Secure implementation in Shapley-Scarf housing markets*

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Abstract

This paper considers the object allocation problem introduced by Shapley and Scarf (1974). We study secure implementation (Saijo, Sjöström, and Yamato, 2007), that is, double implementation in dominant strategy and Nash equilibria. We prove that (i) an *individually rational* solution is securely implementable if and only if it is the no-trade solution, (ii) a *neutral* solution is securely implementable if and only if it is a serial dictatorship, and (iii) an *efficient* solution is securely implementable if and only if it is a sequential dictatorship. Furthermore, we provide a complete characterization of securely implementable solutions in the two-agent case.

Keywords: Secure implementation; Sequential dictatorship; Strict core; Strategy-proofness; Shapley-Scarf housing markets.

JEL codes: C72; C78; D61; D63; D71.

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1 Introduction

We consider the object allocation problem introduced by Shapley and Scarf (1974) with strict preferences. There is a group of agents, each of whom initially owns one object.[1] A solution reallocates the objects with the condition that each agent consumes one and only one object. Important real-life examples of this model are the assignment of campus housing to students (Abdulkadiroğlu and Sönmez, 1999; Chen and Sönmez, 2002, 2004; and Sönmez and Ünver, 2005) and kidney exchange (Roth, Sönmez, and Ünver, 2004).

In this context, the "strict core solution" is a central one since it satisfies various desirable properties. Some characterizations of the solution can be found in Ma (1994), Svensson (1999), Takamiya (2001), and Miyagawa (2002). Furthermore, the solution is dominant strategy implementable (Mizukami and Wakayama, 2007) and Nash implementable when there are at least three agents (Sönmez, 1996). However, these results do not guarantee that the solution is *securely implementable* (Saijo, Sjöström, and Yamato, 2007); note that here, the notion of implementation signifies double implementation in the two equilibrium concepts. Thus, it is natural to raise the following question: Can the strict core solution be securely implemented? In fact, the answer to this question is no (Saijo, Sjöström, and Yamato, 2004, 2007). Based on the result, this paper seeks solutions that can be securely implemented in our model.

Our main results consist of two parts. We first focus on the two-agent case. In this case, we provide a complete characterization of securely implementable solutions; a solution is securely implementable if and only if it is either a constant solution or a "serial dictatorship." By a serial dictatorship, we mean that one agent chooses her best object from among the set of objects, then the second agent chooses his best object from among the set of remaining objects, then the third agent chooses, and so on; the order in which agents make their choices is fixed in advance.

Next, we consider the general case where there are more than two agents. In contrast to the two-agent case, it is hard to characterize the class of securely implementable solutions in the general case. Thus, in the general case, we then pin down smaller classes of securely implementable solutions by adding some properties.

[1] In this paper, the sets of agents and objects are fixed. Some studies consider object allocation problems where either the set of agents or the set of objects varies; for instance, Ergin (2000), Ehlers, Klaus, and Pápai (2002), and Ehlers and Klaus (2003a) consider house allocation problems where each agent consumes at most one object, and Klaus and Miyagawa (2001) and Ehlers and Klaus (2003b) consider multiple assignment problems where agents may consume more than one object.

First, we show that the "no-trade solution" is the unique securely implementable one that satisfies *individual rationality* (no agent is worse off after trading with other agents). The no-trade solution is the one that selects the initial endowments for each preference profile. Second, we prove that a securely implementable solution satisfies *neutrality* (symmetric treatment of objects) if and only if it is a serial dictatorship. Finally, we establish that an *efficient* solution is securely implementable if and only if it is a "sequential dictatorship." For any sequential dictatorship, there exists the first dictator in every preference profile. However, in contrast to serial dictatorships, in the sequential dictatorship, the second agent, who chooses his best object from among the set of remaining objects, is decided by the choice of the first dictator. Similarly, the third agent is decided by the choices of the previous agents, and so on. As far as we know, ours is the first result that characterizes the class of sequential dictatorships in Shapley-Scarf housing markets.

Our model has a close relationship with multiple assignment problems. Klaus and Miyagawa (2001) show that serial dictatorships are the only ones that satisfy *efficiency* and *strategy-proofness* in the two-agent case. In the general case, Pápai (2001) and Ehlers and Klaus (2003b) characterize sequential dictatorships by means of *efficiency*, *strategy-proofness*, and *non-bossiness*. Their characterizations still hold even if *strategy-proofness* and *non-bossiness* are replaced by secure implementability. On the other hand, it should be noted that the results of Klaus and Miyagawa (2001), Pápai (2001), and Ehlers and Klaus (2003b) do not hold in our model. This is because the strict core solution satisfies *efficiency*, *strategy-proofness*, and *non-bossiness*. Therefore, results in multiple assignment problems cannot directly apply to our model.

The rest of the paper is organized as follows: Section 2 provides basic notation and definitions. Section 3 examines the implementability of the strict core solution. Section 4 addresses the two-agent case. Section 5 analyzes the general case. Section 6 concludes the paper. Appendix contains the proofs of the results omitted from the main text.

2 Preliminaries

2.1 The model

We denote the set of *agents* by $N = \{1, 2, \ldots, n\}$, where $2 \leq n < +\infty$. Each agent $i \in N$ owns one object, denoted by i. Thus, N also stands for the set of *objects*.

Each agent $i \in N$ has a complete and transitive binary relation $\succsim_i$ over N, i.e., a *preference relation.* We denote the associated strict preference relation by $\succ_i$ and indifference relation by $\sim_i$. We assume that all preferences are *strict*; i.e., for each $h, k \in N$, if $h \sim_i k$, then $h = k$. Let $\mathscr{P}$ denote the set of all strict preferences. A *preference profile* is a list of preferences $\succsim \equiv (\succsim_1, \succsim_2, \ldots, \succsim_n) \in \mathscr{P}^N$. We often denote $N \setminus \{i\}$ by "$-i$." With this notation, $(\succsim'_i, \succsim_{-i})$ is the preference profile where agent i has $\succsim'_i$ and agent $j \neq i$ has $\succsim_j$. Similarly, given $S \subseteq N$, we denote $N \setminus S$ by "$-S$," and $(\succsim'_S, \succsim_{-S})$ is the preference profile where each agent $i \in S$ has $\succsim'_i$ and each agent $i \notin S$ has $\succsim_i$. We often represent $\succsim_i$ by an ordered list of objects as follows:

$$\succsim_i\colon h_1, h_2, h_{3,} \ldots$$

This means that agent i prefers object h_1 the most; further, i prefers h_1 to h_2, h_2 to h_3, and so on.

An *allocation* is a bijection $x\colon N \to N$. Let $x(i)$ denote the object allocated to agent $i \in N$. For convenience, we use the notation x_i instead of $x(i)$. Let X be the set of allocations.

2.2 Solutions

A *solution* is a function $f\colon \mathscr{P}^N \to X$ that associates an allocation $x \in X$ with each preference profile $\succsim \in \mathscr{P}^N$. Let $f_i(\succsim)$ denote the object allocated to agent i at $\succsim$.

Let $x, y \in X$ and $S \subseteq N$ with $S \neq \emptyset$. Then, x *weakly dominates* y via S at $\succsim \in \mathscr{P}^N$ if $S = \bigcup_{i \in S}\{x_i\}$, and $x_i \succsim_i y_i$ for each $i \in S$ and $x_j \succ_j y_j$ for some $j \in S$. The *strict core* for $\succsim \in \mathscr{P}^N$ is the set of all allocations that are not weakly dominated by any other allocation at $\succsim \in \mathscr{P}^N$. The *strict core solution* is the solution $C\colon \mathscr{P}^N \to X$ such that for each $\succsim \in \mathscr{P}^N$, $C(\succsim)$ is the strict core for $\succsim$.[2]

A solution f is *constant* if there exists $x \in X$ such that for each $\succsim \in \mathscr{P}^N$, $f(\succsim) = x$. In particular, we term the constant solution that selects the initial endowments for each preference profile as the *no-trade solution.*

A *permutation* π on N is a bijection $\pi\colon N \to N$. Let Π^N denote the set of all permutations on N. Given that $i \in N$ and $S \subseteq N$, let $b(\succsim_i, S)$ be agent i's most preferred object under $\succsim_i$ in S, i.e., $b(\succsim_i, S) \in S$ and for each $h \in S$, $b(\succsim_i, S) \succsim_i h$. A solution f is a *sequential choice function* if for each $\succsim \in \mathscr{P}^N$, there exists a

