(x_1') = r_2^+(x_1') = x_2^*$ whenever $x_1^* < x_1' < x_1$, so that (x_1', x_2^*) is a quasiequilibrium whenever $x_1^* < x_1' < r_1^+(x_2^*)$, contrary to the definition of x^*. Symmetrically, $r_1^-(x_2) > x_1^*$ whenever $x_2 > x_2^*$.

We need to show that the game is $\mathcal{X}$-secure at an arbitrary x. There are two main cases:

Case 1: $x \in \Omega$. Let $\alpha = (\alpha_1, \alpha_2)$ where α_1 and α_2 be lower bounds for u_1 and u_2 on $[0,1]^2$. Then each agent i can use any element of $\mathcal{X}_i(x)$ to $\mathcal{X}$-secure α_i at x. We claim that there is a neighborhood U of x such that there is at least one i such that for all $z \in U$ we have $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$ because $\mathcal{X}_i(z) \subset (z_i, 1]$, so that the game is $(\mathcal{X}, \alpha, U)$-secure at x. This is obviously the case when x is in the relative interior of Ω. If x is in the relative boundary of Ω, then $x_1 \geq x_1^*$ and $x_2 \geq x_2^*$, and at least one of these inequalities holds with equality. Suppose that $x_1 > x_1^*$ and $x_2 = x_2^*$. (The case $x_2 > x_2^*$ and $x_1 = x_1^*$ is symmetric.) Since $r_2^-(x_1') > x_2^*$ whenever $x_1' > x_1^*$, and r_2^- is weakly increasing, there is a $\delta > 0$ such that $r_2^-(x_1') > x_2^* + \delta$ for all x_1' in some neighborhood of x_1, and it follows that $\mathcal{X}_2(z) \subset [r_2^-(z_1), 1] \subset (z_2, 1]$ for all z in some neighborhood of x. Finally suppose that $x = x^*$. Then either $r_1^+(x_2^*) > x_1^*$ or $r_2^+(x_1^*) > x_2^*$ because otherwise x^* is a Nash equilibrium. By symmetry we may assume that $r_1^+(x_2^*) > x_1^*$. Then $\mathcal{X}_1(z) \subset (z_1, 1]$ for all z in some neighborhood of x, either because $z \in \Omega$ or because $z_2 > x_2^*$, so that $\mathcal{X}_1(z) \subset [r_1^+(x_2^*), 1]$.

Case 2: $x \notin \Omega$. Since x is not a quasiequilibrium, for some i we have either $x_i < r_i^-(x_{-i})$ or $x_i > r_i^+(x_{-i})$. We will consider the case $x_i < r_i^-(x_{-i})$ (the other case is handled similarly) and by symmetry we may suppose that $i = 1$. Let α_1 be a number less than the best response utility $u_1(r_1^-(x_2), x_2)$ but large enough that for some $\delta > 0$, $u_1(x') < \alpha_1$ for all x' with $x_1' \geq x_1 - \delta$ and $x_2 - \delta \leq x_2' \leq x_2 + \delta$. (Such an α_1 exists because u_1 is upper semicontinuous.) Then $z_1 \notin C_1^{\mathcal{X}}(z, \alpha_1)$ for all z near x. Since $u_1(r_1^-(x_2), \cdot)$ is continuous, player 1 can $\mathcal{X}$-secure α_1 at x by playing $r_1^-(x_2)$. Let $\alpha = (\alpha_1, \alpha_2)$ where α_2 is a lower bound on u_2. Then player 2 can $\mathcal{X}$-secure α_2 by playing anything in $\mathcal{X}_2(x)$, so the game is $(\mathcal{X}, \alpha)$-secure at x.

5 The Proof of Theorem 3.3

It is not hard to show that the game is restrictionally secure when it has a Nash equilibrium, so we do this first. Recall that a topological space is *regular* if each point has a neighborhood base of closed sets. Topological vector spaces are regular topological spaces, even if they are not Hausdorff (e.g., Schaefer (1971, p. 16)). It is easy to see that any subspace of a regular space is regular, and that finite cartesian products of regular spaces are regular, so each X_i, each X_{-i}, and X are all regular. Suppose $x^* \in X$ is a Nash equilibrium. For each i we define $\mathcal{X}_i : X \to X_i$ by setting $\mathcal{X}_i(x) = \{x_i^*\}$. Consider an $x \in X$ such that $[x]$ does not contain a Nash equilibrium. Then $x^* \notin [x]$, and regularity implies that $X \setminus [x]$ contains a closed neighborhood of x^* that is disjoint from $[x]$, so $x \notin [x^*]$. Let U be any neighborhood of x that does not contain x^*. For each i let $\alpha_i = \inf_{x' \in U} u_i(x_i^*, x'_{-i})$, and let $\alpha = (\alpha_1, \dots, \alpha_N)$. Evidently each player i can $\mathcal{X}$-secure α_i at x. For each i and $z \in U$ we have $B_i^{\mathcal{X}}(z, \alpha_i) = C_i^{\mathcal{X}}(z, \alpha_i) = \{x_i^*\}$, so $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$. We have shown that the game is $\mathcal{X}$-secure at x.

