Games with Discontinuous Payoffs: a Strengthening of Reny's Existence Theorem

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Abstract: We provide a Nash equilibrium existence theorem for games with discontinuous payoffs whose hypotheses are in a number of ways weaker than those of the theorem of Reny (1999). Our result subsumes prior existence results of Nishimura and Friedman (1981) and Novshek (1985) that are not covered by his theorem.

1 Introduction

Many important and famous games in economics (e.g, the Hotelling location game, Bertrand competition, Cournot competition with fixed costs, and various auction models) have discontinuous payoffs, and consequently do not satisfy the hypotheses of Nash's existence proof or its infinite dimensional generalizations, but nonetheless have a nonempty set of pure Nash equilibria. Using an argument that is quite ingenious and involved, Reny (1999) establishes a result that explains equilibrium existence for many such examples, and which has been applied in novel settings many times since then. (See for example Monteiro and Page (2007) and Monteiro and Page (2008).) His theorem's hypotheses are simple and weak, and in many cases easy to verify.

As Reny explains, many earlier results can be obtained as corollaries of his result. However, he describes the results of Nishimura and Friedman (1981) and Roberts and Sonnenschein (1976) as seemingly having a different character. In this paper we provide a generalization of Reny's theorem that implies both the Nishimura and Friedman result and the two firm case of an existence result of Novshek (1985), which is the most refined result asserting existence of Cournot equilibrium in the stream of literature following Roberts and Sonnenschein. In both cases the arguments are considerably simpler than those in the original papers. Our method of proof of our main result is also novel, yielding a demonstration that is straightforward and brief, at least in comparison with Reny's.

The next section reviews Reny's theorem and states a result whose hypotheses are less restrictive than Reny's, but more restrictive than Theorem 3.3, which suffices to imply the Nihimura-Friedman existence theorem. Section 3 states Theorem 3.3, which is our main result. In Section 4 we show that Theorem 3.3 implies Novshek's result, that there in a pure Nash equilibrium in his variant of the Cournot model, in the case of two firms. Section 5 gives the proof of Theorem 3.3.

2 Preliminaries: Reny and Nishimura-Friedman

Our system of notation is largely taken from Reny (1999). There is a fixed **normal form game** with N players, which is a $2N$-tuple

$$G = (X_1, \dots, X_N, u_1, \dots, u_N)$$

where, for each $i = 1, \dots, N$, X_i is a non-empty set called the i-th player's **strategy set** and each u_i is a function from the set of **strategy profiles** $X = \prod_{i=1}^N X_i$ to $\mathbb{R}$ called the i-th player's **payoff function**. Let

$$u = (u_1, \dots, u_N) : X \to \mathbb{R}^N.$$

We assume throughout that u is bounded. For $x \in X$ let $A(x)$ be the set of $\alpha \in \mathbb{R}^n$ such that (x, α) is in the closure of the graph of u. Since u is bounded, each $A(x)$ is compact.

We also assume throughout that each X_i is a nonempty compact convex subset of a topological vector space, and X is endowed with the product topology. Unless otherwise indicated, all topological notions refer to the relative topology of X. We adopt the usual notation for "all players other than i." Let $X_{-i} = \prod_{j \neq i} X_j$. If $x \in X$ is given, x_{-i} denotes the projection of x on X_{-i}. For each $x_{-i} \in X_{-i}$ (or $x \in X$) and $y_i \in X_i$ we write (y_i, x_{-i}) for the strategy profile $z \in X$ satisfying $z_i = y_i$ and $z_j = x_j$ if $j \neq i$. A **Nash equilibrium** of G is a point $x^* \in X$ satisfying $u_i(x^*) \geq u_i(y_i, x^*_{-i})$ for all i and all $y_i \in X_i$.

We now review Reny's theorem, to provide a context for the conditions we introduce. For each player i let $B_i \colon X \times \mathbb{R} \twoheadrightarrow X_i$ be the set valued mapping

$$B_i(x, \alpha_i) = \{y_i \in X_i \colon u_i(y_i, x_{-i}) \geq \alpha_i\}.$$

Definition 2.1. A player i can **secure** a payoff $\alpha_i \in \mathbb{R}$ at $x \in X$ if there is a neighborhood U of x in X such that

$$\bigcap_{z \in U} B_i(z, \alpha_i) \neq \emptyset.$$

That is, there is some $y_i \in X_i$ such that $u_i(y_i, z_{-i}) \geq \alpha_i$ for all $z \in U$. The game G is **better reply secure** at $x \in X$ if, for any $\alpha \in A(x)$, there is some player i and $\varepsilon > 0$ such that player i can secure $\alpha_i + \varepsilon$ at x. The game G is **better reply secure** if it is better reply secure on every strategy profile that is not a Nash equilibrium.