[2]Under strict preferences, the strict core is a singleton for every preference profile (Roth and Postlewaite, 1977). Thus, the strict core solution C is well-defined.

permutation $\pi_{\succsim} \in \Pi^N$ such that

$$\begin{aligned}
f_{\pi_{\succsim}(1)}(\succsim) &= b(\succsim_{\pi_{\succsim}(1)}, N);\\
f_{\pi_{\succsim}(2)}(\succsim) &= b(\succsim_{\pi_{\succsim}(2)}, N \setminus \{f_{\pi_{\succsim}(1)}(\succsim)\});\\
f_{\pi_{\succsim}(3)}(\succsim) &= b(\succsim_{\pi_{\succsim}(3)}, N \setminus [\{f_{\pi_{\succsim}(1)}(\succsim)\} \cup \{f_{\pi_{\succsim}(2)}(\succsim)\}]);\\
&\vdots\\
f_{\pi_{\succsim}(n)}(\succsim) &= b\left(\succsim_{\pi_{\succsim}(n)}, N \Bigg\backslash \left[\bigcup_{i=1}^{n-1} \left\{f_{\pi_{\succsim}(i)}(\succsim)\right\}\right]\right).
\end{aligned}$$

We then say that $\pi_{\succsim}(i)$ is the *i-th dictator* at $\succsim$.

The class of sequential dictatorships is a subclass of sequential choice functions. For any sequential dictatorship, there exists a unique first dictator who chooses her best object in every preference profile. However, the second dictator, who chooses his best object from among the set of remaining objects, is decided by the choice of the first dictator. Similarly, the next dictator is decided by the choices of the previous dictators. Formally, a solution f is a *sequential dictatorship* if it is a sequential choice function that satisfies the following properties: for each $\succsim, \succsim' \in \mathscr{P}^N$, (i) $\pi_{\succsim}(1) = \pi_{\succsim'}(1)$ and (ii) for each $j \in N \setminus \{1\}$, if $\pi_{\succsim}(i) = \pi_{\succsim'}(i)$ and $f_{\pi_{\succsim}(i)}(\succsim) = f_{\pi_{\succsim'}(i)}(\succsim')$ for each $i \in \{1, 2, \ldots, j-1\}$, then $\pi_{\succsim}(j) = \pi_{\succsim'}(j)$.

The class of serial dictatorship is a subclass of sequential dictatorships. For any serial dictatorship, the order in which an agent chooses an object from the set of remaining objects is fixed. That is, the order does not depend on the choices of the previous dictators. Formally, a solution f is a *serial dictatorship* if it is a sequential dictatorship and there exists $\bar{\pi} \in \Pi^N$ such that for each $\succsim \in \mathscr{P}^N$, $\pi_{\succsim} = \bar{\pi}$.

2.3 Axioms and implementation

We now define our central axioms. The first axiom is a voluntary participation condition, according to which no agent receives an object that she considers worse than her endowment.

Individual rationality: For each $\succsim \in \mathscr{P}^N$ and each $i \in N$, $f_i(\succsim) \succsim_i i$.

The next axiom states that it is impossible to render an agent better off without rendering someone else worse off.

Efficiency: For each $\succsim \in \mathscr{P}^N$, there does not exist $x \in X$ such that $x_i \succsim_i f_i(\succsim)$ for each $i \in N$ and $x_j \succ_j f_j(\succsim)$ for some $j \in N$.

The last axiom states that a solution is defined independently of the names of the objects. For each $\succsim \in \mathscr{P}^N$ and each $\pi \in \Pi^N$, let $T(\succsim, \pi)$ be a preference profile $\succsim'$ such that for each $i, j, k \in N$,

$$j \succsim_i k \iff \pi(j) \succsim'_i \pi(k).$$

Neutrality: For each $\succsim \in \mathscr{P}^N$, each $\pi \in \Pi^N$, and each $i \in N$, $f_i(T(\succsim, \pi)) = \pi(f_i(\succsim))$.

We next define some notions of implementation. Let $M \equiv M_1 \times M_2 \times \cdots \times M_n$ be the *message space* where M_i is agent i's message space. A *mechanism* is a pair $\Gamma = (M, g)$, where $g\colon M \to X$ is an *outcome function*. For each $\succsim \in \mathscr{P}^N$, let $\mathbf{NE}^\Gamma(\succsim)$ and $\mathbf{DSE}^\Gamma(\succsim)$ denote the sets of Nash equilibrium and dominant strategy equilibrium allocations of Γ at $\succsim \in \mathscr{P}^N$ respectively.

A solution f is *Nash implementable* if there is a mechanism Γ such that for each $\succsim \in \mathscr{P}^N$, $f(\succsim) = \mathbf{NE}^\Gamma(\succsim)$. By $L(h, \succsim_i) \equiv \{k \in N \colon h \succsim_i k\}$, we denote that agent i's *lower contour set* of object $h \in N$ at $\succsim_i \in \mathscr{P}$. Maskin (1999) shows that the following axiom is necessary and almost sufficient for Nash implementation.

Monotonicity: For each $\succsim, \succsim' \in \mathscr{P}^N$, if $L(f_i(\succsim), \succsim_i) \subseteq L(f_i(\succsim), \succsim'_i)$ for each $i \in N$, then $f(\succsim) = f(\succsim')$.

That is, *monotonicity* states that if an allocation x is chosen for $\succsim$ and another preference profile $\succsim'$ is obtained by expanding each agent's lower contour set at x_i, then x is also chosen for $\succsim'$.

A solution f is *dominant strategy implementable* if there is a mechanism Γ such that for each $\succsim \in \mathscr{P}^N$, $f(\succsim) = \mathbf{DSE}^\Gamma(\succsim)$. Mizukami and Wakayama (2007) show that the following axiom is necessary and sufficient for dominant strategy implementation in many economic environments.

Strategy-proofness: For each $\succsim \in \mathscr{P}^N$, each $i \in N$, and each $\succsim'_i \in \mathscr{P}$, $f_i(\succsim) \succsim_i f_i(\succsim'_i, \succsim_{-i})$.

That is, *strategy-proofness* states that no agent can obtain a benefit by misrepresenting her preferences.

Although *strategy-proofness* and dominant strategy implementability are desirable requirements in the light of robust mechanism design, Saijo, Sjöström, and Yamato (2007) show that many *strategy-proof* (and dominant strategy implementable) solutions admit multiple Nash equilibrium outcomes that are different from the "true" outcome, thereby making these solutions somewhat ineffective. In other words, the existence of "bad" Nash equilibria prevents *strategy-proof* solutions from working effectively. The results of the experiments conducted by Cason, Saijo, Sjöström, and Yamato (2006) support this fact.[3] Therefore, Saijo, Sjöström, and Yamato (2006) developed a new concept, namely, *secure implementation.* It states that a solution is securely implementable if there exists a mechanism that implements it through dominant strategy equilibria and if the set of dominant strategy equilibrium outcomes coincides with the set of Nash equilibrium outcomes. Formally, a solution is *securely implementable* if there exists a mechanism Γ such that for each $\succsim \in \mathscr{P}^N$, $f(\succsim) = \mathbf{NE}^\Gamma(\succsim) = \mathbf{DSE}^\Gamma(\succsim)$. Saijo, Sjöström, and Yamato (2007) provide a characterization of the class in the abstract setting on the basis of *strategy-proofness* and the following additional axiom:[4]

Rectangular property: For each $\succsim, \succsim' \in \mathscr{P}^N$, if $f_i(\succsim') = f_i(\succsim_i, \succsim'_{-i})$ for each $i \in N$, then $f(\succsim) = f(\succsim')$.

Proposition 1 (Saijo, Sjöström, and Yamato, 2007). *A solution is securely* implementable *if and only if it satisfies* strategy-proofness *and the* rectangular property.

Before closing this section, we will discuss the relationships among axioms that are often studied in the literature. It is well known that *monotonicity* and *strategy-proofness* are related to each other in many economic environments. In fact, *monotonicity* implies *strategy-proofness.* However, the converse does not hold. Takamiya (2001) shows that the gap between the two aforementioned axioms is filled by the next axiom. This axiom states that when each agent unilaterally changes her preference report, she cannot influence the total allocation without changing her own consumption.