What this argument points to is that, in practice, the value of the result is not that it gives conditions that are necessary and sufficient. Rather, it is useful to the extent that one can find restriction operators that are easily shown to satisfy the hypotheses even though the existence of equilibrium would not otherwise be obvious.

In preparation for the main body of the argument we present three lemmas, the first of which is a fixed point theorem. Unlike Kakutani's fixed point theorem and its various infinite dimensional extensions, it holds in topological vector spaces that are neither Hausdorff nor locally convex.

Lemma 5.1. *Let X be a nonempty compact convex subset of a topological vector space Y and let $P \colon X \to X$ be a set valued mapping. If there is a finite closed cover $G_1, \dots, G_m$ of X such that $\bigcap_{z \in G_j} P(z) \neq \emptyset$ for each $j = 1, \dots, m$, then there exists $x^* \in X$ such that $x^* \in \operatorname{con} P(x^*)$.*

Proof. For each $j = 1, \dots, m$ choose a $y_j \in \bigcap_{z \in G_j} P(z)$. Let $H = \{\omega \in \mathbb{R}^m : \sum_j \omega_j = 1\}$, and let $\tilde{\pi} : H \to Y$ be the function $\omega \mapsto \sum_j \omega_j y_j$. Then $\tilde{\pi}$ is continuous, because addition and scalar multiplication are continuous operations in any topological vector space. Let $\tilde{J}$ be the image of $\tilde{\pi}$, let J be $\tilde{J}$ endowed with the usual Euclidean topology (this is unambiguous because $\tilde{J}$ is finite dimensional) and let π be $\tilde{\pi}$ interpreted as a map from H to J. Then π is evidently an open mapping. For any open $V \subset \tilde{J}$ the corresponding subset of J is $\pi(\tilde{\pi}^{-1}(V))$, which is open, so the identity map from J to $\tilde{J}$ is continuous, which is to say that the topology of J is at least as fine as the topology of $\tilde{J}$.

Let $\tilde{C}$ be the convex hull of $y_1, \dots, y_m$ endowed with the subspace topology inherited from $\tilde{J}$, and let C be $\tilde{C}$ with the topology inherited from J. Each $G_j \cap \tilde{C}$ is closed in $\tilde{C}$, so it is also closed in C. Define a correspondence $Q : C \to C$ by setting $Q(x) = \mathrm{con}\{ y_j : x \in G_j\}$; this is nonempty because the G_j cover X, and of course it is compact. Since each $G_j \cap C$ is closed, Q is upper semicontinuous, and applying Kakutani's fixed point theorem gives an x^* such that

$$x^* \in Q(x^*) \subset \mathrm{con}\Big(\bigcup_{j : x^* \in G_j} \big(\bigcap_{x \in G_j} P(x)\big)\Big) \subset \mathrm{con}\ P(x^*).$$

■

Lemma 5.2. *Suppose that $x \in X$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}^N$, $U_1, \ldots, U_\ell$ are neighborhoods of x, and, for each $h = 1, \ldots, \ell$, the game is $(\mathcal{X}, \alpha_h, U_h)$-secure at x. Let $\alpha = \max_{h=1}^{\ell} \alpha^h$ and $U = \bigcap_{h=1}^{\ell} U^h$. Then the game is $(\mathcal{X}, \alpha, U)$-secure at x.*

Proof. For each h and $z \in U_h$ there are closed sets $F_h^1, \ldots, F_h^{J_h}$ whose union contains U_h such that for each i and j we have $\bigcap_{z \in F_h^j} B_i^{\mathcal{X}}(z, \alpha_i^h) \neq \emptyset$. This condition continues to hold with U_h replaced by U. It also continues to hold if $F_h^1, \ldots, F_h^{J_h}$ is replaced by the collection $G^1, \ldots, G^J$ of all nonempty intersections of the form $F_1^{j_1} \cap \ldots \cap F_\ell^{j_\ell}$. Then $\bigcap_{z \in G^j} B_i^{\mathcal{X}}(z, \alpha_i^h) \neq \emptyset$ for all $i = 1, \ldots, N$, $j = 1, \ldots, J$, and $h = 1, \ldots, \ell$. For each i there is some h such that $\alpha_i = \alpha_i^h$, so for any j we actually have $\bigcap_{z \in G^j} B_i^{\mathcal{X}}(z, \alpha_i) \neq \emptyset$, so i can $\mathcal{X}$-secure α_i at x.