Theorem 2.2 (Reny (1999)). *Suppose that for each i and each $x_{-i} \in X_{-i}$ the function $u_i(\cdot, x_{-i})\colon X_i \to \mathbb{R}$ is quasiconcave. If G is better reply secure, then it has a Nash equilibrium.*

To better understand Reny's result in the context of this note we restate his condition.

Definition 2.3. The game is **A-secure** at $x \in X$ if there is $\alpha \in \mathbb{R}^N$ and $\varepsilon > 0$ such that:

(a) every player i can secure $\alpha_i + \varepsilon$ at x;

(b) there is a neighborhood U of x such that for any $z \in U$ there exists some player i with $u_i(z_i) < \alpha_i$, i.e., $z_i \notin B_i(z, \alpha_i)$.

Lemma 2.4. *For $x \in X$ the game is better reply secure at x if and only if it is A-secure at x.*

Proof. For each $x \in X$ let τ_x be a neighborhood base of x. For each i define the function $\underline{u}_i\colon X \to \mathbb{R}$ as follows:

$$\underline{u}_i(x) = \sup_{U \in \tau_x} \inf_{z \in U} u_i(x_i, z_{-i}) .$$

This function is automatically lower semicontinuous as a function of the strategies of the other players. For each i and $x_{-i} \in X_{-i}$ let

$$\delta_i(x_{-i}) = \sup_{y_i \in X_i} \underline{u}_i(y_i, x_{-i}) .$$

As the pointwise supremum of a collection of lower semicontinuous functions, δ_i is lower semicontinuous. Notice that each $\alpha_i < \delta_i(x_{-i})$ can be strongly secured by player i at x.

Fixing an x, assume that the game is better reply secure at x. Then for each $\alpha \in A(x)$ there is some player i such that $\delta_i(x_{-i}) > \alpha_i$, which implies that $\delta_i(x_{-i}) > \alpha'_i + \varepsilon$ hold for some $\varepsilon > 0$ and all α' in some neighborhood of α. Since $A(x)$ is compact, it is covered by finitely many such neighborhoods so there is $\varepsilon > 0$ such that whenever $\alpha \in A(x)$ there exists some player i such that $\delta_i(x_{-i}) > \alpha_i + \varepsilon$.

Let $\alpha_i = \delta_i(x_{-i}) - \varepsilon/2$ for all i. Each $\alpha_i + \varepsilon/2$ can be strongly secured by player i at x. If (b) is false, then for each $U \in \tau_x$ there is some $z_U \in U$ such that $u_i(z_U) \geq \alpha_i$ for all i. Since τ_x is a directed set (ordered by reverse inclusion) the compactness of the closure of the range of u implies that there is a convergent subnet, so there is $\alpha' \in A(x)$ such that $\alpha'_i \geq \alpha_i = \delta_i(x_{-i}) - \varepsilon/2$ for all i. This is impossible, so we conclude that the game is A-secure at x.

Assume that the game is A-secure at x, with α, ε and U as in the definition. If we replace α with $\alpha + (\varepsilon/2, \dots, \varepsilon/2)$, then replace ε with $\varepsilon/2$, it will still be the case that each player can secure $\alpha_i+\varepsilon$, but now for each $z \in U$ there is some i such that $u_i(z) < \alpha_i - \varepsilon$. For each $\alpha' \in A(x)$ there is some i with $\alpha'_i \le \alpha_i - \varepsilon$, and this i can secure $\alpha' + \varepsilon$ at x, so the game is better reply secure at x. ■

For each i let $C_i \colon X \times \mathbb{R} \twoheadrightarrow X_i$ be the set valued mapping

$$C_i(x, \alpha_i) = \operatorname{con} B_i(x, \alpha_i)\,.$$

Part (a) of the following definition is more easily satisfied than its analogue above. On the other hand, in part (b) we replace $B_i(z, \alpha_i)$ with $C_i(z, \alpha_i)$, which makes it harder to satisfy, but $B_i(z, \alpha_i) = C_i(z, \alpha_i)$ when each $u_i(\cdot, x_{-i})$ is quasiconcave, so the net effect is to make the hypotheses of Proposition 2.6 below (which is a relatively simple special case of Theorem 3.3) weaker than those of Theorem 2.2.