Non-bossiness: For each $\succsim \in \mathscr{P}^N$, each $i \in N$, and each $\succsim'_i \in \mathscr{P}$, if $f_i(\succsim) = f_i(\succsim'_i, \succsim_{-i})$, then $f(\succsim) = f(\succsim'_i, \succsim_{-i})$.

It has been shown that *non-bossiness* and *strategy-proofness* together imply a coalitional version of *strategy-proofness*, which states that no group of agents can

[3] The laboratory experiment demonstrates that subjects play dominant strategies more frequently in a securely implementable solution, rather than in a non-securely implementable solution.

[4] Mizukami and Wakayama (2009) provide an alternative characterization of securely implementable solutions.

gain by collusively misrepresenting their preferences. This can be formally explained as follows:

***Coalitionally strategy-proofness*:** There exists no $S \subseteq N$ with $S \neq \emptyset$, $\succsim \in \mathscr{P}^N$, and $\succsim'_S \in \mathscr{P}^S$ such that (i) $f_i(\succsim'_S, \succsim_{-S}) \succsim_i f_i(\succsim)$ for each $i \in S$, and (ii) $f_j(\succsim'_S, \succsim_{-S}) \succ_j f_j(\succsim)$ for some $j \in S$.

Takamiya (2001) shows that the converse holds in Shapley-Scarf housing markets with strict preferences. The above discussions can be summarized as follows:

Fact 1 (Takamiya, 2001). *The following three statements are equivalent:*

- *A solution f satisfies* strategy-proofness *and* non-bossiness.
- *A solution f satisfies* coalitionally strategy-proofness.
- *A solution f satisfies* monotonicity.

It should be noted that *non-bossiness* is much weaker than the *rectangular property.*

Fact 2 (Saijo, Sjöström, and Yamato, 2007). *If a solution satisfies the* rectangular property, *then it satisfies* non-bossiness.

From Proposition 1 and Facts 1 and 2, one might infer that the class of securely implementable solutions is equivalent to the class of *coalitionally strategy-proof* solutions. Clearly, secure implementability implies *coalitionally strategy-proofness* and *monotonicity.*

Fact 3. *If a solution is securely implementable, then it satisfies* coalitionally strategy-proofness *and* monotonicity.

Proof. It immediately follows from Proposition 1 and Facts 1 and 2. □

However, the converse of Fact 3 does not hold. The strict core solution does not satisfy the *rectangular property*, although it satisfies *coalitionally strategy-proofness* and *monotonicity.*[5]

[5] The fact that the strict core solution does not satisfy the *rectangular property* will be established in the next section.

3 Implementability of the strict core solution

This section examines the implementability of the strict core solution, which is the central solution in our model because it satisfies the desirable axioms introduced in Section 2 (except for *neutrality* and the *rectangular property*) (Roth, 1982; Bird, 1984). Nash implementability of the strict core solution has been studied by Sömmez (1996). He establishes that the strict core solution is Nash implementable whenever there are more than three agents.[6] Thus, we discuss the dominant strategy implementability and secure implementability of the solution.

We first consider the dominant strategy implementability. To the best of our knowledge, no one has previously attempted to explicitly identify the dominant strategy implementability of the strict core solution. In order to establish it, we will exploit the result of Mizukami and Wakayama (2007). They show that if a solution satisfies *strategy-proofness* and the following axiom, then it is dominant strategy implemented by its associated direct revelation mechanism (see Theorem 2 in Mizukami and Wakayama, 2007).

***Quasi-strong-non-bossiness*:** For each $\succsim \in \mathscr{P}^N$, each $i \in N$, and each $\succsim_i' \in \mathscr{P}$, if $f_i(\succsim_i, \succsim_{-i}'') \sim_i f_i(\succsim_i', \succsim_{-i}'')$ for each $\succsim_{-i}'' \in \mathscr{P}^{N\setminus\{i\}}$, then $f(\succsim) = f(\succsim_i', \succsim_{-i})$.

Proposition 2. *The strict core solution is dominant strategy implemented by its associated direct revelation mechanism.*

Proof. It suffices to show that the strict core solution C satisfies *quasi-strong-non-bossiness*. Let $\succsim \in \mathscr{P}^N$, $i \in N$, and $\succsim_i' \in \mathscr{P}$ be such that $C_i(\succsim_i, \succsim_{-i}'') \sim_i C_i(\succsim_i', \succsim_{-i}'')$ for each $\succsim_{-i}'' \in \mathscr{P}^{N\setminus\{i\}}$. Since preferences are strict, $C_i(\succsim_i, \succsim_{-i}'') = C_i(\succsim_i', \succsim_{-i}'')$ for each $\succsim_{-i}'' \in \mathscr{P}^{N\setminus\{i\}}$. Thus, $C_i(\succsim) = C_i(\succsim_i', \succsim_{-i})$. Since C satisfies *non-bossiness*, $C(\succsim) = C(\succsim_i', \succsim_{-i})$. □

The solution is not only dominant strategy implementable, but also Nash implementable, when there are at least three agents. Thus, one might conjecture that it is securely implementable. However, Saijo, Sjöström, and Yamato (2004) show that the strict core solution is *not* securely implementable.[7] To see this, consider the following example:

[6] In the two-agent case, the strict core solution cannot be Nash implementable. The proof of this result is available upon request.

[7] Saijo, Sjöström, and Yamato (2004) illustrate this for the two-agent case. However, we can see that the strict core solution is not even Nash implementable in the two-agent case. On the other hand, it is Nash implementable when there are at least three agents. Thus, it is not clear as to whether the strict core solution is securely implementable when there are three or more agents.

Example 1. Suppose that $N = \{1, 2, 3\}$. Let $\succsim \in \mathscr{P}^N$ and $\succsim_1', \succsim_2' \in \mathscr{P}$ be such that

$$\succsim_1: 1, 2, 3; \qquad \succsim_1': 2, 1, 3;$$
$$\succsim_2: 1, 2, 3; \qquad \succsim_2': 2, 1, 3;$$
$$\succsim_3: 3, 2, 1.$$

Then,

$$C(\succsim_1, \succsim_2, \succsim_3) = C(\succsim_1, \succsim_2', \succsim_3) = C(\succsim_1', \succsim_2', \succsim_3) = (1, 2, 3);$$
$$C(\succsim_1', \succsim_2, \succsim_3) = (2, 1, 3).$$

Since $C(\succsim_1, \succsim_2', \succsim_3) = C(\succsim_1', \succsim_2', \succsim_3)$ and $C(\succsim_1, \succsim_2', \succsim_3) = C(\succsim_1, \succsim_2, \succsim_3)$, the *rectangular property* requires that $C(\succsim_1, \succsim_2', \succsim_3) = C(\succsim_1', \succsim_2, \succsim_3)$. However, since $C(\succsim_1, \succsim_2', \succsim_3) \neq C(\succsim_1', \succsim_2, \succsim_3)$, the strict core solution violates the *rectangular property* and is thus not securely implementable.[8] ■

Thus, this paper seeks to identify the solutions that are securely implementable.

4 The two-agent case

In this section, we consider the two-agent case. For each $i \in N$, let

$$\succsim_i^{12}: 1, 2;$$
$$\succsim_i^{21}: 2, 1.$$

Proposition 3 provides a complete characterization of the class of solutions satisfying *strategy-proofness* and the *rectangular property* in the two-agent case.

Proposition 3. *Assume $n = 2$. A solution satisfies strategy-proofness and the rectangular property if and only if it is either a constant solution or a serial dictatorship.*

Proof. It is easy to verify the "if" part. We prove the "only if" part below. Let f be a solution satisfying the two axioms. We now discuss the following two cases:

[8] This can be directly derived from Theorem 2 in our paper.

Case 1: $f(\succsim_1^{12}, \succsim_2^{12}) = (1,2)$. If $f(\succsim_1^{12}, \succsim_2^{21}) = (2,1)$, then $f_2(\succsim_1^{12}, \succsim_2^{21}) \succ_2^{12} f_2(\succsim_1^{12}, \succsim_2^{12})$, which is in violation of *strategy-proofness*. Therefore, $f(\succsim_1^{12}, \succsim_2^{21}) = (1,2)$.

We first consider the case $f(\succsim_1^{21}, \succsim_2^{12}) = (1,2)$. By the *rectangular property*, $f(\succsim_1^{21}, \succsim_2^{21}) = (1,2)$. Hence, f is constant.