For any $z \in U$ there are h and i such that $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i^h)$. Since $\alpha_i \geq \alpha_i^h$, this implies that $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$. ■

Lemma 5.3. *If, for each $x \in X$, there is some $\alpha_x \in \mathbb{R}^N$ such that the game is $(\mathcal{X}, \alpha_x)$-secure at x, then there is a function $\psi : X \to \mathbb{R}^N$, each of whose component functions $\psi_i : X \to \mathbb{R}$ is upper semicontinuous and takes on finitely many values, such that the game is $(\mathcal{X}, \psi(x))$-secure at each $x \in X$.*

Proof. For each $x \in X$ there is an open neighborhood $U_x \subset X$ such that the game is $(\mathcal{X}, \alpha_x, U_x)$-secure at x. Since X is regular, for each x there is a closed set $F_x \subset U_x$ containing x in its interior, and since X is compact it is covered by the interiors of some finite subcollection, say $F_{x_1}, \ldots, F_{x_m}$. For each $x \in X$ let

$$\psi(x) = \max_{x \in F_{x_j}} \alpha_{x_j}.$$

Then each component $\psi_i \colon X \to \mathbb{R}$ of ψ is upper semicontinuous and finite valued. If $x \in F_{x_j}$, then $x \in U_{x_j}$ and the game is $(\mathcal{X}, \alpha_{x_j}, U_{x_j})$-secure at x. By Lemma 5.2 there is a neighborhood U of x such that the game is $(\mathcal{X}, \psi(x), U)$-secure at x. ■

We now have the tools we need to complete the proof of our theorem. Aiming at a contradiction, suppose that G is restrictionally secure but has no Nash equilibrium, so that there is a restriction operator $\mathcal{X}$ such that G is $\mathcal{X}$-secure at each $x \in X$. Lemma 5.3 gives a function $\psi \colon X \to \mathbb{R}^N$, each of whose component functions is finite valued and upper semicontinuous, such that each x is $(\mathcal{X}, \psi(x))$-secure. For each i and x let $P_i(x) = B_i^{\mathcal{X}}(x, \psi_i(x))$.

Consider a particular agent i. For each $x \in X$ there is a neighborhood U of x and a finite collection of closed sets $F_1, \ldots, F_m$ whose union contains U such that $\bigcap_{z \in F_j} B_i^{\mathcal{X}}(z, \psi_i(x)) \neq \emptyset$ for each j. Since X is regular, we can replace U with a smaller closed neighborhood K, and in fact (because ψ_i is upper semicontinuous and finite valued) we can choose K small enough that $\psi_i(z) \leq \psi_i(x)$ for all $z \in K$. Replacing each F_j with $F_j \cap K$, we then have

$$\bigcap_{z \in F_j} P_i(z) = \bigcap_{z \in F_j} B_i^{\mathcal{X}}(z, \psi_i(z)) \supseteq \bigcap_{z \in F_j} B_i^{\mathcal{X}}(z, \psi_i(x)) \neq \emptyset\,.$$

Since X is compact, it has a finite cover $F^{i1}, \ldots, F^{ik_i}$ where, for each $j = 1, \ldots, k_i$, F^{ij} is a neighborhood of a point $x^{ij} \in X$ such that the description above is satisfied.

For each x let $P(x) = P_1(x) \times \cdots \times P_N(x)$. If $G_1, \ldots, G_m$ are the nonempty intersections of the form $F^{1j_1} \cap \ldots \cap F^{Nj_N}$, then $\bigcap_{z \in G_h} P(z) \neq \emptyset$ for each $h = 1, \ldots, m$. Thus the hypotheses of Lemma 5.1 are satisfied by $P \colon X \to X$, so there is an $x^* \in X$ satisfying $x^* \in \operatorname{con} P(x^*)$, which is to say that $x_i^* \in C_i^{\mathcal{X}}(x^*, \psi_i(x^*))$ for all i. But the game is $\mathcal{X}$-secure at x^* for $\psi(x^*)$, so for some i we have $x_i^* \notin C_i^{\mathcal{X}}(x^*, \psi_i(x^*))$. This contradiction completes the proof.

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