Definition 2.5. The game is B**-secure** at $x \in X$ if there is $\alpha \in \mathbb{R}^N$ such that:

(a) every player i can secure α_i at x;

(b) there is a neighborhood U of x such that for any $z \in U$ there exists some player i with $z_i \notin C_i(z, \alpha_i)$.

Proposition 2.6. *If the game is B-secure at each $x \in X$ that is not a Nash equilibrium, then G has a Nash equilibrium.*

Nishimura and Friedman (1981) prove the existence of a Nash equilibrium when each X_i is a nonempty, compact, convex subset of a Euclidean space, u is continuous (but not necessarily quasiconcave) and for any x that is not a Nash equilibrium there is an agent i, a coordinate index k, and an open nieghborhood U of x, such that

$$(y^1_{ik} - x^1_{ik})(y^2_{ik} - x^2_{ik}) > 0$$

whenever $x^1, x^2 \in U$ and y^1_i and y^2_i are best responses for i to x^1 and x^2 respectively. Using compactness, and the continuity of u_i, it is not difficult to show that this is equivalent to $y_{ik} > x_{ik}$ for all best responses y_i to x. A more general condition that does not depend on the coordinate system is that there is a hyperplane that strictly separates x_i from the set of i's best responses to x.

We now show that G satisfies the conditions of Proposition 2.6 when it satisfies the hypotheses of Nishimura and Friedman's result. Consider an $x \in X$ that is not a Nash equilibrium. For each $j = 1, \ldots, N$ let β_j be the utility for j when other agents play their components of x_{-j} and j plays a best response to x. Since u is continuous, for any $\varepsilon > 0$ player j can secure $\beta_j - \varepsilon$ at x by playing such a best response. For any neighborhood V of $B_i(x, \beta_i)$ it is the case that $B_i(x, \beta_i - \varepsilon) \subset V$ when ε is sufficiently small, and in turn it follows that there is a neighborhood U of x such that $B_i(z, \beta_i - \varepsilon) \subset V$ for all $z \in U$. It follows that if there is a hyperplane strictly separating x_i from $B_i(x, \beta_i)$, then for sufficiently small $\varepsilon > 0$ and a sufficiently small neighborhood U of x there is a hyperplane strictly separating z_i and $B_i(z, \beta_i - \varepsilon)$ for all $z \in U$, in which case $z_i \notin C_i(z, \beta_i)$. Setting $\alpha = (\beta_1 - \varepsilon, \ldots, \beta_N - \varepsilon)$ gives the required property.

3 The Main Result

Roughly, the idea of Reny's theorem is to exploit properties of the "better reply correspondence." The refinement that allows us to subsume Novshek's theorem is a matter of pointing out that it is enough if some subcorrespondence enjoys the relevant properties.

For each i let $\mathcal{X}_i \colon X \twoheadrightarrow X_i$ be a given set valued mapping and let $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_N)$. We call $\mathcal{X}$ a **restriction operator**. We say that $\mathcal{X}$ is **universal** if $\mathcal{X}_i(x) = X_i$ for all $x \in X$. Fix a restriction operator $\mathcal{X}$. For each i define the set valued mappings $B_i^{\mathcal{X}} \colon X \times \mathbb{R} \twoheadrightarrow X_i$ and $C_i^{\mathcal{X}} \colon X \times \mathbb{R} \twoheadrightarrow X_i$ by setting

$$B_i^{\mathcal{X}}(x, \alpha_i) = \{y_i \in \mathcal{X}_i(x) \colon u_i(y_i, x_{-i}) \geq \alpha_i\} \ \text{ and } \ C_i^{\mathcal{X}}(x, \alpha_i) = \operatorname{con} B_i^{\mathcal{X}}(x, \alpha_i).$$

In addition to taking advantage of the restriction operator, the next definition incorporates one more refinement of A-security: we allow a different strategy to be used in response to each element of a finite closed cover of a neighborhood of the given strategy.