Next, we consider the case $f(\succsim_1^{21}, \succsim_2^{12}) = (2,1)$. If $f(\succsim_1^{21}, \succsim_2^{21}) = (1,2)$, then, by the *rectangular property*, $f(\succsim_1^{21}, \succsim_2^{12}) = (1,2)$. This is a contradiction. Therefore, $f(\succsim_1^{21}, \succsim_2^{21}) = (2,1)$. Then, $f_1(\succsim) = b(\succsim_1, N)$ for each $\succsim \in \mathscr{P}^N$. This implies that f is a serial dictatorship.

Case 2: $f(\succsim_1^{12}, \succsim_2^{12}) = (2,1)$. By an argument similar to that in Case 1, we have that f is either a constant solution or a serial dictatorship. □

The two axioms in Proposition 3 are independent. It is easily verifiable that the strict core solution satisfies *strategy-proofness* but violates the *rectangular property*. The following solution satisfies the *rectangular property* but violates *strategy-proofness*: for each $\succsim \in \mathscr{P}^N$,

$$f(\succsim) = \begin{cases} (2,1) & \text{if } \succsim = (\succsim_1^{12}, \succsim_2^{21}); \\ C(\succsim) & \text{otherwise.} \end{cases}$$

By Proposition 1, we immediately obtain the characterization of the class of securely implementable solutions in the two-agent case.

Theorem 1. *Assume $n = 2$. A solution is securely implementable if and only if it is either a constant solution or a serial dictatorship.*

Considering other axioms, we obtain the following corollary:

Corollary 1. *Assume $n = 2$.*

1. *An individually rational solution is securely implementable if and only if it is the no-trade solution.*

2. *A neutral solution is securely implementable if and only if it is a serial dictatorship.*

3. *An efficient solution is securely implementable if and only if it is a serial dictatorship.*

5 The general case

In contrast to the two-agent case, in the general case where there are more than two agents, there exists a securely implementable solution other than constant solutions and serial dictatorships. To verify this, consider the following example:

Example 2. Let $N = \{1, 2, 3\}$. Let f be a solution satisfying the following: for each $\succsim \in \mathscr{P}^N$,

$$f(\succsim) = \begin{cases} (2,1,3) & \text{if } 1 \succ_2 3; \\ (2,3,1) & \text{if } 3 \succ_2 1. \end{cases}$$

It is easy to see that the solution is securely implementable. ■

It would be expected that there are a lot of securely implementable solutions in the general case. In fact, as we will see later, in the general case, there are several different types of securely implementable solutions. Thus, the main purpose of this section is to characterize the class of securely implementable solutions satisfying a certain property.

5.1 Individual rationality and neutrality

This subsection first considers the class of securely implementable solutions that satisfy *individual rationality*. The next proposition would be helpful in characterizing the class.

Proposition 4. *A solution satisfies* individual rationality *and the* rectangular property *if and only if it is the no-trade solution.*

Proof. Since the "if" part is obvious, it will suffice to show the "only if" part. Let f be a solution satisfying the two axioms. Let $\succsim' \in \mathscr{P}^N$ be such that for each $i \in N$, $b(\succsim'_i, N) = i$. By *individual rationality*, $f_i(\succsim') = i$ for each $i \in N$. Let $\succsim \in \mathscr{P}^N$. Then, *individual rationality* implies that $f_i(\succsim_i, \succsim'_{-i}) = i$ for each $i \in N$. Hence, by the *rectangular property*, $f(\succsim') = f(\succsim)$. This implies that f is the no-trade solution. □

It is easy to check that none of the axioms in Proposition 4 are redundant. The strict core solution satisfies *individual rationality* but violates the *rectangular property*. A constant solution that is not the no-trade solution satisfies the *rectangular property* but violates *individual rationality*.

Interestingly, Proposition 4 enables us to pin down the class of securely implementable solutions satisfying *individual rationality* without using *strategy-proofness.* Thus, we immediately obtain the following result.

Theorem 2. *An individually rational solution is securely implementable if and only if it is the no-trade solution.*

Next, we consider the class of securely implementable solutions that satisfy *neutrality.* Svensson (1999) establishes that a solution is *strategy-proof*, *non-bossy*, and *neutral* if and only if it is a serial dictatorship.[9] From the logical relationship between the *rectangular property* and *non-bossiness*, we obtain the following result:

Theorem 3. *A* neutral *solution is securely implementable if and only if it is a serial dictatorship.*

5.2 Efficiency

In this subsection, we characterize the class of securely implementable solutions that satisfy *efficiency.* We first provide a characterization of the class of solutions that satisfy *strategy-proofness*, the *rectangular property*, and *efficiency.*

Proposition 5. *A solution satisfies* strategy-proofness, *the* rectangular property, *and* efficiency *if and only if it is a sequential dictatorship.*

Proof. Throughout the proof, we often use the following notation: for each $k \in N$, let $N_k \equiv \{1, 2, \ldots, k\}$.

The "if" part. Let f be a sequential dictatorship. Since it is obvious that f satisfies *efficiency*, we show that f satisfies *strategy-proofness* and the *rectangular property.*

• *Strategy-proofness*: Pick any $\succsim \in \mathscr{P}^N$. Without loss of generality, we assume that $\pi_{\succsim}(i) = i$ for each $i \in N$. Let $j \in N$ and $\succsim'_j \in \mathscr{P}$. First, let $j = 1$. Then, obviously, agent 1 cannot manipulate at $\succsim$. Next, let $j \geq 2$. Note that for each agent $i \in N_{j-1}$, she reveals the same preference relation $\succsim_i$ at both $\succsim$ and $(\succsim'_j, \succsim_{-j})$. By the definition of the sequential dictatorship, this implies that for each agent $i \in N_{j-1}$, $\pi_{\succsim}(i) = \pi_{(\succsim'_j, \succsim_{-j})}(i) = i$ and $f_i(\succsim) = f_i(\succsim'_j, \succsim_{-j})$. Therefore,

[9] Svensson (1999) considers a situation where the total number of objects is at least as great as the number of agents. Therefore, Theorem 3 holds in this situation.

$\pi_{\succsim}(j) = \pi_{(\succsim'_j, \succsim_{-j})}(j) = j$. Note that

$$f_j(\succsim) = b\left(\succsim_j, N \bigg\backslash \left[\bigcup_{i=1}^{j-1} f_i(\succsim)\right]\right);$$
$$f_j(\succsim'_j, \succsim_{-j}) = b\left(\succsim'_j, N \bigg\backslash \left[\bigcup_{i=1}^{j-1} f_i(\succsim'_j, \succsim_{-j})\right]\right) = b\left(\succsim'_j, N \bigg\backslash \left[\bigcup_{i=1}^{j-1} f_i(\succsim)\right]\right).$$

Hence agent j cannot manipulate at $\succsim$.

• *Rectangular property*: Let $\succsim, \succsim' \in \mathscr{P}^N$ be such that $f_i(\succsim') = f_i(\succsim_i, \succsim'_{-i})$ for each $i \in N$. Without loss of generality, we assume that $\pi_{\succsim'}(i) = i$ for each $i \in N$. We now use an induction argument.

• **Basic step:** Let $j = 1$. Note that $\pi_{\succsim'}(1) = \pi_{(\succsim_1, \succsim'_{-1})}(1) = \pi_{\succsim}(1) = 1$. Then, $f_1(\succsim') = f_1(\succsim_1, \succsim'_{-1})$ implies that $b(\succsim'_1, N) = b(\succsim_1, N)$. Therefore,

$$f_1(\succsim') = b(\succsim'_1, N) = b(\succsim_1, N) = f_1(\succsim).$$

• **Induction hypothesis:** When $j = k-1$, $\pi_{\succsim'}(i) = \pi_{\succsim}(i) = i$ and $f_i(\succsim') = f_i(\succsim)$ for each agent $i \in N_{k-1}$.

• **Induction step:** Let $j = k$. Note that for each agent $i \in N_{k-1}$, she reveals the same preference relation $\succsim'_i$ at both $\succsim'$ and $(\succsim_k, \succsim'_{-k})$. Then, by the definition of the sequential dictatorship, for each agent $i \in N_{k-1}$,

$$\pi_{\succsim'}(i) = \pi_{(\succsim_k, \succsim'_{-k})}(i) = i \quad \text{and} \quad f_i(\succsim') = f_i(\succsim_k, \succsim'_{-k}). \tag{1}$$

Thus, $\pi_{\succsim'}(k) = \pi_{(\succsim_k, \succsim'_{-k})}(k) = k$. Then, (1) and $f_k(\succsim') = f_k(\succsim_k, \succsim'_{-k})$ together imply that

$$\begin{aligned} b\left(\succsim'_k, N \bigg\backslash \left[\bigcup_{i=1}^{k-1} f_i(\succsim')\right]\right) &= b\left(\succsim_k, N \bigg\backslash \left[\bigcup_{i=1}^{k-1} f_i(\succsim_k, \succsim'_{-k})\right]\right) \\ &= b\left(\succsim_k, N \bigg\backslash \left[\bigcup_{i=1}^{k-1} f_i(\succsim')\right]\right). \end{aligned} \tag{2}$$

Furthermore, by the induction hypothesis and the definition of the sequential dictatorship, $\pi_{\succsim'}(k) = \pi_{\succsim}(k) = k$. Hence, the induction hypothesis and (2) together imply that

$$f_k(\succsim') = b\left(\succsim'_k, N \Big\backslash \left[\bigcup_{i=1}^{k-1} f_i(\succsim')\right]\right) = b\left(\succsim_k, N \Big\backslash \left[\bigcup_{i=1}^{k-1} f_i(\succsim')\right]\right)$$
$$= b\left(\succsim_k, N \Big\backslash \left[\bigcup_{i=1}^{k-1} f_i(\succsim)\right]\right) = f_k(\succsim).$$

The "only if" part. Let f be a solution satisfying the three axioms. We begin by proving that there exists the first dictator. For each $i \in N$, let $\hat{\succsim}_i$ be such that

$$\hat{\succsim}_i \colon n, n-1, \ldots, k+1, k, k-1, \ldots, 2, 1.$$

Let $\hat{\succsim} \equiv (\hat{\succsim}_1, \hat{\succsim}_2, \ldots, \hat{\succsim}_n)$. Without loss of generality, assume that for each $i \in N$, $f_i(\hat{\succsim}) = i$. We establish the following claim:

Claim 1. *For each $k \in N$ and each $\succsim_{N_k} \in \mathscr{P}^{N_k}$,*

$$f_i(\succsim_{N_k}, \hat{\succsim}_{-N_k}) = i \quad \forall\, i \in N \setminus N_k;$$
$$f_k(\succsim_{N_k}, \hat{\succsim}_{-N_k}) = b(\succsim_k, N_k).$$

The proof for Claim 1 can be found in Appendix. When $k = n$, Claim 1 implies that for each $\succsim \in \mathscr{P}^N$, $f_n(\succsim) = b(\succsim_n, N)$. Therefore, agent n is the first dictator.

Now, we show that f is a sequential dictatorship. Since agent n is the first dictator, we can set $\pi_{\succsim}(1) = n$ and $f_n(\succsim) = b(\succsim_n, N)$ for each $\succsim \in \mathscr{P}^N$. In what follows, we will establish that there is the second dictator decided by the choice of the first dictator, i.e., for each $\succsim \in \mathscr{P}^N$ and each $a \in N$, if $b(\succsim_n, N) = a$, there is an agent $j(a) \in N \setminus \{n\}$ such that $f_{j(a)}(\succsim) = b(\succsim_{j(a)}, N \setminus \{a\})$.

Fix $\succsim \in \mathscr{P}^N$ and $a \in N$ such that $b(\succsim_n, N) = a$. Let $\mathscr{P}|_{N\setminus\{a\}}$ denote the set of all strict preferences $\succsim_i|_{N\setminus\{a\}}$ over $N \setminus \{a\}$. Then, let $f^a \colon \left(\mathscr{P}|_{N\setminus\{a\}}\right)^{N\setminus\{n\}} \to N \setminus \{a\}$ be a solution such that for each $\succsim|_{N\setminus\{a\}} \in \left(\mathscr{P}|_{N\setminus\{a\}}\right)^{N\setminus\{n\}}$ and each $i \in N \setminus \{n\}$, $f^a_i(\succsim|_{N\setminus\{a\}}) \equiv f_i(\succsim^a)$ where $\succsim^a \in \mathscr{P}^N$ is a preference profile such that:

- $\succsim^a_n$: $a, n, n-1, \ldots, a+1, a-1, \ldots, 2, 1$;
- for each $i \in N \setminus \{n\}$, $b(\succsim^a_i, N) = a$ and

$$c \succsim_i|_{N\setminus\{a\}} d \iff c \succsim^a_i d \quad \forall\, c, d \in N \setminus \{a\}.$$

Since f satisfies *strategy-proofness*, the *rectangular property*, and *efficiency*, f^a also satisfies the three axioms. Therefore, by adopting an argument similar to that for proving that there is the first dictator of f, we can prove that there is a dictator of f^a. Let $j(a) \in N \setminus \{n\}$ be the dictator of f^a. It should be noted that who becomes $j(a)$ depends only on a due to the definition of f^a. Let us consider a preference profile $\succsim' \in \mathscr{P}^N$ such that (i) $\succsim'_n = \succsim^a_n$, and (ii) for each $i \in N \setminus \{n\}$, $b(\succsim'_i, N) = a$ and

$$c \succsim_i d \iff c \succsim'_i d \quad \forall c, d \in N \setminus \{a\}.$$

We now establish the following claim:

Claim 2. $f(\succsim) = f(\succsim')$.

The proof for Claim 2 can be found in Appendix. Then, we have

$$f_{j(a)}(\succsim) = f_{j(a)}(\succsim') = f^a_{j(a)}(\succsim|_{N\setminus\{a\}}) = b(\succsim_{j(a)}|_{N\setminus\{a\}}, N \setminus \{a\}) = b(\succsim_{j(a)}, N \setminus \{a\}),$$

where the first equation follows from Claim 2; for each $i \in N \setminus \{n\}$, $\succsim_i|_{N\setminus\{a\}}$ is a preference relation over $N \setminus \{a\}$ such that

$$c \succsim_i|_{N\setminus\{a\}} d \iff c \succsim_i d \iff c \succsim'_i d \quad \forall c, d \in N \setminus \{a\}.$$

Hence, we observe that $f_{j(a)}(\succsim) = b(\succsim_{j(a)}, N \setminus \{a\})$.

By repeating a similar argument, we can establish that f is a sequential dictatorship. □

Remark. The proof of Proposition 5, particularly Claim 1, relies on the assumption that each object is received by only one agent. In the proof, we often infer from the assumption that since an object is received by an agent, the other agents cannot receive it. Therefore, our proof does not work in a situation where there is a *null object*, which refers to "not receiving any real object" because some agents may receive the null object simultaneously. However, the existence of the null object does not matter in a situation where any real object is always preferred to the null object. Our proof can be extended in an appropriate way to such a situation.[10]

It is easy to verify that the "only if" part of Proposition 5 does not hold when any of the three axioms—*efficiency*, *strategy-proofness*, and the *rectangular property*—

[10] The proof of this result is provided in the supplementary note that is available at the following webpage: `http://www.geocities.jp/takuma_wakayama/housingnote.pdf`

is dropped. The strict core solution satisfies *efficiency* and *strategy-proofness* but violates the *rectangular property*. The no-trade solution satisfies *strategy-proofness*, and the *rectangular property* but violates *efficiency*. Finally, the following solution satisfies *efficiency* and the *rectangular property* but violates *strategy-proofness*: let f be a sequential choice solution such that for each $\succsim \in \mathscr{P}^{\{1,2,3\}}$,

$$(\pi_{\succsim}(1), \pi_{\succsim}(2), \pi_{\succsim}(3)) = \begin{cases} (1,2,3) & \text{if } b(\succsim_i, N) = b(\succsim_j, N) \quad \forall\, i, j \in N; \\ (2,3,1) & \text{otherwise.} \end{cases}$$

The following result is a characterization of securely implementable solutions satisfying *efficiency* and follows easily from Proposition 5.

Theorem 4. *An efficient solution is securely implementable if and only if it is a sequential dictatorship.*

It is well-known that *strategy-proofness* together with *non-bossiness* implies *efficiency* as long as no alternative is excluded in advance (Takamiya, 2001); this is an axiom called *ontoness*. This axiom can be expressed as follows:

***Ontoness*:** For each $x \in X$, there exists $\succsim \in \mathscr{P}^N$ such that $f(\succsim) = x$.

Since *ontoness* is a necessary condition for *efficiency*, *ontoness* deems a minimal efficiency condition. Then, we have the following corollary:

Corollary 2. *An onto solution is securely implementable if and only if it is a sequential dictatorship.*

5.3 Other securely implementable solutions

Thus far, we have considered securely implementable solutions satisfying certain properties in the general case. Now, we present other securely implementable solutions in the general case.