Definition 3.1. Player i can **$\mathcal{X}$-secure** a payoff $\alpha_i \in \mathbb{R}$ at $x \in X$ if there is a finite closed cover $F^1, \ldots, F^J$ of a neighborhood of x such that for each j we have $\bigcap_{z \in F^j} B_i^{\mathcal{X}}(z, \alpha_i) \neq \emptyset$.

Definition 3.2. The game G is **$\mathcal{X}$-secure** at $x \in X$ if there is $\alpha \in \mathbb{R}^N$ and a neighborhood U of x such that

(1) Each player i can $\mathcal{X}$-secure α_i at x.

(2) For any $z \in U$ there is some player i for whom $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$.

As in Reny (1999) X need not be a Hausdorff space. Thus, for $x \in X$ the set $\{x\}$ need not be closed, and we write $[x]$ for the closure of $\{x\}$. The game G is **restrictionally secure** if there is a restriction operator $\mathcal{X}$ such that the game is $\mathcal{X}$-secure at any $x \in X$ such that $[x]$ does not contain a Nash equilibrium. Our main theorem is:

Theorem 3.3. *The game G is restrictionally secure if and only if it has a Nash equilibrium.*

4 Novshek's Cournot Model

We now explain the variant of the Cournot duopoly model considered in Novshek (1985), for the case of two firms, and show that Theorem 3.3 implies that it has an equilibrium. Consider the market for a single homogeneous good with inverse demand function $P \colon \mathbb{R} \to \mathbb{R}_+$ and two firms with cost functions $C_1, C_2 \colon \mathbb{R} \to \mathbb{R}_+$.

We assume that P is continuous and that C_1, C_2 are lower semicontinuous. The payoff function of firm i is its profit:

$$u_i(y) = P(y_1 + y_2)y_i - C_i(y_i).$$

This function is upper semicontinuous, and for a given y_i it is continuous as a function of y_{-i}. A Nash equilibrium of this game is called a *Cournot Equilibrium.*

For each i the **reaction mapping** $r_i \colon \mathbb{R} \twoheadrightarrow \mathbb{R}$ is the set valued mapping.

$$r_i(y_{-i}) = \operatorname*{argmax}_{z_i \in \mathbb{R}} u_i(z_i, y_{-i})\,,$$

The assumptions on C_1, C_2, and P imply that r_i has a closed graph.

We assume that there is a number Q^* such that setting $y_i = 0$ gives a larger payoff than playing any quantity greater than or equal to Q^*. It follows that for each i andn y_{-i}, $r_i(y_{-i})$ is nonempty and compact. Let

$$r_i^+(y) = \max r_i(y) \quad \text{and} \quad r_i^- = \min r_i(y)\,.$$

The key assumption is that if $y_{-i} < y'_{-i}$, then $r_i^-(y_{-i}) \geq r_i^+(y'_{-i})$. Novshek (1985) discusses assumptions on the cost and inverse demand functions that imply this.

Since our goal is to show that a Cournot equilibrium exists, there is no loss of generality in assuming there are no Cournot equilibria in which one of the firms produces nothing. What this means concretely is that 0 is not a best response for either firm to the other firm's maximal monopoly output: $r_2^-(r_1^+(0)) > 0$ and $r_1^-(r_2^+(0)) > 0$.

We will say that (y_1, y_2) is a **quasiequilibrium** if $r_1^-(y_2) \leq y_1 \leq r_1^+(y_2)$ and $r_2^-(y_1) \leq y_2 \leq r_2^+(y_1)$. For each i the correspondence mapping y_{-i} to the convex hull of $r_i(y_{-i})$ is upper semicontinuous with compact range, and the set of quasiequilibria is compact because it is the intersection of the graphs of these two correspondences. Kakutani's fixed point theorem implies that this set is nonempty. If (y'_1, y'_2) is a second quasiequilibrium with $y'_1 > y_1$, then $y'_2 \leq y_2$ (if $y'_2 = y_2$, then $r_2^+(y_1) = y_2 = y'_2 = r_2^-(y'_1)$) and this holds also with the two agents reversed, so there is a quasiequilibrium y^* such that $y_1 \leq y_1^*$ and $y_2 \geq y_2^*$ for all quasiequilibria (y_1, y_2).