Example 2 (continued). It can easily be verified that f is securely implementable but satisfies none of the other axioms. ■

Example 3. Let $N = \{1, 2, 3, 4\}$. Let f be a solution satisfying the following: for each $\succsim \in \mathscr{P}^N$,

$$f_1(\succsim) = b(\succsim_1, \{1, 2, 3\});$$

$$f_2(\succsim) = b(\succsim_2, N \setminus \{f_1(\succsim)\});$$
$$f_3(\succsim) = b(\succsim_3, N \setminus \{f_1(\succsim), f_2(\succsim)\});$$
$$f_4(\succsim) = N \setminus \{f_1(\succsim), f_2(\succsim), f_3(\succsim)\}.$$

This solution is securely implementable but satisfies none of the other axioms. ■

It follows from Examples 2 and 3 that the class of securely implementable solutions is expected to be of complicated form. Thus, the characterization of the class of securely implementable solutions remains for future research.

6 Concluding remarks

To end our discussion, we mention some open questions that should be addressed in future research.

1. Other axioms. We succeeded in classifying the securely implementable solutions that satisfy a certain property such as *individual rationality*, *neutrality*, and *efficiency* in Shapley-Scarf housing markets. Studies of characterizations with regard to other desirable properties are also interesting; for example, *anonymity* (Miyagawa, 2002) states that a solution does not depend on the names of agents and objects, and *reallocation-proofness* (Pápai, 2000) states that a solution is robust to pairwise manipulations through reallocations of assignments. Concerning *reallocation-proofness*, since it immediately follows from the definition that the *rectangular property* implies *reallocation-proofness*, Pápai (2000) and Theorem 4 together imply that any "hierarchical exchange solution" other than sequential dictatorships is not securely implementable.[11] As mentioned in the above text, identifying the entire class of securely implementable solutions in the general case still remains an open issue.

2. Coalitional stability. We studied double implementation through dominant strategy equilibria and Nash equilibria. To study other weak notions of "double implementation" is an important issue that should be addressed.[12] In our model, Takamiya (2009) shows that the strict core solution is implemented by its associated direct revelation mechanism in strict strong Nash equilibria. This finding and Proposition 2 together imply that the solution is doubly implemented through domi-

[11]See Pápai (2000) for the formal definition of a hierarchical exchange solution. She fully characterizes the solution by four axioms, including *reallocation-proofness*.

[12]This approach comes from Bochet and Sakai (2009), who study secure implementation in allotment economies.

nant strategy and strict strong Nash equilibria.[13] This fact encourages us to provide the characterization of such solutions in a future research.

3. Random allocation models. This paper discussed a deterministic object allocation model and proved that while a serial dictatorship is securely implementable, the strict core solution is not. In contrast, in a random allocation model, two solutions related to a serial dictatorship and the strict core solution are equivalent: Abdulkadiroğlu and Sönmez (1998) establish the equivalence between the random serial dictatorship and the core solution from random endowment.[14] Determining whether or not the solution is securely implementable and identifying the securely implementable solutions in the random allocation model are interesting issues.

[13]Wako (1999) establishes that the strict core solution is strong Nash implementable by constructing a "natural" mechanism. However, the mechanism does not implement the solution via dominant strategy equilibria.

[14]See Abdulkadiroğlu and Sönmez (1998) for the formal definitions of the two solutions.

Appendix: Proofs of claims

Proof of Claim 1

In the proof of Claim 1, we often use the following notation: for each $S \subseteq N$ and each $\succsim_S \in \mathscr{P}^S$, $f(\succsim_S) \equiv f(\succsim_S, \hat{\succsim}_{-S})$, i.e., $f(\succsim_S)$ denotes the allocation at the preference profile $(\succsim_S, \hat{\succsim}_{-S})$, where each agent $i \in S$ has $\succsim_i$ and each agent $i \notin S$ has $\hat{\succsim}_i$. We now prove Claim 1 by using an induction argument.

- **Basic step:** When $k = 1$, the claim holds: Pick any $\succsim_1 \in \mathscr{P}$. Note that $f_1(\hat{\succsim}_{N_1}) = 1$. Since $L(1, \hat{\succsim}_1) = \{1\}$, $L(1, \hat{\succsim}_1) \subseteq L(1, \succsim_1)$. Thus, by *monotonicity* (Fact 3), $f(\succsim_{N_1}) = f(\hat{\succsim}_{N_1})$. Therefore,

$$
\begin{aligned}
f_i(\succsim_{N_1}) &= i \quad \forall\, i \in N \setminus N_1;\\
f_1(\succsim_{N_1}) &= 1 = b(\succsim_1, N_1).
\end{aligned}
$$

- **Induction hypothesis:** When $k = \ell - 1$, it holds that for each $\succsim_{N_{\ell-1}} \in \mathscr{P}^{N_{\ell-1}}$,

$$
f_i(\succsim_{N_{\ell-1}}) = i \quad \forall\, i \in N \setminus N_{\ell-1}; \tag{3}
$$

$$
f_{\ell-1}(\succsim_{N_{\ell-1}}) = b(\succsim_{\ell-1}, N_{\ell-1}). \tag{4}
$$

- **Induction step:** Let $k = \ell$. In order to show that the claim for $k = \ell$ holds, we proceed in three steps.

Step 1: For each $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$, if $b(\succsim_\ell, N_\ell) = \ell$, then $f_i(\succsim_{N_\ell}) = i$ for each $i \in N \setminus N_\ell$ and $f_\ell(\succsim_{N_\ell}) = b(\succsim_\ell, N_\ell)$.

Let $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$ be such that $b(\succsim_\ell, N_\ell) = \ell$. Since $f_\ell(\succsim_{N_{\ell-1}}) = \ell$ and $N_\ell = L(\ell, \hat{\succsim}_\ell) \subseteq L(\ell, \succsim_\ell)$, by *monotonicity* (Fact 3), $f(\succsim_{N_\ell}) = f(\succsim_{N_{\ell-1}})$. Therefore,

$$
\begin{aligned}
f_i(\succsim_{N_\ell}) &= i \quad \forall\, i \in N \setminus N_\ell;\\
f_\ell(\succsim_{N_\ell}) &= \ell = b(\succsim_\ell, N_\ell).
\end{aligned}
$$

Step 2: For each $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$, $f_\ell(\succsim_{N_\ell}) = b(\succsim_\ell, N_\ell)$.

Let $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$. To simplify notation, let $\bar{b} \equiv b(\succsim_\ell, N_\ell)$. By Step 1, it suffices to consider the case where $\bar{b} \neq \ell$.

Substep 2-1: $f_\ell(\succsim_{N_\ell}) \in N_\ell$. By the induction hypothesis and *strategy-proofness*, $f_\ell(\succsim_{N_{\ell-1}}) = \ell \mathrel{\hat{\succsim}_\ell} f_\ell(\succsim_{N_\ell})$. This implies that $f_\ell(\succsim_{N_\ell}) \in N_\ell$.