Let

$$\Omega = \{\, y \in [0, Q^*]^2 : y_1 \leq y_1^* \text{ or } y_2 \geq y_2^* \,\}.$$

Let $\Omega^c = [0, Q^*]^2 \setminus \Omega$ be the complement of Ω, and let Ω° and $\partial\Omega$ be the interior and the boundary of Ω, respectively, in the relative topology of $[0, Q^*]^2$. We define the restriction operator $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2)$ by setting

$$\mathcal{X}_1(y) = \begin{cases} [r_1^+(y_2^*), Q^*] & \text{if } y \in \Omega^c\,, \\ (y_1, Q^*] & \text{if } y \in \Omega \text{ and } y_1 \neq Q^*\,, \\ \{Q^*\} & \text{if } y \in \Omega \text{ and } y_1 = Q^* \end{cases}$$

and

$$\mathcal{X}_2(y) = \begin{cases} [0, r_2^-(y_1^*)] & \text{if } y \in \Omega^c, \\ [0, y_2) & \text{if } y \in \Omega \text{ and } y_2 \neq 0, \\ \{0\} & \text{if } y \in \Omega \text{ and } y_2 = 0. \end{cases}$$

We now need to show that the game is $\mathcal{X}$-secure. Fix $y \in [0, Q^*]^2$, which we assume is not a Nash equilibrium. First suppose that $y \in \Omega^\circ$. Since Ω° is open we can choose an open neighborhood $U \subset \Omega^\circ$ of y. If α_1 and α_2 are lower bounds for u_1 and u_2 on $[0, Q^*]^2$, then each i can secure α_i at y. In addition, $(Q^*, 0) \notin \Omega$, so either $y_1 < Q^*$, in which case $z_1 \notin \mathcal{X}_1(z) = (z_1, Q^*]$ for all $z \in U$ (proviided U is sufficiently small) or $y_2 > 0$, in which case $z_2 \notin \mathcal{X}_2(z) = [0, z_2)$ for all $z \in U$. Thus the game is $\mathcal{X}$-secure at y.

Next suppose that $y \in \Omega^c$. Since y is not a quasiequilibrium it must be the case that $y_1 < r_1^-(y_2)$, $y_1 > r_1^+(y_2)$, $y_2 < r_2^-(y_1)$, or $y_2 > r_2^+(y_1)$. In each of the four cases it is easy to choose an α such that each i can secure α_i at y and for one i we have $z_i \notin C_i(z, \alpha_i)$ for all z in some neighborhood of y, so that the game is $\mathcal{X}$-secure at y.

The remaining case is that $y \in \partial\Omega$. We may assume that y^* is not a Nash equilibrium, so either $u_1(y^*) < u_1(r_1^+(y_2^*), y_2^*)$, in which case $r_1^+(y_2^*) > y_1^*$, or $u_2(y^*) < u(y_1^*, r_2^-(y_1^*))$, in which case $r_2^-(y_1^*) < y_2^*$. Suppose that $u_1(y^*) < u_1(r_1^+(y_2^*), y_2^*)$. (The argument in the other case is similar.)

First suppose that $y_1 < r_1^+(y_2^*)$. Then we can choose α_1 slightly below the payoff agent 1 receives if she plays a best response to y_2, but greater than $u_1(y)$, and agent 1 can $\mathcal{X}$-secure α_1 because $r_1^+(y_2) \in B_1^{\mathcal{X}}(z, \alpha_1)$ for all z in some neighborhood U of y. If U is small enough, then $z_1 \notin C_1^{\mathcal{X}}(z, \alpha_1)$ for all $z \in U$. If α_2 is any utility that player 2 can $\mathcal{X}$-secure at y, then the game in $(\mathcal{X}, \alpha, U)$-secure at y.

Now suppose that $y_1 \geq r_1^+(y_2^*)$. Then $y_2 = y_2^*$ because $y_1 > y_1^*$. We must have $r_2^-(y_1) > y_2^*$ because if $y_1^* < y_1' \leq r_1^+(y_2^*)$ and $r_2^-(y_1') = y_1^*$, then (y_1', y_2^*) would be a quasiequilibrium, contrary to the definition of y^*. We can choose α_2 slightly below the payoff agent 2 receives if she plays a best response to y_1, but greater than $u_2(y)$, and agent 2 can $\mathcal{X}$-secure α_2 because $r_2^+(y_1) \in B_2^{\mathcal{X}}(z, \alpha_2)$ for all z in some neighborhood U of y. If U is small enough, then $z_2 \notin C_2^{\mathcal{X}}(z, \alpha_2)$ for all $z \in U$. If α_1 is any utility that player 1 can $\mathcal{X}$-secure at y, then the game in $(\mathcal{X}, \alpha, U)$-secure at y. This completes the proof that the game is $\mathcal{X}$-secure at any y that is not an equilibrium.

5 The Proof of Theorem 3.3

It is not hard to show that the game is directionally secure when it has a Nash equilibrium, so we do this first. Recall that a topological space is regular if each point has a neighborhood base of closed sets. Topological vector spaces are regular topological spaces, even if they are not Hausdorff (e.g., Schaefer (1971, Lemma 1.3, p. 16)) and it is easy to see that any subspace of a regular space is regular. Suppose $x^* \in X$ is a Nash equilibrium. For each i we define $\mathcal{X}_i : X \to X_i$ by setting $\mathcal{X}_i(x) = \{x_i^*\}$. Consider an $x \in X$ such that $[x]$ does not contain a Nash equilibrium. Then $[x]$ does not contain x^* and regularity implies that $X \setminus [x]$ contains a closed neighborhood of x^*, so $x \notin [x^*]$. Applying regularity again, there is a closed neighborhood U of x that is disjoint from $[x^*]$. For each i let $\alpha_i = \inf_{x' \in U} u_i(x_i^*, x'_{-i})$, and let $\alpha = (\alpha_1, \dots, \alpha_N)$. Evidently each player i can $\mathcal{X}$-secure α_i at x. For each i and $z \in U$ we have $B_i^{\mathcal{X}}(z, \alpha_i) = C_i^{\mathcal{X}}(z, \alpha_i) = \{x_i^*\}$, so $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$. We have shown that the game is $\mathcal{X}$-secure at x.

What this argument points to is that, in practice, the value of the result is not that it gives conditions that are necessary and sufficient. Rather, it is useful to the extent that one can find restriction operators that are easily seen to satisfy the hypotheses even though the existence of equilibrium would not otherwise be obvious.

Now suppose that G is restrictionally secure, so that there is a restriction operator $\mathcal{X}$ such that G is $\mathcal{X}$-secure at each x such that $[x]$ does not contain a Nash equilibrium. The remainder of this section presents the proof that G has a Nash equilibrium. The following system of terminology will be useful.

Definition 5.1. If $x \in X$, $U \subset X$, and $\alpha \in \mathbb{R}^N$, then the game G is $(\mathcal{X}, \alpha, U)$-**secure** at x if

(1) Each player i can $\mathcal{X}$-secure α_i at x.

(2) For any $z \in U$ there is some player i for whom $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$.

We say that the game is $(\mathcal{X}, \alpha)$-**secure** at x if the game is $(\mathcal{X}, \alpha, U)$-secure at x for some neighborhood U of x.

In preparation for the main body of the argument we develop several lemmas.

Lemma 5.2. *Suppose that $\alpha_1, \dots, \alpha_\ell \in \mathbb{R}^N$ and $U_1, \dots, U_\ell$ are neighborhoods of x, and let $\alpha = \sup_{h=1}^{\ell} \alpha^h$ and $U = \bigcap_{h=1}^{\ell} U^h$. If, for each $h = 1, \dots, \ell$, the game is $(\mathcal{X}, \alpha_h, U_h)$-secure at x for some $\alpha^h \in \mathbb{R}^N$ and some neighborhood U^h of x, then it is $(\mathcal{X}, \alpha, U)$-secure at x.*

Proof. For each h and $z \in U$ there are closed sets $F_h^1, \dots, F_h^{J_h}$ whose union contains U_h such that for each i, j we have $\bigcap_{z \in F_h^j} B_i^{\mathcal{X}}(z, \alpha_i^h) \neq \emptyset$. This condition continues to hold with U_h replaced by U if each $F_h^1, \dots, F_h^{J_h}$ is replaced with the collection $F^1, \dots, F^J$ of all nonempty intersections of the form $F_1^{j_1} \cap \ldots \cap F_\ell^{j_\ell}$. In addition, for each h and $z \in U$ there is some i such that $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i^h)$.