Substep 2-2: $f_\ell(\succsim_{N_\ell}) \neq \ell$. Suppose, by contradiction, that $f_\ell(\succsim_{N_\ell}) = \ell$. Since $\bar{b} \neq \ell$, then $\bar{b} \in N_{\ell-1}$. Therefore, by the induction hypothesis, there exists $j \in N_{\ell-1}$ such that $f_j(\succsim_{N_{\ell-1}}) = \bar{b}$. Let $\succsim_j^{\ell\bar{b}} \in \mathscr{P}$ be a preference relation of agent j such that

$$\succsim_j^{\ell\bar{b}}\colon\ \ell, \bar{b}, \ldots \tag{5}$$

We now establish that by applying the *rectangular property*,

$$f(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}, \succsim_\ell) = f(\succsim_{N_{\ell-1}}). \tag{6}$$

Let

$$\begin{aligned} \succsim' &\equiv (\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}, \succsim_\ell, \hat{\succsim}_{-N_\ell}); \\ \succsim'' &\equiv (\succsim_{N_{\ell-1}}, \hat{\succsim}_{-N_{\ell-1}}). \end{aligned}$$

For agent ℓ, since we suppose that $f_\ell(\succsim_{N_\ell}) = \ell$,

$$f_\ell(\succsim'') = f_\ell(\succsim_{N_{\ell-1}}) = \ell = f_\ell(\succsim_{N_\ell}) = f_\ell(\succsim'_\ell, \succsim''_{-\ell}), \tag{7}$$

where the second equality follows from the induction hypothesis. By the induction hypothesis, $f_\ell(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}) = \ell$. Thus,

$$f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}) \neq \ell. \tag{8}$$

On the other hand, by *strategy-proofness* and the fact that $f_j(\succsim_{N_{\ell-1}}) = \bar{b}$,

$$f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}) \succsim_j^{\ell\bar{b}} f_j(\succsim_{N_{\ell-1}}) = \bar{b}. \tag{9}$$

By (5), (8), and (9), we have $f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}) = \bar{b}$, which implies that

$$f_j(\succsim'') = f_j(\succsim_{N_{\ell-1}}) = \bar{b} = f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}) = f_j(\succsim'_j, \succsim''_{-j}). \tag{10}$$

Note that for each $i \in N \setminus \{\ell, j\}$, $\succsim'_i = \succsim''_i$. Therefore,

$$f_i(\succsim'') = f_i(\succsim'_i, \succsim''_{-i}) \ \ \forall\, i \in N \setminus \{\ell, j\}. \tag{11}$$

By the *rectangular property*, (7), (10), and (11) together imply that $f(\succsim') = f(\succsim'')$. Hence, we obtain (6). Then,

$$\bar{b} \succ_\ell \ell = f_\ell(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}, \succsim_\ell);$$
$$\ell \succ_j^{\ell\bar{b}} \bar{b} = f_j(\succsim_{N_{\ell-1}\setminus\{j\}}, \succsim_j^{\ell\bar{b}}, \succsim_\ell),$$

which is a contradiction to *efficiency*.

Substep 2-3: $f_\ell(\succsim_{N_\ell}) = \bar{b}$. We first consider the case where $b(\succsim_\ell, N_\ell \setminus \{\bar{b}\}) = \ell$. Then, by the induction hypothesis and *strategy-proofness*, $f_\ell(\succsim_{N_\ell}) \succsim_\ell \ell = f_\ell(\succsim_{N_{\ell-1}})$. This result, together with Substeps 2-1 and 2-2, implies that $f_\ell(\succsim_{N_\ell}) = \bar{b}$.

We next consider the case where $b(\succsim_\ell, N_\ell \setminus \{\bar{b}\}) \neq \ell$. Pick any $\succsim_\ell^{\bar{b}\ell} \in \mathscr{P}$ such that

$$\succsim_\ell^{\bar{b}\ell} \colon \bar{b}, \ell, \ldots.$$

Then, by the previous case ($b(\succsim_\ell, N_\ell \setminus \{\bar{b}\}) = \ell$), $f_\ell(\succsim_{N_{\ell-1}}, \succsim_\ell^{\bar{b}\ell}) = \bar{b}$. By *strategy-proofness*,

$$f_\ell(\succsim_{N_\ell}) \succsim_\ell \bar{b} = f_\ell(\succsim_{N_{\ell-1}}, \succsim_\ell^{\bar{b}\ell}). \tag{12}$$

Note that by Substep 2-1, $f_\ell(\succsim_{N_\ell}) \in N_\ell$. Thus, (12) implies that $f_\ell(\succsim_{N_\ell}) = \bar{b}$.

Step 3: For each $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$, $f_i(\succsim_{N_\ell}) = i$ for each $i \in N \setminus N_\ell$.

Let $\succsim_{N_\ell} \in \mathscr{P}^{N_\ell}$ and $\bar{b} \equiv b(\succsim_\ell, N_\ell)$. By Step 1, it suffices to consider the case $\bar{b} \neq \ell$. Let us consider the preference profile $\succsim'_{N_\ell} \in \mathscr{P}^{N_\ell}$ such that:

P1. $\succsim'_\ell : n, n-1, \ldots, \ell+2, \ell+1, \bar{b}, \ell, \ldots;$

P2. For each $i \in N_{\ell-1}$, $\succsim'_i : \bar{b}, f_i(\succsim_{N_\ell}), \ldots$

Note that for each $i \in N_{\ell-1}$, $\succsim'_i$ is well-defined since $f_i(\succsim_{N_\ell}) \neq \bar{b} = f_\ell(\succsim_{N_\ell})$ by Step 2.

Substep 3-1: $f(\succsim_{N_\ell}) = f(\succsim'_{N_\ell})$. In order to establish $f(\succsim_{N_\ell}) = f(\succsim'_{N_\ell})$, we apply the *rectangular property*. For agent ℓ, by Step 2, $f_\ell(\succsim_{N_\ell}) = b(\succsim_\ell, N_\ell)$ and $f_\ell(\succsim_{N_{\ell-1}}, \succsim'_\ell) = b(\succsim'_\ell, N_\ell)$. Therefore,

$$f_\ell(\succsim_{N_\ell}) = \bar{b} = f_\ell(\succsim_{N_{\ell-1}}, \succsim'_\ell). \tag{13}$$

Let $i \in N_{\ell-1}$. Then, by Step 2, $f_\ell(\succsim_{N_\ell \setminus \{i\}}, \succsim'_i) = b(\succsim_\ell, N_\ell) = \bar{b}$. This implies that $f_i(\succsim_{N_\ell \setminus \{i\}}, \succsim'_i) \neq \bar{b}$. By *strategy-proofness*, $f_i(\succsim_{N_\ell \setminus \{i\}}, \succsim'_i) \succsim'_i f_i(\succsim_{N_\ell})$, which in turn implies that

$$f_i(\succsim_{N_\ell}) = f_i(\succsim_{N_\ell \setminus \{i\}}, \succsim'_i). \tag{14}$$

Therefore, by the *rectangular property*, (13) and (14) imply that $f(\succsim_{N_\ell}) = f(\succsim'_{N_\ell})$.

Substep 3-2: $f_i(\succsim'_{N_\ell}) \in N_\ell$ for each $i \in N_\ell$. Suppose, by contradiction, that there exists $j \in N_{\ell-1}$ such that $f_j(\succsim'_{N_\ell}) \notin N_\ell$. Note that by Step 2, $f_\ell(\succsim'_{N_\ell}) = b(\succsim'_\ell, N_\ell) = \bar{b}$. Then, by P1 and P2,

$$f_j(\succsim'_{N_\ell}) \succ'_\ell \bar{b};$$
$$\bar{b} \succ'_j f_j(\succsim'_{N_\ell}),$$

which is a contradiction to *efficiency*. Thus, $f_i(\succsim'_{N_\ell}) \in N_\ell$ for each $i \in N_\ell$.

Substep 3-3: $f_i(\succsim'_{N_\ell}) = i$ for each $i \in N \setminus N_\ell$. Suppose, by contradiction, that there exists $j \in N \setminus N_\ell$ such that $f_j(\succsim'_{N_\ell}) \neq j$. Without loss of generality, we assume that j is the largest index among those such that $f_i(\succsim'_{N_\ell}) \neq i$. Then, $j > f_j(\succsim'_{N_\ell})$ holds because $f_i(\succsim'_{N_\ell}) = i$ for each $i > j$. Note that, by Substep 3-2, $f_j(\succsim'_{N_\ell}) \in N \setminus N_\ell$. Now, let $\succsim^*_j \in \mathscr{P}$ be such that

$$\succsim^*_j : n, n-1, \ldots, j+1, j, \bar{b}, f_j(\succsim'_{N_\ell}), \ldots$$

By *strategy-proofness*, we have $f_j(\succsim'_{N_\ell}, \succsim^*_j) \succsim^*_j f_j(\succsim'_{N_\ell})$. If $f_j(\succsim'_{N_\ell}, \succsim^*_j) \geq j$, then $f_j(\succsim'_{N_\ell}, \succsim^*_j) \hat{\succ}_j f_j(\succsim'_{N_\ell})$, which is a contradiction to *strategy-proofness*. Therefore, we have either $f_j(\succsim'_{N_\ell}, \succsim^*_j) = \bar{b}$ or $f_j(\succsim'_{N_\ell}, \succsim^*_j) = f_j(\succsim'_{N_\ell})$.