We claim that for each j the game is $(\mathcal{X}, \alpha, F_j)$-secure at x. For any i there is some h such that $\alpha_i = \alpha_i^h$, so for any j we have

$$\bigcap_{z \in F^j} B_i^{\mathcal{X}}(z, \alpha_i) = \bigcap_{z \in F^j} B_i^{\mathcal{X}}(z, \alpha_i^h) \neq \emptyset\,.$$

Consider $z \in U$. For any h there is some i such that $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i^h)$, and $B_i^{\mathcal{X}}(z, \alpha_i^h) \supseteq B_i^{\mathcal{X}}(z, \alpha_i)$, so $C_i^{\mathcal{X}}(z, \alpha_i^h) \supseteq C_i^{\mathcal{X}}(z, \alpha_i)$. Consequently $z_i \notin C_i^{\mathcal{X}}(z, \alpha_i)$. ■

By definition a topological space is paracompact if every open cover has an open locally finite refinement. It can be shown that every open cover of a paracompact regular topological space has a closed locally finite refinement. (e.g., Kelley (1955, Theorem 28, p. 156).) In particular, this holds for paracompact subsets of topological vector spaces. We will say that a function $f\colon X \to \mathbb{R}$ is **locally finite valued** if each $x \in X$ has a neighborhood V such that $f(V)$ has finite cardinality.

Lemma 5.3. *Suppose that X is paracompact. If, for each $x \in X$, there is some $\alpha_x \in \mathbb{R}^N$ such that the game is $(\mathcal{X}, \alpha_x)$-secure at x, then there is a function $\psi : X \to \mathbb{R}^N$, each of whose component functions $\psi_i : X \to \mathbb{R}$ is locally finite valued and upper semicontinuous, such that for each x the game is $(\mathcal{X}, \psi(x))$-secure at each $x \in X$.*

Proof. For each $x \in X$ there is an open neighborhood $U_x \subset X$ such that the game is $(\mathcal{X}, \alpha_x)$-secure at x. The open cover $\{U_x\colon x \in X\}$ has a closed locally finite refinements $\{F_\beta\}$. That is, $\{F_\beta\}$ is a collection of closed sets whose union is X, each F_β is contained in some U_x, and for each point in X has a neighborhood V_x for which the set $\{\beta\colon F_\beta \cap V_x \neq \emptyset\}$ is finite. For each β let $\alpha_\beta = \alpha_x$ and $U_\beta = U_x$ for some x such that $F_\beta \subseteq U_x$. For each $x \in X$ let

$$\psi(x) = \sup_{x \in F_\beta} \alpha_\beta\,.$$

Each component $\psi_i\colon X \to \mathbb{R}$ of ψ is upper semicontinuous and locally finite valued because $\{F_\beta\}$ is a closed locally finite cover of X. Notice that if $x \in F_\beta$, then $x \in U_\beta$ and the game is $\mathcal{X}$-secure at x for $\alpha_\beta \in \mathbb{R}^N$ over the neighborhood U_β of x. By Lemma 5.2 there is a neighborhood U of x such that the game is $(\mathcal{X}, \psi(x), U)$-secure at x. ■

Unlike Kakutani's fixed point theorem and its various infinite dimensional extensions, the following fixed point theorem holds true in topological vector spaces that are neither Hausdorff nor locally convex.

Lemma 5.4. *Let X be a nonempty compact convex subset of a topological vector space and let $P\colon X \twoheadrightarrow X$ be a set valued mapping. If there is a finite closed cover $G_1, \ldots, G_m$ of X such that $\bigcap_{z \in G_j} P(z) \neq \emptyset$ for all $j = 1, \ldots, m$, then there exists $x^* \in X$ such that $x^* \in \operatorname{con} P(x^*)$.*

Proof. For each $j = 1, \dots, m$ pick $y_j \in \bigcap_{z \in G_j} P(z)$. Define the correspondence $Q\colon X \twoheadrightarrow X$ by $Q(x) = \operatorname{con}\{y_j \colon x \in G_j\}$. For each $x \in X$ we have $x \in G_j$ for some j, so

$$\emptyset \neq \{y_j \colon x \in G_j\} \subset \bigcup_{j : x \in G_j} \Big(\bigcap_{z \in G_j} P(z) \Big) \subset P(x)$$

and $\emptyset \neq Q(x) \subset \operatorname{con} P(x)$. Let $\tilde{X}$ be the convex hull of $y_1, \dots, y_m$ and let τ be the subspace topology on $\tilde{X}$ induced by the unique Hausdorff linear topology on the affine hull of $\tilde{X}$. This is at least as fine as the subspace topology of $\tilde{X}$, so each $G_j \cap \tilde{X}$ is τ-closed, which implies that $Q|_{\tilde{X}}$ is upper semicontinuous. On the other hand $\tilde{X}$ is the convex hull of a finite set, so it is τ-compact. Applying the Kakutani fixed point theorem to $Q|_{\tilde{X}}$ gives $x^* \in Q(x^*) \subseteq \operatorname{con} P(x^*)$. ∎

We now have the tools we need to complete the proof of our theorem.

Proof of Theorem 3.3. We have already shown that (b) implies (a), so it remains to show that (a) implies (b). For each i and x let $P_i(x) = B_i^{\mathcal{X}}(x, \psi_i(x))$, and let $P(x) = P_1(x) \times \cdots \times P_N(x)$.

Suppose by way of contradiction that there is no Nash equilibrium. Lemma 5.3 gives a function $\psi\colon X \to \mathbb{R}^N$, each of whose component functions is finite valued (because X is compact) and upper semicontinuous, such that each x is $\mathcal{X}$-secure for $\psi(x)$ over some neighborhood of x. Consider a particular $x \in X$ and $i = 1, \dots, N$. There is a closed neighborhood U and a finite collection of closed sets $F_1, \dots, F_m$ whose union contains U such that for each j we have $\bigcap_{z \in F_j} B_i^{\mathcal{X}}(z, \psi_i(x)) \neq \emptyset$. Replacing U with a smaller neighborhood of need be, we may assume (because ψ_i is upper semicontinuous and finite valued) that $\psi_i(z) \leq \psi_i(x)$ for all $z \in U$. We conclude for any F_j we have

$$\bigcap_{z \in F_j} P_i(z) = \bigcap_{z \in F_j} B_i^{\mathcal{X}}(z, \psi_i(z)) \supseteq \bigcap_{z \in F_j} B_i^{\mathcal{X}}(z, \psi_i(x)) \neq \emptyset\,.$$

Since X is compact, there is a finite set of pairs $(x^1, U^1), \dots, (x^k, U^k)$ such that $U^1 \cup \ldots \cup U^k = X$ and, for each j, U^j is a neighborhood of x^j satisfying this description. Thus, for each i, X has a finite closed cover $G_1^i, \dots, G_{m_i}^i$ with $\bigcap_{z \in G_h^i} P_i(z) \neq \emptyset$ for each h. If we let $G_1, \dots, G_m$ be the nonempty intersections of the form $G_{j_1}^1 \cap \ldots, \cap G_{j_n}^N$, then $\bigcap_{z \in G_h} P(z) \neq \emptyset$ for each h.

We see that the hypotheses of Lemma 5.4 are satisfied by $P\colon X \twoheadrightarrow X$. This implies that there is $x^* \in X$ satisfying $x^* \in \operatorname{con} P(x^*)$. In turn, this implies that $x_i^* \in C_i^{\mathcal{X}}(x^*, \psi_i(x^*))$ for all i. But the game is $\mathcal{X}$-secure at x^* for $\psi(x^*)$, so for some i we have $x_i^* \notin C_i^{\mathcal{X}}(x^*, \psi_i(x^*))$. This is a contradiction, so the game has an equilibrium. ∎

Bibliography

J.L. Kelley. *General Topology*. Springer-Verlag, New York, 1955.

P.K. Monteiro and F. Page. Uniform payoff security and Nash equilibrium in compact games. *Journal of Economic Theory*, 134(1):566–575, 2007.

P.K. Monteiro and F. Page. Catalog competition and Nash equilibrium in nonlinear pricing games. *Economic Theory*, 34(3):503–524, 2008.

K. Nishimura and J. Friedman. Existence of Nash equilibrium in n person games without quasiconcavity. *International Economic Review*, 22(3):637–648, 1981.

W. Novshek. On the existence of Cournot equilibrium. *Review of Economic Studies*, 52:85–98, 1985.

P.J. Reny. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, 67(5):1029–1056, 1999.

J. Roberts and H. Sonnenschein. On the existence of Cournot equilibrium without concave profit functions. *Journal of Economic Theory*, 13(1):112–117, 1976.

H.H. Schaefer. *Topological Vector Spaces*. Springer-Verlag, 1971.