Case 1: $f_j(\succsim'_{N_\ell}, \succsim^*_j) = \bar{b}$. By the induction hypothesis and the fact that $j \in N \setminus N_\ell$, $f_j(\succsim'_{N_{\ell-1}}) = j$. Then, $N_j = L(j, \hat{\succsim}_j) \subseteq L(j, \succsim^*_j)$. Thus, by *monotonicity* (Fact 3),

$$f(\succsim'_{N_{\ell-1}}, \succsim^*_j) = f(\succsim'_{N_{\ell-1}}). \tag{15}$$

It follows from (15) and the induction hypothesis that

$$f_\ell(\succsim'_{N_{\ell-1}}, \succsim^*_j) = \ell. \tag{16}$$

By *strategy-proofness*, $f_\ell(\succsim'_{N_\ell}, \succsim^*_j) \succsim'_\ell \ell = f_\ell(\succsim'_{N_{\ell-1}}, \succsim^*_j)$ and $f_\ell(\succsim'_{N_{\ell-1}}, \succsim^*_j) = \ell \; \hat{\succsim}_\ell$

$f_\ell(\succsim'_{N_\ell}, \succsim^*_j)$, which imply that we have either $f_\ell(\succsim'_{N_\ell}, \succsim^*_j) = \bar{b}$ or $f_\ell(\succsim'_{N_\ell}, \succsim^*_j) = \ell$. Note that $f_j(\succsim'_{N_\ell}, \succsim^*_j) = \bar{b}$, and hence, $f_\ell(\succsim'_{N_\ell}, \succsim^*_j) \neq \bar{b}$. Thus,

$$f_\ell(\succsim'_{N_\ell}, \succsim^*_j) = \ell. \tag{17}$$

By (16) and (17), $f_\ell(\succsim'_{N_{\ell-1}}, \succsim^*_j) = f_\ell(\succsim'_{N_\ell}, \succsim^*_j)$. This result, together with *non-bossiness*, implies that

$$f(\succsim'_{N_{\ell-1}}, \succsim^*_j) = f(\succsim'_{N_\ell}, \succsim^*_j). \tag{18}$$

Further, (15) and (18) together imply that

$$f_j(\succsim'_{N_\ell}, \succsim^*_j) = f_j(\succsim'_{N_{\ell-1}}) = j,$$

in contradiction to $f_j(\succsim'_{N_\ell}, \succsim^*_j) = \bar{b}$.

Case 2: $\boldsymbol{f_j(\succsim'_{N_\ell}, \succsim^*_j) = f_j(\succsim'_{N_\ell})}$. By *non-bossiness*, $f(\succsim'_{N_\ell}, \succsim^*_j) = f(\succsim'_{N_\ell})$. Then, since $f_j(\succsim'_{N_\ell}) \in N \setminus N_\ell$ and $f_\ell(\succsim'_{N_\ell}) = \bar{b}$ by Step 2,

$$\begin{aligned} \bar{b} &\succ^*_j f_j(\succsim'_{N_\ell}); \\ f_j(\succsim'_{N_\ell}) &\succ'_\ell \bar{b}, \end{aligned}$$

which is a contradiction to *efficiency*.

By Substeps 3-1, 3-2, and 3-3, we have $f_i(\succsim_{N_\ell}) = i$ for each $i \in N \setminus N_\ell$. □

Proof of Claim 2

Since agent n is the first dictator and $b(\succsim_n, N) = b(\succsim'_n, N) = a$, we have

$$f_n(\succsim) = f_n(\succsim') = a; \tag{19}$$

$$N = L(f_n(\succsim'), \succsim'_n) = L(f_n(\succsim'), \succsim_n). \tag{20}$$

On the other hand, by (19), $f_i(\succsim') \neq a$ for each $i \in N \setminus \{n\}$. This implies that

$$L(f_i(\succsim'), \succsim'_i) \subseteq L(f_i(\succsim'), \succsim_i) \quad \forall\, i \in N \setminus \{n\}. \tag{21}$$

Hence *monotonicity* (Fact 3) together with (20) and (21) implies that $f(\succsim) = f(\succsim')$.
□

References

Abdulkadiroğlu, A. and T. Sönmez (1998) "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, **66**, 689–701.

Abdulkadiroğlu, A. and T. Sönmez (1999) "House allocation with existing tenants," *Journal of Economic Theory*, **88**, 233–260.

Bird, C.G. (1984) "Group incentive compatibility in a market with indivisible goods," *Economics Letters*, **14**, 309–313.

Bochet, O. and T. Sakai (2009) "Secure implementation in allotment economies," forthcoming in *Games and Economic Behavior.*

Cason, T., T. Saijo, T. Sjöström, and T. Yamato (2006) "Secure implementation experiments: do strategy-proof mechanisms really work?" *Games and Economic Behavior*, **57**, 206-235.

Chen, Y. and T. Sönmez (2002) "Improving efficiency of on-campus housing: an experimental study," *American Economic Review*, **92**, 1669–1686.

Chen, Y. and T. Sönmez (2004) "An experimental study of house allocation mechanisms," *Economics Letters*, **83**, 137–140.

Ehlers, L. (2002) "Coalitional strategy-proof house allocation," *Journal of Economic Theory*, **105**, 298–317.

Ehlers, L. and B. Klaus (2003a) "Resource-monotonicity for house allocation problems," *International Journal of Game Theory*, **32**, 545–560.

Ehlers, L. and B. Klaus (2003b) "Coalitional strategy-proofness, resource-monotonicity, and separability for multiple assignment problems," *Social Choice and Welfare*, **21**, 265–280.

Ehlers, L., B. Klaus, and S. Pápai (2002) "Strategy-proofness and population-monotonicity for house allocation problems," *Journal of Mathematical Economics*, **38**, 329–339.

Ergin, H. (2000) "Consistency in house allocation problems," *Journal of Mathematical Economics*, **34**, 77–97.

Klaus, B. and E. Miyagawa (2001) "Strategy-proofness, solidarity, and consistency for multiple assignment problems," *International Journal of Game Theory*, **30**, 421–435.

Ma, J. (1994) "Strategy-proofness and the strict core in a market with indivisibilities," *International Journal of Game Theory*, **23**, 75–83.

Maskin, E. (1999) "Nash equilibrium and welfare optimality," *Review of Economic Studies*, **66**, 23–38.

Miyagawa, E. (2002) "Strategy-proofness and the core in house allocation problems," *Games and Economic Behavior*, **38**, 347–361.

Mizukami, H. and T. Wakayama (2007) "Dominant strategy implementation in economic environments," *Games and Economic Behavior*, **60**, 307–325.

Mizukami, H. and T. Wakayama (2009) "Secure implementation: an alternative characterization," mimeo.

Pápai, S. (2000) "Strategyproof assignment by hierarchical exchange," *Econometrica*, **68**, 1403–1433.

Pápai, S. (2001) "Strategyproof and nonbossy multiple assignments," *Journal of Public Economic Theory*, **3**, 257–271.

Roth, A.E. (1982) "Incentive compatibility in a market with indivisible goods," *Economics Letters*, **9**, 127–132.

Roth, A.E. and A. Postlewaite (1977) "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economies*, **4**, 131–137.

Roth, A.E., T. Sönmez, and U. Ünver (2004) "Kidney exchange," *Quarterly Journal of Economics*, **119**, 457–488.

Saijo, T., T. Sjöström, and T. Yamato (2004) "Secure implementation," COE Discussion Paper Series No. 8, Osaka University.

Saijo, T., T. Sjöström, and T. Yamato (2007) "Secure implementation," *Theoretical Economics*, **2**, 203–229.

Shapley, L. and H. Scarf (1974) "On cores and indivisibility," *Journal of Mathematical Economics*, **1**, 23–37.

Sönmez, T. (1996) "Implementation in generalized matching problems," *Journal of Mathematical Economics*, **26**, 429–439.

Sönmez, T. and U. Ünver (2005) "House allocation with existing tenants: an equivalence," *Games and Economic Behavior*, **52**, 153–185.

Svensson, L.-G. (1999) "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, **16**, 557–567.

Takamiya, K. (2001) "Coalitional strategy-proof and monotonicity in Shapley-Scarf housing markets," *Mathematical Social Sciences*, **41**, 201–213.

Takamiya, K. (2009) "Preference revelation games and strong cores of allocation problems with indivisibilities," *Journal of Mathematical Economics*, **45**, 199–204.

Wako, J. (1999) "Coalition-proofness of the competitive allocations in an indivisible goods market," *Fields Institute Communications*, **23**, 277–283.